A Convergence Test for Sequences

Thm:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = 0$$

In words, this just says that if the absolute value of the *ratio of successive terms* in a sequence $\{a_n\}_n$ approaches a limit L, and if $L < 1$, then the sequence itself converges to 0.

Note that this is a statement about convergence of the *sequence* $\{a_n\}_n$ – it is NOT a statement about the *series* $\sum a_n$.

But we do notice that this statement looks very similar to the Ratio Test for series, which does comment on the convergence of a series. Given this similarity, we might call the theorem above the "Ratio Test for a Sequence".

The difference is that while the Ratio Test for series tells us only that a series converges (absolutely), the theorem above tells us that the *sequence* converges *to zero.*

Rather than write down the proof, let me just motivate why this theorem is true. The idea is that the property above is just a slightly weaker version of the property held by a geometric series – in a geometric series, the ratio of successive terms is EXACTLY a constant, while the condition above only requires that the ratio of successive terms APPROACHES a constant. In some sense then, a sequence satisfying the condition above is basically acting like a geometric series, at least as n approaches ∞. And of course, a geometric series with ratio less than one must converge to zero.

We used this result in class, and at the time I didn't state it as a theorem, but instead just argued its plausibility. Now that we have it stated as a theorem, let me repeat that example from the lecture.

(over)

Ex: Recall that we were attempting to show that $f(x) = T(x) = \lim T_n(x)$ by showing that $\lim R_n(x) = 0$, where $R_n(x) = f(x) - T_n(x)$. This we planned to accomplish by using the Taylor Inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where

$$\left|f^{[n+1]}(x)\right| \leq M \qquad \text{on the interval } |x-a| < R$$

For the function we were interested in, $f(x) = \sin x$, we concluded that we could use $M = 1$ for all n, since all of the derivatives of $\sin x$ are always ≤ 1 in absolute value, for all values of x.

So in this case, we have

$$|R_n(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

To show that $\lim R_n = 0$, it is therefore enough to show that

$$\lim \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

It is at this point that we apply the above Theorem, which we called the Ratio Test for Sequences. To apply this theorem, we must first compute the limit of the absolute value of the ratio of successive terms:

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim \left|\frac{|x-a|^{n+2}/(n+2)!}{|x-a|^{n+1}/(n+1)!}\right| \\ &= \lim \left|\frac{|x-a|}{(n+2)}\right| \\ &= 0 \end{aligned}$$

We conclude that the final limit above is zero because for any value of x, as $n \to \infty$, the numerator remains constant while the denominator approaches infinity.

Since this limit exists and is < 1, the Ratio Test for Sequences then tells us that the original sequence must converge to zero. So

$$\lim \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

and therefore

$$\lim R_n = 0$$

which then allows us to conclude that $f(x) = T(x)$, as desired.