

Cooperative Games

An Introduction to Game Theory – Part II

Lars-Åke Lindahl



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COOPERATIVE GAMES
AN INTRODUCTION TO GAME
THEORY – PART II

Cooperative Games: An Introduction to Game Theory – Part II

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Preface

Non-cooperative game theory focuses on the individual players' strategies and their influence on payoffs and tries to predict what strategies the players will choose. It asks how people should act. Cooperative game theory, on the other hand, abstracts from the individual players' strategies and instead focuses on the coalitions players may form. Cooperative games can be seen as a competition between coalitions of players, rather than between individual players.

The big advantage of the cooperative theory is that it does not need a precisely defined structure for the actual game. It is enough to say what each coalition can achieve; you need not say how. The basic assumption is that the grand coalition, that is the group consisting of all players, will form, and the main question is how to allocate in some fair way the payoff of the grand coalition among the players. The answer to this question is a solution concept which, roughly speaking, is a vector that represents the allocation to each player. Different solution concepts based on different notions of fairness have been proposed, and we will study three of them in this volume, namely the core, the nucleolus and the Shapley solution.

This Part II of An Introduction to Game Theory is essentially independent of Part I on Non-Cooperative Games and references to results in Part I only appear in a few places. Part II can therefore very well be read and studied before Part I.

Lars-Åke Lindahl

Chapter 10

Coalitional Games

In many situations with several people involved, the total result of their efforts is best when everyone cooperates, and conflicts of interest only arise when the collective profit is to be distributed among the individuals. In this chapter we will model such situations as games. We assume that all players use the same unit when measuring their utilities, and that the total payoff that a group can achieve through cooperation can be distributed freely among the members of the group. The main problem is to determine how the payoff should be distributed, and we will study some different solutions.

10.1 Definitions

Definition 10.1.1 A *coalitional game* $\langle N, v \rangle$ (with *transferable utility*) consists of a finite set N of players and a real-valued function v , defined on the set \mathcal{C} of all nonempty subsets of N .

The sets in \mathcal{C} , i.e. the nonempty subsets of N , are called *coalitions*, and the function value $v(S)$ of a coalition S is called the coalition's *value*. The function value $v(N)$ is the game's *total value*. The number of members of a coalition S will be denoted by $|S|$.

In order for some definitions and induction proofs to work, we sometimes need to extend the value function v so that it is also defined for the empty set \emptyset , which we always do by defining $v(\emptyset) = 0$.

The entire set N is a coalition, the so called *grand coalition*. The players in N will in general be numbered $1, 2, \dots, n$, where $n = |N|$. The number of coalitions, i.e. nonempty subsets of N , is $2^n - 1$.

The intuitive interpretation of the value $v(S)$ is that it is the total utility, wealth or power, that players in the S -coalition can achieve by cooperating,

regardless of what players outside the coalition do. In particular, $v(N)$ is the total utility obtained when all players cooperate.

In order to derive interesting results, we need to constrain the class of games through appropriate restrictions on the value function. In the following definitions, two important subclasses are defined.

Definition 10.1.2 A coalitional game $\langle N, v \rangle$ is called *cohesive* if

$$v(S_1) + v(S_2) + \cdots + v(S_k) \leq v(N)$$

for every partition $\{S_1, S_2, \dots, S_k\}$ of N .

Recall, that a partition of a set M is a family of pairwise disjoint subsets whose union is equal to M .

The players of a cohesive game profit from keeping together, which explains the term "cohesive". It is impossible to create greater total utility by splitting the grand coalition into a number of subgroups. In particular,

$$v(S) + v(N \setminus S) \leq v(N)$$

for all coalitions S , and

$$\sum_{i \in N} v(\{i\}) \leq v(N).$$

Cohesiveness will be a natural prerequisite for many results in this chapter.

Superadditive games form an important subclass of cohesive games and are defined as follows.

Definition 10.1.3 A coalitional game is called *superadditive* if

$$v(S) + v(T) \leq v(S \cup T)$$

for all pairwise disjoint coalitions S and T .

Superadditivity implies cohesiveness, but the converse is not true.

EXAMPLE 10.1.1 The coalitional game $\langle N, v \rangle$ with $N = \{1, 2, 3\}$ and value function

$$v(\{1\}) = 0, \quad v(\{2\}) = v(\{3\}) = 2, \quad v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3, \\ v(\{1, 2, 3\}) = 5$$

is cohesive, because there are four partitions of N , namely $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$ and $\{\{3\}, \{1, 2\}\}$, and the left side of the inequality in Definition 10.1.2 is in turn equal to $0 + 2 + 2$, $0 + 3$, $2 + 3$ and $2 + 3$, which in all cases is $\leq 5 = v(N)$.

However, the game is not superadditive, because $v(\{2\}) + v(\{3\}) = 4 > v(\{2, 3\})$. \square

A coalitional game's value function v is a primitive concept. How and in what way the value $v(S)$ depends on the efforts of the individual players in S is irrelevant and unimportant in the context, but of course, this does not preclude us from *defining* a player's marginal contribution to a coalition as follows.

Definition 10.1.4 Let i be a player in the game $\langle N, v \rangle$. The player's *marginal contribution* to the coalition S is the quantity

$$\Delta_i(S) = v(S) - v(S \setminus \{i\}).$$

A player's marginal contribution to a coalition he does not belong to is of course always zero (since $S = S \setminus \{i\}$ if $i \notin S$). A player's marginal contribution is thus only interesting for coalitions to which he belongs.

In superadditive games, $\Delta_i(S) \geq v(\{i\})$ for all $i \in S$. A player's marginal contribution to a coalition to which he belongs is thus greater than or equal to the value that the player can achieve on his own.

EXAMPLE 10.1.2 In the game in Exemple 10.1.1, player 2's marginal contributions to the coalitions to which he belongs are $\Delta_2(\{2\}) = 2 - 0 = 2$, $\Delta_2(\{1, 2\}) = 3 - 0 = 3$, $\Delta_2(\{2, 3\}) = 3 - 2 = 1$, $\Delta_2(\{1, 2, 3\}) = 5 - 3 = 2$. \square

We will sometimes consider the following special type of players.

Definition 10.1.5 A player i in a coalitional game $\langle N, v \rangle$ is called a *null player* if $\Delta_i(S) = 0$ for all coalitions S .

In other words, a null player does not contribute positively or negatively to any coalition.

Definition 10.1.6 Two players i and j in a coalitional game $\langle N, v \rangle$ are *interchangeable* if

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

for all subsets S of N (including $S = \emptyset$) that contain neither i nor j .

EXAMPLE 10.1.3 In the game in Example 10.1.1, players 2 and 3 are interchangeable. \square

Exercises

10.1 A coalitional game $\langle N, v \rangle$ is called *trivial* if $\sum_{i \in N} v(\{i\}) = v(N)$. Prove that $v(S) = \sum_{i \in S} v(\{i\})$ for every coalition S in a trivial superadditive game. Is the same true for every trivial cohesive game?

10.2 A coalitional game is called *convex* if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all coalitions S and T .

Prove that a coalitional game is convex if and only if

$$\Delta_i(S \cup \{i\}) \geq \Delta_i(T \cup \{i\})$$

for all players i and all coalitions $S \supseteq T$ that do not contain player i . Hence, in convex games a player's incentive to join a coalition increases as the coalition grows.

10.2 Imputations

That the grand coalition will form is a basic assumption in coalitional game theory, so the main problem is the question how to distribute the grand coalition's value $v(N)$ among the individual players in a fair way. The utilities of the various players after such a distribution can be described using a vector $x = (x_1, x_2, \dots, x_n)$ in \mathbf{R}^n , where x_i is the utility of player i . We will often

consider the sum of the utilities of all players belonging to a certain coalition S , so we introduce the following compact notation for such sums:

$$x(S) = \sum_{i \in S} x_i.$$

Definition 10.2.1 Let $\Gamma = \langle N, v \rangle$ be a coalitional game with transferable utility. An n -tuple $x = (x_1, x_2, \dots, x_n)$ of real numbers is

- *collectively rational* if $x(N) = v(N)$;
- *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$.

A vector x that is both collectively and individually rational is called an *imputation*.

The set of all imputations in the game Γ is denoted by $\mathcal{I}(\Gamma)$.

Every collectively rational vector corresponds to a possible lossless distribution of the game's total value $v(N)$ to the individual players, and when later on we discuss various solutions to the problem of distributing the game's total value to the individual players, we will always require the solution to be collectively rational.

Individual rationality is also a natural condition for all players to accept the distribution x , because if $x_i < v(\{i\})$ for any player i then there is no incentive for that player to join the grand coalition since he profits from acting alone. (We presume that the players are selfish!) If we define a "solution to the game" to be a distribution of the total value that is accepted by all players and stable in some sense to be specified, we thus have to search for it among the imputations. In the next chapter, however, we will study a solution that is not necessarily individually rational.

Proposition 10.2.1 *The set $\mathcal{I}(\Gamma)$ of imputations is convex and compact in all coalitional games Γ , and it is nonempty in all cohesive games.*

Proof. The sets

$$A = \{x \in \mathbf{R}^n \mid \sum_{i \in N} x_i = v(N)\}$$

of all collectively rational vectors and

$$B = \{x \in \mathbf{R}^n \mid x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

of all individually rational vectors are both closed and convex; the first mentioned set is a hyperplane in \mathbf{R}^n (a line if $n = 2$, a plane if $n = 3$, etc.), and the last mentioned set is obtained by translating the first orthant \mathbf{R}_+^n in \mathbf{R}^n along the vector $(v(\{1\}), \dots, v(\{n\}))$. Since $\mathcal{I}(\Gamma) = A \cap B$ and the intersection of closed sets is closed and the intersection of convex sets is convex, $\mathcal{I}(\Gamma)$ is a closed and convex set.

The set $\mathcal{I}(\Gamma)$ is also bounded, because

$$v(\{i\}) \leq x_i = v(N) - \sum_{k \neq i} x_k \leq v(N) - \sum_{k \neq i} v(\{k\})$$

for all $x \in \mathcal{I}(\Gamma)$ and each $i \in N$.

In a cohesive game, $\sum_{i=1}^n v(\{i\}) \leq v(N)$, so we obtain an individually and collectively rational distribution x by for example defining $x_i = v(\{i\})$ for all players i except player 1, who obtains the remaining $v(N) - \sum_{i=2}^n v(\{i\})$ ($\geq v(\{1\})$) utility units. The set $\mathcal{I}(\Gamma)$ is thus nonempty. \square

The imputation set $\mathcal{I}(\Gamma)$ is by definition the solution set to a finite number of linear equalities and inequalities, and such solution sets are called polyhedra. The imputation set of a cohesive game is in other words a nonempty bounded polyhedron.

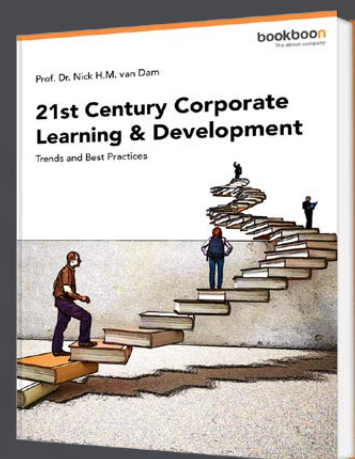
Exercises

10.3 Compute the imputation set of the game in Example 10.1.1.

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10.4 Illustrate in a diagram the imputation set of a three-person coalitional game $\langle \{1, 2, 3\}, v \rangle$ in which

- a) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$,
 $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 2$, $v(\{1, 2, 3\}) = 3$
- b) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$,
 $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 1$.

10.3 Examples

We illustrate the concepts introduced so far with some examples.

EXAMPLE 10.3.1 (Who will have the painting?) Person 1 owns a painting that he values to \$1000. Person 2 values the same painting to \$2000, and for person 3 the painting is worth \$3000. Person 1 is willing to give the painting to one of the other two persons against receiving a sum of money in return.

The situation can be modeled as a coalitional game with transferable utility and with the three persons as players, i.e. with $N = \{1, 2, 3\}$. The value $v(S)$ of a coalition that contains the owner of the painting is defined to be the value of the painting for the person in the coalition who values the painting the most. For coalitions that does not contain the owner of the painting, we define the value to be 0. In the first mentioned case it is possible to change the owner of the painting within the coalition so that the one who values the painting the most gets it; in the last mentioned case no member of the coalition owns anything of value. The value function, with \$1000 as unit, is thus defined as follows: $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$, $v(\{1, 2\}) = 2$, $v(\{1, 3\}) = v(\{1, 2, 3\}) = 3$.

It is easy to verify that the game is superadditive. The imputation set

$$\{x \in \mathbf{R}^3 \mid x_1 + x_2 + x_3 = 3, x_1 \geq 1, x_2 \geq 0, x_3 \geq 0\}$$

is a triangle in \mathbf{R}^3 with vertices at the points $(1, 2, 0)$, $(1, 0, 2)$ and $(3, 0, 0)$. By eliminating $x_3 = 3 - x_1 - x_2$ and using the inequality $x_3 \geq 0$, we obtain the projection in the x_1x_2 -plane of the imputation set; it is given by the inequalities $x_1 \geq 1$, $x_2 \geq 0$, $x_1 + x_2 \leq 3$. See Figure 10.1. \square

EXAMPLE 10.3.2 (A production model.) A person owns a factory that employs n workers. The workers can not produce anything of value without access to the machines at the factory, but at the factory every group of m workers can produce products with a value of $f(m)$, where $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is an increasing, concave function and $f(0) = 0$.

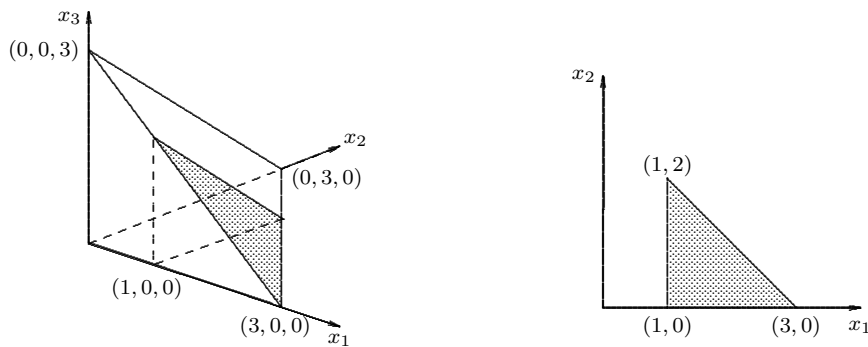


Figure 10.1. The left panel shows the imputation set of the game in Example 10.3.1, and the right panel shows its projection in the x_1x_2 -plane.

The concavity of the function f means that

$$f(k + 1) - f(k) \leq f(k) - f(k - 1).$$

The difference $f(k) - f(k - 1)$ is called, using economic terminology, the workers' "marginal product" when the work force consists of k men, and the concavity assumption therefore means that the marginal product decreases when more workers are employed.

We can model the production at the factory as a coalitional game with $n + 1$ players, with the factory owner as player 0 and the n workers as players $1, 2, \dots, n$. The value $v(S)$ of an arbitrary coalition S of workers and factory owner is defined to be value of the products they can produce, which means that

$$v(S) = \begin{cases} 0 & \text{if } 0 \notin S, \\ f(|S| - 1) & \text{if } 0 \in S. \end{cases}$$

The game is superadditive, because if S_1 and S_2 are two disjoint coalitions and the factory owner belongs to one of them, S_1 say, then

$$v(S_1) + v(S_2) = f(|S_1| - 1) + 0 \leq f(|S_1 \cup S_2| - 1) = v(S_1 \cup S_2),$$

since the function f is increasing, whereas

$$v(S_1) + v(S_2) = 0 + 0 = v(S_1 \cup S_2),$$

if the factory owner does not belong to any of the two coalitions.

The imputation set consists of all vectors $x = (x_0, x_1, \dots, x_n)$ such that $x_i \geq 0$ for all i and $\sum_{i=0}^n x_i = f(n)$. \square

Our next example is about various voting rules.

EXAMPLE 10.3.3 A group of n persons has one unit at its disposal, and the group intends to vote as to whether a certain subgroup S should share the unit. Here are some possible voting rules.

Simple majority: The coalition S shares the unit if $|S| > n/2$. This is a game with $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ if $|S| \leq n/2$.

Unanimity: The coalition S shares the unit if and only if S is the grand coalition. Now, of course, $v(N) = 1$, while $v(S) = 0$ if $S \neq N$.

Dictatorship: Player 1 decides. The coalition S shares the unit if and only if $1 \in S$. Thus, $v(S) = 1$ if $1 \in S$, and $v(S) = 0$, otherwise.

The imputation set consists of all vectors $x \in \mathbf{R}_+^n$ such that $\sum_{i \in N} x_i = 1$ in the majority and the unanimity cases, and of just one point $(1, 0, \dots, 0)$ if player 1 is a dictator. \square

The games in Example 10.3.3 are examples of so called simple games.

Definition 10.3.1 A coalitional game $\langle N, v \rangle$ is called *simple* if the game is superadditive and the value function v only assumes the values 0 and 1.

A coalition S in a simple game is called *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$.



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EXAMPLE 10.3.4 (Transformation of a strategic game to a coalitional game)
We can create a coalitional game $\langle N, v \rangle$ from a strategic n -person game $G = \langle N, (A_i), (u_i) \rangle$ by allowing players to cooperate, defining the value $v(S)$ of the cooperation within a group S of players as the sum of the payoffs that the members of the group would obtain in G if they were acting in an optimal way against the players outside the group, in a sense to be specified below.

For each set S of players, except $S = N$, we define $\bar{S} = N \setminus S$, and consider the following two-person zero-sum game between the row player S and the column player \bar{S} :

- The row player's action set is the product set $\prod_{i \in S} A_i$ of the action sets of the players in S .
- The column player's action set is the product set $\prod_{i \in \bar{S}} A_i$ of the action sets of the players in \bar{S} .
- If the row player chooses the action $(a_i)_{i \in S}$ and the column player chooses the action $(a_i)_{i \in \bar{S}}$, the row player gets

$$\sum_{i \in S} u_i(a_1, \dots, a_n)$$

as payoff from the column player.

This zero-sum game has a value, which by definition is equal to the row player's mixed safety level, and we define the value $v(S)$ of the coalition S as this value.

The value $v(S)$ is thus equal to the total expected value that the coalition S can be guaranteed even if the members in \bar{S} would do their best to keep this value as low as possible.

The value $v(N)$ of the grand coalition N is finally defined as the maximum value of the sum $\sum_{i \in N} u_i(a_1, \dots, a_n)$ over all outcomes (a_1, \dots, a_n) .

In this way we obtain a superadditive coalitional game $\langle N, v \rangle$, because if S and T are two disjoint coalitions, s is a mixed strategy for S that guarantees coalition S a payoff of $v(S)$, and t is a mixed strategy for T that guarantees coalition T a payoff of $v(T)$, then the mixed product strategy $s \times t$ guarantees the coalition $S \cup T$ a payoff of at least $v(S) + v(T)$. Of course, there may be mixed strategies with an even bigger payoff, but this proves that $v(S \cup T) \geq v(S) + v(T)$.

Let us now consider a specific example. Table 10.1 describes the payoffs in a strategic three-person game, where each player has two possible actions, 1 and 2.

We will determine the value function v in the corresponding coalitional game. The values $v(\{1\})$, $v(\{2\})$ and $v(\{3\})$ are obtained by considering the

(a_1, a_2, a_3)	$u_1(a_1, a_2, a_3)$	$u_2(a_1, a_2, a_3)$	$u_3(a_1, a_2, a_3)$
(1, 1, 1)	0	3	3
(1, 1, 2)	1	1	4
(1, 2, 1)	2	2	2
(1, 2, 2)	1	2	1
(2, 1, 1)	1	2	3
(2, 1, 2)	2	2	1
(2, 2, 1)	1	1	2
(2, 2, 2)	0	3	3

Table 10.1. Payoff functions in a strategic three-person game.

three zero-sum games that are obtained by letting coalition $\{1\}$ play against coalition $\{2, 3\}$, coalition $\{2\}$ play against coalition $\{1, 3\}$, and coalition $\{3\}$ play against coalition $\{1, 2\}$. These games have the following payoff matrices:

		Player $\{2, 3\}$			
		(1, 1)	(1, 2)	(2, 1)	(2, 2)
Player $\{1\}$	1	0	1	2	1
	2	1	2	1	0

		Player $\{1, 3\}$			
		(1, 1)	(1, 2)	(2, 1)	(2, 2)
Player $\{2\}$	1	3	1	2	2
	2	2	2	1	3

		Player $\{1, 2\}$			
		(1, 1)	(1, 2)	(2, 1)	(2, 2)
Player $\{3\}$	1	3	2	3	2
	2	4	1	1	3

In the first game above, columns no. 2 and 3 are strictly dominated, and in the second game, columns no. 1 and 4 are strictly dominated. The games therefore have the same values as the games with the payoff matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{och} \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

i.e. 0.5 and 1.5, respectively. The matrix element in the first row and second column of the third payoff matrix is a saddle point, so the value of the third game is 2. Hence,

$$v(\{1\}) = 0.5, \quad v(\{2\}) = 1.5, \quad v(\{3\}) = 2.$$

The values $v(\{1, 2\})$, $v(\{1, 3\})$ and $v(\{2, 3\})$ are obtained as values of the three zero-sum games in which coalition $\{1, 2\}$ plays against $\{3\}$ with $v_1 + v_2$ as payoff function, $\{1, 3\}$ plays against $\{2\}$ with $v_1 + v_3$ as payoff function, and $\{2, 3\}$ plays against $\{1\}$ with $v_2 + v_3$ as payoff function. This gives us the following three games:

	1	2
{1, 1}	3	2
{1, 2}	4	3
{2, 1}	3	4
{2, 2}	2	3

	1	2
{1, 1}	3	4
{1, 2}	5	2
{2, 1}	4	3
{2, 2}	3	3

	1	2
{1, 1}	6	5
{1, 2}	5	3
{2, 1}	4	3
{2, 2}	3	6

The rows 1 and 4 are strictly dominated in the first of these games, row 4 is weakly dominated and row 3 is equal to half the sum of rows 1 and 2 in the second game, and rows 2 and 3 are strictly dominated in the last game.

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The values of the games are therefore the same as the values of the games with payoff matrices

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} \quad \text{och} \quad \begin{bmatrix} 6 & 5 \\ 3 & 6 \end{bmatrix},$$

i.e. 3.5, 3.5 and 5.25. Therefore,

$$v(\{1, 2\}) = 3.5, \quad v(\{1, 3\}) = 3.5, \quad v(\{2, 3\}) = 5.25.$$

The grand coalition's value is equal to the maximum value of $v_1 + v_2 + v_3$, which is 6 and is assumed by for example $a = (1, 1, 1)$. This means that

$$v(\{1, 2, 3\}) = 6. \quad \square$$

Exercises

10.5 Prove for simple games that every sub-coalition of a losing coalition is losing, and that every coalition that contains a winning coalition is winning.

10.6 The payoffs in a strategic three-person game are given by the following table. Transform the game to a coalitional game.

(a_1, a_2, a_3)	$u_1(a_1, a_2, a_3)$	$u_2(a_1, a_2, a_3)$	$u_3(a_1, a_2, a_3)$
(1, 1, 1)	1	2	0
(1, 1, 2)	2	1	3
(1, 2, 1)	1	2	1
(1, 2, 2)	1	3	4
(2, 1, 1)	4	2	1
(2, 1, 2)	3	3	3
(2, 2, 1)	2	3	1
(2, 2, 2)	4	2	1

10.4 The core

The Nash equilibrium is a stable outcome of a strategic game since no player profits from a unilateral change of action. The core plays a similar role for coalitional games – an imputation belongs to the core if no group of players profits from breaking away from the grand coalition to form a coalition that can distribute its own value among its members.

Definition 10.4.1 Let $\Gamma = \langle N, v \rangle$ be a coalitional game with transferable utility. The *core* of the game $\mathcal{K}(\Gamma)$ consists of all imputations x such that

$$x(S) = \sum_{i \in S} x_i \geq v(S)$$

for all coalitions S .

If an imputation x does not belong to the core, then there is a coalition S such that $x(S) < v(S)$, and, by increasing x_i somewhat for each player i in S , it is possible to obtain a vector y such that $y_i > x_i$ for all $i \in S$ and $y(S) = v(S)$. This means that all players in S would profit from leaving the grand coalition, sharing what they can achieve on their own. Therefore, distributions of the game's total value that do not belong to the core are unstable; there is always a coalition that is dissatisfied and has something to gain from breaking away.

Proposition 10.4.1 *The core of a coalitional game is a closed convex set.*

Proof. The core is by definition the solution set to a system of linear equalities and inequalities, namely the equality

$$\sum_{i \in N} x_i = v(N)$$

and the inequalities

$$\sum_{i \in S} x_i \geq v(S)$$

for all coalitions S . The core is thus equal to the intersection of a number of closed halfspaces in \mathbf{R}^n , i.e. a convex polyhedron. \square

Proposition 10.4.2 *Coalitional games with nonempty core are cohesive.*

Proof. Let \hat{x} be an imputation in the core of the game $\langle N, v \rangle$. Then, for each partition $\{S_1, S_2, \dots, S_k\}$ of the set N of players,

$$v(S_1) + v(S_2) + \dots + v(S_k) \leq \hat{x}(S_1) + \hat{x}(S_2) + \dots + \hat{x}(S_k) = \hat{x}(N) = v(N),$$

which proves that the game is cohesive. \square

The core of a non-cohesive game is therefore empty, but cohesive games may also have an empty core, as the following example shows.

EXAMPLE 10.4.1 The coalitional game $\langle \{1, 2, 3\}, v \rangle$, where

$$\begin{aligned} v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0, \\ v(\{1, 2\}) &= v(\{1, 3\}) = v(\{2, 3\}) = a, \quad v(\{1, 2, 3\}) = 1, \end{aligned}$$

is cohesive if $0 \leq a \leq 1$, but the core is empty if $a > \frac{2}{3}$.

Because if x is an imputation belonging to the core, then $x_1 + x_2 \geq a$, $x_1 + x_3 \geq a$ and $x_2 + x_3 \geq a$, and by adding these inequalities and utilizing the equality $x_1 + x_2 + x_3 = 1$, we obtain $2 = x_1 + x_2 + x_1 + x_3 + x_2 + x_3 \geq 3a$. Hence, $a \leq \frac{2}{3}$ is a necessary condition for nonempty core. \square

EXAMPLE 10.4.2 The core of the game "Who will have the painting?" in Example 10.3.1, with values $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = 0$, $v(\{2, 3\}) = 0$, $v(\{1, 2\}) = 2$ and $v(\{1, 3\}) = v(\{1, 2, 3\}) = 3$, consists of the solutions to the system

$$\begin{cases} x_1 \geq 1, & x_2 \geq 0, & x_3 \geq 0 \\ x_1 + x_2 \geq 2, & x_1 + x_3 \geq 3, & x_2 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = 3. \end{cases}$$

We use the last equation to eliminate $x_3 = 3 - x_1 - x_2$ from the system and obtain the following system of inequalities

$$x_1 \geq 1, \quad x_2 \geq 0, \quad 3 - x_1 - x_2 \geq 0, \quad x_1 + x_2 \geq 2, \quad 3 - x_2 \geq 3, \quad 3 - x_1 \geq 0,$$

which has $x_2 = 0$ and $2 \leq x_1 \leq 3$ as solution. The core consists, in other words, of all imputations of the form $x = (t, 0, 3 - t)$, where $2 \leq t \leq 3$.

We conclude that person 1 should give the painting to person 3 against receiving \$ t (thousand), where t is a number between 2 and 3. Person 2 does not get anything but still plays an important role. Thanks to him person 1 may require to receive at least \$2 (thousand) for the painting. \square

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EXAMPLE 10.4.3 We will now determine the core of the production model in Example 10.3.2: a factory owner (player 0), n workers (players 1, 2, ..., n) and values

$$v(S) = \begin{cases} 0 & \text{if } 0 \notin S, \\ f(|S| - 1) & \text{if } 0 \in S. \end{cases}$$

The function f is assumed to be increasing and concave, and $f(0) = 0$.

By definition, a payoff vector $x = (x_0, x_1, \dots, x_n)$ belongs to the core if and only if the following inequalities and equality are met for all subsets A of the set $N = \{1, 2, \dots, n\}$ of workers:

$$\begin{aligned} \text{(i)} \quad & \sum_{i \in A} x_i \geq 0 \\ \text{(ii)} \quad & x_0 + \sum_{i \in A} x_i \geq f(|A|) \\ \text{(iii)} \quad & x_0 + \sum_{i \in N} x_i = f(|N|). \end{aligned}$$

The inequalities in (i) are satisfied if and only if $x_i \geq 0$ for all workers i , and by combining equality (iii) with inequality (ii) in the case $A = N \setminus \{k\}$, i.e. when A consists of all workers but one and consequently $|A| = n - 1$, we obtain the following inequality

$$f(n) = f(|N|) = x_k + x_0 + \sum_{i \in A} x_i \geq x_k + f(|A|) = x_k + f(n - 1)$$

with

$$\text{(iv)} \quad 0 \leq x_k \leq f(n) - f(n - 1) \quad \text{for all } k \in N$$

as conclusion.

The inequalities in (iv) give a necessary condition for the vector x to belong to the core. We will now show, conversely, that (iv) combined with equality (iii) is also a sufficient condition.

Therefore, suppose (iv) holds and define x_0 using equality (iii). The conditions (i) and (iii) are then fulfilled, and condition (ii) is fulfilled in the case $A = N$, so it only remains to prove that (ii) is also fulfilled when $|A| \leq n - 1$.

By assumption, f is a concave function, and this implies that

$$f(n) - f(n - 1) \leq f(k) - f(k - 1)$$

for all $k \in N$, and by adding these inequalities for $k = m + 1, m + 2, \dots, n$ we obtain the following inequality

$$(n - m)(f(n) - f(n - 1)) \leq \sum_{k=m+1}^n (f(k) - f(k - 1)) = f(n) - f(m)$$

for integers m such that $0 \leq m \leq n - 1$,

Let now A be an arbitrary proper subset of workers so that $|A| \leq n - 1$. By utilizing the previous inequality with $m = |A|$ and assumption (iv), we get the following inequality, which shows that (ii) holds.

$$\begin{aligned} x_0 + \sum_{j \in A} x_j &= x_0 + \sum_{j \in N} x_j - \sum_{j \in N \setminus A} x_j = f(n) - \sum_{j \in N \setminus A} x_j \\ &\geq f(n) - \sum_{j \in N \setminus A} (f(n) - f(n - 1)) \\ &= f(n) - (n - |A|)(f(n) - f(n - 1)) \geq f(n) - (f(n) - f(|A|)) \\ &= f(|A|). \end{aligned}$$

The core of the game thus consists of all imputations (x_0, x_1, \dots, x_n) such that $0 \leq x_k \leq f(n) - f(n - 1)$ for all workers k . The utility that a worker obtains from an imputation in the core is in other words at most equal to the marginal product of the last hired worker.

□

Exercises

- 10.7 Determine the core of a coalitional game $\langle \{1, 2, 3\}, v \rangle$ in which $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(\{3\}) = -1$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 2$, $v(\{1, 2, 3\}) = 4$.
- 10.8 Prove that the core of the game in Example 10.4.1 is nonempty if $0 \leq a \leq \frac{2}{3}$ and determine the core.
- 10.9 Let x be an imputation in the core of a coalitional game. Prove that $x_i = 0$ for all null players i .
- 10.10 Suppose that x belongs to the core of a coalitional game and that players i and j are interchangeable. Prove that the imputation that is obtained from x by interchanging x_i and x_j also belongs to the core.
- 10.11 Determine the core of the voting games in Example 10.3.3 in the case of
a) simple majority b) unanimity c) dictatorship.

- 10.12 A player i in a simple game $\langle N, v \rangle$ is a *veto player* if $v(N \setminus \{i\}) = 0$. Prove that the existence of at least one veto player is a necessary and sufficient condition for the core of a simple game to be nonempty. Describe the core in case of non-emptiness.
- 10.13 A coalitional game $\langle N, v \rangle$ is called *symmetric* if the value $v(S)$ only depends on the number of members of the coalition S , i.e. if $v(S) = f(|S|)$ for some function f . (Each player is obviously interchangeable with every other player in a symmetric game.)
- Determine the values of a that make the core of a symmetric three-person game nonempty, if $f(1) = 0$, $f(2) = a$ and $f(3) = 3$.
 - Determine the values of a and b that make the core of a symmetric four-person game nonempty, if $f(1) = 0$, $f(2) = a$, $f(3) = b$ and $f(4) = 4$.
 - The core of a symmetric n -person game with $f(n) = n$ is nonempty. Determine the possible values of $f(k)$ for $k = 1, 2, \dots, n - 1$.
- 10.14 The set N in the game $\langle N, v \rangle$ consists of two types of players that form two subsets P and Q , i.e. $N = P \cup Q$ and $P \cap Q = \emptyset$. The value function is defined by

$$v(S) = \min\{|S \cap P|, |S \cap Q|\}.$$

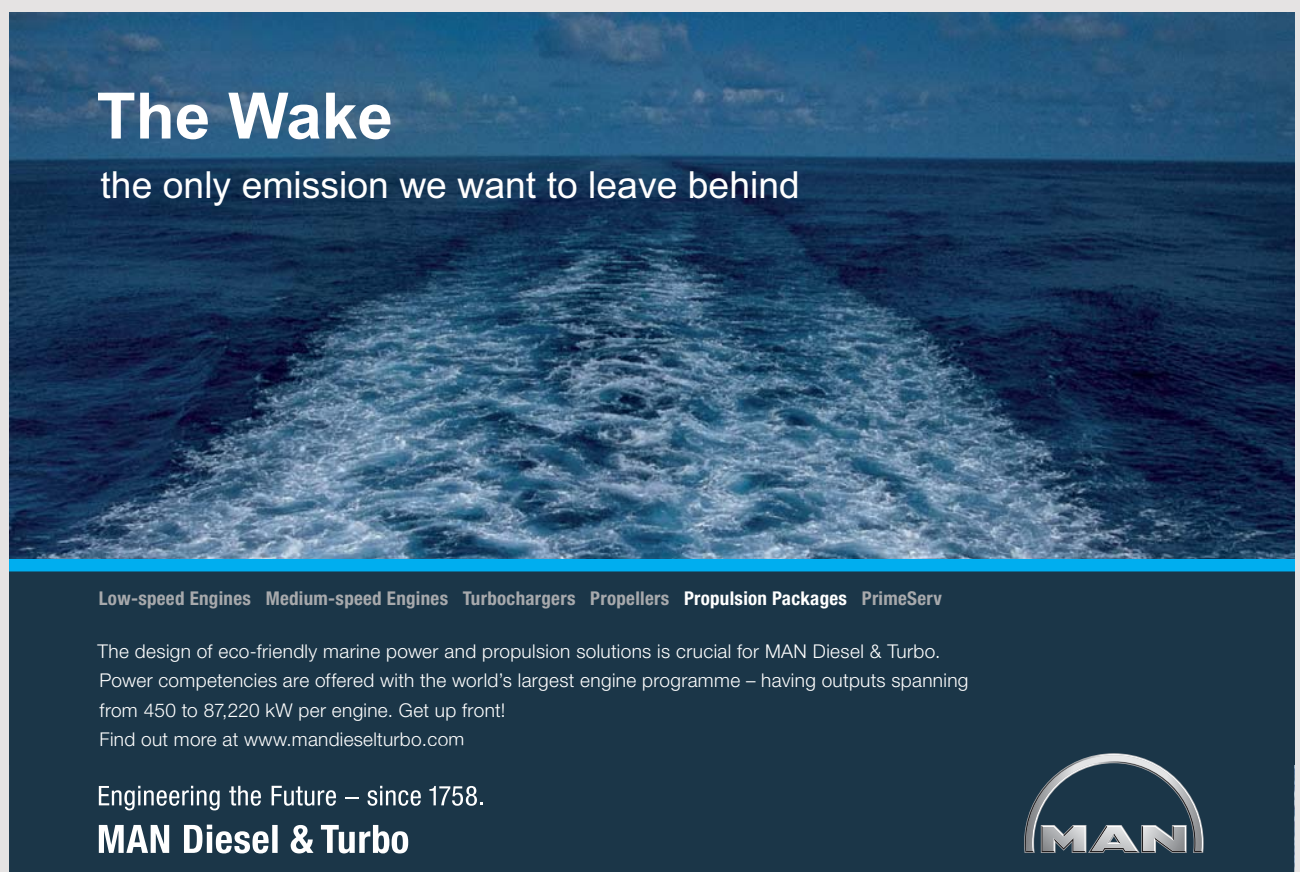
(The game goes under the name *the glove market* because of the following

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interpretation. Each player in P owns a left-hand glove and each player in Q owns a right-hand glove. If j players in P and k players in Q join together, then they possess $\min\{j, k\}$ pairs of gloves, each pair being worth 1. Odd gloves are worthless.)

Determine the core of the game if

- a) $|P| = |Q| = 2$ b) $|P| = 2$ and $|Q| = 3$ c) $|P|$ and $|Q|$ are arbitrary.

10.5 Games with nonempty core

The core of a coalitional game is defined as the solution set to a system of linear inequalities. The problem of solving this system can be rewritten as a linear minimization problem, and by considering the dual maximization problem, one can derive a necessary and sufficient condition for the core to be nonempty. The condition, a kind of convexity condition, was first given and proved by Olga Bondareva and somewhat later and independently by Lloyd Shapley. In order to formulate the condition, we first need a definition.

Definition 10.5.1 Let \mathcal{C} denote the set of all coalitions in the coalitional game $\langle N, v \rangle$, and let $\mathcal{C}(i)$ denote the set of all coalitions that contain player i . A collection $(\lambda_S)_{S \in \mathcal{C}}$ of non-negative real numbers λ_S is called *balanced* if

$$(1) \quad \sum_{S \in \mathcal{C}(i)} \lambda_S = 1 \quad \text{for all players } i.$$

The game $\langle N, v \rangle$ is called *balanced* if

$$\sum_{S \in \mathcal{C}} v(S) \lambda_S \leq v(N)$$

for all balanced weight collections $(\lambda_S)_{S \in \mathcal{C}}$.

EXAMPLE 10.5.1 We get a balanced weight collection in a game with four players by defining $\lambda_{\{1,2\}} = \lambda_{\{1,3\}} = \lambda_{\{1,4\}} = \frac{1}{3}$, $\lambda_{\{2,3,4\}} = \frac{2}{3}$ and $\lambda_S = 0$ for the remaining coalitions S . \square

EXAMPLE 10.5.2 The three-person game of Example 10.4.1, that is the game with values $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = a$ and $v(\{1,2,3\}) = 1$, is not balanced if $a > \frac{2}{3}$, because by defining

$$\lambda_{\{1\}} = \lambda_{\{2\}} = \lambda_{\{3\}} = 0, \quad \lambda_{\{1,2\}} = \lambda_{\{1,3\}} = \lambda_{\{2,3\}} = \frac{1}{2}, \quad \lambda_{\{1,2,3\}} = 0$$

we obtain a balanced weight collection with

$$\sum_{S \in \mathcal{C}} v(S)\lambda_S = 3 \cdot \frac{1}{2} \cdot a = \frac{3}{2}a > v(\{1, 2, 3\})$$

for $a > \frac{2}{3}$. □

Here follows an intuitive interpretation of the notion of a balanced game. Suppose that the players of the game $\langle N, v \rangle$ have to perform certain tasks. If a coalition S of players works with these during λ_S time units, the coalition gets the payoff $\lambda_S v(S)$. Each player has one unit of time at his disposal, which he must distribute among the coalitions of which he is a member. This is obviously possible for all players if and only if the weights λ_S form a balanced collection. The game itself is balanced if there is no way to allocate the time so that the coalitions get a total payoff of more than $v(N)$.

We now proceed by translating the two problems of determining whether a game is balanced and whether the core is nonempty into two dual linear programming problems.

Let $\langle N, v \rangle$ be an arbitrary coalitional game, and denote by Λ the set of all balanced weight collections. Note that for the particular balanced weight collection which is obtained by setting $\lambda_N = 1$ and $\lambda_S = 0$ for all coalitions $S \neq N$, we have $\sum_{S \in \mathcal{C}} v(S)\lambda_S = v(N)$. A coalitional game $\langle N, v \rangle$ is therefore, according to Definition 10.5.1, balanced if and only if the maximization problem

$$(2) \quad \text{Maximize } \sum_{S \in \mathcal{C}} v(S)\lambda_S \text{ as } (\lambda_S)_{S \in \mathcal{C}} \in \Lambda$$

has $v(N)$ as its maximum value.

Problem (2) is a linear programming problem, because the objective function $\sum_{S \in \mathcal{C}} v(S)\lambda_S$ is linear in the variables λ_S , and the constraint set Λ is the solution set to a system consisting of the n linear equations (1) and the $2^n - 1$ linear inequalities $\lambda_S \geq 0$.

In order to write the maximization problem (2) in matrix form, let us number the coalitions in an arbitrary way as $S_1, S_2, \dots, S_{2^n-1}$ and introduce the following notation:

- λ is a column matrix of dimension $2^n - 1$ with matrix elements $\lambda_i = \lambda_{S_i}$;
- \mathbf{v} is a column matrix of dimension $2^n - 1$ with matrix elements $v_i = v(S_i)$;
- $\mathbf{1}$ is a column matrix of dimension n with the number 1 as matrix element everywhere;

- $\mathbf{A} = [a_{ij}]$ is a matrix with n rows, $2^n - 1$ columns and matrix elements

$$a_{ij} = \begin{cases} 1 & \text{if } i \in S_j, \\ 0 & \text{if } i \notin S_j. \end{cases}$$

Using these matrices we now have

$$\sum_{S \in \mathcal{C}} v(S) \lambda_S = \mathbf{v}^t \lambda,$$

where t denotes transposition, and

$$\sum_{S \in \mathcal{C}(i)} \lambda_S = \sum_{j=1}^{2^n-1} a_{ij} \lambda_{S_j},$$

which means that the system (1) can be written as

$$\mathbf{A} \lambda = \mathbf{1}.$$

The maximization problem (2) thus assumes the following canonical form:

$$(2') \quad \begin{aligned} & \text{Maximize } \mathbf{v}^t \lambda \text{ as} \\ & \begin{cases} \mathbf{A} \lambda = \mathbf{1} \\ \lambda \geq 0. \end{cases} \end{aligned}$$

EXAMPLE 10.5.3 If $\langle N, v \rangle$ is a coalitional game with three players and the coalitions are $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$, $S_6 = \{2, 3\}$ and $S_7 = \{1, 2, 3\}$, then \mathbf{A} is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The game is balanced if and only if the optimal value of the maximization problem (2') is equal to $v(S_7)$. \square

Thus, we can determine whether a game is balanced or not by solving the linear programming problem (2'). We can also determine if the game has a nonempty kernel and, if so, determine the kernel by solving another linear programming problem. The core consists of all vectors in the set

$$X = \{x \in \mathbf{R}^n \mid x(S) \geq v(S) \text{ for all } S \in \mathcal{C}\}$$

that also satisfy the collective rationality condition

$$x(N) = v(N).$$

The set X is of course not empty, because it contains all vectors $x \in \mathbf{R}^n$ whose all coordinates are large enough. If the kernel is empty, it is because $x(N) > v(N)$ for all vectors $x \in X$.

Therefore, a necessary and sufficient condition for the core to be nonempty is that the minimum of $x(N)$, when x varies over the set X , be equal to $v(N)$. We can thus decide whether the core is empty or not by studying the minimization problem

$$(3) \quad \text{Minimize } x(N) \text{ as } x \in X.$$

The core is empty if the minimum value is greater than $v(N)$, and the core equals the set of minimum points if the minimum value equals $v(N)$.

The minimization problem is a linear programming problem, since the objective function

$$x(N) = \sum_{i \in N} x_i = \mathbf{1}^t x$$

is linear and the constraint set X consists of all solutions to the linear system

$$x(S_j) = \sum_{i \in S_j} x_i = \sum_{i=1}^n a_{ij} x_i \geq v(S_j), \quad j = 1, 2, \dots, 2^n - 1,$$

which, using matrix notation, can be written as

$$\mathbf{A}^t x \geq \mathbf{v}.$$

The minimization problem (3) therefore has the following matrix form

$$(3') \quad \begin{array}{l} \text{Minimize } \mathbf{1}^t x \text{ as} \\ \mathbf{A}^t x \geq \mathbf{v} \end{array}$$

The two optimization problems (2') and (3') are an example of *dual* linear programming problems, and dual problems with feasible points have the same optimal value, according to an important theorem of linear programming, the *Duality Theorem*. Thus, the maximum value of the maximization problem (2') is equal to $v(N)$ if and only if the minimization problem (3') has $v(N)$ as its minimum value. This proves the following theorem.

Proposition 10.5.1 (Bondareva–Shapley's Theorem) *The core of a coalitional game $\langle N, v \rangle$ is nonempty if and only if the game is balanced.*

Robert Aumann has given an alternative proof of the Bondareva–Shapley theorem by utilizing an interesting connection between the core of a coalitional game and the Nash equilibria of an associated zero-sum game. We will describe this proof and start by noting that we may without loss of generality assume that the coalitional game's value function v is positive, i.e. that $v(S) > 0$ for all coalitions S . This follows from the following simple lemma.

Lemma 10.5.2 *Let $\langle N, v \rangle$ be a coalitional game, let $a = (a_1, a_2, \dots, a_n)$ be a vector in \mathbf{R}^n , and define a new value function w by letting*

$$w(S) = v(S) + a(S)$$

for all coalitions S . Then:

- (i) x is an imputation in the game $\langle N, w \rangle$ if and only if $x - a$ is an imputation in the game $\langle N, v \rangle$;
- (ii) x belongs to the core of $\langle N, w \rangle$ if and only if $x - a$ belongs to the core of $\langle N, v \rangle$;
- (iii) The game $\langle N, w \rangle$ is balanced if and only if the game $\langle N, v \rangle$ is balanced.

Proof. Assertions (i) and (ii) are trivial.

(iii) If $(\lambda_S)_{S \in \mathcal{C}}$ is a balanced weight collection, then

$$\begin{aligned} \sum_{S \in \mathcal{C}} \lambda_S w(S) &= \sum_{S \in \mathcal{C}} \lambda_S \left(v(S) + \sum_{i \in S} a_i \right) = \sum_{S \in \mathcal{C}} \lambda_S v(S) + \sum_{S \in \mathcal{C}} \sum_{i \in S} \lambda_S a_i \\ &= \sum_{S \in \mathcal{C}} \lambda_S v(S) + \sum_{i \in N} \sum_{S \in \mathcal{C}(i)} \lambda_S a_i = \sum_{S \in \mathcal{C}} \lambda_S v(S) + \sum_{i \in N} a_i \sum_{S \in \mathcal{C}(i)} \lambda_S \\ &= \sum_{S \in \mathcal{C}} \lambda_S v(S) + \sum_{i \in N} a_i = \sum_{S \in \mathcal{C}} \lambda_S v(S) + a(N), \end{aligned}$$

and this implies that $\sum_{S \in \mathcal{C}} \lambda_S v(S) \leq v(N)$ if and only if $\sum_{S \in \mathcal{C}} \lambda_S w(S) \leq v(N) + a(N) = w(N)$, which proves statement (iii). \square

The value function w of lemma 10.5.2 is positive if the a_i are sufficiently large positive numbers, so in order to prove the Bondareva–Shapley theorem it suffices to prove the theorem for coalitional games with a positive value function.

Assume therefore that $\langle N, v \rangle$ is a coalitional game with a positive value function v , and denote by $G(N, v)$ the two-person zero-sum game in which N is the set of actions of the row player, the set \mathcal{C} of all coalitions is the set of actions of the column player, and the payoff a_{iS} to the row player i , given

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that he has chosen the action $i \in N$ and the column player has chosen the action $S \in \mathcal{C}$, is given by

$$a_{iS} = \begin{cases} v(N)/v(S) & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Lemma 10.5.3 (i) *The value v^* of the zero-sum game $G(N, v)$ satisfies the inequality $0 < v^* \leq 1$.*

(ii) *The coalitional game $\langle N, v \rangle$ is balanced if and only if $v^* = 1$.*

Proof. (i) The mixed strategy $\bar{x} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ gives the row player a positive expected payoff $U(\bar{x}, S)$, regardless of the action S of the column player, because

$$U(\bar{x}, S) = \sum_{i \in N} a_{iS} \bar{x}_i = \frac{1}{n} \sum_{i \in S} \frac{v(N)}{v(S)} = \frac{|S|v(N)}{nv(S)} > 0.$$

This implies that the row player's mixed safety level, i.e. the value v^* of the zero-sum game, is positive.

The column player can, on the other hand, prevent the row player from getting a higher expected payoff than 1 by choosing the action N , because $a_{iN} = 1$ for all i , and this means that the row player's expected payoff $U(x, N)$ is 1 for all mixed strategies x . Hence, $v^* \leq 1$.

(ii) Suppose that the coalitional game is balanced, and let $y = (y_S)_{S \in \mathcal{C}}$ be an optimal strategy for the column player of the zero-sum game. The expected payoff to the row player is then at most v^* , regardless of the row player's choice of action i , and this means that

$$\sum_{S \in \mathcal{C}} a_{iS} y_S = \sum_{S \in \mathcal{C}(i)} \frac{v(N)y_S}{v(S)} \leq v^*,$$

that is

$$\sum_{S \in \mathcal{C}(i)} \frac{v(N)y_S}{v(S)v^*} \leq 1$$

for all $i \in N$. Now let the weight collection $(\lambda_S)_{S \in \mathcal{C}}$ be defined by

$$\lambda_S = \begin{cases} \frac{v(N)y_S}{v(S)v^*} & \text{if } S \text{ consists of more than one player,} \\ 1 - \sum_{S \in \mathcal{C}(i) \setminus \{i\}} \frac{v(N)y_S}{v(S)v^*} & \text{if } S = \{i\}, i = 1, 2, \dots, n. \end{cases}$$

The weight collection $(\lambda_S)_{S \in \mathcal{C}}$ is balanced, and

$$\lambda_{\{i\}} \geq \frac{v(N)y_{\{i\}}}{v(\{i\})v^*}$$

for all one-member coalitions $\{i\}$. Since the game is assumed to be balanced, we conclude that

$$v(N) \geq \sum_{S \in \mathcal{C}} \lambda_S v(S) \geq \sum_{S \in \mathcal{C}} \frac{v(N)y_S}{v(S)v^*} v(S) = \frac{v(N)}{v^*} \sum_{S \in \mathcal{C}} y_S = \frac{v(N)}{v^*},$$

and this gives us the inequality $v^* \geq 1$, which we combine with part (i) of the lemma to obtain $v^* = 1$.

To prove the converse, we assume that the coalitional game is unbalanced, i.e. that there exists a balanced weight collection $(\lambda_S)_{S \in \mathcal{C}}$ such that

$$A = \sum_{S \in \mathcal{C}} \lambda_S v(S) > v(N),$$

and we will prove that this implies that $v^* < 1$.

Let $y_S = \lambda_S v(S)/A$. Then $y_S \geq 0$ for all $S \in \mathcal{C}$ and $\sum_{S \in \mathcal{C}} y_S = 1$, which means that $y = (y_S)_{S \in \mathcal{C}}$ is a mixed strategy for the column player. The expected payoff $U(i, y)$ to the row player, given that he chooses the action i , is now

$$U(i, y) = \sum_{S \in \mathcal{C}} a_{iS} y_S = \sum_{S \in \mathcal{C}(i)} \frac{v(N)}{v(S)} \frac{v(S)}{A} \lambda_S = \frac{v(N)}{A} \sum_{S \in \mathcal{C}(i)} \lambda_S = \frac{v(N)}{A}$$

for each $i \in N$, and this gives us the inequality $v^* \leq v(N)/A < 1$ for the value of the game. \square

We are now able to prove the following relationship between the core of the coalitional game $\langle N, v \rangle$ and the Nash equilibria of the zero-sum game $G(N, v)$.

Proposition 10.5.4 *Let $\langle N, v \rangle$ be a coalitional game with a positive value function v , and let x be a vector in \mathbf{R}^n . The following two statements are equivalent:*

- (i) *x belongs to the core of the coalitional game.*
- (ii) *The coalitional game is balanced and $v(N)^{-1}x$ is an optimal mixed strategy for the row player of the zero-sum game $G(N, v)$.*

Proof. (i) \Rightarrow (ii): Suppose $x = (x_1, x_2, \dots, x_n)$ belongs to the core. Then $\sum_{i \in S} x_i \geq v(S)$ for all coalitions S , with equality for $S = N$, and we conclude that $z = v(N)^{-1}x$ is a mixed strategy for the row player in the zero-sum game and that

$$\sum_{i=1}^n a_{iS} z_i = \sum_{i \in S} \frac{v(N)}{v(S)} \frac{x_i}{v(N)} = \frac{1}{v(S)} \sum_{i \in S} x_i \geq 1$$

for all coalitions S . The strategy z thus gives the row player an expected payoff that is greater than or equal to 1, regardless of the column player's action, and this means that the value v^* of the zero-sum game is greater than or equal to 1. Lemma 10.5.3 now implies that $v^* = 1$ and that the coalitional game is balanced. Moreover, the strategy z is necessarily optimal since $v^* = 1$.

(ii) \Rightarrow (i): Suppose that the coalitional game is balanced, and that $z = v(N)^{-1}x$ is an optimal strategy for the row player. The value of the zero-sum game is 1, by Lemma 10.5.3, so it follows that

$$\frac{1}{v(S)} \sum_{i \in S} x_i = \sum_{i \in S} \frac{v(N)}{v(S)} z_i = \sum_{i \in N} a_{iS} z_i \geq 1$$

for all coalitions S . This means that $\sum_{i \in S} x_i \geq v(S)$ for all S and proves that the imputation x belongs to the core of the coalitional game. \square


Since there are optimal mixed strategies in any zero-sum game, it now follows immediately from Proposition 10.5.4 (and Lemma 10.5.2) that the core of a coalitional game is nonempty if and only if the game is balanced.

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Exercises

10.15 Show that the coalitional game $\langle \{1, 2, 3, 4\}, v \rangle$, where $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 4\}) = v(\{2, 3, 4\}) = 3$, $v(\{1, 2, 3, 4\}) = 4$, $v(S) = 0$ for all other coalitions S , is unbalanced and that it consequently has an empty core.

10.16 A coalitional game $\langle N, v \rangle$ is convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all coalitions S and T . (See Exercise 10.2.) Prove that the core of a convex game with n players is nonempty by showing that the payoff vector $x = (\Delta_1(S_1), \Delta_2(S_2), \dots, \Delta_n(S_n))$, where S_i is the coalition $\{1, 2, \dots, i\}$, lies in the core.

[Hint: Let $T = \{i_1, i_2, \dots, i_k\}$ be an arbitrary coalition consisting of k members in increasing number order. Prove that $x_{i_k} \geq v(T) - v(T \setminus \{i_k\})$. By iteration, $x(T) = x_{i_k} + x_{i_{k-1}} + \dots + x_{i_1} \geq v(T)$.]

10.6 The nucleolus

The core of a coalitional game can be empty, as we have seen in several examples. On the other hand, it can also be large and equal to the whole imputation set. So the core rarely provides us with a unique solution to the problem of distributing the grand coalition value to the players. In this section we will introduce a unique solution to this problem for all games that have a nonempty set of imputation, the *nucleolus*.

The nucleolus solution is an imputation that minimizes the objections that the coalitions may have to various ways of distributing the game's total value. In order to quantify these objections we first need the following definition.

Definition 10.6.1 Let $\langle N, v \rangle$ be a coalitional game with n players. For each coalition S we define a function $e_S: \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$e_S(x) = v(S) - x(S) = v(S) - \sum_{i \in S} x_i.$$

When x is an imputation, we call $e_S(x)$ the S coalition's *excess* of the imputation.

The excess $e_S(x)$ is what is left over of the coalition value $v(S)$ when all the players in the S coalition have gotten their share, and this excess can be interpreted as a measure of the coalition's discontent or objection to the imputation x . The greater the excess, the greater the reason for the coalition to oppose the imputation. It is therefore in the interest of the coalitions to try to minimize their excesses. The problem is, of course, that when a coalition's

excess is reduced, the excess must increase for some other coalition. We must therefore define how to minimize the excesses.

The grand coalition's excesses are always zero because of the requirement that imputations should be collectively rational, so when we look at the excesses of the various coalitions and minimize these, we do not need to worry about the grand coalition.

Also note that an imputation x belongs to the core if and only if the excess $e_S(x)$ is less than or equal to zero for each coalition S .

EXAMPLE 10.6.1 Consider a coalitional game with three players, coalitions $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$, $S_6 = \{2, 3\}$, $S_7 = \{1, 2, 3\}$, and with the following value function: $v(S_1) = v(S_2) = 0$, $v(S_3) = 3$, $v(S_4) = 6$, $v(S_5) = 9$, $v(S_6) = v(S_7) = 12$.

The coalitions' excesses of an imputation x are given by the following expressions:

$$e_{S_1}(x) = -x_1, \quad e_{S_2}(x) = -x_2, \quad e_{S_3}(x) = 3 - x_3, \quad e_{S_4}(x) = 6 - x_1 - x_2, \\ e_{S_5}(x) = 9 - x_1 - x_3, \quad e_{S_6}(x) = 12 - x_2 - x_3.$$

Let us now compute these excesses for some specific imputations starting with $y = (3, 3, 6)$. We obtain the following table:

Coalition S	S_1	S_2	S_3	S_4	S_5	S_6
Excess $e_S(y)$	-3	-3	-3	0	0	3

Coalition S_6 has the biggest reason to be dissatisfied because it has the greatest excess. Let us increase the allocation to player 3 at the expense of player 1 and examine the excesses of the imputation $z = (2, 3, 7)$:

Coalition S	S_1	S_2	S_3	S_4	S_5	S_6
Excess $e_S(z)$	-2	-3	-4	1	0	2

Now it got better, but the S_6 coalition still has reason to be dissatisfied with the distribution proposal because its excess is greatest of all. Let us be drastic and move player 1's two units to player 2, resulting in the imputation $v = (0, 5, 7)$ and the following excess table:

Coalition S	S_1	S_2	S_3	S_4	S_5	S_6
Excess $e_S(v)$	0	-5	-4	1	2	0

Now, coalition S_6 should be satisfied, but the S_5 coalition can raise as big objections to the distribution proposal v as S_6 could do against z .

Can we compare the two imputations z and v ? Which one is the best? Both are apparently just as bad for the coalitions with the largest excess, S_6 and S_5 , respectively. Both are also equally bad for the coalitions with

the second largest excess, in both cases S_4 , and for the coalitions with the third largest surplus, in the first case S_5 and in the second case S_1 or S_6 . However, z is better than v for the coalition with the fourth largest excess. The imputation z gives S_1 in fourth place the excess -2 , while v gives S_6 (in shared fourth place) the excess 0. On that basis, we judge in favour of z and consider z as a better (fairer) imputation than v .

We now move back one unit from player 2 to player 1 which results in the imputation $w = (1, 4, 7)$ with the following excesses:

Coalition S	S_1	S_2	S_3	S_4	S_5	S_6
Excess $e_S(w)$	-1	-4	-4	1	1	1

The maximum excess is now equal to 1, and it is assumed by the coalitions S_4 , S_5 and S_6 . Can we make the maximum excess even less? No, the only way to reduce the excess that the imputation $w = (w_1, w_2, w_3)$ gives to the S_4 coalition, is to increase $w_1 + w_2$ and then $w_1 + w_3$ or $w_2 + w_3$ has to decrease, which will cause the excesses for S_5 or S_6 to increase. It is also impossible to improve the situation for the three remaining coalitions, because the equation $e_{S_4}(w) = e_{S_5}(w) = e_{S_6}(w) = 1$ determines w uniquely.

The imputation that we have arrived at, $w = (1, 4, 7)$, can be considered to minimize the coalitions' dissatisfaction. It is the nucleolus of the game. \square

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The excess vectors $(e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x))$ of a coalitional game with n players will be compared using the *lexicographic order* $<_l$ on \mathbf{R}^p , where $p = 2^n - 2$. We recall that

$$(\alpha_1, \alpha_2, \dots, \alpha_p) <_l (\beta_1, \beta_2, \dots, \beta_p)$$

if there is an index $k \leq p$ such that $\alpha_i = \beta_i$ for all $i < k$ and $\alpha_k < \beta_k$. In other words, a vector α is less than the vector β in the lexicographic order if $\alpha_k < \beta_k$ for the first k where α_k and β_k differ. For example, $(1, 5, 6) <_l (2, 1, 4) <_l (2, 2, 1) <_l (2, 2, 2)$.

We write $\alpha \leq_l \beta$ if $\alpha <_l \beta$ or $\alpha = \beta$.

We also need to consider the *descending reordering* α^* of a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$; by this is meant the vector $\alpha^* = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_p})$ that is obtained by writing the coordinates $\alpha_1, \alpha_2, \dots, \alpha_p$ in descending order so that $\alpha_{i_1} \geq \alpha_{i_2} \geq \dots \geq \alpha_{i_p}$.

For example, $(4, 8, 3, 9)^* = (9, 8, 4, 3)$.

Definition 10.6.2 Let $\Gamma = \langle N, v \rangle$ be a coalitional game with n players, and let $(S_j)_{j=1}^p$ be the $p = 2^n - 2$ coalitions of the game except the grand coalition, listed in arbitrary order. We define an order relation \prec on the set $\mathcal{I}(\Gamma)$ of imputations by the condition

$$x \prec y \Leftrightarrow (e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x))^* <_l (e_{S_1}(y), e_{S_2}(y), \dots, e_{S_p}(y))^*,$$

and we define $x \preceq y$ to mean that $x \prec y$ or $x = y$, i.e.

$$x \preceq y \Leftrightarrow (e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x))^* \leq_l (e_{S_1}(y), e_{S_2}(y), \dots, e_{S_p}(y))^*.$$

In other words, $x \prec y$ means that the excesses of the coalitions, when arranged in decreasing order, are lexicographically less for the imputation x than for the imputation y .

Definition 10.6.3 The *nucleolus* $\mathcal{N}(\Gamma)$ of a coalitional game Γ is the set of minimal elements in the imputation set $\mathcal{I}(\Gamma)$ with respect to the \preceq ordering, i.e.

$$\mathcal{N}(\Gamma) = \{x \in \mathcal{I}(\Gamma) \mid x \preceq y \text{ for all } y \in \mathcal{I}(\Gamma)\}.$$

According to the definition, an imputation in the nucleolus

- among all imputations, has the smallest largest coalition excess;
- among all imputations with the same smallest largest coalition excess, has the smallest second largest coalition excess;
- among all imputations with the same smallest largest coalition excess and the same smallest second largest coalition excess, has the smallest third largest coalition excess;

- etc.

Thus, in order to compute the nucleolus, we have to solve a series of minimization problems.

EXAMPLE 10.6.2 We return to Example 10.6.1, where we calculated the excess vectors for the imputations $y = (3, 3, 6)$, $z = (2, 3, 7)$, $v = (0, 5, 7)$ and $w = (1, 4, 7)$. The descending reorderings $e(x)^*$ of the excess vectors $e(x) = (e_{S_1}(x), e_{S_2}(x), \dots, e_{S_6}(x))$ are

$$\begin{aligned} e(y)^* &= (3, 0, 0, -3, -3, -3), & e(z)^* &= (2, 1, 0, -2, -3, -4), \\ e(v)^* &= (2, 1, 0, 0, -4, -5), & e(w)^* &= (1, 1, 1, -1, -4, -4). \end{aligned}$$

We see that $e(w)^* <_l e(z)^* <_l e(v)^* <_l e(y)^*$, which translated to the ordering of the corresponding imputations means that $w \prec z \prec v \prec y$. In Example 10.6.1 we also showed that w is the uniquely determined minimal element of \prec , which means that the game's nucleolus is equal to $\{w\}$. \square

Nonemptiness of the imputation set $\mathcal{I}(\Gamma)$ is of course a necessary condition for the existence of a nucleolus solution, and this condition is also sufficient. We have the following interesting result.

Proposition 10.6.1 *The nucleolus of any coalitional game with nonempty set of imputations is nonempty and consists of a single imputation.*

The proof of the proposition rests on the following lemma.

Lemma 10.6.2 *Let f_1, f_2, \dots, f_p be convex functions defined on a convex set X , and let g be the function defined on X by*

$$g(x) = \max\{f_1(x), f_2(x), \dots, f_p(x)\}.$$

Assume that the function g has a minimum m , and denote by M the set of all minimum points of g . Then there is an index i such that $f_i(x) = m$ for all $x \in M$.

Proof. First note that

$$M = \{x \in X \mid f_i(x) \leq m \text{ for all } i \text{ and } f_i(x) = m \text{ for at least one } i\}.$$

Let x_0 be a minimum point of g with the property that of all minimum points, $f_i(x_0) = m$ for as few indices i as possible. Let the number of these indices be k , suppose without loss of generality that $f_i(x_0) = m$ for $i = 1, 2, \dots, k$ and that consequently $f_i(x_0) < m$ for $i = k + 1, \dots, p$. We will prove that this implies that $f_i(x) = m$ for $i = 1, 2, \dots, k$ and all minimum points x of the function g . Of course, this proves our lemma.

For that reason, let x be an arbitrary point in the set M of minimum points of g . Then $f_i(x) \leq m$ for all indices i . Assume there is some index $i \leq k$ with strict inequality, so that for instance $f_k(x) < m$. We will prove that this leads to a contradiction.

Consider for that matter the point $y = \frac{1}{2}x + \frac{1}{2}x_0$, which belongs to the convex set X . Since the functions f_i are convex,

$$f_i(y) \leq \frac{1}{2}f_i(x) + \frac{1}{2}f_i(x_0) \leq \frac{1}{2}m + \frac{1}{2}m = m$$

for all indices i , and

$$f_i(y) \leq \frac{1}{2}f_i(x) + \frac{1}{2}f_i(x_0) < \frac{1}{2}m + \frac{1}{2}m = m,$$

for all indices $i \geq k$, due to our assumptions that $f_i(x_0) < m$ for $i > k$ and $f_k(x) < m$. The point y is therefore a minimum point of g , and the number of indices i with $f_i(y) = m$ is less than k , which contradicts our minimality assumption concerning the number of indices i such that $f_i(x) = m$ at minimum points of g . This completes the proof. \square

Proof of Proposition 10.6.1. Let $\Gamma = \langle N, v \rangle$ be a coalitional game with n players and nonempty imputation set $\mathcal{I}(\Gamma)$. The imputation set is a compact convex set in \mathbf{R}^n . We will prove that the game's nucleolus is nonempty and consists of one single imputation by describing an algorithm that computes the nucleolus.

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Let S_1, S_2, \dots, S_p be the coalitions of the game except the grand coalition, and define the function g_1 on the imputation set $\mathcal{I}(\Gamma)$ by

$$g_1(x) = \max\{e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x)\},$$

where $e_{S_1}, e_{S_2}, \dots, e_{S_p}$ are the excess functions. The function g_1 is continuous and it has therefore a minimum m_1 , since $\mathcal{I}(\Gamma)$ is a compact set. The set M_1 of all minimum points of g_1 is compact and convex, more specifically a compact polyhedron, because it can be written as the intersection of the imputation set and closed half spaces in the following way:

$$\begin{aligned} M_1 &= \{x \in \mathcal{I}(\Gamma) \mid g_1(x) \leq m_1\} \\ &= \mathcal{I}(\Gamma) \cap \{x \in \mathbf{R}^n \mid e_{S_1}(x) \leq m_1\} \cap \dots \cap \{x \in \mathbf{R}^n \mid e_{S_p}(x) \leq m_1\}. \end{aligned}$$

It follows directly from the definition of the set M_1 that the largest coordinate of the excess vector $e(x) = (e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x))$ is greater for imputations x outside M_1 than for imputations x belonging to M_1 . This translates into the following property for the order relation \prec on the imputation set:

$$y \in M_1 \ \& \ x \in \mathcal{I}(\Gamma) \setminus M_1 \Rightarrow y \prec x.$$

The assumptions of Lemma 10.6.2 are now fulfilled by $f_i = e_{S_i}$, $X = \mathcal{I}(\Gamma)$, $g = g_1$, $m = m_1$ and $M = M_1$, which means that there is an index i such that $e_{S_i}(x) = m_1$ for all $x \in M_1$, and we may assume that the coalitions are numbered so that $i = 1$.

The excess function e_{S_1} is, in other words, the largest of all excess functions on the entire set M_1 , so the problem of minimizing the second largest excess function, when the largest excess function is as small as possible, is equivalent to the problem of minimizing the function

$$g_2(x) = \max\{e_{S_2}(x), \dots, e_{S_p}(x)\}$$

over the set M_1 .

The assumptions of Lemma 10.6.2 are fulfilled for this minimization problem, too. If m_2 denotes the minimum value and M_2 denotes the set of minimum points, there is therefore an index i such that $e_{S_i}(x) = m_2$ for all $x \in M_2$, and by renumbering the coalitions we may assume that $i = 2$. The set M_2 is a closed convex subset of M_1 , and

$$y \in M_2 \ \& \ x \in M_1 \setminus M_2 \Rightarrow y \prec x,$$

due to the definition of the set M_2 and of the order relation \prec .

By repeating this procedure $p = 2^{n-2} - 2$ times and renumbering the coalitions, we obtain a sequence of minimization problems, numbers $m_1, m_2,$

\dots, m_p and sets $M_0, M_1, M_2, \dots, M_p$, where $M_0 = \mathcal{I}(\Gamma)$, with the following properties for $k = 1, 2, \dots, p$:

1. m_k is the minimum value and M_k is the set of minimum points of the restriction of the function $g_k(x) = \max\{e_{S_k}(x), e_{S_{k+1}}(x), \dots, e_{S_p}(x)\}$ to the set M_{k-1} .
2. $M_k \subseteq \{x \mid e_{S_k}(x) = m_k\}$
3. $M_k \subseteq M_{k-1}$
4. $y \in M_k \ \& \ x \in M_{k-1} \setminus M_k \Rightarrow y \prec x$

We now claim that the set M_p is the game's nucleolus and that it consists of just one imputation.

Let us start with the last claim. It follows from property 3 that $M_p \subseteq M_k$ for all k , and from property 2 that $e_{S_k}(x) = m_k$ for all k and all $x \in M_p$. In particular, by considering coalitions $S_{k_i} = \{i\}$ consisting of one player i , we conclude that imputations x belonging to M_p have to satisfy the equations

$$e_{\{i\}}(x) = v(\{i\}) - x_i = m_{k_i}$$

for $i = 1, 2, \dots, n$, and this determines x uniquely.

To prove that the singleton set M_p is the nucleolus of the game, let \hat{x} be the element of M_p and let x be an arbitrary imputation different from \hat{x} . It follows from property 3 that there is an index k such that x belongs to the difference set $M_{k-1} \setminus M_k$, and \hat{x} belongs to M_k . But then $\hat{x} \prec x$, by property 4, and this shows that \hat{x} is the uniquely determined minimal element of \prec . This concludes the proof of the Proposition 10.6.1. \square

Remark 1. The problems of minimizing the functions $g_k(x)$ over the polyhedra M_{k-1} can be formulated as linear programming problems, and there are efficient algorithms for such problems, for example the simplex algorithm.

Remark 2. In the generated sequences m_1, m_2, \dots, m_p and M_1, M_2, \dots, M_p of minimum values and minimum sets of the functions g_1, g_2, \dots, g_p , either two successive minimum sets M_k and M_{k+1} are equal, or the dimension of the set M_{k+1} is one less than the dimension of the set M_k . The first mentioned case occurs if at least two excess functions e_{S_k} and $e_{S_{k+1}}$ are constant and equal to m_k on M_k , and then it is not necessary to perform any minimization to determine m_{k+1} and M_{k+1} , since $m_{k+1} = m_k$ and $M_{k+1} = M_k$. The number of necessary minimizations to determine the nucleolus solution, a point of zero dimension, starting from the $(n-1)$ -dimensional set M_0 of imputations, is therefore equal to $n-1$.

Our next example illustrates how to compute the nucleolus by solving a sequence of linear programming problems.

EXAMPLE 10.6.3 We compute the nucleolus of the game $\langle \{1, 2, 3\}, v \rangle$ in which $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 6$, $v(\{1, 2\}) = v(\{1, 3\}) = 8$, $v(\{2, 3\}) = 11$ and $v(\{1, 2, 3\}) = 16$.

The proper coalitions $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$ and $S_6 = \{2, 3\}$ have the following excess functions.

$$e_{S_1}(x) = -x_1, \quad e_{S_2}(x) = -x_2, \quad e_{S_3}(x) = 6 - x_3, \quad e_{S_4}(x) = 8 - x_1 - x_2, \\ e_{S_5}(x) = 8 - x_1 - x_3 \quad \text{and} \quad e_{S_6}(x) = 11 - x_2 - x_3.$$

During step 1 of our algorithm we have to minimize the function

$$g_1(x) = \max\{e_{S_1}(x), e_{S_2}(x), e_{S_3}(x), e_{S_4}(x), e_{S_5}(x), e_{S_6}(x)\}$$

over all imputations x , i.e. all x such that $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 6$, and $x_1 + x_2 + x_3 = 16$. However, determining the largest number of a number of given numbers is the same as determining the smallest number t that is greater than or equal to all the given numbers. Our minimization problem is therefore equivalent to the linear programming problem

$$\begin{array}{l} \text{Minimize } t \text{ as} \\ \left\{ \begin{array}{l} -x_1 \leq t \\ -x_2 \leq t \\ 6 - x_3 \leq t \\ 8 - x_1 - x_2 \leq t \\ 8 - x_1 - x_3 \leq t \\ 11 - x_2 - x_3 \leq t \\ x_1 + x_2 + x_3 = 16 \\ x_1, x_2 \geq 0, x_3 \geq 6. \end{array} \right. \end{array}$$

Such problems can be solved using the simplex algorithm, but we can also solve our problem with the help of a little bit of ingenuity. By adding the third and the fourth constraint inequality and using the equality $x_1 + x_2 + x_3 = 16$, we obtain the inequality

$$2t \geq (6 - x_3) + (8 - x_1 - x_2) = 14 - (x_1 + x_2 + x_3) = -2,$$

and equality prevails if and only if $6 - x_3 = 8 - x_1 - x_2 = -1$, i.e. $t \geq -1$ with equality if and only if $x_1 + x_2 = 9$ $x_3 = 7$.

The minimum value m_1 is therefore equal to -1 provided there are points $x = (x_1, x_2, 7)$ with $x_1 + x_2 = 9$ that also satisfy the remaining constraints when $t = -1$, and the set M_1 of minimum points will then consist of all such points x . When $t = -1$ and $x_3 = 7$, the remaining constraints become $-x_1 \leq -1$, $-x_2 \leq -1$, $1 - x_1 \leq -1$, $4 - x_2 \leq -1$, $x_1 \geq 0$, $x_2 \geq 0$, which

simplifies to $x_1 \geq 2$ and $x_2 \geq 5$, and these two inequalities are compatible with the equality $x_1 + x_2 = 9$. Thus, $m_1 = -1$ and

$$M_1 = \{(x_1, x_2, 7) \mid x_1 \geq 2, x_2 \geq 5, x_1 + x_2 = 9\}.$$

Since the excess functions e_{S_3} and e_{S_4} are both constant and equal to m_1 on the entire minimum set M_1 , we proceed according to Remark 2 by minimizing the function

$$g_3(x) = \max\{e_{S_1}(x), e_{S_2}(x), e_{S_5}(x), e_{S_6}(x)\}$$

over the set M_1 . This amounts to the following linear programming problem:

Minimize t as

$$\left\{ \begin{array}{l} -x_1 \leq t \\ -x_2 \leq t \\ 1 - x_1 \leq t \\ 4 - x_2 \leq t \\ x_1 \geq 2, x_2 \geq 5, x_1 + x_2 = 9. \end{array} \right.$$

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Since $x_2 = 9 - x_1$, we obtain the following simpler problem by eliminating the variable x_2 :

$$\text{Minimize } t \text{ as}$$

$$\begin{cases} -x_1 \leq t \\ x_1 - 9 \leq t \\ 1 - x_1 \leq t \\ x_1 - 5 \leq t \\ 2 \leq x_1 \leq 4. \end{cases}$$

By adding the third and the fourth inequality, we find that $t \geq -2$ with equality for $x_1 = 3$. The remaining constraints are also satisfied when $t = -2$ and $x_1 = 3$, so we have found the minimum, and the corresponding minimum point $\hat{x} = (3, 6, 7)$ is uniquely determined. Hence, \hat{x} is the nucleolus solution of the game.

The descending reordering of the excess vector $e(\hat{x})$ of the nucleolus solution \hat{x} is the vector $(-1, -1, -2, -2, -3, -6)$. \square

Proposition 10.6.3 *The nucleolus of a coalitional game with nonempty core is a subset of the core.*

Proof. The coordinates of the excess vector $e(x) = (e_{S_1}(x), \dots, e_{S_p}(x))$ are all negative or zero for all imputations x in the core. This implies that the excesses $e_S(\hat{x})$ of the imputation \hat{x} , that minimizes lexicographically the descending reordering of the game's excess vectors, have to be negative or zero for all coalitions S . This proves that the nucleolus solution \hat{x} belongs to the core of the game. \square

The nucleolus solution is fair in the sense that interchangeable players get the same payoff and null players, i.e. non-contributing players, get nothing.

Proposition 10.6.4 *Let $\langle N, v \rangle$ be a coalitional game with nucleolus $\{\hat{x}\}$.*

- (a) *If i and j are interchangeable players, then $\hat{x}_i = \hat{x}_j$.*
- (b) *If i is a null player, then $\hat{x}_i = 0$.*

Proof. (a) Suppose that i and j are interchangeable players, and let \bar{x} be the imputation obtained from \hat{x} by interchanging \hat{x}_i and \hat{x}_j , i.e. $\bar{x}_i = \hat{x}_j$, $\bar{x}_j = \hat{x}_i$ and $\bar{x}_k = \hat{x}_k$ for all other players k . Then $e_S(\bar{x}) = e_S(\hat{x})$ for all coalitions S that contain both i and j , or none of the two players, while

$$e_{S \cup \{i\}}(\bar{x}) = e_{S \cup \{j\}}(\hat{x}) \quad \text{and} \quad e_{S \cup \{j\}}(\bar{x}) = e_{S \cup \{i\}}(\hat{x})$$

for all coalitions S that contain neither i nor j . The two excess vectors

$$e(\bar{x}) = (e_{S_1}(\bar{x}), e_{S_2}(\bar{x}), \dots, e_{S_p}(\bar{x})) \quad \text{and} \quad e(\hat{x}) = (e_{S_1}(\hat{x}), e_{S_2}(\hat{x}), \dots, e_{S_p}(\hat{x}))$$

therefore have the same coordinates but in different order, so their descending reorderings $e(\bar{x})^*$ and $e(\hat{x})^*$ are identical. Since \hat{x} minimizes lexicographically the descending reordering of the excess vectors and the minimum point is unique, it follows that $\bar{x} = \hat{x}$, which means that $\hat{x}_i = \bar{x}_i = \hat{x}_j$.

(b) Assume that player 1 is a null player so that $v(\{1\}) = 0$ and $v(S \cup \{1\}) = v(S)$ for all coalitions S . We have to prove that $\hat{x}_1 = 0$.

The conclusion follows if we show that, for each imputation x with $x_1 > 0$, there is an imputation y such that $y \prec x$, where \prec is the order relation in the nucleolus definition.

To this end, let y be the imputation obtained by starting from x and redistributing player 1's positive payoff x_1 uniformly among the other players. Thus,

$$y_1 = 0 \quad \text{and} \quad y_k = x_k + \frac{1}{n-1}x_1 \quad \text{for } k = 2, 3, \dots, n.$$

We now assert the following:

(†) For each coalition S different from N there is a coalition S' different from N such that $e_S(y) < e_{S'}(x)$.

It follows from our claim (†) that the largest coordinate of the excess vector $e(y) = (e_{S_1}(y), e_{S_2}(y), \dots, e_{S_p}(y))$ is strictly less than the largest coordinate of the excess vector $e(x) = (e_{S_1}(x), e_{S_2}(x), \dots, e_{S_p}(x))$. When comparing the descending reorderings $e(y)^*$ and $e(x)^*$ of the two excess vectors, we conclude that the first coordinate of $e(y)^*$ is strictly less than the first coordinate of $e(x)^*$. Hence, $e(y)$ is lexicographically strictly less than $e(x)$, and this means by definition that $y \prec x$.

It remains to prove our claim (†). For coalitions S that do not contain player 1, we have

$$e_S(y) = v(S) - y(S) = v(S) - x(S) - \frac{|S|}{n-1}x_1 = e_S(x) - \frac{|S|}{n-1}x_1 < e_S(x),$$

so our claim holds in these cases with $S' = S$.

If $S = \{1\}$ we take $S' = N \setminus \{1\}$, because $e_{\{1\}}(y) = v(\{1\}) - y_1 = 0$, and

$$e_{N \setminus \{1\}}(x) = v(N \setminus \{1\}) - x(N \setminus \{1\}) = v(N) - x(N) + x_1 = x_1 > 0.$$

Finally, for coalitions S which in addition to player 1 also contain another player and have the form $S = T \cup \{1\}$ with $T \neq \emptyset$, we have

$$\begin{aligned} e_S(y) &= v(T \cup \{1\}) - y(T \cup \{1\}) = v(T) - y(T) = v(T) - x(T) - \frac{|T|}{n-1}x_1 \\ &= e_T(x) - \frac{|T|}{n-1}x_1 < e_T(x), \end{aligned}$$

which means that our claim (†) is valid with $S' = T = S \setminus \{1\}$.

This completes the proof statement (†) and the proposition. \square

EXAMPLE 10.6.4 Let us compute the nucleolus $\{\hat{x}\}$ for the production model in Example 10.3.2, that is the game consisting of a factory owner, player 0, and n workers, players $1, 2, \dots, n$, and with value function v defined by $v(S) = 0$ if $0 \notin S$, and $v(S) = f(|S| - 1)$ if $0 \in S$, where f is an increasing, concave function and $f(0) = 0$. The core of the game was found in Example 10.4.3.

The workers are obviously interchangeable, so we conclude from Proposition 10.6.4 that $\hat{x}_i = \hat{x}_1$ for $i \geq 2$. Hence, it suffices to consider imputations of the form $x = (x_0, x_1, \dots, x_n)$ with $x_i = x_1$ for $i \geq 2$. The excesses of such an imputation is given by the expressions

$$e_A(x) = -mx_1 \quad \text{and} \quad e_{\{0\} \cup A}(x) = f(m) - x_0 - mx_1,$$

if A consists of m workers. The problem of minimizing the maximum of all excesses is therefore equivalent to the following linear programming problem:

$$\begin{array}{l} \text{Minimize } t \text{ as} \\ \left\{ \begin{array}{l} -lx_1 \leq t, \quad 1 \leq \ell \leq n \\ f(m) - x_0 - mx_1 \leq t, \quad 0 \leq m \leq n - 1 \\ x_0 + nx_1 = f(n) \\ x_0, x_1 \geq 0. \end{array} \right. \end{array}$$

By eliminating the variable $x_0 = f(n) - nx_1$ we get the following equivalent problem:

$$\begin{array}{l} \text{Minimize } t \text{ as} \\ \left\{ \begin{array}{l} -lx_1 \leq t, \quad 1 \leq \ell \leq n \\ f(m) - f(n) + (n - m)x_1 \leq t, \quad 0 \leq m \leq n - 1 \\ 0 \leq x_1 \leq f(n)/n. \end{array} \right. \end{array}$$

By adding the first constraint inequality with $\ell = 1$ and the second constraint inequality with $m = n - 1$, we obtain the inequality $f(n - 1) - f(n) \leq 2t$, and conclude that $t \geq (f(n - 1) - f(n))/2$.

Let $t_0 = (f(n - 1) - f(n))/2$. We claim that t_0 is the problem's minimum value. To prove this, we have to find an x_1 -value that satisfies all constraints when $t = t_0$ and does so with equality for at least one of the constraints.

Therefore, take $x_1 = -t_0 = (f(n) - f(n - 1))/2$. Then $x_1 \geq 0$ and $-lx_1 \leq t_0$ for $1 \leq \ell \leq n$ with equality for $\ell = 1$. Since the function f is concave and increasing,

$$f(m) - f(n) = - \sum_{j=1}^{n-m} (f(m+j) - f(m+j-1)) \leq -(n-m)(f(n) - f(n-1)),$$

and it follows that

$$\begin{aligned} f(m) - f(n) + (n - m)x_1 &\leq (n - m)x_1 - (n - m)(f(n) - f(n - 1)) \\ &= -(n - m)(f(n) - f(n - 1))/2 = (n - m)t_0 \leq t_0 \end{aligned}$$

for $0 \leq m \leq n - 1$. Moreover, when $m = 0$ we get the following inequality since $f(0) = 0$:

$$-f(n) + nx_1 \leq nt_0 = -nx_1.$$

Hence, $x_1 \leq f(n)/2n$.

We have solved the minimization problem. The nucleolus solution gives each worker $(f(n) - f(n - 1))/2$ value units, that is one half of the worker's marginal product. The owner of the factory gets $f(n) - n(f(n) - f(n - 1))/2$ value units, and this amount is at least $f(n)/2$. \square

Exercises

- 10.17 Find the nucleolus solution of the game in Example 10.3.1.
 10.18 Find the nucleolus solution of the game in Exercise 10.7.
 10.19 Find the nucleolus solution of the game in Example 10.4.1.

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10.20 Find for each $a \geq 5$ the nucleolus solution of game $\langle \{1, 2, 3\}, v \rangle$ in which $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(\{3\}) = 0$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = v(\{2, 3\}) = 3$ and $v(\{1, 2, 3\}) = a$.

10.21 Find the nucleolus solution of the game $\langle N, v \rangle$ where

$$v(S) = \begin{cases} |S| & \text{if } 1 \in S, \\ 0 & \text{if } 1 \notin S. \end{cases}$$

10.22 (The bankruptcy problem) A person goes bankrupt leaving debts to three persons and the total debt exceeds the assets. How should the assets be divided among the creditors.

The problem is discussed already in Babylonian Talmud, a collection of Jewish laws and traditions from the first five centuries AD, in the following form. A man has three wives with a marriage contract stipulating that they will receive 100, 200 and 300 respectively in the event of the man's death. Talmud recommends that they receive the same amount each, i.e. $33\frac{1}{3}$, if the man leaves 100 at his death. Proportional distribution is recommended if the remaining wealth is 300, i.e. 50, 100, 150. However, if the inheritance amounts to 200, Talmud's recommendation is 50, 75, 75, which may appear as a complete mystery.

Show that Talmud's recommendations coincide with the nucleolus solution of the coalitional game $\langle N, v \rangle$ obtained by letting N be the three wives and the coalition value $v(S)$ be the amount that the S coalition can get without having to take any legal actions, i.e. $v(S)$ is equal to what remains of the assets when players outside the coalition S have all their claims credited if the assets are sufficient for this, whereas $v(S) = 0$ if the assets are not enough for this.

Thus, if the man leaves 300, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = 0$, $v(\{1, 3\}) = 100$, $v(\{2, 3\}) = 200$ and $v(\{1, 2, 3\}) = 300$.

The problem is taken from an article by R.J. Aumann and M. Maschler.

Chapter 11

The Shapley Value

The main challenge in a coalitional game $\langle N, v \rangle$ is to allocate the payoff $v(N)$ of the grand coalition among the players in some fair way. Shapley has described an elegant solution with intuitively appealing features, which is also relatively easy to calculate.

11.1 The Shapley solution

Let \mathcal{G} be a class of n -player coalitional games, such as all coalitional games, or all cohesive games, or all superadditive games for n players. A function $\phi: \mathcal{G} \rightarrow \mathbf{R}^n$ that assigns each game $\langle N, v \rangle$ in the class a unique *collectively rational* payoff vector $\phi(N, v)$ will be called a *solution function*, and the vector $\phi(N, v)$ is called the *solution* of the game $\langle N, v \rangle$ with respect to the current method (function).

The solution concept is undeniably very general, the only requirement being that the vector $\phi(N, v)$ can be used to distribute the grand coalition's value among the players. For a solution to be regarded as fair and relevant, it should of course also have other properties. We have actually already studied such a solution for the class of all games $\langle N, v \rangle$ with nonempty set of imputations, namely the nucleolus which is the solution we get by defining $\phi(N, v)$ to be the game's unique nucleolus vector. The nucleolus solution is a collectively and individually rational vector which (in a precise sense) minimizes the potential objections of the coalitions. In addition, it has the appealing property of treating interchangeable players equally and of allocating zero payoff to null players.

The Shapley solution is based on the latter property, but unlike the nucleolus definition, that refers to a fixed game, Shapley's definition requires that we consider a whole class of games. Hence the need for the concept of

solution function.

We begin by noting that the set of all coalitional games with a given set N of players forms a vector space. We will then determine a special basis for this vector space.

Definition 11.1.1 The sum $\langle N, v \rangle + \langle N, w \rangle$ of two coalitional games with the same set of players is the game $\langle N, v + w \rangle$, and the product $c\langle N, v \rangle$ of a real number c and the game $\langle N, v \rangle$ is the game $\langle N, cv \rangle$.

Note that the sum of two cohesive games is cohesive and that the product of a positive real number and a cohesive game is cohesive. Moreover, the sum of two superadditive games is superadditive, and the product of a positive real number and a superadditive game is superadditive.

The definition of sum and multiplication with scalars makes the set of all coalitional games with N as set of players to a vector space that is isomorphic to the vector space of all functions $v: \mathcal{C} \rightarrow \mathbf{R}$. By numbering the $2^n - 1$ coalitions in the set \mathcal{C} of all coalitions in an arbitrary manner as $S_1, S_2, \dots, S_{2^n-1}$ and setting $v_i = v(S_i)$, we may further identify each function v with the vector $(v_1, v_2, \dots, v_{2^n-1})$ in \mathbf{R}^{2^n-1} . The vector space of all coalitional games is apparently isomorphic to \mathbf{R}^{2^n-1} , and its dimension is equal to $2^n - 1$. Therefore, it has a vector basis consisting of $2^n - 1$ games. We will now construct an explicit such basis that is indexed by the coalitions which are also $2^n - 1$ in number.

Lemma 11.1.1 Let, for each coalition S , $\chi_S: \mathcal{C} \rightarrow \mathbf{R}$ be the value function defined by

$$\chi_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

The games $(\langle N, \chi_S \rangle)_{S \in \mathcal{C}}$ form a basis for the vector space of all coalitional games with N as set of players.

In other words, for each value function v there exist uniquely determined real numbers c_S such that

$$v = \sum_{S \in \mathcal{C}} c_S \chi_S.$$

The numbers c_S are the coordinates of the game $\langle N, v \rangle$ with respect to the specified basis.

Proof. Since the number of value functions of the form χ_S is equal to the dimension of the vector space, it is sufficient to show that the games $\langle N, \chi_S \rangle$ span the space, i.e. that for each value function v there are constants c_S so

that the equality of the lemma holds. To this end, we set $c_\emptyset = 0$ and define c_T recursively for $T \in \mathcal{C}$ using the equation

$$(1) \quad c_T = v(T) - \sum_{S \subsetneq T} c_S.$$

Then

$$\begin{aligned} \sum_{S \in \mathcal{C}} c_S \chi_S(T) &= \sum_{S \subsetneq T} c_S \chi_S(T) + \sum_{S \not\subsetneq T} c_S \chi_S(T) = \sum_{S \subsetneq T} c_S \\ &= c_T + \sum_{S \subsetneq T} c_S = c_T + v(T) - c_T = v(T) \end{aligned}$$

for all coalitions T , and this proves our claim that $v = \sum c_S \chi_S$. \square

EXAMPLE 11.1.1 The coalitional game $\langle \{1, 2, 3\}, v \rangle$, with value function v defined by $v(\{1\}) = 3$, $v(\{2\}) = v(\{3\}) = 2$, $v(\{1, 2\}) = v(\{1, 3\}) = 4$, $v(\{2, 3\}) = 5$ and $v(\{1, 2, 3\}) = 8$, has the following coordinates with respect to the canonical basis of Lemma 11.1.1:

$$\begin{aligned} c_{\{1\}} &= 3, & c_{\{2\}} &= 2, & c_{\{3\}} &= 2, & c_{\{1,2\}} &= 4 - 3 - 2 = -1, \\ c_{\{1,3\}} &= 4 - 3 - 2 = -1, & c_{\{2,3\}} &= 5 - 2 - 2 = 1, \\ c_{\{1,2,3\}} &= 8 - 3 - 2 - 2 - (-1) - (-1) - 1 = 2. \end{aligned} \quad \square$$

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Lemma 11.1.2 *Let $\langle N, v \rangle$ be a coalitional game with value function*

$$v = \sum_{S \in \mathcal{C}} c_S \chi_S.$$

- (i) *Player i is a null player if and only if $c_S = 0$ for all coalitions S that contain i .*
- (ii) *Players i and j are exchangeable if and only if $c_{S \cup \{i\}} = c_{S \cup \{j\}}$ for all subsets S of N that do not contain any of the players i and j .*

Proof. Let T be a subset of N that does not contain player i . Then

$$\begin{aligned} c_{T \cup \{i\}} &= v(T \cup \{i\}) - \sum_{S \subsetneq T \cup \{i\}} c_S = v(T \cup \{i\}) - \sum_{S \subsetneq T} c_{S \cup \{i\}} - \sum_{S \subseteq T} c_S \\ &= v(T \cup \{i\}) - \sum_{S \subsetneq T} c_{S \cup \{i\}} - v(T), \end{aligned}$$

that is

$$(2) \quad v(T \cup \{i\}) - v(T) = c_{T \cup \{i\}} + \sum_{S \subsetneq T} c_{S \cup \{i\}}.$$

The sum in equation (2) is empty if $T = \emptyset$ and $v(\emptyset) = 0$, which means that $v(\{i\}) = c_{\{i\}}$.

(i) If $c_S = 0$ for all coalitions containing player i , then it follows at once from equation (2) that $v(T \cup \{i\}) = v(T)$ for all subsets T of N that do not contain i , and player i is accordingly a null player.

The converse is shown by induction. Suppose that player i is a null player. Then, first, $c_{\{i\}} = v(\{i\}) = 0$. Write coalitions containing i with more than one player as $T \cup \{i\}$, where T is a coalition that does not contain i , and assume inductively that $c_{S \cup \{i\}} = 0$ for all proper subsets S of T . Since $v(T \cup \{i\}) = v(T)$, we now conclude from equation (2) that $c_{T \cup \{i\}} = 0$. This completes the induction step.

(ii) Let T be a subset of N containing neither i nor j . Then

$$v(T \cup \{j\}) - v(T) = c_{T \cup \{j\}} + \sum_{S \subsetneq T} c_{S \cup \{j\}},$$

according to equation (2) with i replaced by j , and by subtracting this from equation (2), we get

$$(3) \quad v(T \cup \{i\}) - v(T \cup \{j\}) = c_{T \cup \{i\}} - c_{T \cup \{j\}} + \sum_{S \subsetneq T} (c_{S \cup \{i\}} - c_{S \cup \{j\}}).$$

We conclude from equation (3) that $v(T \cup \{i\}) = v(T \cup \{j\})$, i.e. that the players i and j are interchangeable, if $c_{S \cup \{i\}} = c_{S \cup \{j\}}$ for all subsets S of N that do not contain i or j .

Conversely, if i and j are interchangeable and T is a subset of N containing neither i nor j so that $v(T \cup \{i\}) = v(T \cup \{j\})$, and $c_{S \cup \{i\}} = c_{S \cup \{j\}}$ for all proper subsets S of T , then equation (3) implies that $c_{T \cup \{i\}} = c_{T \cup \{j\}}$. So it follows by induction that $c_{T \cup \{i\}} = c_{T \cup \{j\}}$ for all subsets T of N that do not contain i or j . \square

Proposition 11.1.3 *There is a unique solution function $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ that is defined for all coalitional games $\langle N, v \rangle$ and has the following properties:*

(i) *If i is a null player, then $\phi_i(N, v) = 0$.*

(ii) *If i and j are exchangeable players, then*

$$\phi_i(N, v) = \phi_j(N, v).$$

(iii) *For all value functions v and w ,*

$$\phi(N, v + w) = \phi(N, v) + \phi(N, w).$$

The payoff $\phi_i(N, v)$ to player i is for $v = \sum_{S \in \mathcal{C}} c_S \chi_S$ given by the equality

$$(4) \quad \phi_i(N, v) = \sum_{S \ni i} \frac{c_S}{|S|}.$$

The summation symbol $\sum_{S \ni i}$ in formula (4) stands for summation over the set $\mathcal{C}(i)$, that is over all coalitions S containing player i .

The three properties (i), (ii) and (iii) in Proposition 11.1.3 are called the *null player, symmetry and additivity properties*.

Proof. Uniqueness: We first prove that if there exists a solution function ϕ with the three stated properties, then it has to be given by formula (4), which implies, of course, that the function is unique.

Suppose therefore that ϕ is a solution function with the three desired properties. We begin by computing $\phi(N, v)$ for functions of the type $v = c\chi_S$, where c is a constant and χ_S is one of the basis functions in Lemma 11.1.1.

First suppose $i \notin S$. Then, $S \subseteq T$ if and only if $S \subseteq T \cup \{i\}$, and hence $\chi_S(T) = \chi_S(T \cup \{i\})$ for all coalitions T . Moreover, $\chi_S(\{i\}) = 0$. Player i is thus a null player in the game $\langle N, c\chi_S \rangle$. Hence, by property (i), $\phi_i(N, c\chi_S) = 0$ for all players i that do not belong to S .

Next suppose that $i, j \in S$. Then, $S \not\subseteq T \cup \{i\}$ and $S \not\subseteq T \cup \{j\}$ for each coalition T that does not contain i or j , which means that

$$\chi_S(T \cup \{i\}) = \chi_S(T \cup \{j\}) = 0.$$

So the two players i and j are exchangeable in the game $\langle N, c\chi_S \rangle$, and it follows from the symmetry property (ii) that $\phi_i(N, c\chi_S) = \phi_j(N, c\chi_S)$. The value $\phi_i(N, c\chi_S)$ is thus the same for all players i belonging to coalition S .

Finally, since $\chi_S(N) = 1$ and $\phi(N, c\chi_S)$ is a collectively rational vector, we get the equality

$$c = c\chi_S(N) = \sum_{k \in N} \phi_k(N, c\chi_S) = \sum_{k \in S} \phi_k(N, c\chi_S) = |S| \phi_i(N, c\chi_S)$$

for all $i \in S$. Consequently,

$$\phi_i(N, c\chi_S) = \begin{cases} c/|S| & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

This determines the values of ϕ for all value functions of the type $c\chi_S$. But each value function v has according to Lemma 11.1.1 a unique expansion of the form $v = \sum_{S \in \mathcal{C}} c_S \chi_S$, and the additivity property (iii) therefore implies that

$$\phi_i(N, v) = \sum_{S \in \mathcal{C}} \phi_i(N, c_S \chi_S) = \sum_{S \ni i} \phi_i(N, c_S \chi_S) + \sum_{S \not\ni i} \phi_i(N, c_S \chi_S) = \sum_{S \ni i} \frac{c_S}{|S|},$$

which proves formula (4) and the uniqueness.

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Existence: To prove the existence of a solution function with the properties (i), (ii) and (iii), we define ϕ by formula (4) and show that ϕ has the desired properties.

We begin by showing that $\phi(N, v)$ is a collectively rational vector. It follows from the definition of $\phi_i(N, v)$ that

$$\sum_{i \in N} \phi_i(N, v) = \sum_{i \in N} \sum_{S \ni i} \frac{c_S}{|S|}.$$

The term $c_S/|S|$ appears in the double sum once for each player i that belongs to the coalition S , i.e. as many times as there are members of S , and it does so for every coalition S . Therefore,

$$\sum_{i \in N} \sum_{S \ni i} \frac{c_S}{|S|} = \sum_{S \in \mathcal{C}} |S| \frac{c_S}{|S|} = \sum_{S \in \mathcal{C}} c_S = v(N),$$

where the last equality follows from the recursive definition of c_N in equation (1). This proves that $\sum_{i \in N} \phi_i(N, v) = v(N)$.

The null player property follows directly from formula (4) and Lemma 11.1.2.

Suppose that i and j are interchangeable players in the game $\langle N, v \rangle$. The coalitions that contain i can be partitioned into coalitions that also contain j and coalitions that do not contain j , and the last mentioned coalitions are of the form $T \cup \{i\}$, where T does not contain j . Using this partition, the definition of ϕ and part (ii) of Lemma 11.1.2, we obtain

$$\begin{aligned} \phi_i(N, v) &= \sum_{S \ni i} \frac{c_S}{|S|} = \sum_{S \ni i, j} \frac{c_S}{|S|} + \sum_{T \not\ni i, j} \frac{c_{T \cup \{i\}}}{|T \cup \{i\}|} \\ &= \sum_{S \ni i, j} \frac{c_S}{|S|} + \sum_{T \not\ni i, j} \frac{c_{T \cup \{j\}}}{|T \cup \{j\}|} = \sum_{S \ni j} \frac{c_S}{|S|} = \phi_j(N, v). \end{aligned}$$

This shows that the function ϕ satisfies the symmetry condition.

The additivity property follows at once from the definition of ϕ , and this concludes the existence proof. \square

Definition 11.1.2 The unique function ϕ in Proposition 11.1.3 is called the *Shapley function*, the vector $\phi(N, v)$ is the *Shapley solution* of the game $\langle N, v \rangle$ and the number $\phi_i(N, v)$ is player i 's *Shapley value*.

To simplify the notation, we will from now on omit the reference to the set N of players by writing $\phi(v)$ and $\phi_i(v)$ instead of $\phi(N, v)$ and $\phi_i(N, v)$.

EXAMPLE 11.1.2 Let us compute the Shapley values of the cohesive, non-superadditive three-person game in Exempel 11.1.1. We use the formula in Proposition 11.1.3 to compute $\phi_1(v)$ and get

$$\phi_1(v) = c_{\{1\}} + \frac{1}{2}(c_{\{1,2\}} + c_{\{1,3\}}) + \frac{1}{3}c_{\{1,2,3\}} = 3 + \frac{1}{2}(-1 - 1) + \frac{1}{3} \cdot 2 = \frac{8}{3}.$$

The players 2 and 3 are interchangeable. Hence $\phi_2(v) = \phi_3(v)$, and since the Shapley solution is collectively rational, it now follows that

$$2\phi_2(v) = v(N) - \phi_1(v) = 8 - \frac{8}{3} = \frac{16}{3}.$$

Thus $\phi_1(v) = \phi_2(v) = \phi_3(v) = \frac{8}{3}$.

Note that the Shapley solution $(\frac{8}{3}, \frac{8}{3}, \frac{8}{3})$ is not individually rational because of the inequality $\phi_1(v) < 3 = v(\{1\})$.

The game's core consists of all imputations of the form $(3, t, 5 - t)$ with $2 \leq t \leq 3$, as is easily verified. The imputation $(3, \frac{5}{2}, \frac{5}{2})$ is the game's nucleolus solution, because the nucleolus is a subset of the core and gives the two interchangeable players 2 and 3 the same payoff. \square

Example 11.1.2 teaches us that the Shapley solution does not have to be an imputation and that it is generally different from the nucleolus solution. However, the Shapley solution of a superadditive game is an imputation, as we will show in the next section when we acquired an alternative formula for the Shapley values.

Since the nucleolus solution has the dummy and the symmetry properties but does not coincide with the Shapley solution, we can also conclude that the nucleolus solution lacks the additivity property.

Exercises

11.1 Determine the Shapley solution of the game "Who will have the painting?" in Example 10.3.1, i.e. the three-person game with value function $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$, $v(\{1, 2\}) = 2$ and $v(\{1, 3\}) = v(\{1, 2, 3\}) = 3$.

11.2 Consider an n -person game $\langle N, v \rangle$ in which

$$v(S) = k \quad \text{if } \{1, 2, \dots, k\} \subseteq S \text{ but } k + 1 \notin S.$$

For example, $v(\{2, 4, 5\}) = 0$, $v(\{1, 4, 5\}) = 1$ and $v(\{1, 2, 3, 5\}) = 3$. Compute the game's Shapley solution.

11.2 Alternative characterization of the Shapley value

We now give an alternative characterization of the Shapley solution in terms of the players' marginal contributions to the various coalitions.

We recall that player i 's marginal contribution to the coalition S in the game $\langle N, v \rangle$ is defined as

$$\Delta_i(S) = v(S) - v(S \setminus \{i\}),$$

where $v(\emptyset) = 0$.

Note that in a superadditive game, $\Delta_i(S) \geq v(\{i\})$ for each coalition S that player i takes part in, i.e. the player's marginal contribution is greater than or equal to the value he can achieve on his own.

Let π denote an arbitrary permutation of the numbers $1, 2, \dots, n$, and denote by $S_i(\pi)$ the coalition that consists of player i and the players that precede i in the ordering given by the permutation π .

EXAMPLE 11.2.1 If $n = 4$ and $\pi = (3, 4, 1, 2)$, then $S_1(\pi) = \{1, 3, 4\}$, $S_2(\pi) = \{1, 2, 3, 4\}$, $S_3(\pi) = \{3\}$ and $S_4(\pi) = \{3, 4\}$. \square

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Proposition 11.2.1 *Player i 's Shapley value in the coalitional game $\langle N, v \rangle$ is given by the formula*

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi} \Delta_i(S_i(\pi)) = \frac{1}{n} \sum_{S \ni i} \frac{\Delta_i(S)}{\binom{n-1}{|S|-1}},$$

where the first summation is to be taken over all the $n!$ permutations of the numbers $1, 2, \dots, n$, and the second summation is over all coalitions S that contain the player i .

Proof. Let $\psi = (\psi_1, \psi_2, \dots, \psi_n)$ be the function defined for the value functions v of the coalitional games $\langle N, v \rangle$ by setting

$$(5) \quad \psi_i(v) = \frac{1}{n!} \sum_{\pi} \Delta_i(S_i(\pi))$$

for each i . In order to prove that $\psi_i(v)$ is player i 's Shapley value, it suffices to show that ψ is a solution function, i.e. that $\psi(v)$ is a collectively rational vector, and that ψ has the null player, symmetry and additivity properties, because the Shapley solution is the unique solution function with these properties.

Let $\pi = (i_1, i_2, \dots, i_n)$ be an arbitrary permutation of the numbers in N . Then $\Delta_{i_k}(S_{i_k}(\pi)) = v(\{i_1, \dots, i_{k-1}, i_k\}) - v(\{i_1, \dots, i_{k-1}\})$, and it follows that

$$\begin{aligned} \sum_{i \in N} \Delta_i(S_i(\pi)) &= \sum_{k=1}^n \Delta_{i_k}(S_{i_k}(\pi)) \\ &= \sum_{k=1}^n (v(\{i_1, \dots, i_{k-1}, i_k\}) - v(\{i_1, \dots, i_{k-1}\})) \\ &= v(\{i_1, \dots, i_{n-1}, i_n\}) - v(\emptyset) = v(N). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i \in N} \psi_i(v) &= \frac{1}{n!} \sum_{i \in N} \sum_{\pi} \Delta_i(S_i(\pi)) \\ &= \frac{1}{n!} \sum_{\pi} \sum_{i \in N} \Delta_i(S_i(\pi)) = \frac{1}{n!} \sum_{\pi} v(N) = v(N), \end{aligned}$$

and this shows that $\psi(v)$ is a collectively rational vector.

The null player property is obvious since $\Delta_i(S) = 0$ for all coalitions S if i is a null player.

Suppose that the two players i and j are interchangeable. For each permutation $\pi = (i_1, \dots, i, \dots, j, \dots, i_n)$, we denote by $\pi' = (i_1, \dots, j, \dots, i, \dots, i_n)$ the permutation obtained by interchanging the numbers i and j . Then

$$\Delta_i(S_i(\pi)) = \Delta_j(S_j(\pi')),$$

because of the interchangeability of i and j , regardless of whether i precedes j in the permutation π (as above) or j precedes i . Consequently,

$$\psi_i(v) = \frac{1}{n!} \sum_{\pi} \Delta_i(S_i(\pi)) = \frac{1}{n!} \sum_{\pi'} \Delta_j(S_j(\pi')) = \psi_j(v),$$

which proves the symmetry property.

The additivity property is obvious, because a player's marginal contribution to a coalition in the sum of two games is equal to the sum of the player's marginal contributions to the two games.

Thus we have proved that the Shapley value is also given by formula (5), and it only remains to show how to rewrite the formula so as to get the second of the sums in Proposition 11.2.1.

$S_i(\pi)$ is, by definition, a coalition that contains player i for each permutation π , but for any given coalition S containing i , there are of course many different permutations π for which $S_i(\pi) = S$, and to start with, we will compute the number of such permutations.

Let $k = |S|$ and $T = S \setminus \{i\}$. Then $S_i(\pi) = S$ if and only if π is a permutation of the form

$$\pi = (i_1, \dots, i_{k-1}, i, j_1, \dots, j_{n-k}),$$

where (i_1, \dots, i_{k-1}) and (j_1, \dots, j_{n-k}) are arbitrary permutations of the elements in T and $N \setminus S$, respectively. The number of such permutations π is equal to $(k-1)!(n-k)!$.

Thus

$$\frac{1}{n!} \sum_{\pi} \Delta_i(S_i(\pi)) = \frac{1}{n!} \sum_{S \ni i} \sum_{\substack{\pi \text{ such that} \\ S_i(\pi) = S}} \Delta_i(S_i(\pi)) = \frac{1}{n!} \sum_{S \ni i} (k-1)!(n-k)! \Delta_i(S),$$

and since

$$\frac{n!}{(k-1)!(n-k)!} = n \binom{n-1}{k-1} = n \binom{n-1}{|S|-1},$$

the proof is now complete. \square

The first half of the formula in Proposition 11.2.1 gives us an intuitively appealing interpretation of the Shapley value.

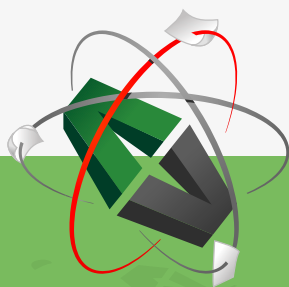
Let $\pi = (i_1, i_2, \dots, i_n)$ be a permutation of N , and suppose that the grand coalition N is formed by the players joining it one after another in the order given by the permutation, i.e. it starts with player i_1 and then player i_2 joins forming the coalition $\{i_1, i_2\}$, and then i_3 joins to this coalition, etc. until all players have joined. In this case, it seems natural to distribute the value $v(N)$ of the grand coalition in such a way that each player i gets its marginal contribution to the coalition that is formed at the moment the player enters the game, i.e. $\Delta_i(\pi)$, which is also possible since the sum of these contributions is equal to $v(N)$.

However, the coalition game concept does not contain any assumption of how the grand coalition is put together, so the Shapley value regards all the $n!$ ways to form the grand coalition as equivalent and assigns players i the average of all his marginal contributions according to the above description.

Proposition 11.2.1 has the following corollary.

Corollary 11.2.2 *The Shapley solution $\phi(v)$ of a superadditive game $\langle N, v \rangle$ is an imputation.*

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Proof. Since $\Delta_i(S_i(\pi)) = v(S_i(\pi)) - v(S_i(\pi) \setminus \{i\}) \geq v(\{i\})$ for all permutations π and all players i , it follows from the first equality in Proposition 11.2.1 that

$$\phi_i(v) \geq \frac{1}{n!} \sum_{\pi} v(\{i\}) = v(\{i\})$$

for all $i \in N$. The Shapley solution $\phi(v)$ is, in other words, individually rational and thus an imputation. \square

EXAMPLE 11.2.2 Let us compute the Shapley solution of the three-person game $\langle \{1, 2, 3\}, v \rangle$, where $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = 2$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 5$, $v(\{1, 2, 3\}) = 8$, using Proposition 11.2.1. The players' marginal contributions to the coalitions that they take part in, are as follows:

$$\begin{aligned} \Delta_1(\{1\}) &= 1, \Delta_1(\{1, 2\}) = 3, \Delta_1(\{1, 3\}) = 4, \Delta_1(\{1, 2, 3\}) = 3; \\ \Delta_2(\{2\}) &= 1, \Delta_2(\{1, 2\}) = 3, \Delta_2(\{2, 3\}) = 3, \Delta_2(\{1, 2, 3\}) = 2; \\ \Delta_3(\{3\}) &= 2, \Delta_3(\{1, 3\}) = 5, \Delta_3(\{2, 3\}) = 4, \Delta_3(\{1, 2, 3\}) = 4. \end{aligned}$$

The players' Shapley values are therefore

$$\begin{aligned} \phi_1(v) &= \frac{1}{3}(1 + \frac{1}{2}(3 + 4) + 3) = \frac{5}{2} \\ \phi_2(v) &= \frac{1}{3}(1 + \frac{1}{2}(3 + 3) + 2) = 2 \\ \phi_3(v) &= \frac{1}{3}(2 + \frac{1}{2}(5 + 4) + 4) = \frac{7}{2}. \end{aligned}$$

As a control, we note that the sum of the Shapley values is equal to the 8, the value of the grand coalition. \square

EXAMPLE 11.2.3 Let us compute the Shapley solution of the production model in Example 10.3.2: one factory owner (player 0), n workers (players 1, 2, ..., n), and value function $v(S) = 0$ if $0 \notin S$ and $v(S) = f(|S| - 1)$ if $0 \in S$. The function f is increasing and concave, and $f(0) = 0$.

It suffices to compute the factory owner's Shapley value $\phi_0(v)$, because the workers are interchangeable, and we use the formula in Proposition 11.2.1. We have $\Delta_0(S) = f(|S| - 1)$ for coalitions all S containing player 0. Thus

$$\begin{aligned} \phi_0(v) &= \frac{1}{n+1} \sum_{S \ni 0} \frac{f(|S| - 1)}{\binom{n}{|S|-1}} = \frac{1}{n+1} \sum_{k=1}^{n+1} \sum_{\substack{S \ni 0 \\ |S|=k}} \frac{f(|S| - 1)}{\binom{n}{|S|-1}} \\ &= \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{f(k-1)}{\binom{n}{k-1}} M_k, \end{aligned}$$

where M_k is the number of subsets S of $\{0, 1, 2, \dots, n\}$ that consist of k numbers and contain the number 0. This number is $M_k = \binom{n}{k-1}$, and by substituting this in the expression above, we obtain

$$\phi_0(v) = \frac{1}{n+1} \sum_{k=0}^n f(k).$$

The workers will share the remaining amount $f(n) - \phi_0(v)$ equally, which means that each worker's Shapley value is

$$\phi_i(v) = \frac{1}{n} \left(f(n) - \frac{1}{n+1} \sum_{k=0}^n f(k) \right).$$

It may be interesting to compare the Shapley solution with the nucleolus solution, which we computed in Example 10.6.4. The nucleolus solution gives each worker $\frac{1}{2}(f(n) - f(n-1))$ value units. Because of the concavity,

$$f(n) - f(k) = \sum_{j=1}^{n-k} (f(j+k) - f(j+k-1)) \geq (n-k)(f(n) - f(n-1)),$$

and it follows that

$$\begin{aligned} \phi_i(v) &= \frac{1}{n(n+1)} \left(nf(n) - \sum_{k=0}^{n-1} f(k) \right) = \frac{1}{n(n+1)} \sum_{k=0}^{n-1} (f(n) - f(k)) \\ &\geq \frac{1}{n(n+1)} \sum_{k=0}^{n-1} (n-k)(f(n) - f(n-1)) = \frac{1}{2}(f(n) - f(n-1)). \end{aligned}$$

Thus, the Shapley solution is better than the nucleolus solution for the workers.

The Shapley solution belongs to the core of the game if the workers' marginal product $f(k) - f(k-1)$ is moderately decreasing when k increases. However, the Shapley solution does not belong to the core if the marginal product decreases rapidly, because the solution gives the workers too much in this case. Compare with Exercise 11.5. \square

Exercises

11.3 Determine the Shapley solution of the game $\langle \{1, 2, 3\}, v \rangle$ with $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(\{3\}) = -1$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 2$, $v(\{1, 2, 3\}) = 4$.

11.4 (A market with one seller and two buyers) Player 1 owns an object that has no value to him and which he therefore wishes to sell. The object is worth a for player 2 and b for player 3, where $b > a$. Formulate this as a coalitional game and determine the Shapley solution. Does the Shapley solution belong to the core?

11.5 Calculate the Shapley solution of the production model in Example 11.2.3 when $n = 4$, $f(0) = 0$, $f(1) = 8$, $f(2) = 12$, $f(3) = 14$ and $f(4) = 15$, and prove that it does not belong to the core.

11.6 Calculate the Shapley solution of the n -person game $\langle N, v \rangle$ when

$$\text{a) } v(S) = \begin{cases} |S| & \text{if } 1 \in S, \\ 0 & \text{otherwise.} \end{cases} \quad \text{b) } v(S) = \begin{cases} |S| & \text{if } 1, 2 \in S, \\ 0 & \text{otherwise.} \end{cases}$$

11.7 Consider the production model in Example 11.2.3 in the case $f(k) = k^\alpha$, where $0 < \alpha \leq 1$. Suppose that the number n of workers is very large. Prove that the Shapley solution is in the core and that the factory owner gets approximately $\frac{1}{\alpha+1}n^\alpha$ while the workers will share the remaining $\frac{\alpha}{\alpha+1}n^\alpha$ equally. Compare with the nucleolus solution, which gives the factory owner approximately $(1 - \frac{\alpha}{2})n^\alpha$ and the workers a total of $\frac{\alpha}{2}n^\alpha$.

[Hint: Approximate the Riemann sum $\frac{1}{n} \sum_{k=1}^n (\frac{k}{n})^\alpha$ with the integral $\int_0^1 x^\alpha dx$.]

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11.3 The Shapley–Shubiks power index

The Shapley solution can be used to describe the voting power of different groups or individuals in voting situations, and this application was developed by Shapley and Shubik.

Consider a vote in, for example, a political assembly, or a union, or at an annual meeting of shareholders with n participants (parties, members or shareholders), where a particular proposal can either be accepted or rejected, and where the outcome depends on the coalition of participants who support the proposal. A coalition is a winning coalition if it can vote through the proposal, otherwise it is a losing coalition. Obviously, we assume that each subset of a losing coalition is losing and that any coalition that includes a winning coalition is also winning.

By defining

$$v(S) = \begin{cases} 1 & \text{if } S \text{ is a winning coalition,} \\ 0 & \text{if } S \text{ is a losing coalition} \end{cases}$$

we can model the vote as a simple coalitional game $\langle N, v \rangle$, and the concepts of winning and losing coalitions get exactly the meanings we gave them in Definition 10.3.1. The *Shapley–Shubik power index* of a voting participant is now simply defined as the Shapley value of the voter in this game.

The formula in Proposition 11.2.1 for the Shapley value is simplified when the game is simple, because a player's marginal contribution $\Delta_i(S)$ can only assume the values 0 and 1. The marginal contribution is 1 if and only if $v(S) = 1$ and $v(S \setminus \{i\}) = 0$, i.e. if S is a winning coalition and $S \setminus \{i\}$ is a losing coalition, so the formula for player i 's Shapley–Shubik power index simplifies to

$$(6) \quad \phi_i(v) = n^{-1} \sum_{\substack{S \text{ winning} \\ S \setminus \{i\} \text{ losing}}} \binom{n-1}{|S|-1}^{-1}.$$

The most common types of voting games are described in the following definition.

Definition 11.3.1 Let w_1, w_2, \dots, w_n be non-negative numbers and let q be a positive number. The simple coalitional game $\langle N, v \rangle$ obtained by defining

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q, \\ 0 & \text{if } \sum_{i \in S} w_i \leq q \end{cases}$$

is called a *weighted voting game* with the numbers w_i as *weights*.

The game is called a *weighted majority game* if $q = \frac{1}{2} \sum_{i \in N} w_i$.

EXAMPLE 11.3.1 Decisions by simple majority are obtained by selecting all the weights equal to 1 and $q = n/2$, qualified majority decisions by selecting all the weights equal to 1 and $q = 2n/3$, if two-thirds majority is required for a proposal to be adopted. For decisions requiring unanimity, $w_i = 1$ for all players i , and $q = n - \frac{1}{2}$. Player 1 is a dictator if $w_1 = 1$, $w_i = 0$ for $i \geq 2$ and $q = \frac{1}{2}$.

In decisions requiring a simple majority, qualified majority or unanimity, all players have Shapley–Shubik power index $1/n$, because the players are pairwise interchangeable. In dictatorial decisions, the dictator has Shapley–Shubik power index 1 and the other players have index 0, because they are null players.

In the above cases, the particular values of the Shapley–Shubik power index are immediate consequences of the definition of the Shapley solution as the unique solution to the conditions given in Proposition 11.1.3, but let us still calculate the Shapley–Shubik power index using formula (6) in the case of simple majority.

Let $n = 2k$ or $n = 2k + 1$. A coalition S that contains player i is winning while the coalition $S \setminus \{i\}$ is losing, if and only if $|S| = k + 1$, and the number of such coalitions is $\binom{n-1}{k}$. It follows therefore from formula (6) that

$$\begin{aligned} \phi_i(v) &= n^{-1} \sum_{\substack{S \text{ winning} \\ S \setminus \{i\} \text{ losing}}} \binom{n-1}{|S|-1}^{-1} = n^{-1} \sum_{|S|=k+1} \binom{n-1}{|S|-1}^{-1} \\ &= n^{-1} \binom{n-1}{k} \binom{n-1}{k}^{-1} = n^{-1}. \quad \square \end{aligned}$$

At shareholders' meetings in a limited liability company that has only issued shares with equal voting rights, each shareholder has as many votes as the number of shares owned, and for a proposal to be approved, it must be supported by a majority of the shares. A general meeting can therefore be regarded as a weighted majority game.

EXAMPLE 11.3.2 The four owners of a company, players 1, 2, 3 and 4, own 10, 20, 30, and 40 shares, respectively. This means that the winning coalitions are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{2, 4\}$ and $\{3, 4\}$.

There is only one winning coalition that contains player 1 and is losing without him, namely $S = \{1, 2, 3\}$, and this means that

$$\phi_1(v) = \frac{1}{4} \binom{3}{2}^{-1} = \frac{1}{12}.$$

The corresponding coalitions for player 2 are $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{2, 4\}$, and therefore

$$\phi_2(v) = \frac{1}{4} \left(2 \binom{3}{2}^{-1} + \binom{3}{1}^{-1} \right) = \frac{1}{4}.$$

The coalitions $\{1, 2, 3\}$, $\{1, 3, 4\}$ and $\{3, 4\}$ are winning for player 3 and losing without him. This means that $\phi_3(v) = \phi_2(v) = \frac{1}{4}$.

All winning coalitions with player 4 as a member, except the grand coalition, are losing if player 4 leaves. Hence,

$$\phi_4(v) = \frac{1}{4} \left(2 \binom{3}{1}^{-1} + 3 \binom{3}{2}^{-1} \right) = \frac{5}{12}.$$

The share holders' Shapley–Shubik power indexes are thus $\frac{1}{12}$, $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{5}{12}$. It is noteworthy that owners 2 and 2 have the same index despite the fact that owner 3 has more shares. \square

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Exercises

- 11.8 Use formula (6) to verify that all players have the same Shapley–Shubik power index in voting games where unanimity is required.
- 11.9 Calculate the Shapley–Shubik power indexes in a company with four shareholders who own 1, 3, 3 and 4 shares.
- 11.10 Calculate the Shapley–Shubik power index of the players of a weighted majority game with $n \geq 3$ players, when $w_1 = 2n - 3$ and $w_i = 2$ for the other players i .
- 11.11 The United Nation Security Council consists of 15 member nations, of which 5 are permanent members with veto rights. In order for a resolution to pass, 9 out of 15 votes are needed, but each of the 5 permanent members has veto power. This situation can be considered as a weighted voting game, in which each of the five permanent member nations gets weight 7 and each of the other 10 member nations gets weight 1, and a total of 39 weights are required to pass a resolution. Determine the Shapley–Shubik power index for a permanent member and for a non-permanent member.

Chapter 12

Coalitional Games without Transferable Utility

In a coalitional game with transferable utility, each coalition S is characterized by a number $v(S)$ representing the worth available to the coalition, and the main problem is to distribute the worth $v(N)$ of the grand coalition N among the players.

In this chapter we will briefly treat coalitional games with non-transferable utility. In such a game, players must agree on an action from a given set X . Each player has its preferences that can be described by preference relations or utility functions. The players can not compensate each other by transferring utility, and one reason why this is not possible may be that they do not measure their utility of a certain action in comparable units. However, the players can form coalitions, and each coalition S controls a subset $V(S)$ of X . This makes it possible for a coalition S to block a proposed action $x \in V(N)$ if there is an action $y \in V(S)$ that all members of S prefer to x . This way of thinking leads to a generalization of the concept of core.

12.1 Coalitional games without transferable utility

Definition 12.1.1 A coalitional game $\langle N, X, V, (\succeq_i) \rangle$ without transferable utility consists of

- a finite set N (of players);
- a set X (of actions);
- for each coalition S (i.e. nonempty subset of N) a subset $V(S)$ of X ;
- for each player $i \in N$ a preference relation \succeq_i on X .

In the following three definitions, $\langle N, X, V, (\succeq_i) \rangle$ is an arbitrary coalitional game without transferable utility.

Definition 12.1.2 The coalition S has a *strong objection* against the action $x \in X$ if there exists an action $y \in V(S)$ such that $y \succ_i x$ for all $i \in S$.

It is, of course, not rational for a coalition to accept a proposed action, if there is an action that the coalition itself controls and that all its members like better. This leads to the concept of core.

Definition 12.1.3 The *core* of the game $\langle N, X, V, (\succeq_i) \rangle$ consists of all actions in $V(N)$ against which there are no strong objections from any coalition.

Thus, an action $x \in V(N)$ belongs to the core if and only if there exists, for each coalition S and each action $y \in V(S)$, at least one coalition member $i \in S$ who likes x at least as much as y (i.e. $x \succeq_i y$).

There is a weaker type of objections that leads to the concept of Pareto optimality.

Definition 12.1.4 The coalition S has a *weak objection* against the action $x \in X$ if there is an action $y \in V(S)$ such that $y \succeq_i x$ for all coalition members $i \in S$ and $y \succ_i x$ for at least one coalition member i .

An action $x \in V(N)$ is *Pareto optimal* if the grand coalition N has no weak objection against x .

In other words, an action in $V(N)$ is Pareto optimal if and only if there is no other action available to the grand coalition that all the players think is at least as good and some player thinks is better.

An action in the core does not have to be Pareto optimal, because the fact that the grand coalition has no strong objection against the action does not exclude that there exists a weak objection. See Exercise 10.1 for an example.

EXAMPLE 12.1.1 Let $\langle N, X, V, (\succeq_i) \rangle$ be the coalitional game without transferable utility in which

$$\begin{aligned} N &= \{1, 2\}, \quad X = \mathbf{R}_+^2, \\ V(\{1\}) &= \{(x_1, 0) \mid 0 \leq x_1 \leq \frac{1}{3}\}, \quad V(\{2\}) = \{(0, x_2) \mid 0 \leq x_2 \leq \frac{1}{3}\}, \\ V(N) &= \{(x_1, x_2) \mid 0 \leq x_2 \leq 1 - x_1, \quad 0 \leq x_1 \leq 1\} \\ (x_1, x_2) \succeq_1 (y_1, y_2) &\Leftrightarrow x_1 \geq y_1 \\ (x_1, x_2) \succeq_2 (y_1, y_2) &\Leftrightarrow x_2 \geq y_2. \end{aligned}$$

See Figure 12.1.

Coalition $\{1\}$ has a strong objection against all actions (x_1, x_2) with $x_1 < \frac{1}{3}$, namely the objection $y = (\frac{1}{3}, 0)$, but it lacks strong objections against actions with $x_1 \geq \frac{1}{3}$.

Coalition $\{2\}$ has a similar strong objection against all actions (x_1, x_2) with $x_2 < \frac{1}{3}$, but it has no strong objection against actions with $x_2 \geq \frac{1}{3}$.

The grand coalition N has strong objections against actions (x_1, x_2) in the triangle $V(N)$ with $x_1 + x_2 < 1$, since there exist points (y_1, y_2) in the triangle such that $x_1 < y_1$ and $x_2 < y_2$, but it has no strong or weak objection if $x_1 + x_2 = 1$, because $y_1 + y_2 > 1$ if $y_1 > x_1$ and $y_2 \geq x_2$ or if $y_1 \geq x_1$ and $y_2 > x_2$.

The core of the game is therefore equal to the set

$$\{(x_1, x_2) \mid x_1 + x_2 = 1, \frac{1}{3} \leq x_1 \leq \frac{2}{3}\},$$

and an action (x_1, x_2) is Pareto optimal if and only if it lies on the side

$$\{(x_1, x_2) \mid x_1 + x_2 = 1, 0 \leq x_1 \leq 1\}$$

of the triangle $V(N)$. □

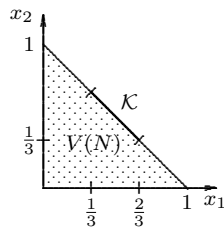


Figure 12.1. The action set $V(N)$ and the core \mathcal{K} of the game in Example 12.1.1.

EXAMPLE 12.1.2 Every coalitional game $\langle N, v \rangle$ with transferable utility can be translated into a coalitional game $\langle N, X, V, (\succeq_i) \rangle$ without transferable utility in the following way:

$$X = \mathbf{R}^n,$$

$$V(S) = \{x \in \mathbf{R}^n \mid \sum_{i \in S} x_i = v(S) \text{ and } x_j = 0 \text{ for all } j \notin S\},$$

$$x \succeq_i y \text{ if and only if } x_i \geq y_i.$$

Coalitional games with transferable utility can therefore be considered as special cases of coalitional games without transferable utility. The core is the same whether we use the core definition for games with transferable utility or the core definition for games without transferable utility. \square

Exercise

12.1 Determine the Pareto optimal actions and the core of the game $\langle N, X, V, (\succeq_i) \rangle$ in which $N = \{1, 2\}$, $X = \mathbf{R}^2$, $V(\{1\}) = V(\{2\}) = \{(0, 0)\}$,

$$(x_1, x_2) \succeq_i (y_1, y_2) \Leftrightarrow x_i \geq y_i,$$

and

a) $V(N) = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$ b) $V(N) = \{(x_1, x_2) \mid \max_{i \in N} |x_i| \leq 1\}$

c) $V(N) = \{(x_1, x_2) \mid |x_1| + |x_2| \leq 1\}.$

12.2 Exchange economies

EXAMPLE 12.2.1 Consider a society without money where all trade takes place through exchange. For simplicity, we assume that the society consists of only two people and that there are only two goods – raspberry jam and strawberry jam. Person 1 initially has 8 kg of raspberry jam and 2 kg of strawberry jam, which we indicate by saying that his basket of goods is

given by the vector $a = (8, 2)$. Person 2 has 2 kg of raspberry jam and 5 kg of strawberry jam, so his initial basket of goods is described by the vector $b = (2, 5)$.

The two people's preferences for jam differ and are described by two utility functions; the utility for person 1 of a basket consisting of x_1 kg of raspberry jam and x_2 kg of strawberry jam is $u_1(x_1, x_2) = x_1x_2$, and person 2's utility of the same basket is $u_2(x_1, x_2) = \min(x_1, x_2)$. It may therefore be beneficial for both to exchange some quantities of jams with each other.

We can model the situation as a two-player coalitional game with action set $X = \mathbf{R}_+^2 \times \mathbf{R}_+^2 (= \mathbf{R}_+^4)$ and the following interpretation of an element $(x, y) = (x_1, x_2, y_1, y_2)$ in X ; the first half (x_1, x_2) is a basket of goods for player 1 and the second half (y_1, y_2) is a basket of goods for player 2.

Of course, not all elements (x, y) of X can be achieved by trading, because together the two players have only 10 kg of raspberry jam and 7 kg of strawberry jam. Therefore, we put

$$V(N) = \{(x_1, x_2, y_1, y_2) \in \mathbf{R}_+^4 \mid x_1 + y_1 = 10, x_2 + y_2 = 7\}.$$

The elements of $V(N)$ describe the results of all possible exchanges.

If the two players do not cooperate nothing happens; the players only have the amounts of jams they had from the beginning. Therefore,

$$V(\{1\}) = V(\{2\}) = \{(a, b)\} = \{(8, 2, 2, 5)\}.$$

The players' preferences for a given outcome are given by the above functions, but we must modify the definitions so that the functions are defined on X . We assume that the players only care about their own holdings, that is

$$u_1(x, y) = x_1x_2 \quad \text{and} \quad u_2(x, y) = \min(y_1, y_2).$$

The game can be illustrated graphically using the *Edgeworth box*. Let R be a rectangle of length $a_1 + b_1 = 10$ and width $a_2 + b_2 = 7$, and choose two opposite corners O and O' as origins of two coordinate systems Ox_1x_2 and $O'y_1y_2$ with coordinate axes along the sides of the rectangle as in Figure 12.2. The connection between the coordinates (x_1, x_2) and (y_1, y_2) in the two coordinate systems of a point is given by the equations $x_1 + y_1 = 10$ and $x_2 + y_2 = 7$. This means that the quadruple (x, y) lies in $V(N)$ if and only if (x_1, x_2) and (y_1, y_2) are coordinates of the same point in the rectangle R . In particular, (a_1, a_2) and (b_1, b_2) are coordinates of the same point.

We can thus identify $V(N)$ with the rectangle R . We can also draw the two players' indifference curves in the rectangle – the hyperbolas $x_1x_2 = k_1$ and the "L-square curves" $\min(y_1, y_2) = k_2$ for various values of k_1 and k_2 . The

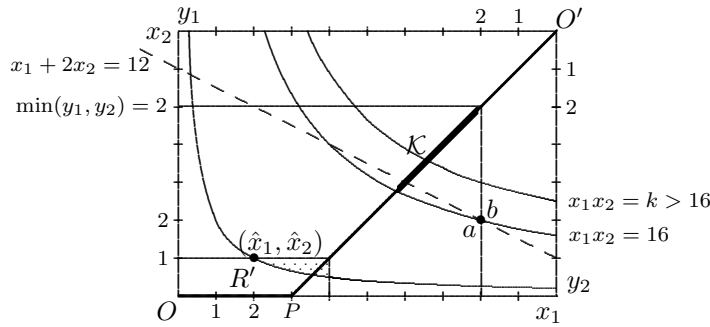


Figure 12.2. The Edgeworth box in Example 12.2.1. The points on the broken line OPO' correspond to the Pareto optimal baskets of goods, and the core \mathcal{K} consists of all Pareto optimal baskets between the hyperbola $x_1x_2 = 16$ and the "L-square curve" $\min(y_1, y_2) = 2$. The dashed line $x_1 + 2x_2 = 12$ is the budget limit for the players when the goods are priced at the equilibrium price. Player 1 can finance all baskets that are below the line and player 2 all baskets that are above the line. The line's intersection with the core, $(6, 3, 4, 4)$, is the competitive equilibrium solution.

two indifference curves that pass through the initial point with x -coordinates $(8, 2)$ and y -coordinates $(2, 5)$, are $x_1x_2 = 16$ and $\min(y_1, y_2) = 2$. These are depicted in Figure 12.2.

The points on the broken line OPO' with the equation

$$x_2 = \begin{cases} 0 & \text{if } 0 \leq x_1 \leq 3, \\ x_1 - 3 & \text{if } 3 < x_1 \leq 10 \end{cases}$$

correspond to the Pareto optimal baskets of goods. To see this, we first note that the utility of player 1 and player 2 at points $(x, y) \in V(N)$ with $x_2 = 0$ and $0 \leq x_1 \leq 3$ are 0 and 7, respectively. It is impossible for player 1 to obtain a positive utility without decreasing player 2's utility, because $x_2 > 0$ implies $y_2 = 7 - x_2 < 7$. And player 2 can not increase his utility either, because 7 is his maximum utility.

At points (x, y) on the broken line, where $x_2 = x_1 - 3$ and $3 < x_1 \leq 10$, we have $y_1 = 10 - x_1 = 7 - x_2 = y_2$. To improve player 2's utility we have to decrease x_1 and x_2 , which decreases player 1's utility x_1x_2 . And to improve player 1's utility, we have to increase x_1 or x_2 , which in both cases leads to a utility decrease for player 2.

This proves that all points on the broken line OPO' are Pareto optimal. No point outside the broken line is Pareto optimal, because for each such point there are points that are better for both players. If a point with x -coordinates (\hat{x}_1, \hat{x}_2) lies above the broken line, then player 2 has a greater

utility at all points inside the rectangle R' bounded by the lines $x_1 = 0$, $x_2 = \hat{x}_2$, $x_1 = \hat{x}_2 + 3$ and $x_2 = 0$ (see Figure 12.2), while player 1's utility is greater at all points above the indifference curve $x_1x_2 = \hat{x}_1\hat{x}_2$, that passes through the point (\hat{x}_1, \hat{x}_2) . Since the intersection of these two regions is nonempty (see the dotted area in the figure), there exist points which give both players a greater utility than the point (\hat{x}_1, \hat{x}_2) . The corresponding applies if (\hat{x}_1, \hat{x}_2) lies below the broken line.

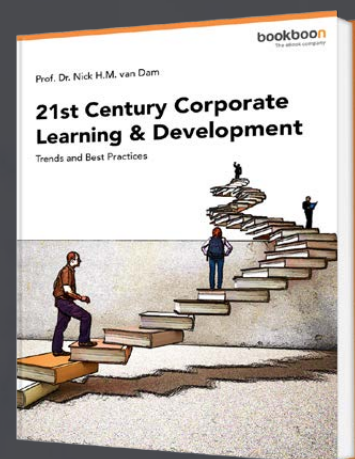
In the present case, the set of Pareto optimal actions coincides with the set of actions against which the grand coalition has no *strong* objections, so the core is a subset of the set of Pareto optimal actions. The original holding $(a, b) = (8, 2, 2, 5)$ is a strong objection for player 1 against any basket (x, y) with $x_1x_2 < 16$ and for player 2 against any basket with $\min(y_1, y_2) < 2$. The core \mathcal{K} of the game is therefore equal to the intersection of the broken line OPO' and the region between the hyperbola $x_1x_2 = 16$ and the "L-square curve" $\min(y_1, y_2) = 2$. The hyperbola intersects the line at a point with x_1 -coordinate $(3 + \sqrt{73})/2 \approx 5.77$. Thus, the core \mathcal{K} consists of all baskets $(x_1, x_1 - 3, 10 - x_1, 10 - x_1)$ with $5.77 \leq x_1 \leq 8$.

Let us see what happens if we introduce money in the exchange economy and price the jam so that the kilo price for raspberry jam is \$1, and the kilo price for strawberry jam is \$ p . This means that player 1's jam fortune is

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$\$(8 + 2p)$ and player 2's jam fortune is $\$(2 + 5p)$. When they trade from each other, each player wants to choose the amounts of jam that are most desirable according to his preferences among all those that are affordable. This means that they should solve the maximization problems

$$\begin{array}{l} \text{Maximize } x_1x_2 \text{ as} \\ \left\{ \begin{array}{l} x_1 + px_2 \leq 8 + 2p \\ x_1, x_2 \geq 0 \end{array} \right. \end{array} \quad \text{and} \quad \begin{array}{l} \text{Maximize } \min(y_1, y_2) \text{ as} \\ \left\{ \begin{array}{l} y_1 + py_2 \leq 2 + 5p \\ y_1, y_2 \geq 0 \end{array} \right. \end{array}$$

The solutions to the two problems are

$$x_1 = 4 + p, x_2 = \frac{p + 4}{p} \quad \text{and} \quad y_1 = y_2 = \frac{5p + 2}{p + 1}.$$

This looks good, but is there enough jam for the exchange, or will there be jam left over. The problem is that they exchange jams with each other, and then the pricing must be such that the above optimal solutions can be implemented. In other words, the price p must be set so that

$$x_1 + y_1 = 4 + p + \frac{5p + 2}{p + 1} = 10 \quad \text{and} \quad x_2 + y_2 = \frac{p + 4}{p} + \frac{5p + 2}{p + 1} = 7.$$

The equations have a unique positive solution, $p = 2$. The corresponding price vector $(1, 2)$ for raspberry and strawberry jam is called the *competitive equilibrium price*. Of course, the solution is not affected by a change of currency, so $(\lambda, 2\lambda)$ is also a competitive equilibrium price for each $\lambda > 0$.

For $p = 2$, the optimal solutions are $x_1 = 6, x_2 = 3, y_1 = 4, y_2 = 4$. At equilibrium, player 1 must sell 2 kg of raspberry jam to player 2 and buy 1 kg of strawberry jam from him.

Note that the competitive equilibrium solution $(6, 3, 4, 4)$ belongs to the core. This is, as we will soon show, no coincidence. \square

The reasoning in Example 12.2.1 can be generalized to exchange markets with more than two agents and two goods. This leads to the following definitions.

Definition 12.2.1 An *exchange economy* consists of

- a finite set N ;
- a positive integer m ;
- for each $i \in N$ a vector $a_i \in \mathbf{R}_+^m$;
- for each $i \in N$ a preference relation \succeq_i on \mathbf{R}_+^m .

The interpretation of the various ingredients of the exchange economy is as follows:

N is a set of n agents, and m is the number of goods. An agent i 's holding of the different goods is described by vectors $x_i = (x_{i1}, x_{i2}, \dots, x_{im}) \in \mathbf{R}_+^m$, where x_{ik} is the amount of item k .

The vector a_i , the endowment of i , is the basket of goods initially owned by i .

$x_i \succeq_i y_i$ means that the basket of goods x_i is perceived by i to be at least as good as y_i .

An *allocation* is a distribution of the total number of goods in the exchange economy among the agents. We describe allocations using n -tuples $x = (x_1, x_2, \dots, x_n)$, where $x_i \in \mathbf{R}_+^m$ is agent i 's basket of goods and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n a_i.$$

It is understood that the goods can be transferred freely between the agents, but they have no way of compensating each other except by exchanging goods.

A *price vector* is a vector $p = (p_1, p_2, \dots, p_m) \in \mathbf{R}_+^m$ with $p \neq 0$. The value of agent i 's basket $x_i = (x_{i1}, x_{i2}, \dots, x_{im})$ of goods at the price vector p is given by the scalar product $p \cdot x_i = \sum_{k=1}^m p_k x_{ik}$.

Definition 12.2.2 A *competitive equilibrium* in the exchange economy is a pair (p^*, x^*) consisting of a price vector $p^* = (p_1^*, p_2^*, \dots, p_m^*)$ and an allocation $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ such that

$$p^* \cdot x_i^* \leq p^* \cdot a_i \quad \text{and} \quad p^* \cdot x_i \leq p^* \cdot a_i \Rightarrow x_i^* \succeq_i x_i.$$

for each agent $i \in N$. In other words, for each agent i , x_i^* is the basket of goods that gives him the greatest satisfaction among all baskets that he can afford at price p^* using the initial endowment a_i .

The following proposition is a special case of a more general theorem due to Arrow and Debreu about competitive equilibria in markets with consumption and production.

Proposition 12.2.1 *There exists a competitive equilibrium (p^*, x^*) in an exchange economy that satisfies the following five conditions for each $i \in N$:*

- (i) $a_i > 0$, that is all agents initially have a positive quantity of all goods.
- (ii) \succeq_i is a continuous preference relation.
- (iii) \succeq_i is increasing, which means that $x_i \succeq_i y_i$ if $x_i \geq y_i$, i.e. "more is better".
- (iv) $\lambda x_i + (1 - \lambda)y_i \succ_i y_i$ if $x_i \succ_i y_i$ and $0 < \lambda < 1$.
- (v) for each $x_i \in \mathbf{R}_+^m$ there is a $y_i \in \mathbf{R}^m$ such that $y_i \succ_i x_i$.

Remark. The competitive allocation is not necessarily unique.

Exchange economies can be perceived as coalitional games without transferable utility. To the exchange economy in Definition 12.2.1 we associate the coalitional game $\langle N, X, V, (\succeq'_i) \rangle$, where

- $X = \mathbf{R}_+^m \times \mathbf{R}_+^m \times \cdots \times \mathbf{R}_+^m$ (n factors);
- $V(S) = \{x \in X \mid \sum_{i \in S} x_i = \sum_{i \in S} a_i \text{ and } x_j = a_j \text{ for all } j \in N \setminus S\}$;
- $(x_1, x_2, \dots, x_n) \succeq'_i (y_1, y_2, \dots, y_n) \Leftrightarrow x_i \succeq_i y_i$.

The second condition means that each coalition can distribute the goods initially available to the coalition freely between its members, and the third condition means that each player only cares about his own consumption.

Definition 12.2.3 The *core* of an exchange economy is the core of the associated coalitional game.

Proposition 12.2.2 *If an exchange economy has a competitive equilibrium (p^*, x^*) , then the equilibrium allocation x^* belongs to the core of the exchange economy.*

Proof. Suppose that x^* is not included in the core. Then there is a coalition S and for each player $i \in S$ a basket of goods y_i so that $\sum_{i \in S} y_i = \sum_{i \in S} a_i$



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and $y_i \succ_i x_i^*$. According to the competitive equilibrium definition, the baskets of goods y_i therefore do not meet the players' budget restrictions. Hence, $p^* \cdot y_i > p^* \cdot a_i$ for all $i \in S$. This implies that $p^* \cdot \sum_{i \in S} y_i > p^* \cdot \sum_{i \in S} a_i$, which violates the fact that $\sum_{i \in S} y_i = \sum_{i \in S} a_i$. \square

It follows, of course, as a corollary of Proposition 12.2.2 that an exchange economy that has a competitive equilibrium has a nonempty core.

Exercise

12.2 Consider an exchange economy with two agents and two goods. Agent 1's endowment is $(1, 1)$ and agent 2's endowment is $(2, 1)$. The preferences for a basket (x_1, x_2) of goods are given by the utility functions $u_1(x_1, x_2) = x_1 + x_2$ and $u_2(x_1, x_2) = x_1 x_2$. Find the core and the competitive equilibrium of the economy.

12.3 The Nash bargaining solution

Consider a negotiation between two parties, and let X denote the set of options negotiated by the parties. A possible outcome is that the parties do not agree and negotiations break down. This outcome will be called D . We assume that the parties' preferences for the different bargaining options can be described by two utility functions u_1 and u_2 , and that there is at least one option in X , which both prefer strictly to the disagreement outcome D .

The negotiation can be described as a coalitional game without transferable utility with $N = \{1, 2\}$, $V(\{1\}) = V(\{2\}) = \{D\}$ and $V(N) = X \cup \{D\}$.

We are looking for reasonable and fair bargaining solutions, but in order to arrive at such a solution we must of course specify what should be meant by reasonable and fair.

As a first step, we begin by transforming the problem of finding an option that both parties can agree on to the problem of choosing a suitable point in a subset of \mathbf{R}^2 by defining

$$S = \{u_1(x), u_2(x) \mid x \in X\} \quad \text{and} \quad d = (u_1(D), u_2(D)).$$

Both parties are indifferent between two options x and x' that are mapped on the same point in S , so we can concentrate on finding a "fair" point in S .

Our negotiation has now been transformed into a coalitional game

$$\langle \{1, 2\}, \mathbf{R}^2, V, (\succeq_i) \rangle$$

without transferable utility with coalition sets $V(\{1\}) = V(\{2\}) = \{d\}$ and $V(\{1, 2\}) = S$, and with the two players' preferences given by

$$(x_1, x_2) \succeq_i (y_1, y_2) \text{ if and only if } x_i \geq y_i.$$

The assumption that there is an agreement that both players prefer strictly to the disagreement option D means that there is a point $x \in S$ with $x > d$. Here, $x = (x_1, x_2) > d = (d_1, d_2)$ means that $x_1 > d_1$ and $x_2 > d_2$.

A *bargaining solution* is a function f that to each set S and each point $d \in S$ with the above properties associates a unique point $f(S, d)$ in S .

A reasonable requirement for a bargaining solution f to be fair is that $f(S, d)$ is a Pareto optimal point in the core of the bargaining game. The problem is, of course, that the core generally consists of more than one point. Coalition $\{i\}$ has strong objections against bargain proposals x with $x_i < d_i$, so the core is a subset of the set $S^*(d) = \{x \in S \mid x \geq d\}$.

The Nash axioms

Nash's idea consisted in defining a solution f to the bargaining game as being "fair and reasonable" if it has four desirable properties, which determine $f(S, d)$ unambiguously for compact, convex sets S .

So from now on, we assume that S is a compact, convex subset of \mathbf{R}^2 , that d is a point in S , and that S contains some point x such that $x > d$. Writing $f_1(S, d)$ and $f_2(S, d)$ for the coordinates of the bargaining function $f(S, d)$, Nash's four axioms are as follows:

Axiom 1 (Pareto optimality) *The point $f(S, d)$ is Pareto optimal, i.e. there is no point $x \in S$ such that $x_i \geq f_i(S, d)$ for $i = 1, 2$ with strict inequality for at least one i .*

Axiom 2 (Symmetry) *If the set S is symmetric around the line $x_1 = x_2$ and $d_1 = d_2$, then $f_1(S, d) = f_2(S, d)$.*

Axiom 3 (Independence of irrelevant alternatives) *If T is a closed, convex subset of S , $d \in T$ and $f(S, d) \in T$, then $f(T, d) = f(S, d)$.*

A mapping $\phi: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is called a *scaling* if

$$\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2)) = (\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2),$$

where α_1, α_2 are positive real numbers and β_1, β_2 are arbitrary real numbers.

Scalings map compact, convex sets onto compact, convex sets, and Nash's fourth axiom gives the relation between the bargaining solutions $f(S, d)$ and $f(\phi(S), \phi(d))$.

Axiom 4 (Scaling invariance) $f(\phi(S), \phi(d)) = \phi(f(S, d))$ for all scalings ϕ .

Pareto optimality is a reasonable requirement, because if a bargaining solution fails to be Pareto optimal, there is room for renegotiations which lead to a better result for either party without making it worse for the other.

The set S is symmetric if

$$(x_1, x_2) \in S \Leftrightarrow (x_2, x_1) \in S.$$

For symmetric sets S and breakdown threats d , the negotiation situation is the same if the two parties change places with each other, and it is a fairness requirement that the outcome of the negotiations should not depend on how the two parties are numbered.

Assume that a negotiation, with bargaining set S and breakdown point d in a subset T of S , leads to the outcome $f(S, d)$ belonging to T . This suggests that the options in $S \setminus T$ are unattractive to the parties and irrelevant in the context. Therefore, by restricting the bargaining set to T , one should obtain the same outcome of the negotiation as before, i.e. $f(T, d) = f(S, d)$, which is the gist of axiom 3.

Against this, one may object that the options outside T may have affected the first negotiation in so far as they served as threats or hopes, and that the

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outcome of the last negotiation may be affected by the loss of these threats or hopes.

Axiom 4 is uncontroversial as it only means that the outcome of the negotiations should be independent of the choice of equivalent utility functions.

The Nash bargaining solution

Proposition 12.3.1 *There is a unique function f that satisfies Nash's four axioms. The bargaining solution $f(S, d)$ is the unique maximum point in the problem of maximizing the product $(x_1 - d_1)(x_2 - d_2)$ over all (x_1, x_2) in the set $S^*(d) = \{x \in S \mid x \geq d\}$.*

The function f in Proposition 12.3.1 is called the *Nash bargaining solution*.

We begin by proving that the maximization problem has a unique solution and that it is invariant under scaling.

Lemma 12.3.2 *Let $M = \max\{(x_1 - d_1)(x_2 - d_2) \mid x \in S^*(d)\}$.*

(i) *The maximum value M is attained at a unique point $\hat{x} = (\hat{x}_1, \hat{x}_2)$ in $S^*(d)$.*

(ii) *The set S is a subset of the halfplane*

$$H = \{x \in \mathbf{R}^2 \mid (\hat{x}_2 - d_2)(x_1 - d_1) + (\hat{x}_1 - d_1)(x_2 - d_2) \leq 2M\}$$

which is bounded by the tangent of the curve $(x_1 - d_1)(x_2 - d_2) = M$ at the point \hat{x} .

(iii) *Let ϕ be an arbitrary scaling, and set $S' = \phi(S)$ and $d' = \phi(d)$. Then, $\phi(\hat{x})$ is the unique solution to the maximization problem*

$$\max \{(y_1 - d'_1)(y_2 - d'_2) \mid y \in S'^*(d')\}.$$

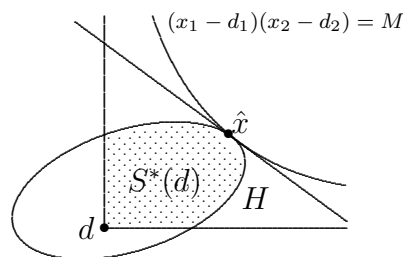


Figure 12.3. Illustration to Lemma 12.3.2.

Proof. (i) and (ii) The maximization problem has a solution $\hat{x} > d$, because the set $S^*(d)$ is compact (and contains points $> d$). To prove that the solution is unique and that S is a subset of H , we transform the general case to the special case $d = (0, 0)$ using the translation $x'_1 = x_1 - d_1$, $x'_2 = x_2 - d_2$.

So we may assume that $d = (0, 0)$. Since the function $x_1 \mapsto M/x_1$ is strictly convex when $x_1 > 0$, the curve $x_2 = M/x_1$ lies above its tangent $\hat{x}_2x_1 + \hat{x}_1x_2 = 2M$ everywhere except the point of tangency \hat{x} . The convex set $\{x \in \mathbf{R}_+^2 \mid x_1x_2 \geq M\}$, with the exception of the point of tangency \hat{x} , is therefore a subset of the open halfplane $U = \{x \in \mathbf{R}^2 \mid \hat{x}_2x_1 + \hat{x}_1x_2 > 2M\}$. If we prove that the set S is a subset of the opposite closed halfplane $H = \{x \in \mathbf{R}^2 \mid \hat{x}_2x_1 + \hat{x}_1x_2 \leq 2M\}$, i.e. assertion (ii), it therefore follows that $x_1x_2 < M$ for all points x in $S^*(0) = S \cap \mathbf{R}_+^2$ except \hat{x} , which proves assertion (i).

Suppose that there is a point $x = (x_1, x_2)$ in $S \cap U$; we will prove that this leads to a contradiction. Because $\hat{x} \in S$, $\hat{x} > 0$ and S is convex, the points

$$tx + (1-t)\hat{x} = (tx_1 + (1-t)\hat{x}_1, tx_2 + (1-t)\hat{x}_2)$$

are in $S^*(0)$ for all sufficiently small positive numbers t , and consequently

$$(1) \quad (tx_1 + (1-t)\hat{x}_1)(tx_2 + (1-t)\hat{x}_2) \leq M$$

for all sufficiently small $t > 0$, due to the definition of M as maximum value.

Since x lies in the open halfplane U , $\hat{x}_2x_1 + \hat{x}_1x_2 = 2M + \varepsilon$ for some number $\varepsilon > 0$. Therefore,

$$\begin{aligned} & (tx_1 + (1-t)\hat{x}_1)(tx_2 + (1-t)\hat{x}_2) \\ &= t^2x_1x_2 + t(1-t)(\hat{x}_1x_2 + \hat{x}_2x_1) + (1-t)^2\hat{x}_1\hat{x}_2 \\ &= t^2x_1x_2 + t(1-t)(2M + \varepsilon) + (1-t)^2M \\ &= M + \varepsilon t - t^2(M + \varepsilon - x_1x_2) > M \end{aligned}$$

for all sufficiently small numbers $t > 0$, which contradicts inequality (1). Hence, $S \cap U = \emptyset$, and S is thus a subset of the halfplane H .

(iii) Let $y_1 = \phi_1(x_1) = \alpha_1x_1 + \beta_1$, $y_2 = \phi_2(x_2) = \alpha_2x_2 + \beta_2$. Then

$$\begin{aligned} \alpha_1\alpha_2(x_1 - d_1)(x_2 - d_2) &= (\alpha_1x_1 + \beta_1 - \alpha_1d_1 - \beta_1)(\alpha_2x_2 + \beta_2 - \alpha_2d_2 - \beta_2) \\ &= (y_1 - d'_1)(y_2 - d'_2). \end{aligned}$$

Maximizing the product $(x_1 - d_1)(x_2 - d_2)$ as x varies over $S^*(d)$ is equivalent to maximizing the product $(y_1 - d'_1)(y_2 - d'_2)$ as y varies over $S'^*(d')$, and the relationship between the two maximum points (\hat{x}_1, \hat{x}_2) and (\hat{y}_1, \hat{y}_2) is given by $\hat{y}_1 = \phi_1(\hat{x}_1)$, $\hat{y}_2 = \phi_2(\hat{x}_2)$, or more briefly $\hat{y} = \phi(\hat{x})$. \square

Proof of Proposition 12.3.1. Define $\phi(S, d)$ as the unique solution \hat{x} of the optimization problem

$$\max \{(x_1 - d_1)(x_2 - d_2) \mid x \in S^*(d)\}.$$

We first show that $\phi(S, d)$ satisfies the four axioms and then show that there is no other solution.

Pareto optimality follows from (ii) in Lemma 12.3.2, because the point x does not belong to the closed halfplane H , and afortiori not to the set S , if $x_i \geq \hat{x}_i$ for $i = 1, 2$ with strict inequality for at least one i .

Suppose that the set S is symmetric and that $d_1 = d_2$. To prove that the symmetry axiom is fulfilled, we have to prove that $\hat{x}_1 = \hat{x}_2$.

The symmetry condition implies that the point (\hat{x}_2, \hat{x}_1) lies in $S^*(d)$, and since S is convex,

$$z = \frac{1}{2}(\hat{x}_1, \hat{x}_2) + \frac{1}{2}(\hat{x}_2, \hat{x}_1) = \left(\frac{1}{2}(\hat{x}_1 + \hat{x}_2), \frac{1}{2}(\hat{x}_1 + \hat{x}_2)\right)$$

is also a point in $S^*(d)$. The inequality of arithmetic and geometric means implies that

$$(z_1 - d_1)(z_2 - d_1) = \left(\frac{(\hat{x}_1 - d_1) + (\hat{x}_2 - d_1)}{2}\right)^2 \geq (\hat{x}_1 - d_1)(\hat{x}_2 - d_1) = M$$

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with equality if and only if $\hat{x}_1 - d_1 = \hat{x}_2 - d_1$, i.e. if and only if $\hat{x}_1 = \hat{x}_2$. Since strict inequality is impossible, due to the definition of M as the maximum value, it follows that $\hat{x}_1 = \hat{x}_2$.

Axiom 3 is trivially fulfilled, because if \hat{x} is a solution to the problem $\max \{(x_1 - d_1)(x_2 - d_2) \mid x \in S^*(d)\}$ and \hat{x} is a point in T , then \hat{x} is obviously also the solution to the problem of maximizing the product over the smaller set $T^*(d)$.

Finally, Axiom 4 is fulfilled according to Lemma 12.3.2 (iii).

Thus, the existence of a bargaining solution that meets the Nash axioms is shown. To show the uniqueness, we first investigate what $f(S, d)$ has to be for certain special sets S .

If the set S is symmetric around the line $x_1 = x_2$, then $f(S, (0, 0))$ is the point in S on the line $x_1 = x_2$ that is furthest from the origin (and in the first quadrant) because of the Pareto and symmetry axioms.

Let now T be an arbitrary compact, convex subset of the halfplane $H_1 = \{(x_1, x_2) \mid x_1 + x_2 \leq 2\}$, and suppose that T contains the point $(1, 1)$. Each such set T is obviously a subset of some compact, convex, symmetric set S that is also a subset of H ; as S we may for example choose a sufficiently big rectangle with one side along the line $x_1 + x_2 = 2$ and symmetric around the line $x_1 = x_2$. Since $f(S, (0, 0)) = (1, 1)$, it now follows from Axiom 3 that $f(T, (0, 0)) = (1, 1)$.

Finally, suppose that S is an arbitrary compact, convex set and define \hat{x} as the maximum point of Proposition 12.3.1. Let ϕ be the scaling that maps \hat{x} onto the point $(1, 1)$ and d onto $(0, 0)$; the coefficients of the scaling are determined by the systems

$$\begin{cases} \alpha_1 \hat{x}_1 + \beta_1 = 1 \\ \alpha_1 d_1 + \beta_1 = 0 \end{cases} \quad \begin{cases} \alpha_2 \hat{x}_2 + \beta_2 = 1 \\ \alpha_2 d_2 + \beta_2 = 0. \end{cases}$$

According to Lemma 12.3.2 (iii), $(1, 1)$ is the optimal solution to the maximization problem $\max \{x_1 x_2 \mid x \in \phi(S)\}$, and according to (ii) in the same lemma, $S' = \phi(S)$ is a convex, compact subset of the halfplane

$$H_1 = \{x \in \mathbf{R}^2 \mid x_1 + x_2 \leq 2\}.$$

Therefore, it follows from the above special case and scaling invariance that

$$\phi(f(S, d)) = f(\phi(S), \phi(d)) = f(S', (0, 0)) = (1, 1) = \phi(\hat{x}),$$

and since the mapping ϕ is invertible, this implies that $f(S, d) = \hat{x}$. The uniqueness is proven \square

Exercise

12.3 Determine the Nash bargaining solution when

$$S = \{(x_1, x_2) \mid 0 \leq x_2 \leq 4 - x_1^2\}$$

and the disagreement point d is the point

- a) (0,0) b) (0,1).

12.4 Charlie and Lisa intend to buy shares for one million dollars. They are interested in three companies, A, B and C, and the choice is between investing the entire sum in one company or distributing it between the companies appropriately. However, Charlie and Lisa have different expectations regarding the future earnings trend in the various companies, and the benefit they see of investing the entire amount in a single company is shown in the following table, where the utility is stated on an individual utility scale.

	Charlie's utility	Lisa's utility
Company A	1	4
Company B	2	4
Company C	4	2

It is now up to Charlie and Lisa to agree on which combination to choose, when both want the greatest utility. If they do not agree, the entire amount is deposited into a bank account, giving each one utility 1. Determine the Nash bargaining solution.

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Appendix 1

Convexity

Convexity plays an important role in game theory, and for those who are not so familiar with this area, we describe some important concepts and results here.

A linear combination $x = \sum_{j=1}^m \lambda_j x_j$ of vectors x_1, x_2, \dots, x_m is a *convex combination* of the vectors if $\sum_{j=1}^m \lambda_j = 1$ och $\lambda_j \geq 0$ för alla j .

The *line segment* between two (distinct) points x_1 and x_2 in \mathbf{R}^n is the set of all convex combinations of x_1 and x_2 .

A subset X of \mathbf{R}^n is *convex* if it contains the line segment between any two of its points, i.e. if

$$x_1, x_2 \in X, 0 \leq \lambda \leq 1 \Rightarrow \lambda x_1 + (1 - \lambda)x_2 \in X.$$

A convex set also contains any convex combination of any number of its points, which is easily proved by induction on the number of points

The set of solutions to a system of finitely many inequalities, i.e. a system of the type

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m, \end{cases}$$

is a convex set, called a *polyhedron*.

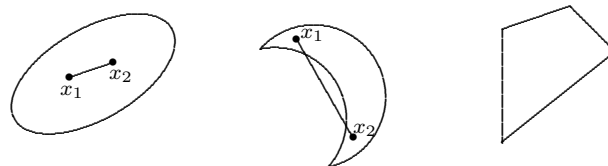


Figure 1. A convex set, a non-convex set and a polyhedron in \mathbf{R}^2 .

Many results in game theory and in mathematical economics regarding utility functions assume that they are concave, or more generally quasiconcave.

A function $f: X \rightarrow \mathbf{R}$ is *concave* if its domain X is convex and

$$(1) \quad x_1, x_2 \in X, 0 < \lambda < 1 \Rightarrow f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

Geometrically, the above condition means that all points on the chord between any two points on the function graph are below the graph.

The function f is *strictly concave* if condition (1) holds with \geq replaced with strict inequality $>$ for all distinct points x_1 and x_2 in X .

It is easily seen that a function $f: X \rightarrow \mathbf{R}$ is concave if and only if the set

$$\{(x, t) \in X \times \mathbf{R} \mid t \leq f(x)\}$$

of all points below the graph of the function is a convex set.

The set $\{x \in X \mid f(x) \geq a\}$ is a convex set if $f: X \rightarrow \mathbf{R}$ is a concave function and a is an arbitrary real number. But the set may be convex for any a without f having to be concave. This observation motivates the following definition:

A function $f: X \rightarrow \mathbf{R}$ with a convex domain X is called *quasiconcave* if the sets $\{x \in X \mid f(x) \geq a\}$ are convex for all constants a .

It is easy to verify that a function f is quasiconcave if and only if

$$(2) \quad f(\lambda x_1 + (1 - \lambda)x_2) \geq \min(f(x_1), f(x_2))$$

for all points $x_1, x_2 \in X$ and all real numbers λ such that $0 < \lambda < 1$.

The function f is called *strictly quasiconcave* if (2) holds with strict inequality except in the case $x_1 = x_2$.

Figure 2 shows the graph of a concave function and the graph of a non-concave but quasiconcave function.

The set of all maximum points of a quasiconcave function $f: X \rightarrow \mathbf{R}$ that assumes a maximum value m , is a convex set, because the set of maximum points is equal to the set $\{x \in X \mid f(x) \geq m\}$. The maximum point is unique if f is strictly quasiconcave.

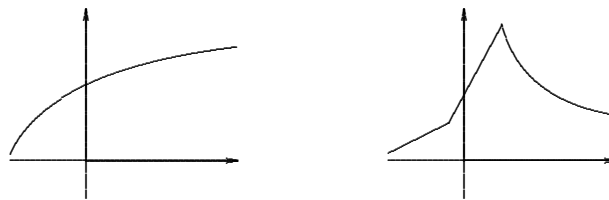


Figure 2. To the left a strictly concave function and to the left a quasiconcave (non-concave) function

An example of a (non-concave) strictly quasiconcave function of two variables is given by the function $g(x, y) = xy$ with the open first quadrant as domain of definition. If X is a compact convex subset of the first quadrant and X contains a point with positive coordinates, then the restriction of g to X attains a maximum value in a unique point. In Section 10.3, we used this fact in the proof of the Nash bargaining solution, but quasiconcavity was not explicitly utilized there.

The advertisement features a central graphic on the left consisting of a circular arrangement of four arrows pointing clockwise, with three stylized human figures in the center and several gears around them. To the right of this graphic, the title 'UNLEASHING CHANGE MANAGEMENT' is written in large, bold, blue capital letters. Below the title, the dates 'OCTOBER 18 & 19, 2018' and the location 'DE RODE HOED AMSTERDAM' are listed in smaller blue capital letters. At the bottom of the advertisement, there is a silhouette of the Amsterdam skyline, including a windmill and various buildings. In the bottom left corner, the text 'Global Executive Events' is displayed. A hand cursor icon is positioned over a green oval button at the bottom right of the advertisement.

Appendix 2

Kakutani's fixed point theorem

The proof in Section 2.3 of Nash's theorem on the existence of equilibrium solutions was based on Kakutani's fixed point theorem for set-valued maps. Here we derive Kakutani's theorem from another classic fixed point theorem, Brouwer's fixed point theorem. We start by formulating the two fixed-point theorems.

Theorem 1 (Brouwer's fixed point theorem) *Every continuous function f from a convex compact subset X of \mathbf{R}^n to X itself has a fixed point, i.e. there is a point $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$.*

Theorem 2 (Kakutani's fixed point theorem) *Let X be a nonempty, compact and convex subset of \mathbf{R}^n , and let $\phi: X \rightarrow \mathcal{P}(X)$ be a set-valued function on X with a closed graph and the property that $\phi(x)$ is nonempty and convex for all $x \in X$. Then there is a point $\bar{x} \in X$ such that $\bar{x} \in \phi(\bar{x})$.*

The proof of Brouwer's fixed point theorem is too complicated and long to be given here. We must content ourselves with the observation that it is enough to prove the theorem in the case when X is the unit ball in \mathbf{R}^n , because each compact convex subset of \mathbf{R}^n is homeomorphic to the unit ball in \mathbf{R}^m for some value of m .

We will deduce Kakutani's theorem from Brouwer's theorem using a technique called *partition of unity*. We start by writing \mathbf{R}^n as a union of small open hypercubes in the following way. Fix a positive integer p and put

$$U = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid \max |x_j| < 1/p\},$$

i.e. U is the open hypercube centered at the origin with side length $2/p$. Consider the family \mathcal{F} of all hypercubes obtained by translating U so that the centers of the new hypercubes are located in points whose coordinates are integer multiples of $1/p$. The union of all these hypercubes covers \mathbf{R}^n , and each point in \mathbf{R}^n lies in at most 2^n such hypercubes. (Points on the boundary of a hypercube are covered by fewer hypercubes.)

Of course, only finitely many of the hypercubes in the family \mathcal{F} intersect the given compact convex set X . Let us assume that there are N intersecting

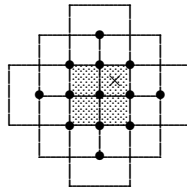


Figure 1. The figure shows a small part of the covering of \mathbf{R}^2 with open squares. The shaded square represents the original square U with center at the origin and side length $2/p$. In the figure there are another 12 squares of the same size obtained through translation of U and whose centers have coordinates $(m/p, n/p)$, where m and n are integers. The point marked \times is covered by exactly 4 such squares.

hypercubes; call these hypercubes U_1, U_2, \dots, U_N , and set $\Omega = \bigcup_{j=1}^N U_j$. We then have the inclusion $X \subseteq \Omega$.

Now choose for each j a continuous function $g_j: \mathbf{R}^n \rightarrow \mathbf{R}$ with the property that $g_j(x) > 0$ for all $x \in U_j$ and $g_j(x) = 0$ for all $x \notin U_j$. The functions g_j can be formed as translates of a single continuous function g which is positive in the original hypercube U and zero outside the hypercube, for example

$$g(x) = \begin{cases} \prod_{j=1}^n (1 - p|x_j|) & \text{if } x \in U, \\ 0 & \text{if } x \notin U, \end{cases}$$

by defining $g_j(x) = g(x - a_j)$, where a_j is the center of the hypercube U_j .

The sum $s(x) = \sum_{j=1}^N g_j(x)$ is positive for all $x \in \Omega$, so we get well-defined functions $f_j: \Omega \rightarrow \mathbf{R}$ by setting $f_j(x) = g_j(x)/s(x)$ for all $x \in \Omega$.

The functions $f_j: \Omega \rightarrow \mathbf{R}$ are continuous, $f_j(x) > 0$ for $x \in U_j$, $f_j(x) = 0$ for $x \notin U_j$, and $\sum_{j=1}^N f_j(x) = 1$ for all $x \in \Omega$. In other words, we have split 1 into a finite sum of non-negative continuous functions, where each function is zero outside an open hypercube (which is small if the number p is large). This is the so called partition of unity.

Now choose, for each $j = 1, 2, \dots, N$, a point $b_j \in X \cap U_j$ and then a point $y_j \in \phi(b_j)$, and define a function $F: X \rightarrow \mathbf{R}^n$ by

$$F(x) = \sum_{j=1}^N f_j(x)y_j.$$

The function F is continuous on X since the functions f_j are continuous. Moreover, $F(x)$ is for each $x \in X$ a convex combination of the points y_1, y_2, \dots, y_N , and since these points all lie in the convex set X , it follows that $F(x)$ is also a point in X . The function F thus fulfills the conditions of Brouwer's fixed point theorem and therefore has a fixed point \hat{x} , which

means that

$$\hat{x} = \sum_{j=1}^N f_j(\hat{x})y_j.$$

Since $f_j(\hat{x}) = 0$ for all indices j except at most 2^n , due to the fact that \hat{x} lies in at most 2^n hypercubes, we can, by renumbering the hypercubes and then setting $\lambda_j = f_j(\hat{x})$, express the fixed point \hat{x} as a convex combination

$$\hat{x} = \sum_{j=1}^M \lambda_j y_j,$$

of M points y_j , where $M \leq 2^n$, $y_j \in \phi(b_j)$, b_j lies in X , and \hat{x} and b_j lie in a common hypercube U_j for $j = 1, 2, \dots, M$. The latter claim means that the distance $\|b_j - \hat{x}\|$ between b_j and \hat{x} for each index $j = 1, 2, \dots, M$ is less than an absolute constant C times half the side length $1/p$. We may further assume that $M = 2^n$, because if $M < 2^n$ we just put $\lambda_j = 0$ and choose $b_j = b_1$ and $y_j = y_1$ for $M < j \leq 2^n$.

The number p has been fixed in the above construction, but we can of course carry out the construction for each positive integer p . The fixed points and the obtained coefficients λ_j and points b_j and y_j will depend on p , and if we express this dependence explicitly, we get fixed points \hat{x}_p in X , coefficients and other points that meet the relationships

$$(1) \quad \hat{x}_p = \sum_{j=1}^M \lambda_{j,p} y_{j,p}$$

$$(2) \quad \lambda_{j,p} \geq 0 \quad \text{and} \quad \sum_{j=1}^M \lambda_{j,p} = 1$$

$$(3) \quad y_{j,p} \in \phi(b_{j,p}), \quad b_{j,p} \in X \quad \text{and} \quad \|b_{j,p} - \hat{x}_p\| \leq C/p.$$

Since the set X is compact, there is due to the Bolzano–Weierstrass theorem a subsequence $(p_\nu)_{\nu=1}^\infty$ of the positive integers such that $\lim_{\nu \rightarrow \infty} p_\nu = \infty$ and the following limits exist:

$$\bar{x} = \lim_{\nu \rightarrow \infty} \hat{x}_{p_\nu}, \quad \bar{\lambda}_j = \lim_{\nu \rightarrow \infty} \lambda_{j,p_\nu} \quad \text{and} \quad \bar{y}_j = \lim_{\nu \rightarrow \infty} y_{j,p_\nu}.$$

The limits \bar{x} and \bar{y}_j lie in X , of course, and by passing to the limit in (2) we conclude that $\bar{\lambda}_j \geq 0$ and $\sum_{j=1}^M \bar{\lambda}_j = 1$. The inequality in (3) implies that

$$\lim_{\nu \rightarrow \infty} b_{j,p_\nu} = \lim_{\nu \rightarrow \infty} \hat{x}_{p_\nu} = \bar{x}$$

for every j . Since $y_{j,p_\nu} \in \phi(b_{j,p_\nu})$ and the map ϕ has a closed graph, we can conclude that $\bar{y}_j \in \phi(\bar{x})$ for every j .

It now only remains to consider the equality (1) for $p = p_\nu$ and to pass to the limit when $\nu \rightarrow \infty$; this results in the equality

$$\bar{x} = \sum_{j=1}^M \bar{\lambda}_j \bar{y}_j,$$

which expresses \bar{x} as a convex combination of the points \bar{y}_j , all of which belong to the convex set $\phi(\bar{x})$. Hence, $\bar{x} \in \phi(\bar{x})$, and this concludes the proof of Kakutani's fixed point theorem. \square

Remark. We have deduced Kakutani's fixed point theorem from Brouwer's theorem. Conversely, Brouwer's fixed point theorem is a special case of Kakutani's theorem, because we get Brouwer's theorem by choosing $\phi(x) = \{f(x)\}$.

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Brief historical notes

The first documented examples of game theory issues can be found in Babylonian Talmud from the centuries AD. The scriptures address the problem of allocating a fortune when there are several people claiming it and the sum of the claims exceeds the wealth. R. Aumann and M. Maschler [1985] have shown that the ingenious solutions suggested in Talmud can be explained by modern game theory concepts.

Several game theoretic concepts occur in embryonic form in some 19th-century economists work on economic equilibrium. Augustin Cournot [1838] treated competition between producers and used in the case of a duopoly a solution concept that is a special case of the Nash equilibrium. Francis Edgeworth [1881] published a work where he, among other things, studied trade and proposed the so-called contract curve as a solution to the problem of determining the outcome of trade. The core is a late generalization of Edgeworth's contract curve.

The first real theorem in game theory was published by Ernst Zermelo [1913] who observed that in chess either White has a strategy that wins against each defense, or Black has a strategy that wins against each defense, or both players can ensure at least a draw.

The concepts of strategic game and mixed strategies were introduced by Émile Borel [1921], who during the years 1921–27 published four short notes on strategic games. Borel proved a special case of the maxminimizing theorem for two-person zero-sum games, but he left the question of the validity of the result in the general case open.

John von Neumann [1928] proved the maxminimizing theorem for general two-person zero-sum games in his paper *Zur Theorie der Gesellschaftsspiele*. The proof uses topological methods. In the article, von Neumann also introduced the extensive form of a game.

In 1944 John von Neumann and Oskar Morgenstern published the book *Theory of Games and Economic Behavior*, which was instrumental in the continued development of game theory. The book presents, among other things, the theory of two-person zero-sum games, introduces the concept of

cooperative games with transferable utility, defines the coalitional form and studies so-called von Neumann–Morgenstern-stable sets in coalition games. In addition, an axiomatic presentation of utility theory is given. Our presentation of the proof of von Neumann–Morgenstern’s theorem (Proposition 1.5.2) is based on an article by I. N. Herstein and John Milnor [1953].

Game theory has much in common with linear programming which was developed in the years following World War II, and several well-known names in optimization theory have also made contributions to game theory. The book *Contributions to the Theory of Games I* with Harold Kuhn and Albert Tucker as editors was published in 1950.

John Nash published three groundbreaking essays on game theory in the years 1950–1951. The existence of a strategic equilibrium solution, now called the Nash equilibrium, is shown for strategic n -person games in Nash [1950a] and Nash [1951], and the bargaining problem is studied in Nash [1950b].

The extensive form of games had already been introduced by von Neumann, but the formulation using information sets that is now used, was introduced in a paper by Harold Kuhn [1953]. Kuhn’s paper also contains the basic theorems for such games.

The core concept was developed by Lloyd Shapley in 1952 and by Donald Gillies in the latter’s doctoral thesis *Some Theorems on N -Person Games* from Princeton University, later published in Gillies [1959].

Shapley introduced the value function for coalitional games that now bears his name in a paper in 1953, and the Shapley value was then used as a measure of the strength of different groups in voting contexts by him and Martin Shubik in 1954.

The fact that coalitional games have a non-empty core if and only if they are balanced was first shown by Olga Bondareva in 1963 with methods from linear programming and then independently of her by Shapley [1967].

The nucleus was introduced and studied by David Schmeidler [1969].

There are many textbooks in game theory. *An introduction to Game Theory* by M. Osborne [2004] is a relatively elementary textbook with many nice examples. *A course in Game Theory* by M. Osborne and A. Rubinstein [1994] is a textbook at graduate level.

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Answers and hints for the exercises

Chapter 10

- 10.1 No, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 1$, $v(\{1, 2, 3\}) = 3$ is a counterexample.
- 10.2 $x_1 \geq 0, x_2 \geq 2, x_1 + x_2 \leq 3, x_3 = 5 - x_1 - x_2$.
- 10.3 $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 3, x_3 = 3 - x_1 - x_2$.
- 10.4 $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, x_3 = 1 - x_1 - x_2$.
- 10.6 $v(\{1\}) = v(\{2\}) = 2, v(\{3\}) = 1, v(\{1, 2\}) = 6, v(\{1, 3\}) = v(\{2, 3\}) = 5, v(\{1, 2, 3\}) = 9$.
- 10.7 $1 \leq x_1 \leq 2, 4 - x_1 \leq x_2 \leq 3, x_3 = 4 - x_1 - x_2$.
- 10.8 The core is defined by the inequalities $0 \leq x_1 \leq 1 - a, 0 \leq x_2 \leq 1 - a, a \leq x_1 + x_2 \leq 1, x_3 = 1 - x_1 - x_2$. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ belongs to the core if $a \leq \frac{2}{3}$.
- 10.11 a) The core is empty (if $n \geq 3$).
 b) $\{x \in \mathbf{R}_+^n \mid x_1 + x_2 + \dots + x_n = 1\}$
 c) With player 1 as dictator the core is equal to $\{(1, 0, \dots, 0)\}$.
- 10.12 The following three assertions are equivalent for simple games:
 (i) Player 1 is a veto player.
 (ii) $(1, 0, \dots, 0)$ belongs to the core
 (iii) The core contains an element x with $x_1 > 0$.
 The core consists of all $x \in \mathbf{R}_+^n$ such that $x_1 + x_2 + \dots + x_n = 1$ and $x_i = 0$ for all non-veto players i .
- 10.13 a) $a \leq 2$ b) $a \leq 2, b \leq 3$.
 c) $f(k) \leq k$ for $k = 1, 2, \dots, n - 1$.
- 10.14 a) $\{(t, t, 1 - t, 1 - t) \mid 0 \leq t \leq 1\}$ if $P = \{1, 2\}$ and $Q = \{3, 4\}$.
 b) $\{(1, 1, 0, 0, 0)\}$ if $P = \{1, 2\}$ and $Q = \{3, 4, 5\}$.
 c) Suppose $P = \{1, 2, \dots, m\}$. If $|P| = |Q|$, the core is equal to the set $\{(t, t, \dots, t, 1 - t, 1 - t, \dots, 1 - t) \mid 0 \leq t \leq 1\}$, where the m players belonging to P get t units each. If $|P| < |Q|$ the core consists of just one element $(1, 1, \dots, 1, 0, 0, \dots, 0)$ which assigns 1 unit to each player in P .
- 10.15 Consider the collection of weights obtained by defining $\lambda_{\{1,2\}} = \lambda_{\{1,3\}} = \lambda_{\{1,4\}} = \frac{1}{3}, \lambda_{\{2,3,4\}} = \frac{2}{3}$ and $\lambda_S = 0$ for the remaining coalitions S .

- 10.16 Note that $S_{i_k} = T \cup S_{i_{k-1}}$ and $T \setminus \{i_k\} = T \cap S_{i_{k-1}}$. By convexity, $v(S_{i_k}) + v(T \setminus \{i_k\}) \geq v(T) + v(S_{i_{k-1}})$, and the required inequality $x_{i_k} = v(S_{i_k}) - v(S_{i_{k-1}}) \geq v(T) - v(T \setminus \{i_k\})$ now follows.
- 10.17 $(\frac{5}{2}, 0, \frac{1}{2})$
 10.18 $(\frac{5}{3}, \frac{8}{3}, -\frac{1}{3})$
 10.19 $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
 10.20 $\frac{1}{4}(a+3, 2a-2, a-1)$ if $5 \leq a \leq 9$, and $\frac{1}{3}(a, a+3, a-3)$ if $a \geq 9$.
 10.21 $(\frac{n+1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$

Chapter 11

- 11.1 $\phi = (\frac{13}{6}, \frac{1}{6}, \frac{4}{6})$
 11.2 $\phi_k = \sum_{j=k}^n \frac{1}{j}$. (Use induction to show that $c_{\{1\}} = c_{\{1,2\}} = c_{\{1,2,3\}} = \dots = c_{\{1,2,3,\dots,n\}} = 1$ and that $c_S = 0$ for all other coalitions S .)
 11.3 $\phi = (\frac{5}{3}, \frac{8}{3}, -\frac{1}{3})$
 11.4 $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$, $v(\{1, 2\}) = a$, $v(\{1, 3\}) = v(\{1, 2, 3\}) = b$.
 $\phi = \frac{1}{6}(a + 3b, a, 3b - 2a)$.
 The Shapley solution does not belong to the core (if $a > 0$).
 11.5 $\phi = (9.8, 1.3, 1.3, 1.3)$
 11.6 a) $\phi = (\frac{n+1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ b) $\phi = (\frac{n+1}{3}, \frac{n+1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$
 11.9 $\phi = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
 11.10 $\phi = (\frac{n-2}{n}, \frac{2}{n(n-1)}, \frac{2}{n(n-1)}, \dots, \frac{2}{n(n-1)})$
 11.11 The power index is $\frac{1}{15} \cdot \binom{9}{3} \cdot \binom{14}{8}^{-1} \approx 0.00186$ for non-permanent members, and $\frac{1}{15} \sum_{k=4}^{10} \binom{10}{k} \cdot \binom{14}{k+4}^{-1} \approx 0.19627$ for permanent members of the security council.

Chapter 12

- 12.1 a) The core coincides with the set of Pareto optimal actions and is equal to $\{x \in \mathbf{R}_+^2 \mid x_1^2 + x_2^2 = 1\}$.
 b) The core is the set $\{(t, 1) \mid 0 \leq t \leq 1\} \cup \{(1, t) \mid 0 \leq t \leq 1\}$, and $(1, 1)$ is the unique Pareto optimal action.
 c) The core coincides with the set of Pareto optimal actions and is equal to the set $\{(t, 1-t) \mid 0 \leq t \leq 1\}$.
 12.2 The core consists of the baskets $(t, t-1)$ and $(3-t, 3-t)$ for the agents 1 and 2, where $\frac{3}{2} \leq t \leq 3 - \sqrt{2}$. The equilibrium price is $(1, 1)$ and the agents' equilibrium baskets are equal to $(\frac{3}{2}, \frac{1}{2})$ and $(\frac{3}{2}, \frac{3}{2})$, respectively.
 12.3 a) $(\frac{2}{\sqrt{3}}, \frac{8}{3})$ b) $(1, 3)$
 12.4 \$0.5 million in company B and \$0.5 million in company C.

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