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MATHEMATICS for Technical Schools

(a practical approach)

Translated from the Russian
by
Leonid Levant



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ПРАКТИЧЕСКИЕ ЗАНЯТИЯ ПО МАТЕМАТИКЕ

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Preface to the English Edition

Students of technical schools frequently encounter difficulties when solving mathematical problems. The principal purpose of this book is to help the student overcome these difficulties, and to teach him how to apply his theoretical knowledge to the solution of problems across the mathematics curriculum.

When solving problems independently, many students need regular consultations concerning the techniques and methods of solution, since it is beyond their power to find the correct way of solving a problem without help of the teacher or an appropriate study aid. The present text is designed to provide just such advice.

This is a revised edition of the book published in 1979. Compared with the first edition, some changes and additions have been made to accommodate the changes in the curriculum.

The chapter on "Systems of Equations and Inequalities" was rewritten and contains all the necessary material envisaged by the curriculum. In the "Elements of Computational Mathematics" section finding the angles and sides of triangles is set forth making use of the slide rule.

Taking into consideration that each section contains a small theoretical piece (one or two questions on a theme), the worked examples are given, in contrast to the first edition, at the beginning of each section and followed by the problems for the student to solve independently. These changes in the presentation enable the student first to get familiar with the methods for solving a typical problem, the classification of the problems and the notations the solutions are presented in. Then he can proceed to acquire the habits for their solution. In the present edition, the number of similar problems is somewhat lessened.

Wherever possible, the problems are classified by types. Each new type is introduced by a worked example followed by several problems of the same type for independent solution by the student. The sections headed "Miscellaneous Problems" contain several problems for the student to solve independently.

As students work their way through this book, they will gain an understanding of the basic concepts and their applications, along

with a proficiency in manipulative skills. To this end, many examples and illustrations stressing geometrical and physical intuition are given. It is hoped that this treatment of combining theory and problem-solving will open up the exciting world of mathematics to the reader.

It is my fond hope that both the instructors and students who use this book will enjoy it as much as I enjoyed its preparation. To my many students—past, present, and future—I sincerely dedicate this book and hope that it may stimulate interest in the study of mathematics.

The Author

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Part I

ELEMENTS OF COMPUTATIONAL MATHEMATICS

Chapter 1

ERRORS OF APPROXIMATE VALUES OF NUMBERS

Sec. 1.1. The Absolute Error of an Approximate Value of a Number. The Absolute Error Bound

The absolute value (or modulus) of the difference between an exact number x and its approximate value a is called the *absolute error* α of the approximate value of the number x:

$$|x - a| = \alpha.$$

The number a is called an *approximate value* of the exact number x to within Δa if the absolute error of the approximate value a does not exceed Δa, i.e. $|x - a| \leqslant \Delta a$.

The number Δa is the *bound of the absolute error* of the approximate number a. There are infinitely many numbers Δa satisfying the given definition; therefore, in practice, we try to choose a number Δa such that is the smallest possible and simplest in notation.

Given the absolute error bound Δa, we find the bounds within which the exact value of the number x is enclosed:

$$(x = a \pm \Delta a) \Leftrightarrow (a - \Delta a \leqslant x \leqslant a + \Delta a).$$

1. Given the following approximate values of the number $x = 2/3$: $a_1 = 0{,}6$; $a_2 = 0.66$; and $a_3 = 0.67$. Which of the three approximations is the best?

○ We find

$$\alpha_1 = \left|\frac{2}{3} - 0.6\right| = \left|\frac{2}{3} - \frac{3}{5}\right| = \frac{1}{15},$$

$$\alpha_2 = \left|\frac{2}{3} - 0.66\right| = \left|\frac{2}{3} - \frac{33}{50}\right| = \frac{1}{150},$$

$$\alpha_3 = \left|\frac{2}{3} - 0.67\right| = \left|\frac{2}{3} - \frac{67}{100}\right| = \frac{1}{300}.$$

The best approximation to the number x is $a_3 = 0.67$. ●

2. The length of a workpiece x (cm) is enclosed within the bounds $33 \leqslant x \leqslant 34$ Find the absolute error bound of measuring the workpiece.

○ Let us take the arithmetic mean of the bounds for the approximate value of the length of the workpiece: $a = (33 + 34)/2 = 33.5$ (cm). Then the bound of the absolute error of the approximate value of the length of the workpiece will not exceed 0.5 (cm). The magnitude of Δa can also be found as the half-difference between the upper and lower bounds, that is, $\Delta a = (34—33)/2 = 0.5$ (cm). The

length of the piece x found with an accuracy to $\Delta a = 0.5$ (cm) lies between the approximate values of the number x:

$$33.5 - 0.5 \leqslant x \leqslant 33.5 + 0.5; \quad x = 33.5 \pm 0.5 \text{ (cm)}. \bullet$$

3. Find the absolute error of rounding-off (to units) the following numbers: (1) 0.8; (2) 7.6; (3) 19.3; (4) 563.58.

4. The bound of the absolute error of the approximate value 386 of the number x is equal to 0.5. Indicate the bounds within which the number x is enclosed.

5. Find the bound of the absolute error of the measurements obtained in the form of the inequality $37 < x < 38$.

6. An ammeter gives an accuracy of $\pm$ 0.02 A. The measurement of the strength of current yields 10.63 A. Indicate the bounds of this number.

7. The atomic mass of hydrogen is 1.0082 ± 0.0005 and that of copper is 63.44 ± 0.15. Indicate the bounds of the approximate values of these numbers.

8. The area of a square is equal to 24.5 ± 0.3 (cm^2). Find the bounds of the measurement of the area of the square.

Sec. 1.2. Correct Digits of a Number. The Notation of an Approximate Value of a Number. Rounding-off

1. Correct and Significant Digits of a Number. The digit m of an approximate number a is said to be *correct in a broad sense* if the absolute error bound of the number a does not exceed the unity of the digital place occupied by the digit m.

The digit m of the approximate number a is said to be *correct in a rigorous sense* if the bound of the absolute error of the number a does not exceed half the unity of the digital place in which the digit m is written.

In the numbers obtained as a result of measurements or computations and used as initial data for further calculations, as well as in the decimal notation of an approximation to a number, all the digits must be correct.

Such a notation of an approximate number (as, for instance, in mathematical tables) in which the digits are correct in a broad sense is the most commonly used.

The absolute error bound Δa is found directly from the notation of the approximate value a of the number x.

If in the notation of an approximate number, it is not known whether its digits are correct, they are called *doubtful*.

The *significant* digits of an approximate number are defined as all the correct digits, except for zeros which stand before the first digit (from left to right) different from zero.

2. Rounding Off Numbers. When rounding off the number a, it is replaced by the number a_1 with a smaller number of significant digits. The absolute value of the difference $|a - a_1|$ is called the *round-off error*.

When rounding off a number to m significant digits, all the digits standing to the right of the mth significant digit are discarded or replaced by zeros if the number of digital places is retained. If the first of the discarded digits is greater than, or equal to, 5, then the last digit kept is increased by unity.

When using this rule, the round-off error does not exceed one half the unity of the decimal place determined by the last significant digit kept.

The rounding off of approximate values of numbers, retaining in the notation only correct digits, is carried out up to the place occupied by the first (from the right) correct digit.

9. Find the bound of the absolute error of the approximation 0.1968 to the number x whose all digits are correct in a rigorous sense.

○ The bound of the absolute error of this number is equal to 0.00005, that is, one-half the unity of the last digital place kept in the notation. ●

10. Indicate the correct digits (in a broad sense) of the following numbers: (1) 3.73 ± 0.056; (2) 3.627 ± 0.0008; (3) 4.732 ± 0.06; (4) 561.274 ± 500.

○ (1) The bound of the error $\Delta a = 0.056$ does not exceed the unity of the tenths digit (the inequality $0.056 < 0.1$ is true). Consequently, 3 and 7 are correct digits.

(2) Since $\Delta a = 0.0008 < 0.001$, all the digits of the approximate number 3.627 are correct.

(3) Since $\Delta a = 0.06 < 0.1$, 4 and 7 are correct digits.

(4) Since $\Delta a = 500 < 1000$, 5, 6, and 1 are correct digits. ●

11. The number 27 is taken for an approximate value of the number 26.7. Are the digits of the number 27 correct?

○ Since $|26.7 - 27| = 0.3 < 1$, the digits 2 and 7 are correct in a rigorous sense. ●

12. Round off the approximate value of the number 9.587 ± 0.03 to the first (from the right) correct digit.

○ The first (from the right) correct digit is found in the tenths digit place, therefore we round off the number 9.587 to tenths: $9.587 \approx 9.6$. The new value of the error bound Δa is equal to the sum of the error bound 0.03 and the round-off error 0.013, that is, $\Delta a = 0.03 + 0.013 = 0.043 < 0.1$. The number 9.6 is an approximation to the number 9.587 with an accuracy to 0.1. The digits 9 and 6 are correct. ●

13. Indicate the correct digits (in a broad sense) of the following numbers: (1) 0.028 ± 0.004; (2) 0.463 ± 0.0008; (3) 0.078 ± 0.002; (4) 12.78 ± 0.0005; (5) 375 ± 20.

14. Mention the correct digits of the number $\pi \approx 3.14$, taking $\pi \approx 3.1416$.

15. The number 1000 is taken for an approximate value of the exact number 999.82. Indicate the correct digits of the number 1000.

16. Retain only correct digits in the notation of the following approximate numbers: (1) 280 ± 10; (2) 8900 ± 100; (3) $530\,000 \pm 100$; (4) 5740 ± 10.

17. Round off to the first (from the right) correct digit the approximate values of the given numbers: (1) 0.3281 ± 0.05; (2) 2.0637 ± 0.0025; (3) 14.0367 ± 0.8; (4) 24.734 ± 0.06.

18. Round off the approximate values of the given numbers to the first (from the right) correct digit and write these numbers in

scientific notation: (1) 12.378 ± 25; (2) 15.763 ± 50; (3) 8724 ± 25; (4) 812 ± 6.

19. Indicate the bound of the round-off error if in the notation of the approximate values of the given numbers all digits are correct (in a broad sense): (1) $x \approx 0.56$; (2) $x \approx 84.3$; (3) $x \approx 5.10$; (4) $x \approx 4.100$.

20. Indicate the bound of the error of approximation to the given numbers written in scientific notation (all the digits are correct (in a broad sense)): (1) $4.28 \cdot 10^2$; (2) $4.2800 \cdot 10^2$; (3) $2.001 \cdot 10^{-4}$; (4) $3.60 \cdot 10^{-3}$.

Sec. 1.3. The Relative Error of an Approximate Value of a Number

The *relative error* δ of an approximate value a of the number x is defined as the ratio of the absolute error α of this approximation to the number a:

$$\delta = \alpha/a.$$

Since the absolute error α is usually unknown, in practice, we estimate the modulus of the relative error by a certain number ε which is necessarily not less than this modulus:

$$|\delta| \leqslant \varepsilon.$$

The number ε is called the *relative error bound.*

The relative error bound ε_a of the approximate value a is defined as the ratio of the absolute error Δa to the modulus of the number a:

$$\varepsilon_a = \frac{\Delta a}{|a|}. \tag{1.1}$$

The smaller the relative error, the higher the quality of measurements or calculations. The relative error is a dimensionless quantity which makes it possible to compare the quality of measurements of quantities having different dimensions.

The dependence of the relative error on the number of significant digits is illustrated in Table 1.

Table 1

Number	Smallest number	Greatest number	Bound of absolute error	Relative error of greatest number	Relative error of smallest number
One-digit	1	9	0.5	0.056=5.6%	0.5=50%
Two-digit	10	99	0.5	0.005=0.5%	0.05=5%
Three-digit	100	999	0.5	0.0005=0.05%	0.005=0.5%
Four-digit	1000	9999	0.5	0.00005=0.005%	0.0005=0.05%

It is seen from the table that three correct significant digits ensure an accuracy of the result (that is, the relative error) from 0.05% to 0.5%. In engineering or other calculations where a very high accuracy is not required, it usually suffices to ensure an accuracy of the result to a tenth of a percent. Therefore in engineering, calculations are carried out with three significant digits. This is why in such calculations a slide rule is widely used—it just ensures three correct significant digits.

In a number of problems, the bound of the absolute error is found by a given relative error and the modulus of an approximate value of a quantity:

$$\Delta a = |a| \varepsilon_a. \qquad (1.2)$$

21. Two measurements have yielded the following results: the length of a pencil is equal to 16 cm, and the length of a room to 730 cm. What can be said about the quality of these two measurements?

○ We shall assume the bound of the absolute error of the measurements to be equal to $\pm$ 0.5 cm. We then find the relative errors of these measurements:

$\varepsilon = 0.5/16 = 0.0312 \approx 3.1\%$ (when measuring the length of the pencil);

$\varepsilon = 0.5/730 = 0.000685 \approx 0.07\%$ (when measuring the length of the room).

Consequently, the quality of measuring the length of the room is considerably higher than that of measuring the length of the pencil. ●

22. Find the relative error of the number 6.8 if both of its digits are correct in a rigorous sense.

○ By hypothesis, $\Delta a = 0.05$; therefore $\varepsilon_a = 0.05/6.8 = 0.00735 = 0.7\%$. ●

23. What digits of the number 4.86 (0.3%) are correct?

○ By formula (1.2), we find

$$\Delta a = 4.86 \cdot 0.003 = 0.0146 < 0.02; \quad a = 4.86 \pm 0.02.$$

The first two digits 4 and 8 are correct. ●

24. When computing a certain quantity X, it became known that $6 < X < 7$. How many correct digits should be taken for the approximate value a to have the relative error not exceeding 0.3%?

○ In order to get the greatest value a, let us take $a = 6.99$. By formula (1.2), we obtain $\Delta a = 6.99 \cdot 0.003 = 0.021$. Consequently, we should take two correct digits. ●

25. Compute the relative error of the number $\pi \approx 3.14$, assuming $\pi \approx 3.1416$.

26. Round off the following exact numbers: (1) 286; (2) 45.8; (3) 7.19 to two significant digits with deficit and with excess. Find the relative error of each rounding.

27. When solving a problem, the sum of the angles of a triangle turned out to be equal to 179°30′. Find the relative error of the obtained approximate value.

28. Find the relative error of the number 2.6 if both of its digits are correct.

29. Which digits of the number 1.28 (0.4%) are correct?

Chapter 2
OPERATIONS WITH APPROXIMATE VALUES OF NUMBERS

Sec. 2.1. Addition of Approximate Values of Numbers

The bound of the absolute error of a sum of approximate values of numbers is equal to the sum of the bounds of the absolute errors of these numbers:

$$\Delta (a + b) = \Delta a + \Delta b, \tag{2.1}$$

where a and b are approximate values of numbers; Δa and Δb are the bounds of the absolute errors of the respective approximations.

The bound of the relative error of a sum is computed by the formula

$$\varepsilon_{a+b} = \frac{\Delta (a+b)}{a+b}. \tag{2.2}$$

1. Find the sum S of the indicated approximate values of numbers: 6.8 ± 0.05, 4.3 ± 0.05, and 3.575 ± 0.0005.

○ We have

$$S = 6.8 + 4.3 + 3.575 = 14.675;$$

$$\Delta S = 0.05 + 0.05 + 0.0005 = 0.1005$$

The bound of the absolute error is enclosed in the range $0.05 < 0.1005 < 0.5$. In the approximate value of the sum, only two digits (occupying the tens and the units places) are correct. We then round off the obtained result to units: $S = 14.675 \approx 15$. ●

2. Find the sum of the approximate values of the given numbers: 6.54 ± 0.005, 16.022 ± 0.0005, and 1.9646 ± 0.00005.

3. Compute the sum $a = \sqrt{5} + \sqrt{11}$, taking the approximate values of the roots to within 0.001. Find a, Δa, and ε_a.

4. Compute the sum $a = \sqrt{3} + \sqrt{5} + \sqrt{7}$ with four significant digits. Find a, Δa, and ε_a.

5. An electric circuit consists of three series-connected conductors having resistances $r_1 = 4.8 \pm 0.05$ (Ohms), $r_2 = 6.25 \pm 0.005$ (Ohms), and $r_3 = 7.725 \pm 0.0005$ (Ohms). Compute the total resistance of the circuit using the formula $R = r_1 + r_2 + r_3$. Find ΔR and ε_R.

Sec. 2.2. Subtraction of Approximate Values of Numbers

The bound of the absolute error of a difference of two approximate values of numbers is equal to the sum of the bounds of their absolute errors:

$$\Delta (a - b) = \Delta a + \Delta b. \tag{2.3}$$

The bound of the relative error of a difference is computed by the formula

$$\varepsilon_{a-b} = \frac{\Delta a + \Delta b}{a - b}. \tag{2.4}$$

6. Compute the difference of two approximate values of the numbers $a = 5.863 \pm 0.0005$ and $b = 2.746 \pm 0.0005$. Find $\Delta(a - b)$ and ε_{a-b}.

○ Using Formula (2.3), we compute the bound of the absolute error of the difference $a - b$:

$$\Delta(a - b) = 0.0005 + 0.0005 = 0.001.$$

In the approximate value of the difference, the digit occupying the thousandths place cannot be correct (since $\Delta(a - b) > 0.0005$). Thus, $a - b = 3.117 \approx 3.12$. The absolute error of the difference equals 0.001. In the approximate number 3.12, all the digits are correct.

By Formula (2.4), we find the relative error of the difference:

$$\varepsilon_{a-b} = 0.001/3.12 = 0.00032 \approx 0.03\%. \bullet$$

7. Compute the difference between the numbers 8.72 and 2.6532, the bounds of whose absolute error are respectively equal to 0.005 and 0.00005.

8. Compute the difference $a = \sqrt{13} - \sqrt{5}$ with four significant digits. Find Δa and ε_a.

Sec. 2.3. Multiplication of Approximate Values of Numbers

The formulas for estimating the bounds of the absolute error of a product (quotient) are rather complicated, therefore, in practice, we first find the relative error of a given product (quotient), and then the bound of its absolute error.

The formulas for computing the bounds of the absolute and relative errors of certain functions are given in Table 2.

9. Find the correct digits of the product of two approximate numbers: $a = 0.3862$ and $b = 0.8$.

○ We have $0.3862 \cdot 0.8 = 0.30896$. The bounds of the absolute error of the multipliers are equal to 0.00005 and 0.05, respectively. Using Formula (2.5), we find the relative error of the product:

$$\varepsilon_{ab} = \frac{0.00005}{0.3862} + \frac{0.05}{0.8} = 0.063.$$

By Formula (1.2), we find the bound of the absolute error of the product:

$$\Delta(ab) = 0.30896 \cdot 0.063 = 0.0195; \quad 0.005 < 0.0195 < 0.05.$$

The obtained result means that there is one correct digit (occupying the tenths place) in the product: $0.30896 \approx 0.3$. ●

10. Compute the volume of the cylinder $V = \pi R^2 H$ if $R = 45.8$ cm and $H = 78.6$ cm. Indicate the correct digits of the answer.

Table 2

Function	Bound of absolute error	Bound of relative error	Formula No.
$y=ab$	$\Delta y=\lvert b\rvert\cdot\Delta a+\lvert a\rvert\cdot\Delta b$	$\varepsilon_y=\frac{\Delta a}{a}+\frac{\Delta b}{b}$	(2.5)
$y=abc$	$\Delta y=\lvert bc\rvert\cdot\Delta a+\lvert ac\rvert\cdot\Delta b+\lvert ab\rvert\cdot\Delta c$	$\varepsilon_y=\frac{\Delta a}{a}+\frac{\Delta b}{b}+\frac{\Delta c}{c}$	(2.6)
$y=a^n$	$\Delta y=na^{n-1}\cdot\Delta a$	$\varepsilon_y=n\frac{\Delta a}{a}$	(2.7)
$y=a^2$	$\Delta y=2a\cdot\Delta a$	$\varepsilon_y=2\frac{\Delta a}{a}$	(2.8)
$y=a^3$	$\Delta y=3a^2\cdot\Delta a$	$\varepsilon_y=3\frac{\Delta a}{a}$	(2.9)
$y=\sqrt{a}$	$\Delta y=\frac{\Delta a}{2\sqrt{a}}$	$\varepsilon_y=\frac{\Delta a}{2a}$	(2.10)
$y=\sqrt[3]{a}$	$\Delta y=\frac{\Delta a}{3\sqrt[3]{a^2}}$	$\varepsilon_y=\frac{\Delta a}{3a}$	(2.11)
$y=\frac{a}{b}$	$\Delta y=\frac{\lvert b\rvert\cdot\Delta a+\lvert a\rvert\cdot\Delta b}{b^2}$	$\varepsilon_y=\frac{\Delta a}{a}+\frac{\Delta b}{b}$	(2.12)

○ We have $V=\pi\cdot 45.8^2\cdot 78.6=517.000$ (cm^3). Using Formulas (2.6) and (2.8) and setting $\pi\approx 3.14$, we find the relative error:

$$\varepsilon_V=\frac{\Delta\pi}{\pi}+\frac{2\Delta R}{R}+\frac{\Delta H}{H}=\frac{0.005}{3.14}+\frac{2\cdot 0.05}{45.8}+\frac{0.05}{78.6}=0.0044.$$

Using Formula (1.2), we find the bound of the absolute error:

$$\Delta V=V\cdot\varepsilon_V=517.000\cdot 0.0044=2270\ (\text{cm}^3).$$

Here, 5 and 1 are correct digits. ●

11. Find the product of the approximate numbers 0.456 ± 0.0005 and 3.35 ± 0.005 and the relative error of this product.

12. The diameter of a circle is equal to 12.5 ± 0.05 (cm). Setting $\pi = 3.14$, compute the length of the circle and find the bound of the absolute error.

13. Compute the volume of a right parallelepiped via the formula $V = abh$ if $a = 7.8$, $b = 4.6$, and $h = 9.3$. How many correct significant digits will the answer contain?

Sec. 2.4. Division of Approximate Values of Numbers

14. Find the absolute error bound of the quotient of two approximate numbers: $a = 8.36 \pm 0.005$ and $b = 3.72 \pm 0.004$.

○ We have $8.36 \div 3.72 = 2.25$. Using Formula (2.12), we find the relative error of the quotient:

$$\varepsilon_{a/b} = \frac{\Delta a}{a} + \frac{\Delta b}{b} = \frac{0.005}{8.36} + \frac{0.004}{3.72} = 0.002 = 0.2\%.$$

The absolute error bound of the quotient is found by Formula (1.2):

$$\Delta\,(a/b) = 2.25 \cdot 0.002 = 0.0045.$$

The obtained result means that all the three digits of the quotient are correct. ●

15. Compute $X = \frac{a}{b+c}$ if it is known that $a = 7.2 \pm 0.05$, $b = 3.46 \pm 0.03$, and $c = 5.09 \pm 0.04$.

○ We find:

$$X = \frac{a}{b+c} = \frac{7.2}{3.46+5.09} = 0.844;$$

$$\varepsilon_X = \frac{\Delta X}{X} = \frac{\Delta a}{a} + \frac{\Delta b + \Delta c}{b+c} = \frac{0.05}{7.2} + \frac{0.03+0.04}{8.55} = 0.015;$$

$$\Delta X = X \cdot \varepsilon_X = 0.844 \cdot 0.015 = 0.0127;$$

$$X = 0.844 \pm 0.0127 \text{ or } X \approx 0.84 \pm 0.01. \bullet$$

16. Find the relative error of the quotient of two approximate numbers: $a = 19.8 \pm 0.05$ and $b = 48.4 \pm 0.03$.

17. Find the correct digits of the quotient of two approximate numbers: $a = 68.4 \pm 0.02$ and $b = 72.8 \pm 0.04$.

18. Compute $X = \frac{a+b}{c}$ if $a = 8.15$, $b = 7.65$, and $c = 6.29$.

Sec. 2.5. Involution and Evolution of Approximate Numbers

19. Find the relative error made in computing the area of a square if the approximate value of the side of the square is equal to 68 ± 0.5.

○ By Formula (2.8), we get

$$\varepsilon_{68^2} = 2 \cdot 0.5/68 = 0.015 = 1.5\%. \quad \bullet$$

20. Compute the relative error made in extracting the square root from the approximate number 76.8 ± 0.05.

○ Using Formula (2.10), we obtain

$$\varepsilon_{\sqrt{76.8}} = 0.05/(2 \cdot 76.8) = 0.0003 = 0.03\%. \quad \bullet$$

21. Compute the absolute error bound in finding the hypotenuse of a right triangle whose legs are: $a = 56.8$ cm and $b = 44.6$ cm.

○ We have

$$c = \sqrt{a^2 + b^2} = \sqrt{56.8^2 + 44.6^2} = 72.21 \text{ (cm)}.$$

The absolute error bound is found by Formulas (2.10) and (2.8):

$$\Delta c = \frac{\Delta (a^2 + b^2)}{2\sqrt{a^2 + b^2}} = \frac{\Delta (a^2) + \Delta (b^2)}{2\sqrt{a^2 + b^2}} = \frac{2a \cdot \Delta a + 2b \cdot \Delta b}{2\sqrt{a^2 + b^2}}$$

$$= \frac{a \cdot \Delta a + b \cdot \Delta b}{\sqrt{a^2 + b^2}} = \frac{56.8 \cdot 0.05 + 44.6 \cdot 0.05}{72.21} = 0.07 \approx 0.1 \text{ (cm)}.$$

Thus, $c = 72.2 \pm 0.1$ (cm). The first two significant digits (7 and 2) are correct. ●

22. Find the relative error in computing the volume of a cube if an approximate value of the length of its edge is equal to 3.8 ± 0.05.

23. Compute the relative error of $\sqrt[3]{26.4}$.

Sec. 2.6. Computations with Preassigned Accuracy

In the previous sections, we considered direct problems in which it was required to estimate the error of the obtained result by the given operations with approximate numbers and given bounds of their errors.

In the inverse problem, it is required to determine what must be the errors of the given approximate numbers so that as a result of computations, a preassigned admissible bound of the error is obtained.

24. With what accuracy is it necessary to measure the length of the side of a square so that, when measuring its area, the absolute error bound of the area does not exceed 1 cm^2? The rough approximate value of the side of the square is equal to 9 cm.

○ Since $S = a^2$, using Formula (2.8), we get $\Delta S = 2a \cdot \Delta a$, whence

$$\Delta a = \frac{\Delta S}{2a} = \frac{1}{2 \cdot 9} = 0.0556 \approx 0.1 \text{ (cm)}.$$

Thus, if the length of a is measured with an error not exceeding 0.1 cm, then the error of the area will not exceed 1 cm^2. ●

25. With what accuracy is it necessary to measure the length of the edge a of a cube so that, when computing its volume, the absolute error bound does not exceed 100 cm^2? The rough approximate length of the edge is equal to 80 cm.

○ Since $V = a^3$, using Formula (2.9), we get $\Delta V = 3a^2 \cdot \Delta a$, that is,

$$\Delta a = \frac{\Delta V}{3a^2} = \frac{100}{3 \cdot 80^2} = 0.005 \text{ (cm)}.$$

Consequently, if a is measured with an error not exceeding 0.005 cm, then the error of the volume will not exceed 100 cm^3. ●

26. With what accuracy is it necessary to measure the radius of the base and the altitude of a right circular cylinder so that when measuring the volume of the cylinder by the formula $V = \pi R^2 H$, the absolute error bound ΔV does not exceed 100 cm^3? Rough approximate values are: $R = 40$ cm and $H = 70$ cm.

27. With what accuracy should the side of a square be measured so that the relative error does not exceed 0.3%? The approximate length of its side is 6 m.

Sec. 2.7. Solving Right Triangles

For solving triangles, it is recommended to make use of a 19-scale 250-mm "Leningrad" slide rule with tenths of a degree marked. The below computations in solving triangles were carried out with the aid of this slide rule*.

Solving a Right Triangle Given the Hypotenuse and an Acute Angle

28. Given: $c = 865$, $A = 38.3°$. Find: B, a, and b.

○ We first compute the angle B: $B = 90° - A = 90° - 38.3° = 51.7°$.

We then compute the leg a:

(1) substituting the numerical values of c and A into the formula $a = c \sin A$, we get $a = 865 \sin 38.3°$;

(2) set the hairline on the scale D at the reading 8-6-5;

(3) bring the final index of the cursor with the hairline;

(4) set the hairline at the angle 38.3° of the scale S;

(5) under the hairline, we read the indication 5-3-6 on the scale D;

(6) the order of reading is equal to 3; $a = 536$.

Now, we compute the leg b. Substituting the numerical values into the formula $b = c \sin B$, we get $b = 865 \sin 51.7°$. Performing the same operations as in the case of the leg a, we find $b = 679$.

* For more detail read *Practical Studies with a Slide Rule* by N.V. Bogomolov (Moscow, 1980, in Russian).

Computing check of the angle A:

(1) substituting the found numerical values of a and b into the formula $\tan A = a/b$, we get $\tan A = 536/679$;

(2) set the hairline on the scale D at the reading 5-3-6;

(3) bring the indication 6-7-9 on the scale C with the hairline;

(4) displace the hairline to align the final line marked on the scale C;

(5) bring into coincidence the final indices of the scale D and the cursor;

(6) under the hairline we read the angle $A = 38.3°$ on the scale T. The problem has been solved correctly.

Answer: $B = 51.7°$, $a = 536$, $b = 679$. ●

29. Given: $c = 358$; $A = 5.5°$. Find B, a, and b.

○ We first compute the angle B: $B = 90° - 5.5° = 84.5°$.

We then compute the leg a:

(1) $a = c \sin A = 358 \sin 5.5°$;

(2) set the hairline at the indication 3-5-8 on the scale D;

(3) align the final mark of the cursor with the hairline;

(4) set the hairline at the angle 5.5° on the scale ST;

(5) under the hairline, we read the indication 3-4-4 on the scale D;

(6) the order of the reading: $\overleftarrow{P}_{358 \sin 52.5°} = P_{358} + P_{\sin 5.5°} = 3 + (-1) = 2$; $a = 34.4$.

Finally, we compute the leg b. Substituting the numerical values into the formula $b = c \sin B$, we get $b = 358 \sin 84.5°$. Performing the same operations as in the case of the leg a (only the angle is read from the scale S), we find $b = 356$.

Answer: $B = 84.5°$, $a = 34.4$, $b = 356$. ●

Solving a Right Triangle Given a Leg and an Acute Angle

30. Given: $a = 0.846$, $A = 39.6°$. Find: B, c, b.

○ Compute the angle B: $B = 90° - A = 90° - 39.6° = 50.4°$.

Compute the hypotenuse c:

(1) substitute the numerical values into the formula $c = a/\sin A$: $c = 0.846/\sin 39.6°$;

(2) set the hairline at the indication 8-4-6 on the scale D;

(3) align the angle 39.6° of the scale S with the hairline;

(4) bring the hairline into coincidence with the initial index of the cursor;

(5) under the hairline, read the indication 1-3-3 from the scale D;

(6) the order of reading is equal to 1 (the cursor escaped to the right); $c = 1.33$.

Compute the leg b:

(1) Substitute the numerical values into the formula $b = a/\tan A$: $b = 0.846/\tan 39.6°$;

(2) set the hairline at the reading 8-4-6 on the scale D;
(3) bring the angle 39.6° of the scale T with the hairline;
(4) align the hairline with the initial index of the cursor;
(5) under the hairline, read the indication 1-0-2 from the scale D;
(6) the order of reading is equal to 1 (the cursor escaped to the right); $b = 1.02$.

Computing check of the angle B: $\sin B = b/c = 1.02/1.33 = 0.77$, $B = 50.4°$. The problem has been solved correctly.

Answer: $B = 50.4°$, $c = 1.33$, $b = 1.02$. ●

Solving a Right Triangle Given the Hypotenuse and a Leg

31. Given: $c = 8.93$, $a = 4.76$. Find: A, B, b.

○ Compute the angle A:
(1) substitute the numerical values into the formula $\sin A = a/c$: $\sin A = 4.76/8.93$;
(2) set the hairline at the reading 4-7-6 on the scale D;
(3) align the reading 8-9-3 on the scale C with the hairline;
(4) bring the hairline into coincidence with the final index of the cursor;
(5) bring into coincidence the final indices of the scale D and the cursor;
(6) under the hairline, read the angle 32.2° from the scale S.

Compute the angle B: $B = 90° - 32.2° = 57.8°$.

Compute the leg b. Substituting the numerical values into the formula $b = c \sin B$, we have $b = 8.93 \sin 57.8°$. Performing the same operations as in solving Problem 28, we get $b = 7.55$.

Computing check of the leg a: $a = b \tan A = 7.55 \tan 32.2° = 4.76$. The problem has been solved correctly.

Answer: $A = 32.2°$, $B = 57.8°$, $b = 7.55$. ●

Solving a Right Triangle Given Two Legs

32. Given: $a = 12.6$, $b = 16.9$. Find: A, B, c.

○ Compute the angle A:
(1) substitute the numerical values into the formula $\tan A = a/b$: $\tan A = 12.6/16.9$ $(0.1 < a/b < 1)$;
(2) set the hairline at the indication 1-2-6 on the scale D;
(3) bring the reading 1-6-9 on the scale C with the hairline;
(4) align the hairline with the final index of the cursor;
(5) bring into coincidence the final division lines of the scale D and the cursor;
(6) under the hairline, read the angle from the scale T: $A = 36.8°$.

Compute the angle B: $B = 90° - 36.8° = 53.2°$.

Compute the hypotenuse c. Substitute the numerical values into

the formula $c = a/\sin A$: $c = 12.6/\sin 36.8°$. Performing the same operations as in solving Problem 30, we get $c = 21.1$.

Computing check of the leg b: $b = c \sin B = 21.1 \sin 53.2° = 16.9$. The problem has been solved correctly.

Answer: $A = 36.8°$, $B = 53.2°$, $c = 21.1$. ●

33. Given: $a = 2.46$, $b = 52.5$. Find: A, B, c.

○ Compute the angle A:

(1) substitute the numerical values into the formula $\tan A = a/b$: $\tan A = 2.46/52.5$ $(0.01 < a/b < 0.1)$, consequently, the angle A is read from the scale ST;

(2) set the hairline at the indication 2-4-6 on the scale D;

(3) bring the reading 5-2-5 on the scale C with the hairline;

(4) align the hairline with the final index of the cursor;

(5) bring into coincidence the final division lines of the scale D and the cursor;

(6) under the hairline, read the angle 2.7° from the scale ST.

Compute the angle B: $B = 90° - 2.7° = 87.3°$.

Compute the hypotenuse c. Using the formula $c = a/\sin A$, we find $c = 52.5$ (for a small angle $A = 2.7°$, the hypotenuse $c = 52.5$, and the leg $b = 52.5$, within the accuracy of taking readings from the slide rule, turned out to be equal). ●

34. Solve a right triangle given its hypotenuse and an acute angle: (1) $c = 26.6$, $A = 63.6°$; (2) $c = 64.3$, $B = 48.9°$; (3) $c = 625$, $A = 28.5°$; (4) $c = 0.586$, $B = 42.7°$.

35. Solve a right triangle given a leg and an acute angle: (1) $a = 356$, $A = 52.3°$; (2) $a = 8.57$, $B = 36.4°$; (3) $b = 0.946$, $B = 72.6°$; (4) $a = 47.9$, $A = 56.8°$.

36. Solve a right triangle given the hypotenuse and one of its legs: (1) $c = 37.6$, $a = 24.8$; (2) $c = 30.6$, $b = 22.8$; (3) $c = 187$, $b = 112$; (4) $c = 0.627$, $a = 0.458$.

37. Solve a right triangle given its legs: (1) $a = 0.820$, $b = 0.650$; (2) $a = 420$, $b = 96.8$; (3) $a = 1.46$, $b = 2.37$; (4) $a = 24.8$, $b = 32.6$.

38. Solve an isosceles triangle (a is the base; b the lateral side; A the angle at the vertex; B the angle at the base; h_a the altitude drawn from the vertex A; and h_b the altitude drawn to the lateral side) given the following elements: (1) $b = 56.3$, $B = 42.2°$; find A, a; (2) $b = 72.4$, $A = 24.5°$; find B, a; (3) $a = 126$, $B = 78.2°$; find A and b; (4) $a = 5.64$, $A = 136.4°$; find B and b; (5) $h_a = 424$, $B = 38.2°$; find A, a, and b; (6) $h_a = 42.6$, $A = 64.6°$; find B, a, and b; (7) $a = 48$, $b = 56$; find A and B; (8) $h_a = 48.4$, $b = 116$; find A, B, and a; (9) $a = 3.18$, $h_a = 8.25$; find A, B, and b; (10) $b = 35.4$, $h_b = 12{,}6$; find A, B, and a; (11) $h_b = 52.8$, $A = 56.2°$; find B, a, and b; (12) $a = 42.8$, $h_b = 32.6$; find A, B, and b.

Sec. 2.8. Solving Oblique Triangles

Solving a Triangle Given Two Sides and the Included Angle

39. Given: $a = 24.6$, $b = 32.8$, $C = 54.2°$. Find: c, A, and B.

○ The side c is computed by the formula $c^2 = a^2 + b^2 - 2ab \cos C$ or $c^2 = a^2 + b^2 - 2ab \sin (90° - C)$, since $\cos C = \sin (90° - C)$. Substituting the numerical values into the second formula, we carry out computations with the aid of the slide rule:

$$c^2 = 24.6^2 + 32.8^2 - 2 \cdot 24.6 \cdot 32.8 \sin (90° - 54.2°)$$

$$= 605 + 1080 - 49.2 \cdot 32.8 \sin 35.8° = 605 + 1080 - 944 = 741;$$

$$c = \sqrt{741} = 27.2.$$

The angles A and B are computed by the formulas $\sin A = (a \sin C)/c$ and $\sin B = (b \sin C)/c$:

$$\sin A = (24.6 \sin 54.2°)/27.2 = 0.735, \quad A = 47.3°;$$

$$\sin B = (32.8 \sin 54.2°)/27.2 = 0.98, \quad B = 78.5°.$$

Computing check: $A + B + C = 47.3° + 78.5° + 54.2° = 180°$. The problem has been solved correctly.

Answer: $c = 27.3$, $A = 47.3°$, $B = 78.5°$. ●

Solving a Triangle Given a Side and Two Angles

40. Given: $a = 76.7$, $B = 78.2°$, $C = 64.6°$. Find: A, b, and c.

○ Compute the angle A: $A = 180° - (B + C) = 180° - (78.2° + 64.6°) = 37.2°$. The sides b and c are computed by the formulas: $b = (a \sin B)/\sin A$ and $c = (a \sin C)/\sin A$:

$$b = (76.7 \sin 78.2°)/\sin 37.2° = 124;$$

$$c = (76.7 \sin 64.6°)/\sin 37.2° = 115.$$

Computing check of the angle B: $\sin B = (b \sin C)/c = (124 \sin 64.6°)/115 = 0.977$, $B = 78.1°$.

The problem has been solved correctly.

Answer: $A = 37.2°$, $b = 124$, $c = 115$. ●

Solving a Triangle Given Three Sides

41. Given: $a = 486$, $b = 475$, $c = 494$. Find: A, B, and C.

○ The angles A, B, and C are computed by the formulas

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}:$$

$$(1) \quad \cos A = \frac{475^2 + 494^2 - 486^2}{2 \cdot 475 \cdot 494} = 0.868; \quad A = 60.2°;$$

$$(2) \quad \cos B = \frac{486^2 + 494^2 - 475^2}{2 \cdot 486 \cdot 494} = 0.848; \quad B = 58°;$$

(3) $\cos C = \frac{486^2+475^2-494^2}{2\cdot 486\cdot 475} = 0.881; \quad C = 61.8°.$

Computing check: $A + B + C = 60.2° + 58° + 61.8° = 180°$. The problem has been solved correctly.

Answer: $A = 60.2°$, $B = 58°$, $C = 61.8°$. ●

Solving a Triangle Given Two Sides and an Angle Opposite One of Them

Given: a, b, A. Find: c, B, and C.

Using the sine law, we find: $\sin B = (b \sin A)/a$. For different values of a, b, and A, the following three cases may occur:

(1) $\sin B > 1$. There is no solution.

(2) $\sin B = 1$. The angle $B = 90°$ (a right triangle).

(3) $\sin B < 1$. The angle B is found on the slide rule. Since there are two angles between 0° and 180° for which the sine has one and the same value (less than unity), there may exist two angles satisfying the conditions of the problem (one angle is acute, and the other is obtuse).

If $a > b$, then the problem has one solution.

If $a < b$, then the problem may have two solutions.

42. Given: $a = 32$, $b = 12$, $A = 78.2°$. Find: B, C, and c.

○ 1. Compute the angle B by the formula $\sin B = (b \sin A)/a$:

$$\sin B = (12 \sin 78.2°)/32 = 0.366; \quad B = 21.5°.$$

2. Compute the angle C: $C = 180° - (A + B) = 180° - (78.2° + 21.5°) = 80.3°$.

3. Find the side c by the formula $c = (a \sin C)/\sin A$:

$$c = (32 \sin 80.3°)/\sin 78.2° = 32.2.$$

4. Carry out a computing check of the angle C by the formula $\sin C = (c \sin B)/b$:

$$\sin C = (32.2 \sin 21.5°)/12 = 0.986; \quad C = 80.3°.$$

The problem has been solved correctly.

Answer: $B = 21.5°$, $C = 80.3°$, $c = 32.2$. ●

43. Given: $a = 48$, $b = 50$, $A = 68.3°$. Find B, C, and c.

○ 1. Compute the angle B by the formula $\sin B = (b \sin A)/a$:

$$\sin B = (50 \sin 68.3°)/48 = 0.968, \quad B = 75.5°$$

Since $b > a$, the problem has two solutions: $B_1 = 75.5°$, $B_2 = 180° - 75.5° = 104.5°$.

The further solution is presented for two cases separately.

First Case: $B_1 = 75.5°$.

2. Compute the angle C: $C = 180° - (A + B_1) = 180° - (68.3° + 75.5°) = 36.2°$.

3. Find the side c by the formula $c = (a \sin C)/\sin A$:

$$c = (48 \sin 36.2°)/\sin 68.3° = 30.5.$$

4. Carry out a computing check of the angle C by the formula $\sin C = (c \sin B)/b$:

$$\sin C = (30.5 \sin 75.5°)/50 = 0.591; \quad C = 36.2°.$$

Second Case: $B_2 = 104.2°$.

2. Compute the angle C: $C = 180° - (A + B_2) = 180° - (68.3° + 104.5°) = 7.2°$.

3. Find the side c by the formula $c = (a \sin C)/\sin A$:

$$c = (48 \sin 7.2°)/\sin 68.3° = 6.48.$$

4. Carry out a computing check of the angle C by the formula $\sin C = (c \sin B)/b$:

$$\sin C = \frac{6.48 \sin 104.5°}{50} = \frac{6.48 \sin (180° - 104.5°)}{50}$$

$$= \frac{6.48 \sin 75.5°}{50} = 0.108; \quad C = 7.2°.$$

The problem has been solved correctly.

Answer: $B = 75.5°$, $C = 36.2°$, $c = 30.5$ or $B = 104.5°$, $C = 7.2°$, $c = 6.48$. ●

44. Solve a triangle given two sides and the included angle: (1) $a = 72.8$, $b = 58.4$, $C = 64.8°$; (2) $b = 658$, $c = 893$, $A = 48.2°$; (3) $a = 5.64$, $c = 7.28$, $B = 63.7°$; (4) $a = 372$, $b = 456$, $C = 112.3°$.

45. Solve a triangle given a side and two angles: (1) $b = 652$, $A = 72.2°$, $C = 68.6°$; (2) $c = 42.8$, $B = 115.2°$, $C = 42.6°$; (3) $a = 5.18$, $A = 39.1°$, $C = 64.3°$; (4) $c = 236$, $A = 27.3°$, $B = 76.4°$.

46. Solve a triangle given three sides: (1) $a = 372$, $b = 356$, $c = 389$; (2) $a = 38.6$, $b = 16.8$, $c = 26.2$; (3) $a = 8.25$, $b = 5.85$, $c = 3.15$; (4) $a = 4.48$, $b = 5.12$, $c = 6.34$.

47. Solve a triangle given two sides and an angle opposite one of them: (1) $a = 256$, $b = 238$, $A = 68.2°$; (2) $a = 36.5$, $b = 40.8$, $A = 62.7°$; (3) $a = 8.75$, $b = 7.15$, $A = 42.5°$; (4) $b = 624$, $c = 638$, $B = 48.2°$.

Sec. 2.9. Miscellaneous Problems

48. Find the relative error of the equality $13/27 \approx 1/2$.

49. The number 8.75 is found with the relative error 0.4%. Determine the bound of the absolute error.

50. Find the relative error of computing the area of a rectangle with the sides 3.86 ± 0.005 and 4.6 ± 0.05.

51. Find the relative error of computing the volume of a right parallelepiped having the following dimensions: $a = 4.48 \pm 0.005$, $b = 5.8 \pm 0.05$ and $h = 6.72 \pm 0.005$.

52. For computing the volume of a cylinder by the formula $V = \pi R^2 H$, it was given: $\pi = 3.14$, $R = 36.7$ (cm), and $H = 86.4$ (cm) (all the digits being correct). How many correct significant digits does the answer contain?

53. Compute the diagonal c of the rectangle whose sides are: $a = 6.24 \pm 0.005$ (cm) and $b = 4.8 \pm 0.05$ (cm). How many correct significant digits are contained in the answer?

54. With what accuracy is it necessary to measure the radius of a circle so that the absolute error of the area of the circle does not exceed 10 cm^2? The rough approximation to R is equal to 8.7 cm.

QUIZ No. 1

Form I

(1) Compute the sum $a = \sqrt{3} + \sqrt{7}$ taking the approximate values of the roots to within 0.001; find ε_a.

(2) Compute the area of a parallelogram if $a = 68.7$ and $h = 52.6$. Indicate the correct digits of the answer.

(3) Find the bound of the absolute error of the product of two approximate numbers $a = 7.36 \pm 0.004$ and $b = 8.61 \pm 0.005$.

(4) Compute the relative error of $\sqrt{38.9}$.

(5) With what accuracy is it necessary to measure the radius of a circle so that the relative error of the area of the circle does not exceed 0.5%? The rough approximate value of $R = 8$ m.

Form II

(1) Compute the difference $a = \sqrt{11} - \sqrt{7}$ with four significant digits; find ε_a.

(2) Compute the area of a triangle if $a = 78.6$ and $h = 48.7$. Indicate the correct digits of the answer.

(3) Compute $X = (a + b)/c$ if $a = 82.6$, $b = 93.8$, and $c = 61.9$. Indicate the bound of the absolute error.

(4) Compute the relative error of $\sqrt[3]{68.4}$.

(5) With what accuracy is it necessary to measure the side of a square so that the relative error of the area of the square does not exceed 1%? The approximate value of the side of the square $a = 9$ m.

Part II

ALGEBRA AND FUNDAMENTALS OF ANALYSIS

Chapter 3

SYSTEMS OF EQUATIONS AND INEQUALITIES

Sec. 3.1. Solving Linear Equations in One Variable

In mathematics, any sentence that can be identified as true or false is called a *statement.*

If from the statement A there follows statement B, then we write $A \Rightarrow B$ (from A follows B).

If from the statement A there follows the statement B and from the statement B there follows the statement A, then these statements are said to be *equivalent* and we write $A \Leftrightarrow B$.

Statements of equality, whether true or false, are called *equations.*

An equality in one variable is called an *equation in one variable* if it is necessary to find those values of the variable for which a true statement of equality is obtained (a true numerical equality).

A *root* (or *solution*) of an equation is defined as a number which, when substituted for the variable in the equation, yields a true statement (a true numerical equality).

Equations are said to be *equivalent* if the sets of their solutions are equal.

A *linear equation in one variable* x is defined as an equation of the form $ax + b = 0$, where a and b are real numbers.

The solution of linear equations and equations reduced to linear ones is based on the following two theorems:

1. *If one and the same number is added to both members (or sides) of an equation, then a new equation is obtained which is equivalent to the original equation.*

2. *If both members of an equation are multiplied or divided by one and the same number, different from zero, then a new equation is obtained which is equivalent to the original equation.*

1. Solve the following equations:

(1) $\frac{1}{4}x + \frac{3}{8} = 0$; (2) $6 - 2x - \frac{2-5x}{3} = \frac{6x-4}{5}$; (3) $\frac{2x-9}{2x-5} - \frac{3x}{2-3x} = 2$;

(4) $\frac{x+1}{x-3} - \frac{x-2}{x+3} = \frac{3(3x-1)}{x^2-9}$; (5) $\frac{2x+1}{x-3} + \frac{5-4x}{3-x} = 6$.

○ (1) First Method.

$$\left(\frac{1}{4}x + \frac{3}{8} = 0\right) \Leftrightarrow \left(\frac{1}{4}x = -\frac{3}{8}\right)$$

$$\Leftrightarrow \left(x = -\frac{3}{8} \div \frac{1}{4}\right) \Leftrightarrow x = -1.5.$$

Second Method. Multiplying both sides of the equation by 8, we get

$$\left(\frac{1}{4}x+\frac{3}{8}=0\right)\Leftrightarrow(2x+3=0)\Leftrightarrow(2x=-3)\Leftrightarrow x=-1.5.$$

Answer: $\{-1.5\}$.

(2) Multiplying both sides of the equation by 15, we obtain

$$\begin{aligned}&(90-30x-5(2-5x)=3(6x-4)\\&\Leftrightarrow(90-30x-10+25x=18x-12)\\&\Leftrightarrow(-30x+25x-18x=-12-90+10)\\&\Leftrightarrow(-23x=-92)\Leftrightarrow x=4.\ \textit{Answer}:\ \{4\}.\end{aligned}$$

(3) This is a fractional rational equation containing the variable in the denominators of the fractions. The denominator of each fraction is not equal to zero (it is impossible to divide by zero). Carrying out the indicated transformations, we get:

$$\begin{cases}\dfrac{2x-9}{2x-5}-\dfrac{3x}{2-3x}=2,\\ 2x-5\neq 0,\\ 2-3x\neq 0\end{cases}$$

$$\Leftrightarrow\begin{cases}\dfrac{(2x-9)(2-3x)-3x(2x-5)-2(2x-5)(2-3x)}{(2x-5)(2-3x)}=0,\\ 2x-5\neq 0,\\ 2-3x\neq 0\end{cases}$$

$$\Leftrightarrow\begin{cases}8x+2=0,\\ 2x-5\neq 0,\\ 2-3x\neq 0,\end{cases}\quad x=-0.25.$$

For $x=-0.25$ the statements $2(-0.25)-5\neq 0$ and $2-3(-0.25)\neq 0$ are true, consequently, $x=-0.25$ is the root of the given equation. *Answer*: $\{-0.25\}$.

(4)
$$\begin{cases}\dfrac{x+1}{x-3}-\dfrac{x-2}{x+3}-\dfrac{3(3x-1)}{(x-3)(x+3)}=0,\\ x-3\neq 0,\\ x+3\neq 0\end{cases}$$

$$\Leftrightarrow\begin{cases}\dfrac{(x+1)(x+3)-(x-2)(x-3)-3(3x-1)}{(x-3)(x+3)}=0,\\ x-3\neq 0,\\ x+3\neq 0,\end{cases}$$

$$\Leftrightarrow\begin{cases}\dfrac{0\cdot x}{(x-3)(x+3)}=0,\\ x-3\neq 0,\\ x+3\neq 0.\end{cases}$$

The equation has infinitely many roots: any real number, except for $x = -3$ and $x = 3$, is a root of this equation. *Answer*: $\{x \mid x \neq -3, x \neq 3\}$.

$$(5)\ \begin{cases} \dfrac{2x+1}{x-3} - \dfrac{5-4x}{x-3} - 6 = 0, \\ x-3 \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{2x+1-5+4x-6x+18}{x-3} = 0, \\ x-3 \neq 0. \end{cases} \Leftrightarrow \begin{cases} \dfrac{14}{x-3} = 0, \\ x-3 \neq 0. \end{cases}$$

The fraction vanishes for no value of the variable x. The equation has no roots. *Answer*: $\varnothing$. ●

2. A piece of alloy of tin and copper, whose mass is 32 kg, contains 55% of tin. How much pure tin should be added to this alloy so that the new alloy contains 60% of tin?

○ Let the mass of tin added to the original alloy be equal to x kg. Then the alloy of mass $(32 + x)$ kg will contain 60% of tin and 40% of copper. The original alloy contained 55% of tin and 45% of copper, that is, it contained $32 \cdot 0.45$ kg of copper. Since the mass of copper in both alloys is one and the same, we get the equation: $0.45 \times 32 = 0.4\,(32 + x)$. Solving this equation, we find $x = 4$, that is, 4 kg of tin should be added to the original alloy. ●

3. A student wrote a two-digit number in which the units digit exceeds the tens digit by 2. If this number is divided by the sum of its digits, then we get 4 as the quotient and 6 as the remainder. What number was written by the student?

○ Let the units digit be equal to x, then the tens digit is equal to $x - 2$ $(x > 2)$; the written number has the form $10\,(x - 2) + x = 11x - 20$. The sum of the digits is $x - 2 + x = 2x - 2$. Consequently, dividing $11x - 20$ by $2x - 2$, we get 4 as the quotient and 6 as the remainder. We then set up the equation $11x - 20 = 4\,(2x - 2) + 6$, since the dividend is equal to the divisor multiplied by the quotient plus the remainder. Solving this equation we get $x = 6$. Thus, the number 46 was written. ●

4. Are the following equations equivalent:

(1) $4x + 6 = 10$ and $2x + 3 = 5$;

(2) $x - 5 = 5 - x$ and $x - 5 + 2x = 5 - x + 2x$;

(3) $\dfrac{x-2}{6} + \dfrac{3x+2}{2} + 6 = 0$ and $x - 4 = 0$?

5. Solve the given equations: (1) $-5x = 0$; (2) $-3x + 2 = 0$; (3) $4x - 1 = 0$; (4) $3x + 5 = 3x - 1$.

6. Solve the indicated equations:

(1) $3x - \dfrac{x+2}{4} - \dfrac{3x-2}{2} + \dfrac{x-1}{3} = 1$; (2) $1 - \dfrac{6-2x}{3} = x - \dfrac{x+3}{2}$;

(3) $x+\frac{x-3}{8}+\frac{x+1}{4}=2x+\frac{5-3x}{2}$; (4) $4-\frac{6-2x}{3}+x=2x-\frac{x+3}{2}$.

7. Solve the following equations:

(1) $\frac{2x}{x-1}-\frac{7}{2}=\frac{x+1}{x-1}+\frac{5}{2-2x}$;

(2) $\frac{x+1}{2-2x^2}-\frac{2x-1}{x^2-1}+\frac{6}{x+1}+\frac{1}{2-2x}=0$;

(3) $\frac{2x-7}{2x^2-4x+2}+\frac{x+1}{x^2-2x+1}-\frac{1}{3-3x}-\frac{2}{x-1}=0$;

(4) $\frac{5x-3}{x^2+3x}-\frac{x+1}{3x^2+9x}-\frac{2}{x+3}+\frac{3}{x}=0$;

(5) $\frac{2-6x}{3-x}-\frac{3x+4}{x-3}=3$; (6) $\frac{x-4}{x+4}-\frac{x+4}{x-4}+\frac{16x}{x^2-16}=0$.

8. A tank of capacity 1800 litres is filled with petrol by two pumps the first of which pumps per minute by 20 litres less than the second. During 15 minutes the tank is filled to three-fourths of its capacity. How many litres of petrol does the first pump supply per minute?

9. The alloy of copper and zinc contains 82% of copper. After 18 kg of zinc had been added to the alloy the copper content reduced to 70%. How much copper and how much zinc does the new alloy contain?

10. How much 26-percent sulfuric acid should be mixed with 40 kg of 68-percent acid to obtain an acid whose concentration is 32 per cent?

11. Divide 850 into two parts so that 8% of the first part plus 24% of the second part amount to 12% of the entire number.

12. In a two-digit number the tens digit exceeds the units digit by 3. If the inverted number is added to this number, we get 143. Find this number.

13. In an isosceles triangle, the base is two-thirds of the lateral side. Find the base of the triangle if its perimeter is equal to 24 cm.

Sec. 3.2. Solving Linear Inequalities in One Variable

1. Number Sets. The set Q of all rational numbers and the set J of all irrational numbers form the set R of real numbers.

A *number set* is defined as any collection of real numbers.

A set is said to be *finite* if it consists of a finite number of elements; otherwise it is called *infinite*.

The set R of all real numbers is called a *number line*, and real numbers themselves are referred to as the *points of the number line*.

The following number sets are encountered most frequently:

a *closed interval* with the end points a and b

$$[a, b] = \{x \in \mathbf{R} \mid a \leqslant x \leqslant b\};$$

an open interval with the end points a and b

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\};$$

half-open intervals with the end points a and b

$$(a, b] = \{x \in \mathbf{R} \mid a < x \leqslant b\}; \ [a, b) = \{x \in \mathbf{R} \mid a \leqslant x \leqslant b\},$$

the number $b - a$ being called the *length* of the interval with the end points a and b;

infinite intervals (rays, half-lines)

$$(a, +\infty) = \{x \in \mathbf{R} \mid x > a\}; \ [a, +\infty) = \{x \in \mathbf{R} \mid x \geqslant a\};$$

$$(-\infty, a) = \{x \in \mathbf{R} \mid x < a\}; \ (-\infty, a] = \{x \in \mathbf{R} \mid x \leqslant a\};$$

the number line

$$(-\infty, +\infty) = \mathbf{R}.$$

The set of ordered pairs of real numbers is called the *number plane* and is denoted by $\mathbf{R}^2$, and any ordered pair of real numbers is termed a *point of the number plane.*

2. Inequalities and Their Properties. The *solution of an inequality* is defined as a value of the variable for which the inequality is true (turns into a true numerical inequality).

To solve an inequality means to find the set of its solutions.

Inequalitites are said to be *equivalent* if the sets of their solutions are equal.

When solving inequalities, use is made of their basic properties:

1°. *If one and the same number is added to both sides of an inequality, then a new inequality is obtained which is equivalent to the original inequality.*

2°. *If both sides of an inequality are multiplied or divided by the same positive number, then a new inequality is obtained which is equivalent to the original inequality.*

3°. *If both sides of an inequality are multiplied or divided by the same negative number, then the sense of the inequality is reversed.*

4°. *Transitivity*: $a < b$ and $b < c \Rightarrow a < c$.

5°. *The rule for adding (subtracting) inequalities*:

$$\begin{array}{r} a < b \\ +\ c < d \\ \hline a + c < b + d \end{array} \qquad \begin{array}{r} a < b \\ -\ d > c \\ \hline a - d < b - c \end{array},$$

that is, inequalities of the same sense may be added, and inequalities of opposite sense may be subtracted.

6°. *The rule for multiplying inequalities*:

$$\left.\begin{array}{l} 0 < a < b \\ 0 < c < d \end{array}\right\} \Rightarrow ac < bd.$$

Corollary: $0 < a < b \Rightarrow a^n < b^n$ (for any $n > 0$).

3. Linear Inequalities. A *linear inequality* is defined as an inequality of the form $ax + b > 0$ (or $ax + b < 0$), where a and b are real numbers.

If $a > 0$, then $(ax + b > 0) \Leftrightarrow (x > -b/a)$.

If $a < 0$, then $(ax + b > 0) \Leftrightarrow (x < -b/a)$.

14. Solve the following inequalities:

(1) $x + 4 > 2 - 3x$; (2) $\dfrac{4 - 3x}{3} < \dfrac{2x - 1}{4} - \dfrac{5x - 2}{6}$;

(3) $(x-3)^2-11 \geqslant (x+2)^2$; (4) $(2x-1)^2-8x < (3-2x)^2$;

(5) $(x-2)^2-2x+10 < (3-x)^2$.

○ (1) $(x+4>2-3x) \Leftrightarrow (4x>-2) \Leftrightarrow (x>-0.5)$.

Answer: $(-0.5, +\infty)$.

(2) Multiplying both sides of the inequality termwise by 12, we get $(4(4-3x)<3(2x-1)-2(5x-2)) \Leftrightarrow (-8x<-15) \Leftrightarrow (x>15/8)$. *Answer*: $(15/8, +\infty)$.

(3) $((x-3)^2-11 \geqslant (x+2)^2) \Leftrightarrow (x^2-6x+9-11 \geqslant x^2+4x+4) \Leftrightarrow (-10x \geqslant 6) \Leftrightarrow (x \leqslant -0.6)$. *Answer*: $(-\infty, -0.6]$.

(4) $((2x-1)^2-8x < (3-2x)^2) \Leftrightarrow$

$(4x^2-4x+1-8x < 9-12x+4x^2) \Leftrightarrow (0 \cdot x < 8)$.

The given inequality is satisfied by any value of x. *Answer*: $(-\infty, +\infty)$.

(5) $((x-2)^2-2x+10 < (3-x)^2) \Leftrightarrow (x^2-4x+4-2x+10 < 9-6x+x^2) \Leftrightarrow (0 \cdot x < -5)$. The given inequality has no solutions. *Answer*: $\varnothing$. ●

In Problems 15 through 19 solve the indicated inequalities:

15. (1) $x+6>2-3x$; (2) $4(x-1) \leqslant 2+7x$;

(3) $3(x-2) \geqslant 4x-9$; (4) $2(3+5x)<3(7x-4)-4$.

16. (1) $\frac{3x}{2}-\frac{3}{5}<4x-3$; (2) $\frac{37-2x}{3}+x \leqslant \frac{3x-8}{4}-9$;

(3) $\frac{7-6x}{2}+10x<\frac{20x+1}{3}+2$; (4) $\frac{5-x}{8}+\frac{3-2x}{4} \geqslant 0$.

17. (1) $(x-1)^2-5 \leqslant (x+4)^2$; (2) $(1+x)^2+3x^2<(2x-1)^2-18$; (3) $(2x+1)^2-8 \geqslant (3-2x)^2$; (4) $8x^2+(x+1)^2>(2-3x)^2+4$.

18. (1) $4x-7<3+4x$; (2) $(3x-1)^2-6x<(2-3x)^2$;

(3) $(x-3)^2>(1+x)^2-8x$.

19. (1) $5x-4>7+5x$; (2) $(x-1)^2-2x+10<(2-x)^2$.

Sec. 3.3. Systems and Combinations of Inequalities in One Variable

1. Intersection and Union of Sets. The *intersection of two sets* A and B is defined as the set M of elements that are members of both of the given sets. The intersection of the sets A and B is denoted as follows: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

The *union of the sets* A and B is defined as the set M of elements that are members of at least one of the given sets. The union of the sets A and B is denoted as follows: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

If the sets A and B have common elements, then such elements enter the union of A and B only once.

2. Systems and Combinations of Sentences. When solving equations and inequalities, we sometimes have to make use of systems of equations or inequalities or of their combinations.

The *system of two sentences* $A(x)$ and $B(x)$ ($A(x)$ and $B(x)$ are equations or inequalities in one variable) is defined as the sentence "$A(x)$ and $B(x)$" which is written with the aid of a brace $\begin{cases} A(x), \\ B(x). \end{cases}$

The number x_0 is said to be a *solution of a system* if it is a solution of each of the sentences $A(x)$ and $B(x)$. The set of solutions of the system is the intersection of the sets of solutions of $A(x)$ and $B(x)$.

The *combination* (or *union*) *of two sentences* $A(x)$ and $B(x)$ is defined as the sentence "$A(x)$ or $B(x)$" which is written with the aid of a bracket: $\left[\begin{array}{l} A(x), \\ B(x). \end{array}\right.$

The number x_0 is said to be a *solution of a combination* if x_0 is a solution of at least one of the sentences: $A(x)$ or $B(x)$. The set of solutions of the combination is the union of the sets of solutions of $A(x)$ and $B(x)$.

We may also consider systems and combinations of three (and more) sentences as well as combinations of systems or systems of combinations of sentences.

Here, the following equivalent transformations may occur:

(1) a passage from an equation to a combination of equations:

$$f(x)\,\varphi(x)=0 \Leftrightarrow \left[\begin{array}{l} f(x)=0, \\ \varphi(x)=0; \end{array}\right.$$

(2) a passage from an inequality to a combination of inequalities:

$$f(x)\,\varphi(x)\geqslant 0 \Leftrightarrow \left[\begin{array}{l} \begin{cases} f(x)\geqslant 0, \\ \varphi(x)\geqslant 0, \end{cases} \\ \begin{cases} f(x)\leqslant 0, \\ \varphi(x)\leqslant 0; \end{cases} \end{array}\right. \qquad f(x)\,\varphi(x)\leqslant 0 \Leftrightarrow \left[\begin{array}{l} \begin{cases} f(x)\geqslant 0, \\ \varphi(x)\leqslant 0, \end{cases} \\ \begin{cases} f(x)\leqslant 0, \\ \varphi(x)\geqslant 0; \end{cases} \end{array}\right.$$

(3) ridding inequalities of fractions:

$$\frac{f(x)}{\varphi(x)}\geqslant 0 \Leftrightarrow \left[\begin{array}{l} \begin{cases} f(x)\geqslant 0, \\ \varphi(x)>0, \end{cases} \\ \begin{cases} f(x)\leqslant 0, \\ \varphi(x)<0; \end{cases} \end{array}\right. \qquad \frac{f(x)}{\varphi(x)}\leqslant 0 \Leftrightarrow \left[\begin{array}{l} \begin{cases} f(x)\geqslant 0, \\ \varphi(x)<0, \end{cases} \\ \begin{cases} f(x)\leqslant 0, \\ \varphi(x)>0. \end{cases} \end{array}\right.$$

20. Solve the following systems of inequalities:

(1) $\begin{cases} 6x+2>3x-4; \\ 2x+1>4x-7; \end{cases}$ (2) $\begin{cases} \dfrac{5-x}{8}+\dfrac{3-2x}{4}>0, \\ \dfrac{3x-5}{2}-\dfrac{2x-1}{3}<-2; \end{cases}$

(3) $\begin{cases} 2(2x-1)\geqslant 3(1+x), \\ 1-\dfrac{1}{2}x<\dfrac{3}{4}x-4; \end{cases}$ (4) $\begin{cases} 3x-8<2x-10, \\ 2-5x\geqslant 6-6x. \end{cases}$

○ (1) $\begin{cases} 6x+2>3x-4, \\ 2x+1>4x-7 \end{cases} \Leftrightarrow \begin{cases} 3x>-6, \\ -2x>-8 \end{cases} \Leftrightarrow \begin{cases} x>-2, \\ x<4, \end{cases}$

that is, the solution of the system is represented by the intersection of the sets: $(-\infty, 4) \cap (-2, +\infty) = (-2, 4)$. *Answer*: $(-2, 4)$.

$$(2)\ \begin{cases} \frac{5-x}{8}+\frac{3-2x}{4}>0, \\ \frac{3x-5}{2}-\frac{2x-1}{3}<-2 \end{cases} \Leftrightarrow \begin{cases} -5x>-11, \\ 5x<1 \end{cases} \Leftrightarrow \begin{cases} x<2.2, \\ x<0.2, \end{cases}$$

that is, the solution of the system is the intersection of the sets: $(-\infty, 0.2) \cap (-\infty, 2.2) = (-\infty, 0.2)$. *Answer*: $(-\infty, 0.2)$.

$$(3)\ \begin{cases} 2(2x-1)\geqslant 3(1+x), \\ 1-\frac{1}{2}x<\frac{3}{4}x-4 \end{cases} \Leftrightarrow \begin{cases} 4x-2\geqslant 3+3x, \\ 4-2x<3x-16 \end{cases} \Leftrightarrow \begin{cases} x\geqslant 5, \\ x>4, \end{cases}$$

that is, the solution of the system is the intersection of the sets: $[5, +\infty) \cap] (4, +\infty) = [5, +\infty)$. *Answer*: $[5, +\infty)$.

$$(4)\ \begin{cases} 3x-8<2x-10, \\ 2-5x\geqslant 6-6x \end{cases} \Leftrightarrow \begin{cases} x<-2, \\ x\geqslant 4. \end{cases}$$

We have $(-\infty, -2) \cap (4; +\infty) = \varnothing$, that is, the intersection of the sets is empty; consequently, the given system has no solution. *Answer*: $\varnothing$. ●

21. Solve the following combinations of inequalities:

$$(1)\ \left[\begin{array}{l} 3x-7>7x+9, \\ x-3>-3x+1; \end{array}\right. \quad (2)\ \left[\begin{array}{l} 4-3x<5(2-x), \\ 2-3x>x-18. \end{array}\right.$$

○ (1) $\left[\begin{array}{l} 3x-7>7x+9, \\ x-3>-3x+1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} -4x>16, \\ 4x>4 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x<-4, \\ x>1. \end{array}\right.$

The solution of the combination is represented by the union of the intervals: $(-\infty, -4) \cup (1, +\infty)$. *Answer*: $(-\infty, -4) \cup (1, +\infty)$.

(2) $\left[\begin{array}{l} 4-3x<5(2-x), \\ 2-3x>x-18 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} -3x+5x<10-4, \\ -3x-x>-18-2 \end{array}\right.$

$$\Leftrightarrow \left[\begin{array}{l} 2x<6, \\ -4x>-20 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x<3, \\ x<5: \end{array}\right.$$

The solution of the given combination is the union of the intervals: $(-\infty, 3) \cup (-\infty, 5) = (-\infty, 5)$. *Answer*: $(-\infty, 5)$. ●

22. Solve the following inequalities: (1) $\frac{2x-3}{3x-6}>0$; (2) $\frac{x-4}{2x-3}\leqslant 0$; (3) $\frac{3-6x}{2x+1}>-5$.

○ (1) The solution of this nonlinear inequality in one variable is reduced to solving a combination of two systems of linear inequalities in one variable, since a fraction is positive if its numerator and denominator have either only positive or only negative value. There-

fore the given inequality is equivalent to a combination of two systems:

$$\frac{2x-3}{3x-6}>0 \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}2x-3>0,\\3x-6>0,\end{array}\right.\\\left\{\begin{array}{l}2x-3<0,\\3x-6<0\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x>1.5,\\x>2\end{array}\right.\\\left\{\begin{array}{l}x<1.5,\\x<2\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x>2,\\x<1.5.\end{array}\right.$$

The set of solutions of the first system is the interval $(2, +\infty)$, and that of the second system is the interval $(-\infty, 1.5)$. The union of these sets $(-\infty, 1.5) \cup (2, +\infty)$ is the set of solutions of the combination of systems, that is, the set of solutions of the given inequality. *Answer*: $(-\infty, 1.5) \cup (2, +\infty)$.

(2) A fraction is negative if its numerator and denominator have opposite signs. Therefore the given inequality is equivalent to the combination of two systems:

$$\frac{x-4}{2x-3}\leqslant 0 \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x-4\geqslant 0,\\2x-3<0\end{array}\right.\\\left\{\begin{array}{l}x-4\leqslant 0,\\2x-3>0\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x\geqslant 4,\\x<1.5\end{array}\right.\\\left\{\begin{array}{l}x\leqslant 4,\\x>1.5\end{array}\right.\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l}\varnothing\\1.5<x\leqslant 4.\end{array}\right. \quad \textit{Answer}: (1.5, 4].$$

$$(3)\ \frac{3-6x}{2x+1}>-5 \Leftrightarrow \frac{3-6x}{2x+1}+5>0 \Leftrightarrow \frac{3-6x+10x+5}{2x+1}$$

$$>0 \Leftrightarrow \frac{4x+8}{2x+1}>0 \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}4x+8>0,\\2x+1>0\end{array}\right.\\\left\{\begin{array}{l}4x+8<0,\\2x+1<0\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x>-2,\\x>-0.5\end{array}\right.\\\left\{\begin{array}{l}x<-2,\\x<-0.5\end{array}\right.\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l}x>-0.5,\\x<-2\end{array}\right. \Leftrightarrow (-\infty, -2) \cup (-0.5, +\infty).$$

Answer: $(-\infty, -2) \cup (-0.5, +\infty)$. ●

In Problems 23 and 24 solve the indicated systems of inequalities:

23. (1) $\left\{\begin{array}{l}3x+7>7x-9,\\x-3>-3x+1;\end{array}\right.$ (2) $\left\{\begin{array}{l}2x>4x+6,\\4x+3<2x+1;\end{array}\right.$

(3) $\left\{\begin{array}{l}6x-7>5x-1,\\3x+6>8x-4;\end{array}\right.$ (4) $\left\{\begin{array}{l}3x+2>x-2,\\x+15>6-2x,\\x-14<5x+14.\end{array}\right.$

24. (1) $\begin{cases} 5x-3>1+x, \\ \frac{1}{2}-3x<\frac{2}{3}x-5; \end{cases}$ (2) $\begin{cases} \frac{7-6x}{2}+10x\leqslant\frac{8x+1}{3}-12, \\ \frac{x+1}{2}>2x-2\frac{1}{2}; \end{cases}$

(3) $\begin{cases} \frac{4x-3}{6}+3>\frac{3x}{2}+\frac{5}{8}, \\ \frac{4x-3}{8}+\frac{x-5}{5}>\frac{x-1}{2}; \end{cases}$ (4) $\begin{cases} \frac{3-2x}{4}\geqslant\frac{5-2x}{8}, \\ \frac{4x-15}{3}>-4\frac{2}{3}. \end{cases}$

In Problems 25 and 26 solve the given combinations of inequalities:

25. (1) $\begin{cases} 4x+7>2x+13; \\ 3x+2<2x+3; \end{cases}$ (2) $\begin{cases} 2-3x<8-5x, \\ 2(4-x)\geqslant x-22; \end{cases}$

(3) $\begin{cases} 5-2x>3x-10, \\ 2(1-2x)>1-5x; \end{cases}$ (4) $\begin{cases} 2(x-2)>5-x, \\ 1-5x\leqslant 4(2-x). \end{cases}$

26. (1) $\begin{cases} \frac{x-26}{3}>-7, \\ \frac{x-10}{4}<-3; \end{cases}$ (2) $\begin{cases} \frac{3x}{4}>\frac{2(x-1)}{3}, \\ \frac{2x+1}{3}>-\frac{3}{4}; \end{cases}$

(3) $\begin{cases} \frac{x-3}{2}>\frac{x-1}{3}, \\ \frac{2-x}{4}-\frac{3-2x}{3} \end{cases}$ (4) $\begin{cases} \frac{2x}{3}+\frac{x}{4}>x-1, \\ \frac{x+1}{2}<\frac{x-1}{3}+x. \end{cases}$

In Problems 27 through 29 solve the indicated inequalities:

27. (1) $\frac{3a+1}{2a+3}>0$; (2) $\frac{5-m}{2-3m}\geqslant 0$; (3) $\frac{2(4-x)}{1-3x}>0$;

(4) $\frac{2-y}{y-4}\geqslant 0$.

28. (1) $\frac{2x+1}{3x+2}<0$; (2) $\frac{1-2y}{4-3y}\leqslant 0$; (3) $\frac{5-a}{a-4}\leqslant 0$;

(4) $\frac{x-3}{4-x}<0$.

29. (1) $\frac{2x+3}{3x+2}\leqslant 2$; (2) $\frac{3x+1}{2x-5}>2$; (3) $\frac{2(1-4a)}{2a+1}\geqslant -6$;

(4) $\frac{3-5m}{2m-5}<-3$; (5) $\frac{x+4}{2x+3}\leqslant -2$.

Sec. 3.4. Inequalities in One Variable Containing the Variable under the Sign of Modulus

The *modulus* (or the absolute value) of a real number x (denoted by $|x|$) is equal to the number x itself if x is positive, it is equal to the number $-x$ if x is negative, and the modulus of zero is equal to zero:

$$|x|=\begin{cases} x & \text{if } x\geqslant 0; \\ -x & \text{if } x<0. \end{cases}$$

Let us mention the following properties of the modulus:

1°. *Two opposite numbers have the same modulus*: $|a| = |-a|$, *that is, the modulus of a number remains unchanged if the number is multiplied by* -1.

2°. *The modulus cannot be a negative number*: $|a| \geqslant 0$.

3°. *A number cannot be greater than its modulus*: $a \leqslant |a|$.

4°. *The modulus of a sum of real numbers is not greater than the sum of their moduli*:

$$|a+b| \leqslant |a| + |b|.$$

5°. *The modulus of a difference of two numbers is not less than these numbers*:

$$|a-b| \geqslant |a| - |b|, \quad |a-b| \geqslant |b| - |a| \text{ or}$$

$$|a-b| \geqslant ||a| - |b||.$$

6°. *The modulus of a product of several numbers is equal to the product of their moduli*:

$$|a \cdot b| = |a| \cdot |b|.$$

7°. *The modulus of an integral power of some number is equal to the power of its modulus*:

$$|a^n| = |a|^n.$$

8°. *The modulus of a quotient of two numbers is equal to the quotient of their moduli*:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0.$$

9°. $|x| < a \Leftrightarrow \begin{cases} x > -a, \\ x < a \end{cases} \Leftrightarrow -a < x < a \Leftrightarrow (-a, a).$

10°. $|x| > a \Leftrightarrow \left[\begin{array}{l} x < -a, \\ x > a \end{array}\right. \Leftrightarrow (-\infty, -a) \cup (a, +\infty).$

30. Find the numerical value of the expression $\left|\frac{3x^3+1}{2x^2+2}\right|$ for $x = -3$.

○ We have $\left|\frac{3x^3+1}{2x^2+2}\right| = \left|\frac{3(-3)^3+1}{2(-3)^2+2}\right| = \left|\frac{-80}{20}\right| = |-4| = 4.$ ●

31. Solve the inequality $|x-5| < 3$.

○ According to Property 9°, we get the system of inequalities:

$$|x-5| < 3 \Leftrightarrow \begin{cases} x-5 > -3, \\ x-5 < 3 \end{cases} \Leftrightarrow \begin{cases} x > 2, \\ x < 8 \end{cases}$$

$$\Leftrightarrow (2 < x < 8) \Leftrightarrow (2, 8).$$

Geometrically, the solution of the given inequality is the set of points the distance from which to the point 5 does not exceed 3. *Answer*: (2, 8). ●

32. Solve the inequality $|x-4| > 5$.

○ According to Property 10°, we arrive at the combination of inequalities:

$$|x-4| > 5 \Leftrightarrow \left[\begin{array}{l} x-4 < -5, \\ x-4 > 5 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x < -1, \\ x > 9. \end{array}\right.$$

The set of solutions of the obtained combination is the union of the intervals $(-\infty, -1) \cup (9, +\infty)$. *Answer*: $(-\infty, -1) \cup (9, +\infty)$. ●

33. Evaluate the following expressions:

(1) $\left|\frac{5x-7}{x-5}\right|$ for $x=3$; (2) $\left|\frac{x^2+5}{x^3-1}\right|$ for $x=-2$;

(3) $\left|\frac{z}{z-2}\right|$ for $z=1$; (4) $\left|\frac{a-1}{a^2+2}\right|$ for $a=-\frac{1}{2}$.

34. Solve the indicated inequalities: (1) $|x-2|<4$; (2) $|x+8|<1$; (3) $|x+7|<5$; (4) $|x-a|<\varepsilon$; (5) $|x-3|\leqslant 1$.

35. Solve the given inequalities: (1) $|x+3|>2$; (2) $|x-5|>8$; (3) $|2x-4|>2$; (4) $|3x+6|\geqslant 3$; (5) $|2x-2|\geqslant 1$.

Sec. 3.5. Solving Systems of Two Linear Equations in Two Variables

1. Equations and Systems of Equations in Two Variables. An *equation in two variables* x and y is written in the form $f(x, y) = 0$ where f is an expression containing the variables x and y. The *solution of* such *equation* is defined as an ordered pair of numbers (x_0, y_0) which, when substituted into the given equation, yields a true numerical equality $f(x_0, y_0) = 0$.

A *system of equations in two variables* x and y in the general form is written as follows:

$$\begin{cases} f(x, y)=0, \\ g(x, y)=0. \end{cases}$$

A *solution of a system of equations* is defined as an ordered pair of numbers which is the solution of each of the equations entering the system.

The set of all the solutions of a system of equations is the intersection of the sets of solutions of the equations entering the system.

Two systems of equations are said to be *equivalent* if the sets of solutions of these systems coincide.

2. Solving Systems of Two Linear Equations in Two Variables. A *determinant of the second order* formed from the numbers a_1, b_1, a_2, b_2 is a number defined by the equality

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The numbers a_1, b_1, a_2, and b_2 are called the *elements* of the determinant. The elements a_1 and b_2 form the *principal* (or *leading*) *diagonal*, and the elements a_2 and b_1, the *secondary diagonal*. Thus, a determinant of the second order is equal to the product of the elements of the principal diagonal less the product of the elements of the secondary diagonal.

The system of two linear equations in two variables

$$\begin{cases} a_1x+b_1y=c_1, \\ a_2x+b_2y=c_2 \end{cases}$$

(provided that the determinant of the system $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$) has a unique solution which is found by the formulas

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta x}{\Delta}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta y}{\Delta}. \tag{3.1}$$

These equalities are called *Cramer's formulas.*

Here Δ_x and Δ_y are determinants obtained from the determinant Δ by replacing the column of the coefficients of the respective variable by the column of constant terms.

And if the determinant of a system $\Delta = 0$, then the system is either *incompatible* (if $\Delta_x \neq 0$ and $\Delta_y \neq 0$) or *indeterminate* (when $\Delta_x = \Delta_y = 0$). In the latter case the system is reduced to one equation, the other being the consequence of this equation.

The condition of incompatibility of a system may be written in the form $a_1/a_2 = b_1/b_2 \neq c_1/c_2$, and the condition of indeterminacy, in the form $a_1/a_2 = b_1/b_2 = c_1/c_2$.

36. Solve the following systems of equations:

(1) $\begin{cases} 3x+4y=18, \\ 2x+5y=19; \end{cases}$ (2) $\begin{cases} 3x-2y=1, \\ 6x-4y=2; \end{cases}$ (3) $\begin{cases} 2x-3y=2, \\ 4x-6y=3. \end{cases}$

○ (1) Since $\Delta = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 15-8=7 \neq 0$, the system has the only solution which is found by Formulas (3.1):

$$x = \frac{\begin{vmatrix} 18 & 4 \\ 19 & 5 \end{vmatrix}}{7} = \frac{90-76}{7} = 2; \quad y = \frac{\begin{vmatrix} 3 & 18 \\ 2 & 19 \end{vmatrix}}{7} = \frac{57-36}{7} = 3.$$

Answer: $\{(2, 3)\}$.

(2) We find $\Delta = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = -12+12=0$. Here the constant terms are proportional to the coefficients of the variables: $3/6 = (-2)/(-4) = 1/2$. Therefore the given system is equivalent to one of the equations, say, to the first, and, consequently, has infinitely many solutions.

(3) We find $\Delta = \begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix} = -12+12=0$. Here the constant terms are not proportional to the coefficients of the variables: $2/4 = -3)/(-6) \neq 2/3$; therefore the given system is incompatible. ●

37. Solve the system of equations $\begin{cases} ax+2y=a, \\ 8x+ay=2a. \end{cases}$

○ We have $\Delta = \begin{vmatrix} a & 2 \\ 8 & a \end{vmatrix} = a^2-16$.

1. For $a \neq \pm 4$ the system has the only solution:

$$x = \frac{\begin{vmatrix} a & 2 \\ 2a & a \end{vmatrix}}{a^2-16} = \frac{a^2-4a}{a^2-16} = \frac{a}{a+4}; \quad y = \frac{\begin{vmatrix} a & a \\ 8 & 2a \end{vmatrix}}{a^2-16} = \frac{2a^2-8a}{a^2-16} = \frac{2a}{a+4}.$$

2. For $a = 4$ the determinant of the system is equal to zero, and the system takes the form

$$\begin{cases} 4x+2y=4, \\ 8x+4y=8. \end{cases}$$

The constant terms are proportional to the coefficients of the variables: $4/8 = 2/4 = 4/8$; therefore for $a = 4$ the system has an infinite set of solutions.

3. For $a = -4$ the determinant of the system is equal to zero, and the system takes the form

$$\begin{cases} -4x+2y=-4, \\ 8x-4y=-8. \end{cases}$$

The constant terms are not proportional to the coefficients of the variables: $(-4)/8 = 2/(-4) \neq (-4)/(-8)$; therefore for $a = -4$ the system is incompatible. ●

In Problems 38 and 39 solve the indicated systems of equations:

38. (1) $\begin{cases} 5x-2y=7; \\ 3x+4y=25; \end{cases}$ (2) $\begin{cases} 8x+4y=7, \\ 4x+2y=9; \end{cases}$

(3) $\begin{cases} 2x-3y=-3; \\ -6x+9y=9; \end{cases}$ (4) $\begin{cases} 2x+3y=13, \\ 5x-y=7; \end{cases}$

(5) $\begin{cases} 2x-4y=14, \\ 4x+3y=-27; \end{cases}$ (6) $\begin{cases} 3x+5y=14, \\ 2x-4y=-20 \end{cases}$

39. (1) $\begin{cases} \frac{2x-y}{3} - \frac{3x-2}{4} = x+y, \\ 5x-4y=-18; \end{cases}$

(2) $\begin{cases} \frac{x+2y}{4} - \frac{x-2y}{2} - \frac{7-2y}{3} = 1-x, \\ 3x-2y=8; \end{cases}$

(3) $\begin{cases} \frac{x+y}{2} - \frac{3x+4}{4} - \frac{5x+2y}{8} = 1, \\ 2x-y=-5; \end{cases}$

(4) $\begin{cases} \frac{2x-6}{3} - \frac{x-2}{2} = 2y, \\ \frac{3x-6}{2} + \frac{y}{2} = x; \end{cases}$ (5) $\begin{cases} \frac{1-2y}{5} - \frac{x}{5} - 2y = 4, \\ 2(1-y)-x=1; \end{cases}$

(6) $\begin{cases} \frac{3x-5y}{3} - \frac{x+2}{6} = 10 \\ 7x-10y=62. \end{cases}$

40. For what value of a does the system $\begin{cases} 2x-ay=3, \\ 6x-9y=9 \end{cases}$ have an infinite set of solutions?

41. For what value of a does the system $\begin{cases} 4x+3y=12, \\ 2x+ay=7 \end{cases}$ have no solutions?

42. Find a two-digit number which, when divided by the sum of its digits, yields 6 as the quotient and 8 as the remainder, and, when divided by the difference of the tens digit and the units digit, yields 24 as the quotient and 2 as the remainder.

43. If the width of a rectangular ground is increased by 4 m, and its length is decreased by 2 m, then its area will increase by 8 m^2, and if the width is decreased by 3 m and the length is increased by 1 m, then its area will be reduced by 23 m^2. Find the width and length of the ground.

44. Two workers earned 255 roubles. The first worked 10 days, and the second 9 days. How much did each of them get daily if it is known that the first worker got for 5 days by 15 roubles more than the second worker for 3 days?

45. The speed of a helicopter exceeds the speed of a car by 70 km/h, and the ratio of their speeds is 15 : 8. Find the speeds of the helicopter and the car.

46. One of the angles of a triangle is equal to 50°, and the difference of two other angles to 10°. Find the angles of the triangle.

47. 5% of one number and 4% of the other amount to 16, and 6% of the first number and 8% of the second to 24. Find these numbers.

Sec. 3.6. Solving Systems of Three Linear Equations in Three Variables

A *determinant of the third order* made up of the numbers a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 is defined as a number determined by the equality

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \qquad (3.2)$$

Formula (3.2) is called the *expansion of the third-order determinant in elements of the first row.*

The system of three linear equations in three variables

$$\begin{cases} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2, \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

(provided that the determinant of the system $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$) has a

unique solution which is found by Cramer's formulas:

$$x=\frac{\Delta_x}{\Delta},\quad y=\frac{\Delta_y}{\Delta},\quad z=\frac{\Delta_z}{\Delta},\tag{3.3}$$

where

$$\Delta_x=\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix},\quad \Delta_y=\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix},\quad \Delta_z=\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}.$$

And if $\Delta=0$, then the system is either indeterminate or incompatible. If a system is homogeneous, that is, if it has the form

$$\begin{cases} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0,\\ a_3x+b_3y+c_3z=0,\end{cases}$$

and $\Delta\neq 0$, then it has a unique solution: $x=0$, $y=0$, $z=0$.

If the determinant of a homogeneous system $\Delta=0$, then the system is reduced either to two independent equations (the third being their consequence) or to one equation (the remaining two equations being its consequence). In both cases, a homogeneous system has an infinite set of solutions.

48. Compute

$$\begin{vmatrix} 3 & 2 & 2\\ 1 & -5 & -8\\ 4 & 2 & 1\end{vmatrix}.$$

○ By Formula (3.2) we get

$$\begin{vmatrix} 3 & 2 & 2\\ 1 & -5 & -8\\ 4 & 2 & 1\end{vmatrix}=3\begin{vmatrix} -5 & -8\\ 2 & 1\end{vmatrix}-2\begin{vmatrix} 1 & -8\\ 4 & 1\end{vmatrix}+2\begin{vmatrix} 1 & -5\\ 4 & 2\end{vmatrix}$$

$$=3(-5+16)-2(1+32)+2(2+20)=33-66+44=11.\ \bullet$$

49. Solve the system of equations

$$\begin{cases} 7x-3y+5z=32,\\ 5x+2y+z=11,\\ 2x-y+3z=14.\end{cases}$$

○ We find:

$$\Delta=\begin{vmatrix} 7 & -3 & 5\\ 5 & 2 & 1\\ 2 & -1 & 3\end{vmatrix}=7\begin{vmatrix} 2 & 1\\ -1 & 3\end{vmatrix}-(-3)\begin{vmatrix} 5 & 2\\ 2 & 3\end{vmatrix}+5\begin{vmatrix} 5 & 2\\ 2 & -1\end{vmatrix}$$

$$=7(6+1)+3(15-2)+5(-5-4)=49+39-45=43;$$

$$\Delta_x=\begin{vmatrix} 32 & -3 & 5\\ 11 & 2 & 1\\ 14 & -1 & 3\end{vmatrix}=32\begin{vmatrix} 2 & 1\\ -1 & 3\end{vmatrix}-(-3)\begin{vmatrix} 11 & 1\\ 14 & 3\end{vmatrix}+5\begin{vmatrix} 11 & 2\\ 14 & -1\end{vmatrix}=86;$$

$$\Delta_y = \begin{vmatrix} 7 & 32 & 5 \\ 5 & 11 & 1 \\ 2 & 14 & 3 \end{vmatrix} = 7 \begin{vmatrix} 11 & 1 \\ 14 & 3 \end{vmatrix} - 32 \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} + 5 \begin{vmatrix} 5 & 11 \\ 2 & 14 \end{vmatrix} = -43;$$

$$\Delta_z = \begin{vmatrix} 7 & -3 & 32 \\ 5 & 2 & 11 \\ 2 & -1 & 14 \end{vmatrix} = 7 \begin{vmatrix} 2 & 11 \\ -1 & 14 \end{vmatrix} - (-3) \begin{vmatrix} 5 & 11 \\ 2 & 14 \end{vmatrix} + 32 \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = 129.$$

By Formulas (3.3), we get

$$x = \frac{\Delta_x}{\Delta} = \frac{86}{43} = 2; \quad y = \frac{\Delta_y}{\Delta} = \frac{-43}{43} = -1; \quad z = \frac{\Delta_z}{\Delta} = \frac{129}{43} = 3.$$

Answer: $\{(2, -1, 3)\}$. ●

50. Compute the following third-order determinants:

(1) $\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & 16 \\ 0 & -1 & 10 \end{vmatrix}$; (2) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 4 & -2 \\ 1 & 3 & -1 \end{vmatrix}$; (3) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$;

(4) $\begin{vmatrix} 1 & 2 & 8 \\ 3 & 2 & 10 \\ 4 & 3 & 4 \end{vmatrix}$.

51. Solve the following systems of equations:

(1) $\begin{cases} x - 2y + 3z = 6, \\ 2x + 3y - 4z = 20, \\ 3x - 2y - 5z = 6; \end{cases}$ (2) $\begin{cases} 4x - 5(y+1) = 1, \\ (5/12)\,y + (1/2)\,z = -1, \\ (5/6)\,x + (1/3)\,y - (3/2)\,z = -1; \end{cases}$

(3) $\begin{cases} 5x + y - 3z = -2, \\ 4x + 3y + 2z = 10, \\ 2x - 3y + z = 17; \end{cases}$ (4) $\begin{cases} 3x - 2y + z = 10, \\ x + 5y - 2z = -13, \\ 2x - 2y - z = 3; \end{cases}$

(5) $\begin{cases} 5x - 3y + 4z = 11, \\ 2x - y - 2z = -6, \\ 3x - 2y + z = 2; \end{cases}$ (6) $\begin{cases} 5x - 3y + 4z = 6, \\ 2x - y - z = 0, \\ x - 2y + z = 0; \end{cases}$

(7) $\begin{cases} 5x + 3y + 3z = 48, \\ 2x + 6y - 3z = 18, \\ 8x - 3y + 2z = 21; \end{cases}$ (8) $\begin{cases} x - 2y - z = 2, \\ 3x - 6y - 3z = 6, \\ 5x - 10y - 5z = 10. \end{cases}$

Sec. 3.7. Solving Quadratic Equations

The equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, and x is a variable, is called a *quadratic equation*. If $a = 1$, then the quadratic equation is said to be *reduced*, if $a \neq 1$, then it is *nonreduced*.

Quadratic equations of the form $ax^2 = 0$, $ax^2 + c = 0$, and $ax^2 + bx = 0$ are called *incomplete*.

The formula of the roots of the quadratic equation $ax^2 + bx + c = 0$ has the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{3.4}$$

where $b^2 - 4ac = D$ is the discriminant of the quadratic equation.

If $D < 0$, then a quadratic equation has no real roots; if $D = 0$, then it has one root: $-b/(2a)$; if $D > 0$, then it has two roots.

If p is an even number, the reduced quadratic equation $x^2 + px + q = 0$ is solved more conveniently by the formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}. \tag{3.5}$$

52. Solve the following pure quadratic equations: (1) $5x^2 = 0$; (2) $x^2 + 4x = 0$; (3) $3x^2 - 27 = 0$; (4) $x^2 + 16 = 0$.

○ (1) $(5x^2 = 0) \Leftrightarrow (x^2 = 0) \Leftrightarrow (x = 0)$. *Answer*: $\{0\}$.

(2) $(x^2 + 4x = 0) \Leftrightarrow (x(x+4) = 0) \Leftrightarrow \left[\begin{array}{l} x = -4, \\ x = 0. \end{array}\right.$ *Answer*: $\{-4, 0\}$.

(3) $(3x^2 - 27 = 0) \Leftrightarrow (x^2 - 9 = 0) \Leftrightarrow (x^2 = 9) \Leftrightarrow \left[\begin{array}{l} x = -3, \\ x = 3. \end{array}\right.$ *Answer*: $\{-3, 3\}$.

(4) There is no value of the variable x such that the sum $x^2 + 16$ attains a value equal to zero. Consequently, the equation has no roots. *Answer*: $\varnothing$. ●

53. Solve the following quadratic equations:

(1) $5x^2 + 7x + 2 = 0$; (2) $x^2 - 5x + 6 = 0$; (3) $x^2 - 6x + 8 = 0$; (4) $9x^2 + 24x + 16 = 0$; (5) $8x^2 - 16x + 9 = 0$.

○ (1) Using Formula (3.4), we find

$$x_{1,2} = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 5} = \frac{-7 \pm \sqrt{9}}{10} = \frac{-7 \pm 3}{10};$$

$$x_1 = \frac{-7-3}{10} = -1; \quad x_2 = \frac{-7+3}{10} = -0.4. \quad \textit{Answer}: \{-1, -0.4\}.$$

(2) By Formula (3.4), we get

$$x_{1,2} = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}, \quad x_1 = \frac{5-1}{2} = 2,$$

$$x_2 = \frac{5+1}{2} = 3. \quad \textit{Answer}: \{2, 3\}.$$

(3) By Formula (3.5), we get

$$x_{1,2} = 3 \pm \sqrt{9 - 8} = 3 \pm 1, \quad x_1 = 2, \quad x_2 = 4. \quad \textit{Answer}: \{2, 4\}.$$

This equation can also be solved by Formula (3.4).

(4) Using Formula (3.4), we get

$$x_{1,2} = \frac{-24 \pm \sqrt{24^2 - 4 \cdot 9 \cdot 16}}{2 \cdot 9} = \frac{-24 \pm \sqrt{576 - 576}}{18}.$$

The discriminant is equal to zero, the equation has one root: $x = -24/18 = -4/3$. *Answer*: $\{-4/3\}$.

(5) We find $D = 16^2 - 4 \cdot 8 \cdot 9 = 256 - 188 < 0$. Hence, the equation has no real roots. *Answer*: $\varnothing$. ●

54. Solve the given pure quadratic equations:

(1) $x^2 - 2 = 0$; (2) $y^2 - y = 0$; (3) $3x^2 + 6x = 8x^2 - 9x$;

(4) $\frac{3z^2 - 11}{8} + \frac{74 - 2z^2}{12} = 10$; (5) $\frac{5y^2 + 9}{6} - \frac{4y^2 - 9}{5} = 3$;

(6) $(5x + 4)(5x - 4) - 10(x - 2) = 4$; (7) $(x + 2)^3 + 19 = (x + 3)^3$;

(8) $(y - 3)^3 + 2y(5y + 1) = y^3 - (2y - 1)^2 - 26$.

55. Solve the indicated quadratic equations:

(1) $x^2 - 6x + 8 = 0$; (2) $x^2 + 9x + 20 = 0$;

(3) $x^2 + x - 12 = 0$; (4) $3x^2 - 8x + 4 = 0$;

(5) $16x^2 + 16x + 3 = 0$; (6) $5x^2 - 26x - 24 = 0$;

(7) $x^2 - 4x + 4 = 0$; (8) $2x^2 - 3x + 8 = 0$.

56. Solve the following quadratic equations:

(1) $(x - 3)^2 + (x - 4)^2 - (x - 5)^2 - x = 24$;

(2) $\frac{(x - 12)^2}{6} - \frac{(x - 14)^2}{2} + \frac{x(x - 9)}{18} - \frac{x}{9} = 5$;

(3) $\frac{x(x - 7)}{3} + \frac{x - 4}{3} - \frac{11x}{10} = 1$;

(4) $\frac{x(2x - 3)}{2} + \frac{(3x - 1)^2}{5} - \frac{(x + 3)^2}{5} = 1$.

Sec. 3.8. The Properties of the Roots of a Quadratic Equation. Factorization of a Quadratic Trinomial

Viète's Theorem (direct). *If the equation* $ax^2 + bx + c = 0$ *has two real roots* x_1 *and* x_2, *then*

$$\begin{cases} x_1 + x_2 = -b/a, \\ x_1 x_2 = c/a. \end{cases} \tag{3.6}$$

If the reduced quadratic equation $x^2 + px + q = 0$ has roots x_1 and x_2, then

$$\begin{cases} x_1 + x_2 = -p, \\ x_1 x_2 = q. \end{cases} \tag{3.7}$$

Viète's Theorem (converse). *If the sum of certain numbers* x_1 *and* x_2 *is equal to* $-b/a$ $(-p)$, *and their product is equal to* c/a (q), *then these numbers are the roots of the quadratic equation* $ax^2 + bx + c = 0$ $(x^2 + px + q = 0)$.

The formula for factoring a quadratic trinomial into linear factors has the form

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \tag{3.8}$$

where x_1 and x_2 are the roots of the quadratic trinomial, and x is a variable.

57. Using Viète's theorem, find the roots of the quadratic equation $x^2 - 9x + 20 = 0$.

○ Applying Viète's theorem to the given equation, we get

$$\begin{cases} x_1 + x_2 = 9, \\ x_1 x_2 = 20. \end{cases}$$

It is obvious that this system is satisfied only by the numbers 4 and 5. *Answer*: {4, 5}. ●

58. Set up the quadratic equations whose roots are: (1) $x_1 = -2$, $x_2 = 4$; (2) $x_1 = -3/4$, $x_2 = -5/6$.

○ (1) Using Viète's theorem (the converse), we get

$$\begin{cases} x_1 + x_2 = -2 + 4 = 2, \\ x_1 x_2 = (-2) \cdot 4 = -8. \end{cases}$$

Consequently, to the given roots there corresponds the quadratic equation $x^2 - 2x + 8 = 0$.

(2) On the basis of Viète's theorem (the converse), we have

$$\begin{cases} x_1 + x_2 = -\frac{3}{4} - \frac{5}{6} = -\frac{19}{12}, \\ x_1 x_2 = \left(-\frac{3}{4}\right) \cdot \left(-\frac{5}{6}\right) = \frac{5}{8}, \end{cases}$$

$$\left(x^2 + \frac{19}{12}x + \frac{5}{8} = 0\right) \Leftrightarrow (24x^2 + 38x + 15 = 0). \quad ●$$

59. Determine the signs of the roots of the given quadratic equation without solving it: (1) $15x^2 + x - 10 = 0$; (2) $5x^2 - x - 10 = 0$; (3) $2x^2 - 13x + 12 = 0$; (4) $6x^2 + 18x + 12 = 0$.

○ (1) Here $a > 0$, $b > 0$, $c < 0$, $D = b^2 - 4ac > 0$, since $c < 0$. Consequently, the equation has two distinct roots. According to Viète's theorem, we have $\begin{cases} x_1 + x_2 = -1/15, \\ x_1 x_2 = -10/15. \end{cases}$ The product of the roots is a negative number, therefore the roots have opposite signs. The sum of the roots is a negative number, consequently, the greater (in modulus) root is negative.

(2) Here $c < 0$, therefore $D = b^2 - 4ac > 0$, and the equation has two roots. According to Viète's theorem, we get

$\begin{cases} x_1 + x_2 = 1/5, \\ x_1 x_2 = -10/5. \end{cases}$ The product of the roots is negative, consequently, the roots have opposite signs. The sum of the roots is a positive number, therefore the greater (in modulus) root is positive.

(3) Since $D = 13^2 - 4 \cdot 2 \cdot 12 > 0$ the equation has two roots. By virtue of Viète's theorem, we get $\begin{cases} x_1 + x_2 = 13/2, \\ x_1 x_2 = 12/2. \end{cases}$ The product and the sum of the roots are positive numbers. Consequently, both roots are positive.

(4) We find $D = 18^2 - 4 \cdot 6 \cdot 12 > 0$, that is, the equation has two roots. By Viète's theorem, we obtain $\begin{cases} x_1 + x_2 = -18/6, \\ x_1 x_2 = 12/6. \end{cases}$ The product of the roots is a positive number, while the sum of the roots is negative; consequently, both roots are negative. ●

60. Factorize the indicated quadratic trinomials into linear factors:

(1) $2x^2 - 5x - 12$; (2) $9x^2 + 6x - 8$; (3) $-20x^2 + 7x + 6$.

○ (1) Find the roots of the trinomial: $-3/2$ and 4. By Formula (3.8), we get

$$2x^2 - 5x - 12 = 2\left(x + \frac{3}{2}\right)(x - 4) = (2x + 3)(x - 4).$$

(2) Here, the roots of the trinomial are equal to $-4/3$ and $2/3$. Hence,

$$9x^2 + 6x - 8 = 9\left(x + \frac{4}{3}\right)\left(x - \frac{2}{3}\right) = (3x + 4)(3x - 2).$$

(3) Here, the roots of the trinomial are equal to $-2/5$ and $3/4$. Therefore

$$-20\left(x + \frac{2}{5}\right)\left(x - \frac{3}{4}\right) = -(5x + 2)(4x - 3) = (5x + 2)(3 - 4x). \bullet$$

61. Solve by inspection the following quadratic equations using Viète's theorem: (1) $x^2 - 4x + 3 = 0$; (2) $x^2 - 7x + 10 = 0$; (3) $x^2 - 2x - 15 = 0$; (4) $x^2 - x - 12 = 0$; (5) $x^2 + 6x + 8 = 0$; (6) $x^2 + 2x - 15 = 0$.

62. Set up the quadratic equations having the following roots: (1) $\{5; 8\}$; (2) $\{-5; 2\}$; (3) $\{-4; -5\}$; (4) $\{2/3; 4/5\}$; (5) $\{-1/4; 3/8\}$; (6) $\{-2/5; -5/6\}$; (7) $\{1/4; 4\}$; (8) $\{-1/3; 3\}$.

63. Determine the signs of the given quadratic equations (without solving the equations): (1) $6x^2 + 2x - 11 = 0$; (2) $4x^2 - x - 9 = 0$; (3) $2x^2 - 15x + 11 = 0$; (4) $x^2 - 7x + 10 = 0$; (5) $3x^2 + 13x + 9 = 0$; (6) $2x^2 - 12x + 9 = 0$.

64. Factorize the given quadratic trinomials into linear factors: (1) $2x^2 - 7x + 3$; (2) $6a^2 + 5a - 6$; (3) $3y^2 - 11y - 20$; (4) $12x^2 + 7x + 1$; (5) $m^2 - 2m - 3$; (6) $72x^2 - 67x + 15$.

65. Reduce the following fractions: (1) $\frac{2x^2 - 9x + 10}{2x^2 + x - 15}$;

(2) $\frac{3y^2 + 8y - 3}{6y^2 + 13y - 5}$; (3) $\frac{6a^2 + 5a - 4}{3a^2 + 19a + 20}$; (4) $\frac{4 + 3a - a^2}{3a^2 + 4a + 1}$;

(5) $\frac{2x^2 + 7x + 6}{12 + 5x - 2x^2}$; (6) $\frac{21y^2 + 11y - 2}{15y^2 + 16y + 4}$.

Sec. 3.9. Solving Equations Reducible to Quadratics

Equations Containing the Variable in the Denominator of a Fraction (Rational-Fractional Equations in One Variable)

66. Solve the following equations:

(1) $\frac{1}{2(x-2)}-\frac{1}{3(3x-7)}-\frac{1}{x}=0$; (2) $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{64}{x^2-16}$.

○ (1) $\frac{1}{2(x-2)}-\frac{1}{3(3x-7)}-\frac{1}{x}=0$

$$\Leftrightarrow \begin{cases} \dfrac{3x(3x-7)-2x(x-2)-6(x-2)(3x-7)}{6x(x-2)(3x-7)}=0, \\ 6x(x-2)(3x-7)\neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 11x^2-61x+84=0, \\ 6x(x-2)(3x-7)\neq 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=2\frac{6}{11}, \\ x=3 \end{array}\right. \\ 6x(x-2)(3x-7)\neq 0. \end{cases}$$

For $x=2\frac{6}{11}$ the statement $6\cdot 2\frac{6}{11}\left(2\frac{6}{11}-2\right)\left(3\cdot 2\frac{6}{11}-7\right)\neq 0$ is true; for $x=3$ the statement $6\cdot 3\cdot(3-2)\cdot(3\cdot 3-7)\neq 0$ is true. Thus we get the *answer*: $\left\{2\frac{6}{11};\ 3\right\}$.

(2) $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{64}{x^2-16} \Leftrightarrow \begin{cases} \dfrac{(x+4)^2+(x-4)^2-64}{(x-4)(x+4)}=0, \\ (x-4)(x+4)\neq 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x^2-16=0, \\ (x-4)(x+4)\neq 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=-4, \\ x=4 \end{array}\right. \\ (x-4)(x+4)\neq 0 \end{cases}$$

For $x=\pm 4$ both sides of the original equation have no sense. Therefore the roots of the quadratic equation $x^2-16=0$ are not the roots of the given equation. *Answer*: $\varnothing$. ●

Biquadratic Equations

67. Solve the equation: $x^4-13x^2+36=0$.

○ To solve the given biquadratic equation, we make a substitution: $x^2=z$. We then obtain the quadratic equation $z^2-13z+36=0$ whose roots are $z_1=4$ and $z_2=9$. In order to find the roots of the original equation, we solve the combination of equa-

tions:

$$(x^4-13x^2+36=0) \Leftrightarrow \left[\begin{array}{l} x^2=4, \\ x^2=9 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x_{1,2}=\pm 2, \\ x_{3,4}=\pm 3. \end{array}\right.$$

Answer: $\{-3;\ -2;\ 2;\ 3\}$.

Equations Whose Left Member Is Factorizable

68. Solve the following equations: (1) $x^3-2x^2-8x=0$; (2) $x^3-5x^2-x+5=0$.

○ (1) It is obvious that the left-hand member is factorizable. To find the roots of the original equation, we equate each factor to zero and solve the combination of equations

$$(x^3-2x^2-8x=0) \Leftrightarrow (x(x^2-2x-8)=0)$$

$$\Leftrightarrow \left[\begin{array}{l} x=0, \\ x^2-2x-8=0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=0, \\ x=-2, \\ x=4. \end{array}\right.$$ *Answer*: $\{-2;\ 0;\ 4\}$.

(2) Factorize the left-hand member of the equation and equate each factor to zero. Solving the obtained combination of equations, we get the roots of the original equation:

$$x^3-5x^2-x+5=0 \Leftrightarrow (x^2(x-5)-(x-5)=0)$$

$$\Leftrightarrow ((x-5)(x+1)(x-1)=0) \Leftrightarrow \left[\begin{array}{l} x-5=0, \\ x+1=0, \\ x-1=0 \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x=5, \\ x=-1, \\ x=1. \end{array}\right.$$ *Answer*: $\{-1;\ 1;\ 5\}$. ●

Binomial Equations

69. Solve the given equations: (1) $x^4-16=0$; (2) $x^3-8=0$.

○ (1) $(x^4-16=0) \Leftrightarrow ((x^2-4)(x^2+4)=0) \Leftrightarrow ((x+2)(x-2)$

$$(x^2+4)=0 \Leftrightarrow \left[\begin{array}{l} x+2=0, \\ x-2=0, \\ x^2+4=0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=-2, \\ x=2, \\ \varnothing. \end{array}\right.$$ *Answer*: $\{-2;\ 2\}$.

(2) $(x^3-8=0) \Leftrightarrow ((x-2)(x^2+2x+4)=0) \Leftrightarrow \left[\begin{array}{l} x-2=0, \\ x^2+2x+4=0 \end{array}\right.$

$$\Leftrightarrow \left[\begin{array}{l} x=2, \\ \varnothing. \end{array}\right.$$ *Answer*: $\{2\}$. ●

70. Solve the following equations:

(1) $\frac{14}{x^2-9}+\frac{1}{3-x}+\frac{4-x}{x+3}-\frac{7}{x+3}=0$;

(2) $\frac{3}{x-2}+\frac{2}{x-3}=\frac{4}{x-1}+\frac{1}{x-4}$;

(3) $\frac{2}{x^2-x+1}-\frac{1}{x+1}=\frac{2x-1}{x^3+1}$;

(4) $\frac{1}{x^2}+\frac{1}{x}+\frac{6}{x}=\frac{4(1+x^2)}{x^2}$;

(5) $\frac{1}{2(1-y)}+\frac{5}{4y^2+4y+4}=\frac{3y}{y^3-1}$;

(6) $\frac{18z+7}{z^3-1}+\frac{30}{1-z^2}+\frac{13}{z^2+z+1}=0$;

(7) $\frac{6-x}{3x^2-12}+\frac{1}{2-x}-\frac{1}{x-2}=1$;

(8) $\frac{6}{x^2-9}+\frac{13-x}{3+x}-\frac{3}{x+3}+\frac{2}{3-x}=0$.

71. Solve the indicated equations:

(1) $x^4-5x^2+4=0$; (2) $x^4-34x^2+225=0$;

(3) $9x^4-37x^2+4=0$; (4) $4x^4-65x^2+16=0$.

72. Factor into linear factors: (1) $4x^4-17x^2+4$; (2) x^4-125x^2+484.

73. Solve the following equations:

(1) $x^3-4x^2-11x+30=0$; (2) $x^4-6x^3+3x^2+26x-24=0$;

(3) $x^4+2x^3-13x^2-14x+24=0$; (4) $2x^3-7x^2+2x+3=0$; (5) $x^4-13x^2-12x=0$; (6) $x^3-2x^2-5x+6=0$.

74. Set up the equations having the indicated roots:

(1) $\{-1; 2; -3\}$; (2) $\{-1/2; -1/3; 1\}$.

75. Solve the following equations: (1) $x^4-81=0$; (2) $125x^3+8=0$; (3) $16x^4-625=0$; (4) $125x^3-27=0$.

Sec. 3.10. Problems on Setting up Quadratic Equations

76. The product of two successive natural numbers is equal to 552. Find these numbers.

77. The numerator of a fraction is less than its denominator by 2. The addition of this fraction with its inverse yields 34/15. Find this fraction.

78. Find the two-digit number if it is known that the units digit of the desired number is greater than its tens digit by 2, and that the product of the number by the sum of its digits is equal to 144.

79. The perimeter of a rectangle is equal to 42 cm, and its diagonal to 15 cm. Find the lengths of the sides of the rectangle.

80. The area of a rectangle is equal to 192 cm^2, and its perimeter is equal to 56 cm. Find the sides of the rectangle.

81. The perimeter of a rectangle is equal to 40 cm, and the sum of the area of the squares constructed on adjacent sides of the rectangle is equal to 208 cm^2. Find the lengths of the sides of the rectangle.

82. If each participant of a chess tournament plays one game with each of the remaining participants, then 153 games will be played during the tournament. Find the number of the participants.

83. Two teams, working jointly, complete harvesting a plot for four days. For how many days would each team complete the job, working separately, if one of the teams might do the work six days faster than the other?

84. Working together, two road-repairing teams repaired 4.5 km of the road per day. The second team worked two days less than the first team, but each of them repaired 20 km of the road. How many kilometres of the road were repaired by each team per day?

85. When testing two engines of equal power for economical operation, it was established that one of them consumed 600 g of petrol, while the other, which operates two hours less than the first engine, consumed 384 g of fuel. If the consumption of petrol per hour of the first engine were equal to that of the second, and vice versa, then, for the same time of operation, the fuel consumption in both engines would be the same. How much petrol was consumed by each engine per hour?

86. The sum of the circumferences of the front and rear wheels of a vehicle is equal to 5 m. Over a distance of 60 m, the front wheel made nine revolutions more than the rear wheel over a distance of 63 m. Find the circumferences of the wheels.

87. Two skaters run in the same direction along a circular path 2 km long, meeting each 20 minutes. Find the speed (in km/h) of each skater if the first of them completes the circle one minute faster than the second.

88. The distance between two towns is equal to 960 km. A passenger train covers this distance with a speed 20 km/h higher than a goods train does. Find the speed of each train if the passenger train covers the whole path four hours faster than the goods train.

89. A cyclist had to ride a distance of 30 km. Having started 3 minutes ahead of schedule, the cyclist rode with a speed 1 km/h less than the prescribed speed, and reached the place of destination in time. Find the speed with which the cyclist rode.

Sec. 3.11. Solving Quadratic Inequalities

A *quadratic inequality* is defined as an inequality of the form $ax^2 + bx + c > 0$ (or $ax^2 + bx + c < 0$), where $a \neq 0$.

Let it be required to solve the inequality $ax^2 + bx + c > 0$. Depending on the sign of the discriminant $D = b^2 - 4ac$ three possibilities may occur.

1. If $D < 0$, then the graph of the quadratic trinomial $f(x) = ax^2 + bx + c$ does not intersect the x-axis and lies above this axis (for $a > 0$) or below it (for

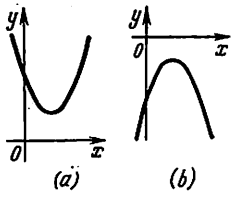


Fig. 1

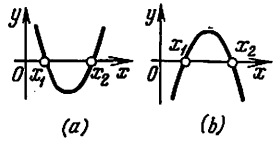


Fig. 2

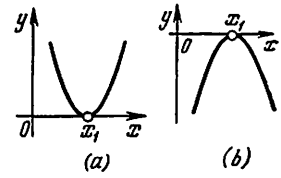


Fig. 3

$a < 0$). In the first case, the set of solutions of the inequality is the set of all real numbers (Fig. 1 (*a*)), while in the second case the set of solutions is empty (Fig. 1 (*b*)).

2. If $D > 0$, then the graph of the quadratic trinomial intersects the x-axis at the points x_1 and x_2 ($x_1 < x_2$) which are the roots of the equation $ax^2 + bx + c = 0$. The points x_1 and x_2 separate the entire number line into three intervals: $(-\infty, x_1)$, (x_1, x_2) and $(x_2, +\infty)$.

The sign of the quadratic trinomial coincides with the sign of the coefficient a at all points of the intervals $(-\infty, x_1)$ and $(x_2, +\infty)$ and is opposite to the sign of the coefficient a at all points of the interval (x_1, x_2) (Fig. 2(*a*), (*b*)).

3. If $D = 0$, then the graph of the quadratic trinomial touches the x-axis at the point x_1 which is the only root of the equation $ax^2 + bx + c = 0$. The point x_1 divides the number line into two intervals: $(-\infty, x_1)$ and $(x_1, +\infty)$. Here, the sign of the quadratic trinomial coincides with the sign of the coefficient a for all $x \neq x_1$ (Fig. 3, (*a*) (*b*)).

90. Solve the following inequalities: (1) $2x^2 + 3x - 2 > 0$; (2) $x^2 - 5x + 8 > 0$; (3) $-3x^2 - x + 2 > 0$; (4) $9x^2 + 6x + 1 > 0$.

○ (1) Here $f(x) = 2x^2 + 3x - 2$, $D = 9 + 16 = 25 > 0$; we have $(2x^2 + 3x - 2 = 0) \Leftrightarrow \begin{cases} x_1 = -2, \\ x_2 = 1/2. \end{cases}$ Thus, we obtain three intervals: $(-\infty, -2)$, $(-2, 1/2)$ and $(1/2, +\infty)$. Since $a = 2 > 0$, we have $f(x) > 0$ in the intervals $(-\infty, -2)$ and $(1/2, +\infty)$. Consequently, the set of solutions of the given inequality is the union of the intervals $(-\infty, -2) \cup (1/2, +\infty)$.

(2) Here $f(x) = x^2 - 5x + 8$, $D = 25 - 32 = -7 < 0$. Since $a = 1 > 0$, we have $f(x) > 0$ for all $x \in \mathbf{R}$, that is, the given inequality holds true on the entire number line.

(3) We have $(-3x^2 - x + 2 > 0) \Leftrightarrow (3x^2 + x - 2 < 0)$, $f(x) = 3x^2 + x - 2$, $D = 1 + 24 = 25 > 0$. Further, $(3x^2 +$

$+ x - 2 = 0) \Leftrightarrow \left[\begin{array}{l} x_1 = -1, \\ x_2 = 2/3. \end{array}\right.$ Consequently, we get three intervals: $(-\infty, -1)$, $(-1, 2/3)$ and $(2/3, +\infty)$. Since $a = 3 > 0$, we have $f(x) < 0$ in the interval $(-1, 2/3)$. Hence, the interval $(-1, 2/3)$ is the solution of the given inequality.

(4) Here $f(x) = 9x^2 + 6x + 1$, $D = 6^2 - 4 \cdot 9 = 0$. We find $(9x^2 + 6x + 1 = 0) \Leftrightarrow (x_1 = -1/3)$. Since $a = 9 > 0$, we have $f(x) > 0$ for all $x \neq -1/3$. Consequently, the union of the intervals $(-\infty, -1/3) \cup (-1/3, +\infty)$ is the solution of the given inequality. ●

91. Solve the inequality $\frac{x-1}{2x+1} > 0$.

○ Multiplying both sides of the given inequality by the positive number $(2x + 1)^2$ which equals the square of the denominator, we get an equivalent inequality $(x - 1)(2x + 1) > 0$. This is a quadratic inequality whose left-hand member has been already factored into linear factors. It is obvious that $f(x) = (x - 1)(2x + 1) > 0$ in the intervals $(-\infty, -1/2)$ and $(1, +\infty)$. Therefore the solution of the given inequality is represented by the union of the intervals $(-\infty, -1/2) \cup (1, +\infty)$. ●

92. Solve the following inequalities: (1) $x^2 - 8x - 20 < 0$; (2) $-x^2 - 6x + 27 < 0$; (3) $2x^2 - 13x + 20 > 0$; (4) $2x^2 - x + 4 < 0$; (5) $-3x^2 + 5x + 2 > 0$; (6) $4x^2 - 4x + 1 > 0$; (7) $2x^2 - 4x + 13 > 0$; (8) $-x^2 + 12x - 36 < 0$.

93. Solve the indicated inequalities: (1) $\frac{2x-3}{3x-2} > 0$; (2) $\frac{2-3x}{2x+7} > 0$; (3) $\frac{8-3x}{7x-2} < 0$; (4) $\frac{5x+8}{3x-7} < 0$.

Sec. 3.12. Irrational Equations

An equation in which the variable is contained under the radical sign is called an *irrational equation*. Or in other words: if an equation contains an irrational expression, it is called an *irrational equation*.

When solving irrational equations it is recommended to observe the following rules: (1) raising both sides of an equation to an even power yields an equation which is the consequence of the original equation. In this case, a mixed system consisting of the derived equation and the constraints of the variable is equivalent to the original equation; (2) raising both sides of an equation to an odd power yields an equation equivalent to the original equation.

Solving irrational equations involves the same steps involved in solving other equations. However, one additional step is necessary. First we eliminate the radical by raising both sides of the equation to a higher power. Then the new derived equation can usually be solved. However, when the equation has been solved, all roots must be checked in the original equation.

94. Solve the indicated irrational equations:

(1) $\sqrt{x-4} = 2$; (2) $\sqrt{x^2-7} = 3$; (3) $\sqrt{x} = x - 6$; (4) $\sqrt{x^2-12} = \sqrt{x}$.

(5) $\sqrt{x-5}=\sqrt{3-x}$; (6) $\sqrt{x-3}=x-9$; (7) $\sqrt{x-1}+\sqrt{2x-1}=5$.

○ (1) $(\sqrt[3]{x-4}=2)\Leftrightarrow((\sqrt[3]{x-4})^3=2^3)\Leftrightarrow(x-4=8)\Leftrightarrow x=12$. *Answer*: {12}.

(2) $((\sqrt{x^2-7})^2=3^2)\Leftrightarrow(x^2-7=9)\Leftrightarrow(x^2=16)\Leftrightarrow\left[\begin{array}{l}x=-4,\\x=4.\end{array}\right.$

Substituting the values -4 and 4 into the original equation, we get $\sqrt{(-4)^2-7}=3$ and $\sqrt{4^2-7}=3$, that is, the found roots satisfy the given equation. *Answer*: $\{-4, 4\}$.

$$(3)\ \sqrt{x}=x-6\Leftrightarrow\left\{\begin{array}{l}x=(x-6)^2,\\x-6\geqslant 0,\\x>0\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x^2-13x+36=0,\\x\geqslant 6\end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l}\left[\begin{array}{l}x=4,\\x=9\end{array}\right.\\x\geqslant 6\end{array}\right.\Leftrightarrow x=9.$$

In solving this equation, we have used equivalent passages, taking into consideration that the root of an even power of a positive number is a positive number. *Answer*: {9}.

$$(4)\left\{\begin{array}{l}(\sqrt{x^2-12})^2=(\sqrt{x})^2,\\x\geqslant 0,\\x\geqslant\sqrt{12}\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x^2-x-12=0,\\x\geqslant\sqrt{12}\end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l}\left[\begin{array}{l}x=-3,\\x=4\end{array}\right.\\x\geqslant\sqrt{12}\end{array}\right.\Leftrightarrow x=4.$$

Answer: {4}.

(5) Let us show that the given equation has no solutions. By the definition of the root of an even power, we get

$$\left\{\begin{array}{l}x-5\geqslant 0,\\3-x\geqslant 0\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x\geqslant 5,\\x\leqslant 3.\end{array}\right.$$

This system of inequalities has no solution. *Answer*: ∅.

$$(6)\ \sqrt{x-3}=x-9\Leftrightarrow\left\{\begin{array}{l}x-3=(x-9)^2\\x-9\geqslant 0\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x^2-19x+84=0,\\x\geqslant 9\end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l}\left[\begin{array}{l}x=7,\\x=12\end{array}\right.\\x\geqslant 9\end{array}\right.\Leftrightarrow x=12.$$

Answer: {12}.

$$(7)\ \begin{cases} (\sqrt{x-1}+\sqrt{2x-1})^2=25, \\ x\geqslant 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2\sqrt{(x-1)(2x-1)}=27-3x, \\ x\geqslant 1, \\ 27-3x\geqslant 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (2\sqrt{(x-1)(2x-1)})^2=(27-3x)^2, \\ x\geqslant 1, \\ x\leqslant 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-150x+725=0, \\ 1\leqslant x\leqslant 9 \end{cases} \Leftrightarrow \left[\begin{array}{l} x=5, \\ x=145 \end{array}\right. \Leftrightarrow x=5.$$

Answer {5}. ●

95. Without solving the indicated equations, explain why each of them cannot have roots:

(1) $\sqrt{x+1}+\sqrt{x+3}=0$; (2) $\sqrt{4x-3}=-4$;

(3) $\sqrt{x^2+2}+\sqrt{2x-1}=-2$; (4) $\sqrt{3-x}+\sqrt{2-x}=-1$;

(5) $\sqrt{x-7}-\sqrt{6-x}=3$; (6) $\sqrt{x}-\sqrt{-3-x}=1$.

In Problems 96 through 98 solve the given equations:

96. (1) $\sqrt[3]{x(x+6)}=x$; (2) $\sqrt{x}=x-2$;

(3) $\sqrt{x+2}=x-4$; (4) $\sqrt{x^2+9}=2x-3$;

(5) $\sqrt{x^2-9}=3x-11$; (6) $\sqrt{x^2+5x+1}=2x-1$;

(7) $\sqrt{x^2-9}=x^2-21$; (8) $\sqrt{x^2-1}=\sqrt{3}$.

97. (1) $\sqrt{2x-1}-\sqrt{x-1}=1$; (2) $\sqrt{5x+20}-\sqrt{x+8}=2$; (3) $\sqrt{1-x}+\sqrt{1+x}=1$; (4) $\sqrt{4-x}+\sqrt{5+x}=3$; (5) $\sqrt{25-x}+\sqrt{9+x}=2$; (6) $\sqrt{4x+2}+\sqrt{4x-2}=4$; (7) $\sqrt[3]{8x+4}-\sqrt[3]{8x-4}=2$; (8) $\sqrt{x-3}\sqrt{2x+2}=x+1$.

98. (1) $\sqrt{x+3}-\sqrt{7-x}=\sqrt{2x-8}$; (2) $\sqrt{x+7}+\sqrt{x+2}=\sqrt{3x+19}$; (3) $\sqrt{3x+1}+\sqrt{4x-3}=\sqrt{5x+4}$; (4) $\sqrt{5x-4}+\sqrt{2x-1}=\sqrt{3x+1}$.

Sec. 3.13. Irrational Inequalities in One Variable

The solution of an irrational inequality in one variable is reduced to solving an equivalent system of rational inequalities or a combination of systems of rational inequalities.

These systems are solved by imposing some constraints on the variable and raising both sides of the inequalities to the same power.

Let us consider the methods of solving simplest irrational inequalities in one variable.

99. Solve the following irrational inequalities:

(1) $\sqrt{3x+13} < x+1$; (2) $\sqrt{x^2-1} > x-3$; (3) $\sqrt{1-x} - \sqrt{x} \geqslant 1/\sqrt{5}$.

○ (1) The given inequality is equivalent to a system of rational inequalities:

$$\sqrt{3x+13} < x+1 \Leftrightarrow \begin{cases} x+1>0, \\ 3x+13 \geqslant 0, \\ 3x+13 < x^2+2x+1 \end{cases} \Leftrightarrow \begin{cases} x>-1 \\ x \geqslant -13/3, \\ x^2-x-12>0 \end{cases} \Leftrightarrow \begin{cases} x>-1 \\ \left[\begin{array}{l} x<-3, \\ x>4. \end{array}\right. \end{cases}$$

The solution of this system is the set of solutions of two systems:

$$\begin{cases} x>-1, \\ \left[\begin{array}{l} x<-3, \\ x>4 \end{array}\right. \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x>-1, \\ x<-3 \end{cases} \\ \begin{cases} x>-1, \\ x>4 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \varnothing, \\ x>4 \end{array}\right. \Leftrightarrow x>4.$$

Answer: $(4, +\infty)$.

(2) This inequality is equivalent to the combination of two systems:

$$\sqrt{x^2-1} > x-3 \Leftrightarrow \left[\begin{array}{l} \begin{cases} x-3 \geqslant 0, \\ x^2-1 > x^2-6x+9. \end{cases} \\ \begin{cases} x-3<0, \\ x^2-1 \geqslant 0. \end{cases} \end{array}\right.$$

We then find the solution of this combination:

$$\left[\begin{array}{l} \begin{cases} x-3 \geqslant 0, \\ x^2-1 > x^2-6x+9 \end{cases} \\ \begin{cases} x-3<0, \\ x^2-1 \geqslant 0 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \begin{cases} x \geqslant 3, \\ x>5/3 \end{cases} \\ \begin{cases} x<3, \\ \left[\begin{array}{l} x \leqslant -1, \\ x \geqslant 1 \end{array}\right. \end{cases} \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x\geqslant 3, \\ \left\{\begin{array}{l} x<3, \\ x\leqslant -1 \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x\geqslant 3, \\ x\leqslant -1, \\ 1\leqslant x<3 \end{array}\right.$$

$$\Leftrightarrow (-\infty,\ -1]\cup[1,\ 3)\cup[3,\ +\infty)=(-\infty,\ -1]\cup[1,\ +\infty).$$

Answer: $(-\infty,\ -1]\cup[1,\ +\infty)$.

(3) The given inequality is equivalent to the system

$$\sqrt{1-x}-\sqrt{x}\geqslant 1/\sqrt{5} \Leftrightarrow \left\{\begin{array}{l} x\geqslant 0, \\ 1-x>0, \\ \sqrt{1-x}>\sqrt{x}, \\ (\sqrt{1-x}-\sqrt{x})^2\geqslant (1/\sqrt{5})^2 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} x\geqslant 0, \\ x<1, \\ 1-x>x, \\ 1-x-2\sqrt{x(1-x)}+x\geqslant 1/5 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x\geqslant 0, \\ x<1, \\ x<1/2, \\ 4/5\geqslant 2\sqrt{x(1-x)} \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} 0\leqslant x<1/2, \\ 2/5\geqslant \sqrt{x(1-x)} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} 0\leqslant x<1/2, \\ 4/25\geqslant x(1-x) \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} 0\leqslant x<1/2, \\ x^2-x+4/25\geqslant 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} 0\leqslant x<1/2, \\ x\leqslant 1/5, \\ x\geqslant 4/5 \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} \left\{\begin{array}{l} 0\leqslant x<1/2, \\ x\leqslant 1/5 \end{array}\right. \\ \left\{\begin{array}{l} 0\leqslant x<1/2, \\ x\geqslant 4/5 \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 0\leqslant x\leqslant 1/5, \\ \varnothing \end{array}\right. \Leftrightarrow 0\leqslant x\leqslant 1/5.$$

Answer: $[0,\ 1/5]$. ●

In Problems 100 through 102 solve the indicated inequalities:

100. (1) $\sqrt{x+12}<x$; (2) $\sqrt{x+3}<x+1$; (3) $\sqrt{2x+9}<3-x$; (4) $\sqrt{x^2-1}<5-x$; (5) $\sqrt{3x-x^2}<4-x$; (6) $\sqrt{x^2-3x-10}<8-x$.

101. (1) $\sqrt{x}>-1$; (2) $\sqrt{2x-5}>7$; (3) $\sqrt{x+2}>x$;

(4) $\sqrt{x^2-3x-10} > x-2$; (5) $\sqrt{x^2+4x} > 2-x$; (6) $\sqrt{2x+1} > 1-x$; (7) $\sqrt{3x+1} > \sqrt{2-x}$; (8) $\sqrt{2x+1} > \sqrt{3-x}$.

102. (1) $\sqrt{2x+1} - \sqrt{x-8} > 3$; (2) $\sqrt{9-x^2} > 3 - \sqrt{6x-x^2}$; (3) $\sqrt{20-x} - \sqrt{10-x} > 2$.

Sec. 3.14. Nonlinear Systems of Equations and Inequalities in Two Variables

An *inequality in two variables* x and y has the form $f(x, y) > 0$. The *solution of* such an *inequality* is defined as an ordered pair of numbers (x_0, y_0) the substitution of which into the given inequality results in a true statement $f(x_0, y_0) > 0$.

A system of inequalities in two variables x and y has the form

$$\begin{cases} f(x, y) > 0, \\ g(x, y) > 0. \end{cases}$$

The *solution of a system of inequalities* is defined as an ordered pair of numbers satisfying each of the inequalities of the system.

It is clear that the set of all the solutions of a system of equations or inequalities (or a mixed system of equations and inequalities) is the intersection of the sets of the solutions of the equations or inequalities (or both of them) entering this system.

Two systems of equations or inequalities are said to be *equivalent* if the sets of their solutions coincide.

For solving systems of equations the following rules are used: (1) one of the equations entering a system may be replaced by an equivalent equation; (2) if one of the equations of a system has the form $x = A$, where A is an expression not containing x, then in the remaining equations of the system the variable x may be replaced by its expression A; (3) any equation of a system may be replaced by the equation obtained as the result of addition of this equation and any other equation of the system; (4) any equation of a system may be multiplied by a nonvanishing expression.

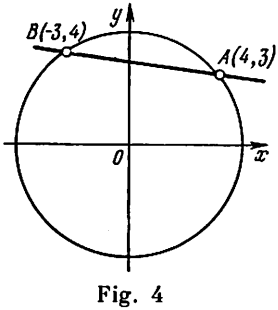


Fig. 4

103. Give a geometrical illustration of the solutions of the indicated systems:

(1) $\begin{cases} x^2+y^2=25, \\ x+7y-25=0; \end{cases}$

(2) $\begin{cases} x^2+y^2 \leqslant 25, \\ x+7y-25=0. \end{cases}$

○ (1) The set of the solutions of the first equation is represented by the set of points belonging to the circle of radius 5 centered at the origin (Fig. 4). The set of the points belonging to the straight line (AB) is the set of solutions of the second equation. Consequently, the set of all the solutions of the given system consists of two points of intersection of the circle and a straight line, that is, of the points (4, 3), and (−3, 4).

(2) It is obvious that the set of all the solutions of the given system is represented by the set of points of the line segment $[AB]$ which is the intersection of the circle $x^2 + y^2 \leqslant 25$ and the straight line (AB). ●

104. Solve the following systems of equations:

(1) $\begin{cases} x^2+y^2=41, \\ y-x=1; \end{cases}$ (2) $\begin{cases} x^2+y^2=13, \\ xy=6; \end{cases}$

(3) $\begin{cases} \frac{x}{y}+\frac{y}{x}=\frac{26}{5}, \\ x^2-y^2=24; \end{cases}$ (4) $\begin{cases} x^3-y^3=37, \\ x-y=1. \end{cases}$

○ (1) $\begin{cases} x^2+y^2=41, \\ y-x=1 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2=41, \\ y=x+1 \end{cases}$

$\Leftrightarrow \begin{cases} x^2+(x+1)^2=41, \\ y=x+1 \end{cases} \Leftrightarrow \begin{cases} 2x^2+2x-40=0, \\ y=x+1 \end{cases}$

$\Leftrightarrow \begin{cases} x^2+x-20=0, \\ y=x+1 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=-5, \\ x=4 \end{array}\right. \\ y=x+1 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=-5, \\ y=-4 \end{cases} \\ \begin{cases} x=4, \\ y=5. \end{cases} \end{array}\right.$

Answer: $\{(-5, -4); (4, 5)\}$.

(2) Adding the first equation of the system to the doubled second equation, we get

$\begin{cases} x^2+y^2=13, \\ xy=6 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2=13, \\ 2xy=12 \end{cases} \Leftrightarrow \begin{cases} (x+y)^2=25, \\ xy=6 \end{cases}$

$\Leftrightarrow \left[\begin{array}{l} \begin{cases} x+y=5, \\ xy=6 \end{cases} \\ \begin{cases} x+y=-5, \\ xy=6 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=2, \\ y=3 \end{cases} \\ \begin{cases} x=3, \\ y=2 \end{cases} \\ \begin{cases} x=-2, \\ y=-3 \end{cases} \\ \begin{cases} x=-3, \\ y=-2. \end{cases} \end{array}\right.$

Answer: $\{(-3, -2); (-2, -3); (2, 3); (3, 2)\}$.

(3) Here $x \neq 0$ and $y \neq 0$. Let us set $z = x/y$; then the first equation of the system will take the form $z + \frac{1}{z} = \frac{26}{5}$, whence $z_1 = 5$, $z_2 = 1/5$. Thus, the given system falls into the composition of two

systems each of which is solved by the substitution method:

$$\left[\begin{array}{l}\left\{\begin{array}{l}x/y=5,\\ x^2-y^2=24\end{array}\right.\\ \left\{\begin{array}{l}x/y=1/5,\\ x^2-y^2=24\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x=5y,\\ 24y^2=24\end{array}\right.\\ \left\{\begin{array}{l}x=y/5,\\ -24y^2/25=24\end{array}\right.\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x=5y,\\ \left[\begin{array}{l}y=-1,\\ y=1\end{array}\right.\end{array}\right.\\ \varnothing\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x=-5,\\ y=-1\end{array}\right.\\ \left\{\begin{array}{l}x=5,\\ y=1.\end{array}\right.\end{array}\right.$$

Answer: $\{(-5, -1); (5; 1)\}$.

(4) $\left\{\begin{array}{l}x^3-y^3=37,\\ x-y=1\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}(x-y)(x^2+xy+y^2)=37,\\ x-y=1\end{array}\right.$

$$\Leftrightarrow \left\{\begin{array}{l}x^2+xy+y^2=37,\\ x=y+1\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}(y+1)^2+(y+1)y+y^2=37,\\ x=y+1\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l}y^2+y-12=0,\\ x=y+1\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\left[\begin{array}{l}y=-4,\\ y=3\end{array}\right.\\ x=y+1\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}y=-4,\\ x=-3\end{array}\right.\\ \left\{\begin{array}{l}y=3,\\ x=4.\end{array}\right.\end{array}\right.$$

Answer: $\{(-3, -4); (4, 3)\}$. ●

105. Give a geometrical illustration of the solutions of the following systems:

(1) $\left\{\begin{array}{l}x^2+y^2=4,\\ x^2-y^2=0;\end{array}\right.$ (2) $\left\{\begin{array}{l}x^2+y^2\leqslant 4,\\ x^2-y^2=0;\end{array}\right.$

(3) $\left\{\begin{array}{l}x^2+y^2\geqslant 4,\\ x^2-y^2\geqslant 0;\end{array}\right.$ (4) $\left\{\begin{array}{l}x^2+y^2=4,\\ x^2-y^2\leqslant 0.\end{array}\right.$

106. Solve the following systems of equations:

(1) $\left\{\begin{array}{l}x^2+6xy+8y^2=91,\\ x+3y-10=0;\end{array}\right.$ (2) $\left\{\begin{array}{l}x^2+3xy=18,\\ xy+4y^2=7;\end{array}\right.$

(3) $\left\{\begin{array}{l}x^2+xy+2y^2=74,\\ 2x^2+2xy+y^2=73;\end{array}\right.$ (4) $\left\{\begin{array}{l}\frac{x}{y}-\frac{y}{x}=\frac{3}{2},\\ x^2+y^2=45;\end{array}\right.$

(5) $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{3}{8}, \\ x+y=12; \end{cases}$ (6) $\begin{cases} 5x^2-6xy+5y^2=29, \\ 7x^2-8xy+7y^2=43; \end{cases}$

(7) $\begin{cases} \frac{x}{y}-\frac{y}{x}=\frac{5}{6}, \\ x^2+y^2=13; \end{cases}$ (8) $\begin{cases} \frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}, \\ x^2+y^2=20; \end{cases}$

(9) $\begin{cases} x^3-y^3=19, \\ x^2y-xy^2=6; \end{cases}$ (10) $\begin{cases} x^2+2y^2=17, \\ x^2-2xy=-3. \end{cases}$

Sec. 3.15. Problems on Setting up Systems of Equations

107. The arithmetic mean of two numbers is equal to 20, and their geometric mean equals 12. Find these numbers.

108. The length of the hypotenuse of a right triangle is equal to 37 cm, and its area amounts to 210 cm^2. Find the lengths of the legs.

109. The area of a rectangle is equal to 972 cm^2, and the length of its diagonal is equal to 45 cm. Find the lengths of the sides of the rectangle.

110. The perimeter of a right triangle is equal to 90 cm, and its area to 270 cm^2. Find the lengths of the sides of the triangle.

111. The area of a rectangle is equal to 1080 cm^2, and its perimeter equals 138 cm. Find the lengths of the sides and the diagonal of the rectangle.

112. The division of a two-digit number by the sum of its digits yields 4 as a quotient and 12 as a remainder. And if this number is divided by the product of its digits, then the quotient will be equal to 1 and the remainder to 20. Find this number.

Sec. 3.16. Simplest Problems of Linear Programming in Two Variables

113. Find the greatest value of the linear form $z = 2x_1 + x_2$ subject to the linear constraints

$$\begin{cases} 3x_1+x_2\leqslant 9, \\ 2x_1+4x_2\leqslant 16, \\ x_1\geqslant 0, \\ x_2\geqslant 0. \end{cases} \quad (*)$$

○ Taking into consideration that $x_1\geqslant 0$ and $x_2\geqslant 0$, we construct the straight lines $3x_1 + x_2 = 9$ and $2x_1 + 4x_2 = 16$ only in Quad-

rant I (Fig. 5). The set of points whose coordinates satisfy the system of inequalities (*) is represented by the convex polygon $OAED$ (polygon of solutions). The vertices A, E, and D of the polygon are found by solving the systems of equations:

$$\begin{cases} 3x_1 + x_2 = 9, \\ x_2 = 0; \end{cases} \quad A(3,\ 0); \quad \begin{cases} 3x_1 + x_2 = 9, \\ 2x_1 + 4x_2 = 16; \end{cases} \quad E(2,\ 3);$$

$$\begin{cases} 2x_1 + 4x_2 = 16, \\ x_1 = 0; \end{cases} \quad D(0,\ 4).$$

Now, given the set of these points, it is necessary to find such points at which the function $z = 2x_1 + x_2$ takes on the greatest value. Let us construct the straight line $2x_1 + x_2 = 0$, that is, $x_2 = -2x_1$. With an increase in z, this line displaces parallelly to itself. The maximum value of z is attained at one of the vertices of the polygon.

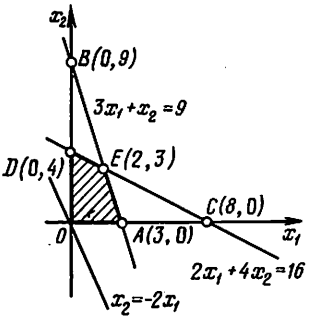


Fig. 5

We then find the values of the function $z = 2x_1 + x_2$ at the vertices of the polygon:

$$z_O = 0, \quad z_A = 2 \cdot 3 + 1 \cdot 0 = 6,$$

$$z_E = 2 \cdot 2 + 1 \cdot 3 = 7, \quad z_D = 2 \cdot 0 + 1 \cdot 4 = 4.$$

Thus, for $x_1 = 2$, $x_2 = 3$ the function $z = 2x_1 + x_2$ takes on the maximum value $z_{\max} = 7$. ●

114. It is required to work out a plan of manufacturing articles of two kinds for four bays of a shop in order to get a maximum profit from realizing these articles. Here, the following constraints are to be imposed: the operation time at the first bay should not exceed 16 hours, at the second bay 30 hours, at the third bay 16 hours, and at the fourth bay 12 hours.

The table given below indicates the time (in hours) necessary for manufacturing each article of the two kinds at each bay. The number zero means that the article is not manufactured at the appropriate bay.

The shop gets profits at the following rate: one article of kind I realizes 3 roubles and one piece of kind II realizes 4 roubles.

○ Let us denote the number of articles of kind I by x_1, and the number of articles of kind II by x_2.

The first bay spends $4x_1$ hours to manufacture the articles of kind I and $2x_2$ hours to manufacture the articles of kind II, that is, the total number of hours spent is $4x_1 + 2x_2$. Since the operation time of the first bay does not exceed 16 hours, we have $4x_1 + 2x_2 \leqslant 16$.

Articles	Bays			
	1	2	3	4
I	4	3	0	2
II	2	6	4	0
Possible operation time of a bay, hrs	16	30	16	12

The second bay spends $3x_1$ hours for manufacturing the articles of kind I and $6x_2$ for manufacturing the articles of kind II. Since the total operation time does not exceed 30 hours, we have $3x_1 + 6x_2 \leqslant 30$.

The third bay spends 0 hours for the articles of kind I and $4x_2$ hours for the articles of kind II, that is, $4x_2 \leqslant 16$.

At the fourth bay, it takes $2x_1$ hours to manufacture the articles of kind I and 0 hours to manufacture the articles of kind II, that is, $2x_1 \leqslant 12$.

From the realization of x_1 articles of kind I the shop makes a profit of $3x_1$ roubles, and from the realization of x_2 articles of kind II it makes a profit of $4x_2$ roubles. Thus, the total profit of the shop amounts to $3x_1 + 4x_2$ (roubles), where $x_1 \geqslant 0$ and $x_2 \geqslant 0$.

The mathematical model of the problem is described by the system of linear inequalities:

$$\begin{cases} 4x_1 + 2x_2 \leqslant 16, \\ 3x_1 + 6x_2 \leqslant 30, \\ 4x_2 \leqslant 16, \\ 2x_1 \leqslant 12, \\ x_1 \geqslant 0, \\ x_2 \geqslant 0. \end{cases}$$

On the set of solutions of this system of inequalities, it is required to find the maximum value of the linear form $z = 3x_1 + 4x_2$.

Constructing straight lines $4x_1 + 2x_2 = 16$, $3x_1 + 6x_2 = 30$, $4x_2 = 16$, and $2x_1 = 12$, we get a closed polygon $OABCD$ (Fig. 6). We then compute the coordinates of its vertices:

$$\begin{cases} 2x_1 = 12, \\ x_2 = 0; \end{cases} \quad A(6,\ 0); \quad \begin{cases} 2x_1 = 12, \\ 3x_1 + 6x_2 = 30; \end{cases} \quad B(6,\ 2);$$

$$\begin{cases} 3x_1 + 6x_2 = 30, \\ 4x_2 = 16; \end{cases} \quad C(2,\ 4); \quad \begin{cases} 4x_2 = 16, \\ x_1 = 0; \end{cases} \quad D(0,\ 4).$$

Substituting the coordinates of the vertices into the expression of the linear form, we get

$$z_A = 3\cdot 6 + 4\cdot 0 = 18; \qquad z_B = 3\cdot 6 + 4\cdot 2 = 26;$$

$$z_C = 3\cdot 2 + 4\cdot 4 = 22; \qquad z_D = 3\cdot 0 + 4\cdot 4 = 16.$$

At the point B (6, 2), the linear form reaches its maximum: $z_{\max} = 26$.

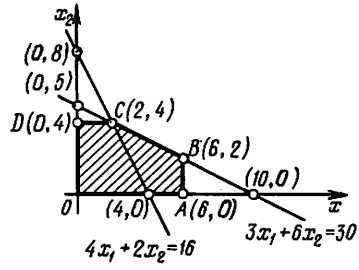


Fig. 6

Thus, the maximum profit from realizing the two kinds of articles amounts to 26 roubles. It will be obtained if the shop manufactures 6 pieces of kind I and two pieces of kind II. ●

115. (1) Find the maximum value of the linear form $z = 4x_1 + 3x_2$ with the constraints
$$\begin{cases} 3x_1 + x_2 \leqslant 9, \\ x_1 + 2x_2 \leqslant 8, \\ x_1 \geqslant 0, \\ x_2 \geqslant 0 \end{cases}$$ imposed.

(2) Find the maximum value of the linear form $z = 2x_1 + 3x_2$ with the constraints
$$\begin{cases} x_1 + 2x_2 \leqslant 10, \\ 2x_1 + x_2 \leqslant 8, \\ x_1 \geqslant 0, \\ x_2 \geqslant 0 \end{cases}$$ imposed.

116. A shop manufactures products of two kinds. One piece of kind I consumes 5 kg of copper and 1 kg of aluminium, and one piece of kind II consumes 3 kg of copper and 2 kg of aluminium. One piece of kind I realizes 2 roubles, and one piece of kind II, 3 roubles. How many pieces of each kind must the shop manufacture to get a maximum profit if the shop has a supply of 45 kg of copper and 16 kg of aluminium?

QUIZ No. 1

Solve the given equations, inequalities, systems of equations and inequalities.

Form I

(1) $\begin{cases} 5-x>2x-4, \\ 3x-7<3-2x. \end{cases}$

(2) $\begin{cases} 2x-7y=-8, \\ 3x+2y=13. \end{cases}$

(3) $x^3-2x^2-5x+6=0,$

(4) $5x^2-24x+16\geqslant 0.$

(5) $\begin{cases} x^2+y^2+x+y=68, \\ x^2-y^2+x-y=44. \end{cases}$

Form II

(1) $\begin{cases} \frac{2x}{3}-\frac{3}{4}>\frac{x}{2}+\frac{5}{3}, \\ 7x-3>4x+2. \end{cases}$

(2) $\begin{cases} 7x-5y=13, \\ 4x-3y=7. \end{cases}$

(3) $x^3-2x^2-x+2=0.$

(4) $3x^2-13x-10\leqslant 0.$

(5) $\begin{cases} x^2-xy=4, \\ y^2-xy=-3. \end{cases}$

Chapter 4
FUNCTIONS. LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Sec. 4.1. Function. Domain of Definition and Range of Function

A *numerical function* f is defined as a correspondence of two sets D and E, where $D \in \mathbf{R}$ and $E \in \mathbf{R}$, in which to each element $x \in D$ there corresponds only one element $f(x) \in E$.

The set D is called the *domain of definition* of function, and the set E is termed the *range of function*. The domain of definition of the function f is denoted by $D(f)$, and its range by $E(f)$.

A function f is completely defined by specifying the set of pairs $(x, f(x))$, where x runs over the entire set $D(f)$, and $f(x)$ through the corresponding values of the function. The set of these pairs is symbolized in the following way: $\{(x, y) \mid y = f(x)\}$.

To define a function, it is necessary and sufficient to give a rule of correspondence f, according to which for each value of the argument it is possible to indicate exactly one value of the function and the domain of definition $D(f)$.

A function can be represented by a mathematical formula, a table, a functional scale (say, a logarithmic scale), a graph or in some other way.

The numerical value of a function corresponding to the assigned numerical value of the argument is called a *particular value* of this function. For instance, for $x = a$ the function $y = f(x)$ has the value $y = f(a)$.

1. Given the function $f(x) = x^3 - 2x^2 + x - 1$. Find $f(0)$, $f(1)$, $f(-1)$, and $f(2)$.

○ To compute the value $f(0)$, it is necessary to replace the argument x by its value $x = 0$ in the given function. We have $f(0) = 0^3 - 2 \cdot 0^2 + 0 - 1 = -1$. ●

2. Find the domain of definition for each of the given functions

$$(1)\ y = x^2;\quad (2)\ y = \frac{1}{x};\quad (3)\ y = \frac{1}{2x-6};\quad (4)\ y = \frac{1}{x^2-5x+6}.$$

○ (1) Here, no constraints are imposed on x, therefore the function $y = x^2$ is defined on the set $\mathbf{R} = (-\infty; +\infty)$.

(2) If $x = 0$, then y has no numerical values, since it is impossible to divide by zero. For all values (except for zero), y takes on real values, therefore the desired domain of definition is represented by the union of the intervals: $(-\infty; 0) \cup (0; +\infty)$.

(3) The function is defined for all values of x, except those for which the denominator of the fraction vanishes. Solving the equation $2x - 6 = 0$, we find its root $x = 3$. Thus, the domain of definition $D(y) = (-\infty; 3) \cup (3; +\infty)$.

(4) The function is defined for all values of the argument except those for which the denominator vanishes. Solving the equation $x^2 - 5x + 6 = 0$, we find its roots: $x_1 = 2$ and $x_2 = 3$. Consequently, $D(y) = (-\infty; 2) \cup (2; 3) \cup (3; +\infty)$. ●

3. Find the domain of definition for each of the following functions:

(1) $y=\sqrt{x}$; (2) $y=\sqrt{2x-4}$; (3) $y=\sqrt{x}+\sqrt{x-1}$;

(4) $y=\sqrt{\dfrac{3x-2}{2x+6}}$.

○ (1) Square roots are defined for nonnegative numbers. Therefore the function $y=\sqrt{x}$ is defined for all the values of x which satisfy the inequality $x\geqslant 0$, that is, $D(y)=[0, +\infty)$.

(2) Solving the inequality $2x-4\geqslant 0$, we get $x\geqslant 2$, i.e. $D(y)=[2, +\infty)$.

(3) Let us find the domain of definition for each of the terms; the common region of these domains will just represent the domain of definition of the given function. For the first term $x\geqslant 0$, and for the second term $x\geqslant 1$. Then the domain of definition of the sum $\sqrt{x}+\sqrt{x-1}$ will be represented by the intersection of the intervals $D(y)=[0; +\infty)\cap[1, +\infty)=[1, +\infty)$.

(4) The function is defined for all the values of x satisfying the inequality $\dfrac{3x-2}{2x+6}\geqslant 0$. Thus,

$$\frac{3x-2}{2x+6}\geqslant 0 \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{l}x\geqslant 2/3,\\ x>-3\end{array}\right.\\ \left\{\begin{array}{l}x\leqslant 2/3,\\ x<-3\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x\geqslant 2/3,\\ x<-3.\end{array}\right.$$

Consequently, the domain of definition of the given function is represented by the union of the intervals $D(y)=(-\infty; -3)\cup[2/3; +\infty)$. ●

4. (1) Given the function $F(x)=x^4-x^3+2x^2+4$. Find $F(0)$, $F(-1)$, and $F(2)$.

(2) Given the function $s(t)=t^2-6t+8$. Find $s(0)$, $s(2)$, and $s(-1)$.

5. (1) Given the function $f(x)=x^4-x^2+1$. Show that $f(1)=f(-1)$.

(2) Given the function $f(x)=x^4+x^2+5$. Show that $f(2)=f(-2)$.

6. (1) Given the function $f(x)=x^3+x$. Show that $f(1)=-f(-1)$.

(2) Given the function $f(x)=x^5+x^3$. Show that $f(2)=-f(-2)$.

In Problems 7 through 14 find the domain of definition for each of the indicated functions:

7. (1) $y=x^2$; (2) $y=x^2-1$; (3) $y=x^3+1$.

8. (1) $y=\dfrac{1}{4x-2}$; (2) $y=\dfrac{x+2}{2x-8}$; (3) $y=\dfrac{x^2-4}{x+2}$.

9. (1) $y=\frac{1}{1-x^2}$; (2) $y=\frac{1}{x^2-x-12}$; (3) $y=\frac{4x-1}{3x^2-5x-2}$; (4) $y=\frac{x-1}{x^2-9x+20}$.

10. (1) $y=\sqrt{1-x}$; (2) $y=\sqrt{18-6x}$; (3) $y=\sqrt{3x-12}$.

11. (1) $y=\sqrt{x}+\sqrt{4-x}$; (2) $y=\sqrt{x-2}+\sqrt{x-5}$.

12. (1) $y=3\sqrt{5-x}-\frac{4}{\sqrt{x-3}}$; (2) $y=\sqrt{7-x}+\frac{1}{x-1}$.

13. (1) $y=\sqrt{x^2-2x-8}$; (2) $y=\sqrt{x^2+8x+15}$; (3) $y=\sqrt{(2-x)(5+x)}$.

14. (1) $y=\sqrt{\frac{x-8}{12-x}}$; (2) $y=\sqrt{\frac{4x-8}{3-6x}}$.

Sec. 4.2. The Logarithmic Function

1. The Logarithmic Function. The *logarithm* of a number N ($N \in \mathbf{R}_+$) to a base a ($a \in (0; 1) \cup (1; +\infty)$) is the exponent x indicating the power to which a must be raised to obtain N, that is

$$\log_a N = x \Leftrightarrow a^x = N. \tag{4.1}$$

The equality (4.1), expressing the definition of the logarithm can be rewritten in the form

$$a^{\log_a N} = N. \tag{4.2}$$

This equality is called the *fundamental logarithmic identity.*

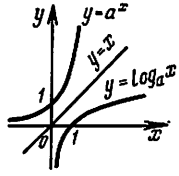


Fig. 7

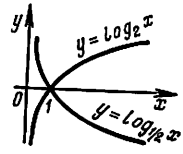


Fig. 8

Logarithms of numbers to the base $a = 10$ are called *common logarithms.*

The function $y = \log_a x$ ($x \in \mathbf{R}_+$, $a \in (0; 1) \cup (1; +\infty)$)is called the *logarithmic function.*

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$ ($x \in \mathbf{R}$, $a \in (0; 1) \cup (1; +\infty)$). Therefore their graphs are symmetric with respect to the bisector of Quadrants I and III (Fig. 7).

Let us indicate the basic properties of the logarithmic function.

1°. *The domain of definition*: $D(y) = \mathbf{R}_+$.

2°. *The range of the function*: $E(y) = \mathbf{R}$, *that is, the whole number line.*

3°. *The logarithm of unity is equal to zero, the logarithm of the base is equal to unity*: $\log_a 1=0$, $\log_a a=1$.

4°. *The function* $y = \log_a x$ ($a \in (1; +\infty)$) *increases in the interval* (0;

$+\infty$) (Fig. 7). *If* $a \in (1; +\infty)$, *then* $\log_a x > 0$ *for* $x \in (1; +\infty)$ *and* $\log_a x < 0$ *for* $x \in (0; 1)$, *that is, for* $a \in (1; +\infty)$ *the logarithms of the numbers greater than unity are positive, and the logarithms of the numbers less than unity are negative.*

5°. *The function* $y = \log_a x$ ($a \in (0, 1)$) *decreases in the interval* $(0; +\infty)$ (Fig. 8). *If* $a \in (0; 1)$, *then* $\log_a x < 0$ *for* $x \in (1; +\infty)$ *and* $\log_a x > 0$ *for* $x \in (0; 1)$, *that is, for* $a \in (0; 1)$ *the logarithms of the numbers less than unity are positive, and the logarithms of the numbers greater than unity are negative.*

2. Algebraic Operations with Logarithms.

1°. *The logarithm of the product of positive numbers is equal to the sum of the logarithms of the factors*:

$$\log_a (MN) = \log_a M + \log_a N. \tag{4.3}$$

2°. *The logarithm of the quotient of two positive numbers is equal to the logarithm of the dividend minus the logarithm of the divisor*:

$$\log_a \frac{M}{N} = \log_a M - \log_a N. \tag{4.4}$$

3°. *The logarithm of the power of a positive number is equal to the exponent times the logarithm of the base*:

$$\log_a M^n = n \log_a M. \tag{4.5}$$

4°. *The logarithm of a root of a positive number is equal to the logarithm of the number divided by the index of the root*:

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M. \tag{4.6}$$

3. Taking Logarithms and Antilogarithms. If a number x is represented by an algebraic expression containing numbers a, b, c, . . ., then to find the logarithm of this expression means to express the logarithm of the number x in terms of the logarithms of the numbers a, b, c, ... Finding a positive number by its logarithm is called *taking the antilogarithm.*

4. Relations Between the Logarithms of Numbers to Different Bases.

1°. *Change of base of logarithms can be accomplished by using the formula*:

$$\log_a M = \frac{\log_b M}{\log_b a}. \tag{4.7}$$

2°. *The interrelation between the bases a and b is expressed by the formula*:

$$\frac{1}{\log_a b} = \log_b a. \tag{4.8}$$

3°. *The following relationship takes place*:

$$\log_{a^m} M = \frac{1}{m} \log_a M. \tag{4.9}$$

15. Find $\log_{1/6} 36$.

○ First method. $\log_{1/6} 36 = \log_{1/6} 6^2 = \log_{1/6} (1/6)^{-2} = -2$.

Second method. $(\log_{1/6} 36 = x) \Leftrightarrow ((1/6)^x = 36) \Leftrightarrow (6^{-x} = 6^2)$; $x = -2$. ●

16. Solve the following equations: (1) $\log_6 x = -2$; (2) $\log_x 8 = -1/2$.

○ (1) $(\log_6 x = -2) \Leftrightarrow (x = 6^{-2})$; $x = 1/36$;

(2) $(\log_x 8 = -1/2) \Leftrightarrow (x^{-1/2} = 8) \Leftrightarrow ((x^{-1/2})^{-2} = 8^{-2})$; $x = 8^{-2} = 1/64$. ●

17. Find the domain of definition of each of the following functions:

(1) $y = \log_4 (8 - 2x)$; (2) $y = \log_{1/3} (2x + 6)$; (3) $y = \log_3 (x + 6) + \log_{1/3} (6 - x)$.

○ (1) Here $8 - 2x > 0$, $x < 4$, i.e. $D(y) = (-\infty, 4)$.

(2) We have $2x + 6 > 0$, $2x > -6$, $x > -3$, i.e. $D(y) = (-3, +\infty)$.

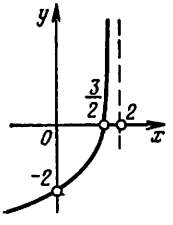


Fig. 9

(3) We have $\left\{ \begin{array}{l} x+6>0, \\ 6-x>0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x>-6, \\ x<6, \end{array} \right.$ i.e. $D(y)=(-6, 6)$. ●

18. Construct the graph of the function $y = \log_{1/2} (4 - 2x)$.

○ The domain of definition of the given function is represented by the infinite interval $(-\infty, 2)$. Let us find the points at which the graph intersects the coordinate axes. Setting $y = 0$, we get the equation $\log_{1/2} (4 - 2x) = 0$, whence $x = 3/2$. For $x = 0$ we have $y = \log_{1/2} 4 = -2$. The graph of the given function is represented in Fig. 9. ●

19. Compute x:

(1) $\log_{1/\sqrt{2}} (1/8) = x$; (2) $\log_{3\sqrt{3}} (1/27) = x$;

(3) $\log_x 0.125 = -3$; (4) $\log_x 4 = -1/2$;

(5) $\log_{16} x = 3/4$; (6) $\log_5 x = -3$.

20. Find the domains of definition of each of the following functions:

(1) $y = \log_2 (6 - 4x)$; (2) $y = \log_{1/5} (4x - 5)$;

(3) $y = \log_5 (x + 8) + \log_5 (4 - x)$.

21. Graph the following functions:

(1) $y = \log_2 (x - 2)$; (2) $y = \log_2 |x - 2|$; (3) $y = \log_3 (3 - x)$.

Sec. 4.3. Exponential Equations

An *exponential equation* is an equation in which the variable appears in the exponent.

When solving exponential equations of the form $a^{f(x)} = a^{\varphi(x)}$ (where $a > 0$, $a \neq 1$), we use the following property: $(a^{f(x)} = a^{\varphi(x)}) \Leftrightarrow (f(x) = \varphi(x))$.

An exponential equation is transformed to the form $a^{f(x)} = a^{\varphi(x)}$ in many ways. In what follows, we consider some of these methods.

The Method of Equating the Bases

22. Solve the following equations:

(1) $2^{x^2-7x+12} = 1$; (2) $(1/0.125)^{2x} = 128$; (3) $2^{x-2} = 5^{2-x}$; (4) $2^{x+5} - 2^x = 112$.

○ (1) By the definition of the zero exponent, we get

$$(x^2-7x+12=0) \Leftrightarrow \left[\begin{array}{l} x=3, \\ x=4. \end{array}\right. \textit{Answer}: \{3, 4\}.$$

(2) $((1/0.125)^{2x}=128) \Leftrightarrow ((2^3)^{2x}=2^7) \Leftrightarrow (2^{6x}=2^7) \Leftrightarrow (6x=7) \Leftrightarrow (x=7/6)$. *Answer*: $\{7/6\}$.

(3) Writing the equation in the form $2^{x-2}\cdot 5^{x-2}=1$, we obtain $(2^{x-2}\cdot 5^{x-2}=1) \Leftrightarrow ((2\cdot 5)^{x-2}=1) \Leftrightarrow (x-2=0) \Leftrightarrow (x=2)$. *Answer*: $\{2\}$.

(4) $(2^{x+3}-2^x=112) \Leftrightarrow (2^x(2^3-1)=112) \Leftrightarrow (2^x=112/7) \Leftrightarrow (2^x=16) \Leftrightarrow (x=4)$. *Answer*: $\{4\}$. ●

Taking the Logarithms of Both Sides of an Equation. Application of the Fundamental Logarithmic Identity

23. Solve the following equations: (1) $3^{2x-3}=11^{1-x}$; (2) $3^x=8$.

○ (1) Taking the logarithms of both sides of the equation to the base 10, we get

$$(3^{2x-3}=11^{1-x}) \Leftrightarrow ((2x-3)\log 3=(1-x)\log 11) \Leftrightarrow (2x\log 3+x\log 11=\log 11+3\log 3) \Leftrightarrow (x(2\log 3+\log 11)=\log 11+3\log 3)$$

$$\Leftrightarrow \left(x=\frac{\log 11+3\log 3}{2\log 3+\log 11}\right). \textit{Answer}: \left\{\frac{\log 11+3\log 3}{\log 11+2\log 3}\right\}.$$

(2) According to identity (4.2), we have $8=3^{\log_3 8}$; then $(3^x=8) \Leftrightarrow (3^x=3^{\log_3 8}) \Leftrightarrow (x=\log_3 8)$.

We can obtain the same result by taking the logarithms of both sides of the equation to the base 3: $(3^x=8) \Leftrightarrow (x\log_3 3=\log_3 8) \Leftrightarrow (x=\log_3 8)$. *Answer*: $\{\log_3 8\}$.

Taking the logarithms of both sides of the equation to the base 10, we compute the approximate value of the root: $(3^x=8) \Leftrightarrow (x\log 3=\log 8) \Leftrightarrow (x=\log 8/\log 3) \Leftrightarrow (x=0.9031/0.4771)$; $x\approx 1.89$. ●

Transforming to a Quadratic Equation

24. Solve the following equations:

(1) $4^x+2^{x+4}-8=0$; (2) $5^x+\frac{125}{5^x}=30$;

(3) $6\cdot 2^{2x}-13\cdot 6^x+6\cdot 3^{2x}=0$.

○ (1) $(4^x+2^{x+1}-8=0) \Leftrightarrow (2^{2x}+2\cdot 2^x-8=0)$.

We solve the quadratic equation with respect to the variable 2^x

$$(2^{2x}+2\cdot 2^x-8=0) \Leftrightarrow \left[\begin{array}{l} 2^x=-4, \\ 2^x=2 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \varnothing, \\ x=1. \end{array}\right. \textit{Answer}: \{1\}.$$

$$(2)\ \left(5^x+\frac{125}{5^x}=30\right) \Leftrightarrow (5^{2x}-30\cdot 5^x+125=0) \Leftrightarrow \left[\begin{array}{l} 5^x=5, \\ 5^x=25, \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x=1, \\ x=2. \end{array}\right. \textit{Answer}: \{1, 2\}.$$

(3) $(6\cdot 2^{2x}-13\cdot 6^x+6\cdot 3^{2x}=0) \Leftrightarrow (6\cdot 2^{2x}-13\cdot 2^x\cdot 3^x+6\cdot 3^{2x}=0)$.

Dividing all the terms of the equation by 3^{2x} ($3^{2x}\neq 0$), we get a quadratic equation with respect to the variable $(2/3)^x$:

$$(6\cdot 2/3)^{2x}-13\cdot(2/3)^x+6=0) \Leftrightarrow \left[\begin{array}{l} (2/3)^x=2/3, \\ (2/3)^x=3/2 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=1, \\ x=-1 \end{array}\right.$$

Answer: $\{-1, 1\}$. ●

Grouping Method

25. Solve the equation $5^{2x+1}+7^{x+1}-175^x-35=0$.

○ $(5^{2x+1}+7^{x+1}-175^x-35=0) \Leftrightarrow (5\cdot 25^x+7\cdot 7^x-25^x\cdot 7^x-35=0)$
$\Leftrightarrow (25^x(5-7^x)-7(5-7^x)=0) \Leftrightarrow (5-7^x)(25^x-7)=0$

$$\Leftrightarrow \left[\begin{array}{l} 5-7^x=0, \\ 25^x-7=0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=\log_7 5, \\ x=\log_{25} 7. \end{array}\right. \textit{Answer}: \{\log_7 5,\ \log_{25} 7\}.$$ ●

26. Solve graphically the equation $3^x=2x+3$.

○ Let us construct the graphs of the functions $y=3^x$ and $y=2x+3$ (Fig. 10). With the aid of the figure, we find the abscissas of two points of intersection of the graphs. One of the roots is contained in the interval $(-2; -1)$, and the other in the interval $(1; 2)$. The approximate values of the roots are: $x_1\approx -1.4$ and $x_2\approx 1.7$. ●

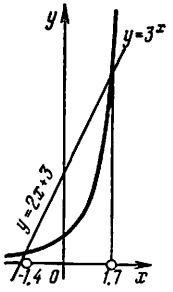


Fig. 10

In Problems 27 through 31 solve the given equations:

27. (1) $(0.5)^{\sqrt{x-3}}=1$; (2) $5^{x^2-8x+12}=1$;
(3) $2^{x^2}:4^x=8$;
(4) $9^{2\sqrt{x}}=3^{2x-6}$; (5) $1000\sqrt[x]{0.1}=100^x$.

28. (1) $5^{x-4}=6^{x-4}$; (2) $8^{5-x}=7^{x-5}$; (3) $4^{2x-3}=7^{x-1.5}$;
(4) $9^{x+2}=13^{2x-1}$.

29. (1) $3^{x+4}-5^{x+3}=3^x-5^{x+2}$; (2) $5^{x+1}+5^x=750$;
(3) $2^x-2^{x-2}=3$; (4) $27^{x-2/3}-9^{x-1}=2\cdot 3^{2x-1}-2\cdot 3^{3x-1}$.

30. (1) $7^{2x}-6\cdot 7^x+5=0$; (2) $2^{3-2x}-3\cdot 2^{1-x}+1=0$;
(3) $3^{x+1}+\frac{18}{3^x}-29=0$; (4) $3\cdot 4^x-5\cdot 6^x+2\cdot 9^x=0$;
(5) $4^x+6^x-9^x=0$; (6) $3^{2x+1}+8^{x+1}-72^x-24=0$.

31. Solve graphically the following equations: (1) $2^x=x^2$; (2) $2-x=2^{-x}$.

Sec. 4.4. Systems of Exponential Equations

32. Solve the following systems of equations:

(1) $\begin{cases} 3^x\cdot 5^y=75; \\ 3^y\cdot 5^x=45; \end{cases}$ (2) $\begin{cases} 4^x\cdot 5^y=16; \\ 2\cdot 3^x=18; \end{cases}$ (3) $\begin{cases} 3^x\cdot 3^y=27, \\ 3^x+3^y=12. \end{cases}$

○ (1) We rewrite the given system as follows

$$\begin{cases} 3^x\cdot 5^y=3\cdot 5^2, \\ 3^y\cdot 5^x=3^2\cdot 5. \end{cases}$$

Multiplying the equations, we have

$$(3^{x+y}\cdot 5^{x+y}=3^3\cdot 5^3) \Leftrightarrow (15^{x+y}=15^3) \Leftrightarrow (x+y=3).$$

Dividing the first equation by the second, we get

$$(3^{x-y}\cdot 5^{y-x}=3^{-1.5}) \Leftrightarrow ((3/5)^{x-y}=(3/5)^{-1} \Leftrightarrow (x-y=-1).$$

The solution of the given system is reduced to solving an equivalent system:

$\begin{cases} x+y=3, \\ x-y=-1. \end{cases}$ As a result, we get the *answer*: $\{(1, 2)\}$.

(2) Taking the logarithms of each of the equations, we get

$$\begin{cases} x\log 4+y\log 5=\log 16, \\ \log 2+x\log 3=\log 18 \end{cases} \Leftrightarrow \begin{cases} 2x\log 2+y\log 5=4\log 2, \\ \log 2+x\log 3=\log 2+2\log 3. \end{cases}$$

From the second equation we have $x\log 3 = 2\log 3$, that is, $x = 2$. Substituting the found value $x = 2$ into the first equation, we get

$$4\log 2 + y\log 5 = 4\log 2 \text{ or } y\log 5 = 0, \text{ that is, } y = 0.$$

Thus, we get the *answer*: $\{(2, 0)\}$. Using this method, we could solve the preceding system as well.

(3) According to the properties of the roots of a quadratic equation, 3^x and 3^y are the roots of the equation $z^2 - 12z + 27 = 0$. Solving this equation, we find $z_1 = 3$; $z_2 = 9$. Consequently, $3^x = 3$, $x = 1$, and $3^y = 9$, $y = 2$, and vice versa, $3^x = 9$, $x = 2$, and $3^y = 3$, $y = 1$. *Answer*: $\{(1, 2); (2, 1)\}$. ●

In problems 33 through 35 solve the given systems of equations:

33. (1) $\begin{cases} 2^x \cdot 3^y = 12, \\ 2^y \cdot 3^x = 18; \end{cases}$ (2) $\begin{cases} 2^x \cdot 3^y = 108, \\ 2^x + 3^y = 31; \end{cases}$ (3) $\begin{cases} xy = y^x, \\ x^3 = y^2; \end{cases}$

(4) $\begin{cases} 2^{2x} - 3^y = 55, \\ 2^x - 3^{y/2} = 5. \end{cases}$

34. (1) $\begin{cases} 3 \cdot 2^x + 2 \cdot 3^y = 11/4, \\ 2^x - 3^y = -3/4; \end{cases}$ (2) $\begin{cases} 9^y \sqrt[x]{100} = 27/10, \\ 25^y \sqrt[x]{10\,000} = 5/4; \end{cases}$

(3) $\begin{cases} 2^x + 3^y = 17, \\ 2^{x+2} - 3^{y+1} = 5; \end{cases}$ (4) $\begin{cases} 9^{x+y} = 729, \\ 3^{x-y-1} = 1. \end{cases}$

35. (1) $\begin{cases} 3^x - 4^y = 77, \\ 3^{x/2} - 2^y = 7; \end{cases}$ (2) $\begin{cases} 3^y \cdot 2^{6/x} = 36, \\ 5^y \cdot 2^{9/x} = 200. \end{cases}$

Sec. 4.5. Exponential Inequalities

Inequalities of the form $a^x > c$, $a^x < c$, $f(x)^{\varphi_1(x)} > f(x)^{\varphi_2(x)}$, where $a > 0$, $a \neq 1$, $c > 0$ are called *simplest exponential inequalities.*

There occur the following equivalent transformations:

$$a^x > c \Leftrightarrow \left[\begin{array}{l} \begin{cases} a > 1, \\ x > \log_a c \end{cases} \\ \begin{cases} 0 < a < 1, \\ x < \log_a c; \end{cases} \end{array}\right. \tag{4.10}$$

$$a^x < c \Leftrightarrow \left[\begin{array}{l} \begin{cases} a > 1, \\ x < \log_a c \end{cases} \\ \begin{cases} 0 < a < 1, \\ x > \log_a c; \end{cases} \end{array}\right. \tag{4.11}$$

$$f(x)^{\varphi_1(x)} > f(x)^{\varphi_2(x)} \Leftrightarrow \begin{cases} f(x) > 1, \\ \varphi_1(x) > \varphi_2(x) \\ 0 < f(x) < 1, \\ \varphi_1(x) < \varphi_2(x). \end{cases} \tag{4.12}$$

36. Solve the following inequalities:

(1) $3^x > 4$; (2) $6^{x^2-7x+12} > 1$; (3) $(1/3)^{x^2-5x+8} < 1/9$;

(4) $(x+3)^{x^2-5x+6} > 1$.

○ (1) Using the transformation (4.10) for $a = 3$, $c = 4$, we get $(3^x > 4) \Leftrightarrow (x > \log_3 4)$. *Answer*: $(\log_3 4, +\infty)$;

(2) $(6^{x^2-7x+12} > 1) \Leftrightarrow (6^{x^2-7x+12} > 6^0) \Leftrightarrow (x^2 - 7x + 12 > 0)$

$\Leftrightarrow \left[\begin{array}{l} x < 3, \\ x > 4. \end{array}\right.$ *Answer*: $(-\infty, 3) \cup (4, +\infty)$.

(3) $((1/3)^{x^2-5x+8} < 1/9 \Leftrightarrow ((1/3)^{x^2-5x+8} < (1/3)^2 \Leftrightarrow (x^2-5x+8 > 2) \Leftrightarrow (x^2-5x+6>0) \Leftrightarrow \left[\begin{array}{l} x<2, \\ x>3. \end{array}\right.$ *Answer*: $(-\infty, 2) \cup (3, +\infty)$.

(4) According to the transformation (4.12), the given inequality is equivalent to the combination of two systems

$$((x+3)^{x^2-5x+6} > 1) \Leftrightarrow ((x+3)^{x^2-5x+6} > (x+3)^0)$$

$$\Leftrightarrow \left[\begin{array}{l} \left\{\begin{array}{l} x+3>1, \\ x^2-5x+6>0 \end{array}\right. \\ \left\{\begin{array}{l} 0<x+3<1, \\ x^2-5x+6<0 \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \left\{\begin{array}{l} x>-2, \\ \left[\begin{array}{l} x<2, \\ x>3 \end{array}\right. \end{array}\right. \\ \left\{\begin{array}{l} -3<x<-2, \\ 2<x<3. \end{array}\right. \end{array}\right.$$

The first system is equivalent to two systems, while the second system has no solution, that is,

$$\left[\begin{array}{l} \left\{\begin{array}{l} x>-2, \\ \left[\begin{array}{l} x<2, \\ x>3 \end{array}\right. \end{array}\right. \\ \left\{\begin{array}{l} -3<x<-2, \\ 2<x<3 \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \left\{\begin{array}{l} x>-2, \\ x<2 \end{array}\right. \\ \left\{\begin{array}{l} x>-2 \\ x>3 \end{array}\right. \\ \varnothing \end{array}\right. \Leftrightarrow \left[\begin{array}{l} -2<x<2, \\ x>3. \end{array}\right.$$

Answer: $(-2, 2) \cup (3, +\infty)$. ●

In Problems 37 and 38 solve the indicated inequalities:

37. (1) $(1/3)^x < 1/27$; (2) $3^x > 27$; (3) $2^{x^2-8x+18} > 8$.

38. (1) $(x^2-8x+16)^{x-6} < 1$; (2) $(x-2)^{x^2-6x+8} > 1$.

Sec. 4.6. Logarithmic Equations

An equation containing the logarithm of the variable is called a *logarithmic equation*.

39. Solve the following equations:

(1) $\log_3 (x-12) = 2$; (2) $\log_x 16 - \log_x 2 = 1/2$;

(3) $\log (x-3) + \log (x-2) = 1 - \log 5$;

(4) $\log^2 x + \log x^2 = \log^2 2 - 1$; (5) $x^{\log x} = 100x$.

○ (1) Using the definition of the logarithm and taking into account the domain of definition, we get

$$(\log_3 (x-12)=2) \Leftrightarrow \left\{\begin{array}{l} x-12=3^2, \\ x-12>0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=21, \\ x>12 \end{array}\right. \Leftrightarrow (x=21).$$

Answer: {21}.

$$(2)\ (\log_x 16 - \log_x 2 = 1/2) \Leftrightarrow \begin{cases} \log_x (16/2) = 1/2, \\ x > 0,\ x \neq 1 \end{cases} \Leftrightarrow (x^{1/2} = 8)$$

$\Leftrightarrow (x = 64)$.

Answer: {64}.

(3) Taking into consideration that $1 = \log 10$, we find the antilogarithms:

$$\log (x-3) + \log (x-2) = 1 - \log 5$$

$$\Leftrightarrow \begin{cases} \log (x-3) + \log (x-2) = \log 10 - \log 5, \\ x-3 > 0, \\ x-2 > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \log [(x-3)(x-2)] = \log (10/5), \\ x > 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x-3)(x-2) = 2, \\ x > 3 \end{cases} \Leftrightarrow \begin{cases} x^2 - 5x + 4 = 0, \\ x > 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} x = 1, \\ x = 4 \end{array}\right. \Leftrightarrow x = 4. \\ x > 3. \end{cases}$$

Answer: {4}.

(4) We transform the given equation to a quadratic equation. Solving the latter with respect to the variable $\log x$, we get

$$(\log^2 x + \log x^2 = \log^2 2 - 1) \Leftrightarrow \begin{cases} \log^2 x + 2 \log x - \log^2 2 + 1 = 0, \\ x > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \log x = -1 - \log 2, \\ \log x = -1 + \log 2, \end{array}\right. \Leftrightarrow \begin{cases} \left[\begin{array}{l} \log x + \log 2 = -1, \\ \log x - \log 2 = -1, \end{array}\right. \\ x > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \log (2x) = -1, \\ \log (x/2) = -1, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 2x = 10^{-1}, \\ x/2 = 10^{-1} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = 0.05, \\ x = 0.2. \end{array}\right. \\ x > 0 \end{cases}$$

Answer: {0.05, 0.2}.

(5) Taking the logarithms of both sides of the equation to the base 10 and solving then the obtained quadratic equation, we find

$$\begin{cases} \log x \cdot \log x = \log 100 + \log x, \\ x > 0, \\ x \neq 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \log^2 x - \log x - 2 = 0, \\ x > 0, \\ x \neq 1 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} \log x = -1, \\ \log x = 2, \end{array}\right. \\ x > 0, \\ x \neq 1 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} x = 0.1, \\ x = 100. \end{array}\right.$$ *Answer*: {0.1, 100}. ●

40. Solve the following equations:

(1) $\log_3 x + \log_{\sqrt{3}} x + \log_{1/3} x = 6$; (2) $\log_{x^2} 16 + \log_{2x} 64 = 3$.

○ (1) Here $x > 0$. Using Formula (4.9), we transform the left-hand side of the equation to the base 3:

$$\log_{\sqrt{3}} x = \log_{3^{1/2}} x = 2 \log_3 x; \quad \log_{1/3} x = \log_{3^{-1}} x = -\log_3 x.$$

Thus,

$$(\log_3 x + 2 \log_3 x - \log_3 x = 6) \Leftrightarrow (2 \log_3 x = 6)$$

$$\Leftrightarrow (x = 3^3) \Leftrightarrow (x = 27).$$

(2) Here, $x > 0$. Using Formula (4.7), we change the left-hand side of the equation to the base 2:

$$\log_{x^2} 16 = \frac{\log_2 16}{\log_2 x^2} = \frac{4}{2 \log_2 x};$$

$$\log_{2x} 64 = \frac{\log_2 64}{\log_2 2x} = \frac{6}{\log_2 2 + \log_2 x} = \frac{6}{1 + \log_2 x}.$$

Then

$$\left(\frac{4}{2 \log_2 x} + \frac{6}{1 + \log_2 x} = 3\right)$$

$$\Leftrightarrow \begin{cases} 3 \log_2^2 x - 5 \log_2 x - 2 = 0, \\ x > 0, \\ x \neq 1, \\ x \neq 0.5 \end{cases} \Leftrightarrow \left[\begin{array}{l} \log_2 x = -1/3, \\ \log_2 x = 2 \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x = 2^{-1/3}, \\ x = 4. \end{array}\right.$$ *Answer*: $\{2^{-1/3}, 4\}$. ●

In Problems 41 through 44 solve the indicated equations:

41. (1) $\log_5 (x+10)=2$; (2) $\log_x 2 \div \log_x 3 = 1/3$;
(3) $\log (7x-9)^2 + \log (3x-4)^2 = 2$;
(4) $\log (x-1)^3 - 3\log (x-3) = \log 8$.

42. (1) $\frac{1}{5-\log x} + \frac{2}{1+\log x} = 1$; (2) $\log_3 \log_4 \log_3^2 (x-3) = 0$.

43. (1) $x^{\log x} = 1$; (2) $x^{\sqrt{x}} = (\sqrt{x})^x$; (3) $x^x = x$.

44. (1) $\log_2 x + \log_8 x = 8$; (2) $2\log_x 25 - 3\log_{25} x = 1$.

Sec. 4.7. Systems of Logarithmic Equations

45. Solve the system of equations $\begin{cases} \log_y x - 3\log_x y = 2, \\ \log_2 x = 4 - \log_2 y. \end{cases}$

○ Here $x>0$, $y>0$. We have

$$\begin{cases} \log_y x - 3\dfrac{1}{\log_y x} = 2, \\ \log_2 x = \log_2 16 - \log_2 y \end{cases} \Leftrightarrow \begin{cases} \log_y^2 x - 2\log_y x - 3 = 0, \\ x = 16/y \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \log_y x = -1, \\ \log_y x = 3, \end{array}\right. \\ x = 16/y \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} y^{-1} = x, \\ y^3 = x \end{array}\right. \\ x = 16/y \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} y^{-1} = x, \\ x = 16/y \end{cases} \\ \begin{cases} y^3 = x, \\ x = 16/y \end{cases} \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} \varnothing, \\ \begin{cases} x = 8, \\ y = 2 \end{cases} \end{array}\right. \Leftrightarrow \begin{cases} x = 8, \\ y = 2. \end{cases}$$

Answer: $\{(8, 2)\}$. ●

In Problems 46 and 47 solve the indicated systems of equations:

46. (1) $\begin{cases} y - x = 9, \\ \log x - \log y = -1; \end{cases}$ (2) $\begin{cases} \log_x y - \log_y x = 3/2, \\ x + y = 3/4; \end{cases}$

(3) $\begin{cases} x + y = 10.1 \\ \log x - \log y = 2; \end{cases}$ (4) $\begin{cases} 5x + 2y = 100, \\ \log x - \log y = \log 16 - 1. \end{cases}$

47. (1) $\begin{cases} x + y = 29, \\ \log x + \log y = 2\log 10; \end{cases}$

(2) $\begin{cases} \log_2 (x-y) = 5 - \log_2 (x+y), \\ \log x - \log 4 = \log 3 - \log y; \end{cases}$

(3) $\begin{cases} \log_{1/2} (y-x) + \log_2 (1/y) = -2; \\ x^2 + y^2 = 25; \end{cases}$

(4) $\begin{cases} \log_2 x \cdot \log_x (x-3y) = 2, \\ xy^{\log_x y} = y^{5/2}; \end{cases}$

(5) $\begin{cases} \log_x \log_3 \log_x y = 0, \\ \log_y 27 = 1; \end{cases}$ (6) $\begin{cases} 3^x \cdot 2^y = 576, \\ \log_{\sqrt{2}} (y-x) = 4; \end{cases}$

(7) $\begin{cases} 2^{\sqrt{x}+\sqrt{y}} = 256, \\ \log \sqrt{xy} - \log 1.5 = 1. \end{cases}$

Sec. 4.8. Logarithmic Inequalities

Inequalities of the form $\log_a x > c$, $\log_a x < c$, where $a > 0$ and $a \neq 1$, are called *simplest logarithmic inequalities.*

The following equivalent transformations are valid:

$$\log_a x > c \Leftrightarrow \left[\begin{array}{l} \begin{cases} a > 1, \\ x > a^c \end{cases} \\ \begin{cases} 0 < a < 1, \\ 0 < x < a^c; \end{cases} \end{array}\right. \tag{4.13}$$

$$\log_a x < c \Leftrightarrow \left[\begin{array}{l} \begin{cases} a > 1, \\ 0 < x < a^c \end{cases} \\ \begin{cases} 0 < a < 1, \\ x > a^c; \end{cases} \end{array}\right. \tag{4.14}$$

$$\log_{f(x)} \varphi_1(x) > \log_{f(x)} \varphi_2(x) \Leftrightarrow \left\{\begin{array}{l} \varphi_1(x) > 0, \\ \varphi_2(x) > 0, \\ \quad \left[\begin{array}{l} \begin{cases} f(x) > 1, \\ \varphi_1(x) > \varphi_2(x) \end{cases} \\ \begin{cases} 0 < f(x) < 1, \\ \varphi_1(x) < \varphi_2(x). \end{cases} \end{array}\right. \end{array}\right. \tag{4.15}$$

48. Solve the following inequalities:
(1) $\log^2_{1/2} x > 36$; (2) $2 + \log_2 (x+1) > 1 - \log_{1/2} (4 - x^2)$.

○ (1) It is obvious that

$$(\log^2_{1/2} x > 36) \Leftrightarrow (|\log_{1/2} x| > 6) \Leftrightarrow \left[\begin{array}{l} \log_{1/2} x > 6, \\ \log_{1/2} x < -6. \end{array}\right.$$

Applying the equivalent transformations (4.13) and (4.14), we get

$$\left[\begin{array}{l} \log_{1/2} x > 6, \\ \log_{1/2} x < -6 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 0 < x < (1/2)^6, \\ x > (1/2)^{-6} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 0 < x < 1/64, \\ x > 64. \end{array}\right.$$

Answer: $(0, 1/64) \cup (64, +\infty)$.

(2) We have

$(2+\log_2(x+1)>1-\log_{1/2}(4-x^2)) \Leftrightarrow (1+\log_2(x+1)>\log_2(4-x^2))$

$\Leftrightarrow (\log_2 2+\log_2(x+1)>\log_2(4-x^2))$

$\Leftrightarrow (\log_2 2(x+1)>\log_2(4-x^2))$.

Using then the equivalent transformations (4.15), we get

$$\log_2 2(x+1)>\log_2(4-x^2)$$

$$\Leftrightarrow \begin{cases} x+1>0, \\ 4-x^2>0, \\ 2(x+1)>4-x^2 \end{cases} \Leftrightarrow \begin{cases} x>-1, \\ x^2-4<0, \\ x^2+2x-2>0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x>-1, \\ -2<x<2, \\ \left[\begin{array}{l} x<-\sqrt{3}-1, \\ x>\sqrt{3}-1 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} -1<x<2, \\ \left[\begin{array}{l} x<-\sqrt{3}-1, \\ x>\sqrt{3}-1 \end{array}\right. \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} -1<x<2, \\ x<-\sqrt{3}-1 \end{cases} \\ \begin{cases} -1<x<2, \\ x>\sqrt{3}-1 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \varnothing, \\ \sqrt{3}-1<x<2 \end{array}\right.$$

$\Leftrightarrow (\sqrt{3}-1<x<2)$. *Answer*: $(\sqrt{3}-1,\ 2)$. ●

49. Solve the following inequalities:

(1) $\log_5(2x^2-3x-1)>0$; (2) $\log_{15}(x^2-4x+3)<1$;

(3) $\log_{x+7}25>2$; (4) $\log_{x+1}9<2$; (5) $\log_{3x-3}x>1$;

(6) $\log_{3x+3}x<1$; (7) $\log_{x-3}(x^2+4x-5)>\log_{x-3}(x-1)$;

(8) $\log_2(x+1)+\log_2(11-x)<5$.

Sec. 4.9. Miscellaneous Problems

50. Find the domain of definition for each of the following functions:

(1) $y=\frac{4}{1-x}$; (2) $y=\frac{1}{3x-12}$; (3) $y=\sqrt{x^2-3}$;

(4) $y=\frac{1}{x^2+x-20}$; (5) $y=\sqrt{\log(x^2-8x+13)}$;

(6) $y=\log(x-1)-\sqrt{x-5}$;

(7) $y=\log\frac{x^2-9}{x^2-3x+4}$; (8) $y=x-\log x$.

51. Solve the following equations:

(1) $\left(\frac{1}{5}\right)^{-x-\frac{1}{2}+\frac{2}{x}} = 5\sqrt{5}$; (2) $4^{x-1} - 0.5^{3-x} = 62$;

(3) $3\sqrt[x]{81} - 10\sqrt[x]{9} + 3 = 0$.

52. Solve the following equations:

(1) $5^{2x-1} + 2^{2x} = 5^{2x} - 2^{2x+2}$; (2) $\left(\frac{9}{4}\right)^{x} \cdot \left(\frac{8}{27}\right)^{x-1} = \frac{\log 9}{\log 27}$;

(3) $4^{\frac{x+1}{x}} + 6^{\frac{x+1}{x}} = 2 \cdot 9^{\frac{x+1}{x}}$; (4) $2^{3x} \cdot 3^{x} - 2^{3x+1} \cdot 3^{x-1} = 192$;

(5) $3^{2x-1} - 3^{2x} + 3^{2x+3} = 237$; (6) $\sqrt[2x]{2} \cdot \sqrt[3]{8^{x-2}} = \sqrt[8x]{0.25}$;

(7) $7 \cdot 4^{x^2} - 9 \cdot 14^{x^2} + 2 \cdot 49^{x^2} = 0$.

53. Solve the following inequalities:

(1) $2^{x} + 2^{1-x} < 3$; (2) $(\sqrt{23}/5)^{x^2-5} < (23/25)^{2x}$;

(3) $(x^2 - 8x + 16)^{x-6} > 1$.

54. Solve the following equations:

(1) $\log(3x^2 + 28) - \log(3x - 2) = 1$;

(2) $1 + \log(x + 1) - \log(x^3 + 7x + 8) = 0$;

(3) $\log\sqrt{3x + 1} + \log\sqrt{x + 4} = \log 12$;

(4) $\log(x - 4) - \log\sqrt{2x - 11} = \log 2$;

(5) $\log(x - 2) - \log\sqrt{x - 4} = \log 3$;

(6) $\log(x^2 + 1) - \log(x - 2) = 1$;

(7) $2\log\sqrt{5 - x} + \log(x - 3) = 0$;

(8) $\log\sqrt{x - 7} + \log\sqrt{3x - 8} = 1$;

(9) $\log(8 - x) + 2\log\sqrt{x - 6} = 0$;

(10) $2(\log 2 - 1) + \log(5^{\sqrt{x}} + 1) = \log(5^{1-\sqrt{x}} + 5)$.

55. Solve the following equations:

(1) $\sqrt{1 + \log_2 x} + \sqrt{4\log_4 x - 2} = 4$; (2) $\log(64\sqrt[24]{2^{x^2-40x}}) = 0$;

(3) $2^{\log(x^2-6x+10\sqrt{10})} = 2\sqrt{2}$; (4) $0.4^{\log^2 x + 1} = (6.25)^{2-\log x^3}$;

(5) $\log^2 x - \log x^2 = \log^2 3 - 1$; (6) $\log\log x = \log(3 - 2\log x)$;

(7) $(\sqrt{x})^{\log_5 x - 1} = 5$; (8) $2\log_2\log_2 x + \log_{1/2}\log_2(2\sqrt{2x}) = 1$.

56. Solve the following inequalities:

(1) $\log_{3/10}|2x + 1| > 1$; (2) $\log_{1/5}(3x - 5) > \log_{1/5}(x + 1)$;

(3) $\log_{2/3}|x - 2| > \log_{2/3} 6$; (4) $\log_2 \frac{4x-1}{x+3} < 1$;

(5) $\log_{1/2}(x + 8) > \log_{1/2}(x - 3) + \log_{1/2}(3x)$;

(6) $\log_{1/3}(3^{x+2} - 9) \cdot \log_3(3^{x} - 1) > -3$.

QUIZ No. 1

Form I

(1) Solve the equation
$$3^{2x-3}-9^{x-1}+3^{2x}=675.$$

(2) Solve the equation
$$\frac{\log_2^2 x-\log_2 x-2}{\log_2 x+1}=1.$$

(3) Solve the inequality
$$2^x+2^{1-x}<3.$$

(4) Solve the inequality
$$\log_3 |2x-7|<1.$$

(5) Given: $\log_7 2=m$. Find $\log_{49} 28$.

Form II

(1) Solve the equation
$$4^{x+1.5}+7\cdot 2^{x+1}=4.$$

(2) Solve the equation
$$\log_3 x+\log_9 x+\log_{27} x=\frac{11}{12}.$$

(3) Solve the inequality
$$(\sqrt{10}/3)^{3x^2-3}<(0.81)^{-2x}.$$

(4) Solve the inequality
$$\log_{1/2}\frac{2x-1}{x+1}>1.$$

(5) Given: $\log 3=a$, $\log 5=b$. Find $\log_{15} 30$.

Chapter 5
INFINITE SEQUENCES OF NUMBERS. THE LIMIT OF A SEQUENCE

Sec. 5.1. An Infinite Sequence of Numbers

An infinite *sequence of numbers* is a numerical function defined on the set N of natural numbers.

A sequence (x_n) is said to be *increasing* (*decreasing*) if each of its terms, beginning with the second, is greater (less) than the preceding term (or predecessor), that is, if for any n the inequality $x_{n+1} > x_n (x_{n+1} < x_n)$ is fulfilled.

A sequence (x_n) is called *nonincreasing* (*nondecreasing*) if each of its terms, beginning with the second, is not greater (not less) than its predecessor, that is, if for any n the following inequality is fulfilled: $x_{n+1} \leqslant x_n$ $(x_{n+1} \geqslant x_n)$.

Decreasing, increasing, nondecreasing, and nonincreasing sequences are called *monotonic* (or *monotone*).

A sequence (x_n) is said to be *bounded above* (*bounded below*) if it is possible to indicate a number M (a number m) such that for all the terms of this sequence the inequality $x_n \leqslant M$ $(x_n \geqslant m)$ is fulfilled. The numbers M and m are termed, respectively, an *upper bound* and a *lower bound* of the sequence (x_n). The fact that a sequence is bounded above by the number M (below by the number m) means geometrically that no point x_n lies on the right of the point M (on the left of the point m).

A sequence (x_n) is said to be *bounded* if there are two numbers m and M such that for all n the inequality $m \leqslant x_n \leqslant M$ is fulfilled. The fact that a sequence is bounded by the numbers m and M geometrically means that all of its terms are contained in the interval $[m, M]$.

A sequence (x_n) is called *constant* if all of its terms coincide.

A sequence is usually given by a formula expressing the general term of the sequence in terms of n. Sometimes, a rule is indicated with the aid of which it is possible to compute the nth term of a sequence if its previous terms are known. Such method of representing a sequence is called the *induction* (or *recurrent*) method.

1. Compute the first five terms of the sequence $x_n = \dfrac{n-1}{n+1}$.

○ Replacing n successively by 1, 2, 3, 4, 5, we get $x_1 = 0$, $x_2 = 1/3$, $x_3 = 1/2$, $x_4 = 3/5$, $x_5 = 2/3$. ●

2. Write the general term of the sequence of natural numbers each of which, when divided by 3, yields a remainder equal to 1.

○ In order for a number to yield 1 as a remainder after the division by 3, it must have the form $3n + 1$; consequently, the general term of the sequence $x_n = 3n + 1$. ●

3. A sequence is given by the recurrence relation $x_n = 3x_{n-1} + 1$. Find its first several terms.

○ Let us specify the first term of the sequence: let $x_1 = 2$. Setting $n = 2$ in the recurrence relation, we get $x_2 = 3x_{2-1} + 1 = 3x_1 + 1 = 3 \cdot 2 + 1 = 7$. For $n = 3, 4, 5$ we find respectively: $x_3 = 3x_2 + 1 = 3 \cdot 7 + 1 = 22$, $x_4 = 3x_3 + 1 = 3 \cdot 22 + 1 = 67$, $x_5 = 3x_4 + 1 = 3 \cdot 67 + 1 = 202$. As a result, we obtain the sequence 2, 7, 22, 67, 202, ... ●

4. Prove that the sequence with the general term $x_n = 1/(n^2 - 1)$ is monotonic decreasing.

○ For a decreasing sequence the following inequality is fulfilled: $x_{n+1} < x_n$ or $x_{n+1}/x_n < 1$. Let us write the $(n + 1)$th term of the

sequence:

$$x_{n+1}=\frac{1}{(n+1)^2-1}=\frac{1}{n^2+2n+1-1}=\frac{1}{n^2+2n}.$$

Then $x_{n+1}/x_n=(n^2-1)/(n^2+2n)<1$, since $n^2-1<n^2+2n$ for any natural n. Consequently, the given sequence is decreasing. ●

5. Prove that the sequence $x_n=\frac{n+1}{n}$ is bounded both below and above.

○ Obviously, $x_n=\frac{n+1}{n}>1$, that is, the sequence is bounded below. On the other hand, we have $\frac{n+1}{n}=1+\frac{1}{n}$, where $\frac{1}{n}$ is a proper fraction, and, consequently, $1+\frac{1}{n}<2$, that is, the given sequence is bounded above. ●

6. Compute the first five terms of each of the following sequences:

(1) $x_n=2n+5$; (2) $x_n=\frac{1}{2n-1}$; (3) $x_n=\frac{(-1)^n}{n}$;

(4) $x_n=4$; (5) $x_n=\frac{1}{(-1)^n}$; (6) $x_n=2n$;

(7) $x_n=\frac{1}{2^n}+2^n$; (8) $x_n=4n^2+3n+1$;

(9) $x_n=\begin{cases}\frac{1}{n^2-1} & \text{for an even } n;\\ \frac{n-1}{n} & \text{for an odd } n;\end{cases}$ (10) $x_n=\frac{1}{n(n+2)}$.

7. Write the general term of the sequence of natural numbers each of which, when divided by 5, yields a remainder equal to 3.

8. Write the general term of the indicated sequence: (1) 1, 1/4, 1/9, 1/16, ...; (2) 1, 7, 13, 19, ...; (3) 2, 4, 8, 16, 32, ...; (4) 1, 7, 17, 31, ...

9. Given the sequences:

(1) $x_n=\frac{n}{n+1}$; (2) $x_n=\frac{n^2}{n^2+2}$; (3) $x_n=\frac{2n}{n^2+1}$;

(4) $x_n=\frac{3n+5}{2n+1}$.

Prove that sequences (1) and (2) are increasing, and (3) and (4) are decreasing.

10. Given the sequences:

(1) $x_n=3n-1$; (2) $x_n=\frac{1}{n^2}$; (3) $x_n=\frac{n(n+1)}{3}$;

(4) $x_n=\frac{1}{n^3-1}$; (5) $x_n=\frac{2}{n(n+1)}$; (6) $x_n=\frac{n}{n+1}$;

(7) $x_n=\left(-\frac{1}{2}\right)^n$; (8) $x_n=\frac{2^n}{2^n-1}$.

Which of them are bounded?

Sec. 5.2. The Limit of a Sequence of Numbers

1. Basic Definitions. A number a is said to be the *limit of the sequence* x_n if for any arbitrary $\varepsilon > 0$ all the terms of the sequence x_n, except possibly for their finite number, lie in the ε-neighbourhood $(a - \varepsilon;\ a + \varepsilon)$ of the point a, that is, there is a natural number N such that for $n > N$ there will be fulfilled the inequality $| x_n - a | < \varepsilon$.

A sequence can have only one limit. A sequence which has a limit is *convergent*; otherwise, it is *divergent*.

If a sequence (x_n) has a limit a, then we write $\lim\limits_{n\to\infty} x_n = a$. In this case the sequence is said to converge to the number a.

2. Weierstrass' Theorem. *If a sequence is monotonic and bounded, then it has a limit.*

The boundedness of a sequence is a necessary condition for convergence.

3. Theorem on Limits. Theorem 1. *If the sequences* (x_n) *and* (y_n) *converge, then*

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Theorem 2. *If the sequences* (x_n) *and* (y_n) *converge, then*

$$\lim_{n\to\infty} (x_n \cdot y_n) = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} y_n).$$

Corollary. *A constant factor may be taken outside the limit:*

$$\lim_{n\to\infty} (kx_n) = k \cdot \lim_{n\to\infty} x_n.$$

Theorem 3. *If the sequences* (x_n) *and* (y_n) *converge and the limit of the sequence* (y_n) *is different from zero, then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}.$$

4. Infinitely Small and Infinitely Large Sequences. A sequence is called *infinitely small* if its limit is equal to zero.

Theorem 1. *The sum of two infinitely small sequences is an infinitesimal.*

Theorem 2. *The product of a bounded sequence and an infinitesimal is an infinitesimal.*

Corollary. *The product of two infinitesimals is again an infinitesimal.*

Theorem 3. *In order for the equality* $\lim\limits_{n\to\infty} x_n = a$, *to be fulfilled, it is necessary and sufficient that* $x_n = a + \alpha_n$, *where* $\lim\limits_{n\to\infty} \alpha_n = 0$.

A sequence is called *infinitely large* if for an arbitrary $M > 0$ there is a natural number N such that for $n \geqslant N$ the inequality $| a_n | > M$ is fulfilled. In this case we write $\lim\limits_{n\to\infty} a_n = \infty$.

If $\lim\limits_{n\to\infty} a_n = \infty$ and all the numbers a_n, beginning with a certain number N, are positive, then the sequence (a_n) tends to $+\infty$: $\lim\limits_{n\to\infty} a_n = +\infty$; if all the numbers a_n, beginning with a certain N, are negative, then the sequence tends to $-\infty$: $\lim\limits_{n\to\infty} a_n = -\infty$.

If (a_n) is an infinitely large sequence, then $(1/a_n)$ is an infinitely small sequence. Vice versa, if (α_n) is an infinitely small sequence, then $(1/\alpha_n)$ is infinitely large.

11. Prove that the sequence $x_n = 5n/(n+1)$ converges to the number 5.

○ According to the definition, the number 5 is the limit of the sequence (x_n) if for any $\varepsilon > 0$ it is possible to indicate a number N such that for all the terms of the sequence with numbers $n > N$ the following inequality will be fulfilled:

$$\left|\frac{5n}{n+1} - 5\right| < \varepsilon.$$

Let there be given an arbitrary positive number ε; then from the last inequality we get

$$\left|\frac{5n-5n-5}{n+1}\right| < \varepsilon \quad \text{or} \quad \frac{5}{n+1} < \varepsilon.$$

Solving this inequality with respect to n, we find $n > \frac{5}{\varepsilon} - 1$.

Thus, if for N we take any natural number not less than $\frac{5}{\varepsilon} - 1$, then for all $n > N$ for any $\varepsilon > 0$ the inequality $\left|\frac{5n}{n+1} - 5\right| < \varepsilon$ will be fulfilled. Then, by the definition of the limit, it follows that $\lim\limits_{n\to\infty} \frac{5n}{n+1} = 5$.

Let, for instance, $\varepsilon = 0.01$; then $\frac{5}{\varepsilon} - 1 = \frac{5}{0.01} - 1 = 499$. Let us take any term of the sequence (x_n) with the number greater than 499, say, $n = 500$; then $x_{500} = \frac{5 \cdot 500}{500+1} = \frac{2500}{501}$. We then find the quantity

$$|x_{500} - 5| = \left|\frac{2500}{501} - 5\right| = \left|-\frac{5}{501}\right| = \frac{5}{501} < 0.01,$$

that is, $|x_{500} - 5| < \varepsilon = 0.01$. Thus, all the terms of the sequence, beginning with the 500th term, are found in the ε-neighbourhood of the number 5, that is, in the interval (4.99, 5.01).

Similarly, for any assigned number $\varepsilon > 0$ we can find a number N beginning with which all the terms of the sequence will be contained in the ε-neighbourhood of the number 5. ●

12. Prove that the sequence $x_n = n$ is divergent.

○ Let us assume the contrary: suppose that the sequence $x_n = n$ converges, and its limit is equal to the number a, that is, $\lim\limits_{n\to\infty} x_n = a$. Let the natural number N exceed a: $N > a$. For any $n > N$ we have

$$|x_n - a| = |n - a| = n - a > n - N \geqslant 1,$$

which contradicts the definition of the limit, since for all $\varepsilon < 1$ the inequality $|x_n - a| < \varepsilon$ must be fulfilled. ●

13. Compute the limit for each of the following sequences:

(1) $\lim\limits_{n\to\infty}\dfrac{3n+1}{7n-1}$; (2) $\lim\limits_{n\to\infty}5\cdot\dfrac{n+1}{n}$; (3) $\lim\limits_{n\to\infty}\dfrac{2n^2+3n-4}{3n^2-4n+1}$;

(4) $\lim\limits_{n\to\infty}\dfrac{1-n}{n^2+n+1}$; (5) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+3n+1}+n}{2n+3}$.

○ (1) The numerator and denominator have no limits, since these are unbounded sequences; consequently, the theorem on the limit of a quotient cannot be applied directly. Dividing both the numerator and denominator by n, and applying then the theorem on the limit of a quotient, we get

$$\lim_{n\to\infty}\frac{3n+1}{7n-1}=\lim_{n\to\infty}\frac{3+1/n}{7-1/n}=\frac{\lim\limits_{n\to\infty}(3+1/n)}{\lim\limits_{n\to\infty}(7-1/n)}$$

$$=\frac{3+\lim\limits_{n\to\infty}(1/n)}{7-\lim\limits_{n\to\infty}(1/n)}=\frac{3+0}{7-0}=\frac{3}{7}.$$

The remaining limits are computed in a similar way:

(2) $\lim\limits_{n\to\infty}5\cdot\dfrac{n+1}{n}=5\lim\limits_{n\to\infty}\dfrac{n+1}{n}=5\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)=5(1+0)=5;$

(3) $\lim\limits_{n\to\infty}\dfrac{2n^2+3n-4}{3n^2-4n+1}=\lim\limits_{n\to\infty}\dfrac{2+3/n-4/n^2}{3-4/n+1/n^2}=\dfrac{\lim\limits_{n\to\infty}(2+3/n-4/n^2)}{\lim\limits_{n\to\infty}(3-4/n+1/n^2)}$

$$=\frac{2+0-0}{3-0+0}=\frac{2}{3};$$

(4) $\lim\limits_{n\to\infty}\dfrac{1-n}{n^2+n+1}=\lim\limits_{n\to\infty}\dfrac{1/n^2-1/n}{1+1/n+1/n^2}=\dfrac{0-0}{1+0+0}=0;$

(5) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+3n+1}+n}{2n+3}=\lim\limits_{n\to\infty}\dfrac{\sqrt{1+3/n+1/n^2}+1}{2+3/n}=\dfrac{1+1}{2}=1.$ ●

14. Prove that:

(1) $\lim\limits_{n\to\infty}\dfrac{1}{n}=0$; (2) $\lim\limits_{n\to\infty}\dfrac{n+1}{n}=1$; (3) $\lim\limits_{n\to\infty}\dfrac{5n-2}{7n+3}=\dfrac{5}{7}$.

15. Do the indicated sequences: (1) $x_n=(-1)^n$; (2) $x_n=(-1)^n+\dfrac{1}{n}$ have a limit?

16. Prove that the sequences: (1) $\alpha_n=\dfrac{1}{n^2}$; (2) $\alpha_n=\dfrac{1}{n^2+1}$; (3) $\alpha_n=\dfrac{5}{n^2+4}$ are infinitesimals.

17. Prove that the sequences: (1) $\alpha_n=\dfrac{(-1)^n+2}{n}$; (2) $\alpha_n=\dfrac{1}{2^n+1}$; (3) $\alpha_n=\dfrac{n^2}{n^4+4}$ are infinitesimals.

18. Compute the limit for each of the following sequences:

(1) $\lim\limits_{n\to\infty} \dfrac{5+n-3n^2}{4-n+2n^2}$; (2) $\lim\limits_{n\to\infty} \dfrac{3n+5}{3n^2+2n+4}$;

(3) $\lim\limits_{n\to\infty} \dfrac{3n^2-5n+1}{2n^3+3n^2}$; (4) $\lim\limits_{n\to\infty} \dfrac{(n+1)(n+2)}{n^2+1}$;

(5) $\lim\limits_{n\to\infty} \dfrac{(n+2)(n+3)(n+4)}{n^3+n+1}$; (6) $\lim\limits_{n\to\infty} \dfrac{(2n-1)(n-2)(n-3)}{3n^3+2n^2+n}$.

Chapter 6

LIMIT OF A FUNCTION

Sec. 6.1. Computing the Limit of a Function

1. Limit of a Function. The number A is said to be the *limit of the function* $f(x)$ as $x \to a$ if for any number $\varepsilon > 0$ there is $\delta > 0$ such that for any $x \neq a$, satisfying the inequality $0 < |x - a| < \delta$, the inequality $|f(x) - A| < \varepsilon$ is fulfilled. In this case, we write: $\lim\limits_{x \to a} f(x) = A$.

If the number A_1 (number A_2) is the limit of the function $y = f(x)$ as x tends to a so that x takes on only values less (greater) than a, then A_1 (A_2) is called the *left-hand (right-hand) limit* of the function $f(x)$ at the point a. And we write, respectively: $\lim\limits_{x \to a-0} f(x) = A_1$, $\lim\limits_{x \to a+0} f(x) = A_2$.

2. Infinitely Small and Infinitely Large Functions. A function $f(x)$ is said to be *infinitely small (an infinitesimal)* as $x \to a$ if $\lim\limits_{x \to a} f(x) = 0$.

A function $f(x)$ is said to be *infinitely large (an infinitely large magnitude)* as $x \to a$ if $\lim\limits_{x \to a} f(x) = \infty$, or $\lim\limits_{x \to a} f(x) = +\infty$, or $\lim\limits_{x \to a} f(x) = -\infty$.

Let us note the properties of infinitely small and infinitely large functions.

1°. *If the functions $f(x)$ and $\varphi(x)$ are infinitesimals as $x \to a$, then their sum $f(x) + \varphi(x)$, as $x \to a$, is also an infinitesimal.*

2°. *If the function $f(x)$ is an infinitesimal as $x \to a$, and $F(x)$ is a bounded function, then their product $f(x) \cdot F(x)$ is an infinitely small function (or an infinitesimal).*

Corollary. *The product of a finite number of infinitely small functions is an infinitesimal.*

3°. *If, as $x \to a$, the function $f(x)$ has a finite limit $\lim\limits_{x \to a} f(x) = A$ and the function $\varphi(x)$ is infinitely large, then*

$$\lim_{x \to a} [f(x) + \varphi(x)] = \infty, \quad \lim_{x \to a} \frac{f(x)}{\varphi(x)} = 0.$$

4°. *If a function $f(x)$ is an infinitesimal as $x \to a$, then the function $1/f(x)$ is infinitely large, it being assumed here that in the neighbourhood of the point a the function $f(x)$ does not vanish.* And vice versa, *if a function $\varphi(x)$ tends to infinity as $x \to a$, then the function $1/\varphi(x)$ is an infinitesimal.*

There is the following relation between an infinitely small function and a function having a finite limit.

If a function $f(x)$ has a finite limit as $x \to a$, then it can be represented as a sum of a constant, equal to that limit, and an infinitesimal as $x \to a$. Reciprocally, *if a function $f(x)$ is representable as a sum of a constant and an infinitesimal function as $x \to a$, then this function has a finite limit as $x \to a$ which is equal to the value of the constant.*

3. Theorem on Limit Uniqueness. *If a function $f(x)$ has a limit as x tends to a, then this limit is unique.*

4. Theorems on Limits. Theorem 1. *If there exist limits of functions $f(x)$ and $\varphi(x)$, then there also exists a limit of their sum which is equal to the sum of the limits of the functions $f(x)$ and $\varphi(x)$.*

More concisely: *the limit of a sum is equal to the sum of the limits*:

$$\lim_{x \to a} [f(x) + \varphi(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} \varphi(x).$$

Theorem 2. *If there exist limits of functions $f(x)$ and $\varphi(x)$ as $x \to a$, then there also exists a limit of their product which is equal to the product of the limits of the functions $f(x)$ and $\varphi(x)$.*

More concisely: *the limit of the product of several factors is equal to the product of their limits*:

$$\lim_{x \to a} [f(x) \cdot \varphi(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \varphi(x).$$

Theorem 3. *If there exist limits of functions $f(x)$ and $\varphi(x)$ as $x \to a$, and the limit of the function $\varphi(x)$ is different from zero, then there also exists a limit of the quotient $f(x)/\varphi(x)$ equal to the quotient of the limits of the functions $f(x)$ and $\varphi(x)$.*

More concisely: *the limit of a quotient is equal to the quotient of the limits if the limit of the divisor is not equal to zero:*

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} \varphi(x)}.$$

Corollary 1. *A constant factor may be taken outside the sign of the limit*

$$\lim_{x \to a} [k \cdot f(x)] = k \cdot \lim_{x \to a} f(x).$$

Corollary 2. *If n is a natural number, then*

$$\lim_{x \to a} x^n = a^n, \quad \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

Corollary 3. *The limit of a polynomial (of an integral rational function)*

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

as $x \to a$ is equal to the value of this polynomial for $x = a$, that is, $\lim\limits_{x \to a} P(x) = P(a)$.

Corollary 4. *The limit of a fractional rational function*

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n}{b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m}$$

as $x \to a$ is equal to the value of this function for $x = a$ if a belongs to the domain of definition of the function, that is, $\lim\limits_{x \to a} R(x) = R(a)$.

In Problems 1 through 4 we shall compute the indicated limits:

1. (1) $\lim\limits_{x \to 2} (5x^3 - 6x^2 + x - 5)$; (2) $\lim\limits_{x \to 2} \dfrac{x^2 - x + 1}{x - 3}$.

○ According to the rule for finding the limit of a polynomial, we get

$$\lim_{x \to 2} (5x^3 - 6x^2 + x - 5) = 5 \cdot 2^3 - 6 \cdot 2^2 + 2 - 5 = 13.$$

(2) Since for $x = 2$ the denominator of the given fraction is different from zero, by the rule for finding the limit of a fractional rational function, we obtain

$$\lim_{x \to 2} \frac{x^2 - x + 1}{x - 3} = \frac{2^2 - 2 + 1}{2 - 3} = -3. \quad \bullet$$

2. (1) $\lim\limits_{x \to 2} \dfrac{5}{4x - 8}$; (2) $\lim\limits_{x \to 0} \dfrac{3x^2 - 2x}{2x^2 - 5x}$; (3) $\lim\limits_{x \to 3} \dfrac{x^2 - 5x + 6}{3x^2 - 9x}$.

○ (1) Here, the limit of the divisor is equal to zero: $\lim\limits_{x \to 2} (4x - 8) = 4 \cdot 2 + 8 = 0$. Consequently, it is impossible to apply the theorem on the limit of a quotient. Since $\lim\limits_{x \to 2} (4x - 8) = 0$,

as $x \to 2$, $4x-8$ is an infinitesimal, and its reciprocal $\frac{1}{4x-8}$ is an infinitely large quantity. Therefore, as $x \to 2$, the product $\frac{1}{4x-8} \cdot 5$ is an infinitely large quantity, that is, $\lim\limits_{x \to 2} \frac{5}{4x-8} = \infty$.

(2) Here, the limits of both the numerator and denominator, as $x \to 0$, are equal to zero. It is impossible to compute the limit by a direct replacement of the argument by its limiting value, since for $x \to 0$ we get a quotient of two infinitesimals.

Let us factorize both the numerator and denominator into linear factors in order to reduce the fraction by a common factor which tends to zero and, consequently, to make the application of Theorem 3 possible. It should be borne in mind that we do not reduce the fraction by zero here which is simply impossible. By the definition of the limit of a function, the argument x tends to its limiting value, never attaining this value; therefore, prior to passing to the limit, it is possible to carry out the reduction by the factor tending to zero. We have

$$\lim_{x \to 0} \frac{3x^2-2x}{2x^2-5x} = \lim_{x \to 0} \frac{x(3x-2)}{x(2x-5)} = \lim_{x \to 0} \frac{3x-2}{2x-5} = \frac{3 \cdot 0-2}{2 \cdot 0-5} = \frac{2}{5}.$$

(3) The limits of both the numerator and denominator, as $x \to 3$ are equal to zero: $\lim\limits_{x \to 3} (x^2-5x+6) = 3^2-5 \cdot 3+6 = 0$, $\lim\limits_{x \to 3} (3x^2-9) = 3 \cdot 3^2-9 \cdot 3 = 0$. Let us factorize the quadratic trinomial in the numerator into linear factors using the formula $ax^2+bx+c = a(x-x_1)(x-x_2)$, where x_1 and x_2 are roots of the trinomial. We then factor the denominator and reduce the fraction by $x-3$. Using Corollary 4, we get

$$\lim_{x \to 3} \frac{x^2-5x+6}{3x^2-9x} = \lim_{x \to 3} \frac{(x-3)(x-2)}{3x(x-3)} = \lim_{x \to 3} \frac{x-2}{3x} = \frac{3-2}{3 \cdot 3} = \frac{1}{9}. \bullet$$

3. (1) $\lim\limits_{x \to 0} \frac{x}{\sqrt{5-x}-\sqrt{5+x}}$; (2) $\lim\limits_{x \to -2} \left(\frac{1}{x+2} - \frac{12}{x^3+8}\right)$.

○ (1) The limits of both the numerator and denominator, as $x \to 0$, are equal to zero. Multiplying the numerator and denominator by the factor $\sqrt{5-x}+\sqrt{5+x}$ and reducing then the fraction by x, we get

$$\lim_{x \to 0} \frac{x}{\sqrt{5-x}-\sqrt{5+x}} = \lim_{x \to 0} \frac{x(\sqrt{5-x}+\sqrt{5+x})}{(\sqrt{5-x}-\sqrt{5+x})(\sqrt{5-x}+\sqrt{5+x})}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{5-x}+\sqrt{5+x})}{-2x}$$

$$= \lim_{x \to 0} \frac{\sqrt{5-x}+\sqrt{5+x}}{-2} = \frac{\sqrt{5}+\sqrt{5}}{-2} = -\sqrt{5}.$$

(2) It is obvious that for $x \to -2$ the function represents a difference of two infinitely large quantities. Performing the subtraction of the fractions, we get a fraction whose numerator and denominator tend to zero for $x \to -2$. Reducing the fraction by $x+2$, we find

$$\lim_{x\to -2}\left(\frac{1}{x+2}-\frac{12}{x^3+8}\right)=\lim_{x\to -2}\frac{x^2-2x-8}{x^3+8}=\lim_{x\to -2}\frac{(x+2)(x-4)}{(x+2)(x^2-2x+4)}$$

$$=\lim_{x\to -2}\frac{x-4}{x^2-2x+4}=\frac{-2-4}{(-2)^2-2(-2)+4}=-\frac{1}{2}. \bullet$$

4. (1) $\lim\limits_{x\to\infty}(x^3-6x^2+5x-1)$; (2) $\lim\limits_{x\to\infty}\frac{5}{4x+1}$; (3) $\lim\limits_{x\to\infty}\frac{2x+3}{5x+1}$;

(4) $\lim\limits_{x\to\infty}\frac{x^4-2x^2+3}{3x^3-5}$; (5) $\lim\limits_{x\to\infty}(x-\sqrt{x^2-4x})$.

○ (1) The first three terms have no limits, as $x \to \infty$, therefore it is impossible to take advantage of Corollary 3 directly. Factoring out x^3, we get

$$\lim_{x\to\infty}\left[x^3\left(1-\frac{6}{x}+\frac{5}{x^2}-\frac{1}{x^3}\right)\right]=(\lim_{x\to\infty}x)^3\lim_{x\to\infty}\left(1-\frac{6}{x}+\frac{5}{x^2}-\frac{1}{x^3}\right)=\infty$$

(for $x \to \infty$ the quantities $6/x$, $5/x^2$, and $1/x^3$ are infinitesimals, and their limits are equal to zero).

(2) As $x \to \infty$, the denominator $4x+1$ increases without bound, that is, it becomes an infinitely large quantity, while its reciprocal $\frac{1}{4x+1}$ is an infinitesimal. The product $\frac{1}{4x+1}\cdot 5$ of an infinitesimal and a bounded quantity (a constant is a particular case of a bounded quantity) is an infinitesimal, and its limit is equal to zero, as $x \to \infty$. Consequently, $\lim\limits_{x\to\infty}\frac{5}{4x+1}=0$.

(3) For $x \to \infty$, both the numerator and denominator are infinitely large quantities. Therefore, a direct application of Theorem 3 yields the expression ∞/∞ which is an indeterminate form. To compute the limit of this function, it is necessary to divide both the numerator and denominator by x:

$$\lim_{x\to\infty}\frac{2x+3}{5x+1}=\lim_{x\to\infty}\frac{2+3/x}{5+1/x}=\frac{2+0}{5+0}=\frac{2}{5}$$

(as $x \to \infty$, the terms $3/x$ and $1/x$ become infinitesimals and, consequently, their limits are equal to zero).

(4) We divide both the numerator and denominator by the highest power of the argument in the denominator, that is, by x^3:

$$\lim_{x\to\infty}\frac{x^4-2x^2+3}{3x^3-5}=\lim_{x\to\infty}\frac{x-2/x+3/x^3}{3-5/x^3}.$$

For $x \to \infty$ we have

$$\lim_{x\to\infty}\left(x-\frac{2}{x}+\frac{3}{x^3}\right)=\infty \quad \text{and} \quad \lim_{x\to\infty}\left(3-\frac{5}{x^3}\right)=3.$$

Since the denominator is a bounded quantity, we have

$$\lim_{x\to\infty}\frac{x^4-2x^2+3}{3x^3-5}=\infty.$$

(5) For $x \to \infty$ the given function represents a difference of two infinitely large quantities (∞, $-\infty$). Multiplying and dividing the function by the expression $x+\sqrt{x^2-4x}$, we get

$$\lim_{x\to\infty}(x-\sqrt{x^2-4x})=\lim_{x\to\infty}\frac{(x-\sqrt{x^2-4x})(x+\sqrt{x^2-4x})}{x+\sqrt{x^2-4x}}$$

$$=\lim_{x\to\infty}\frac{x^2-x^2+4x}{x+\sqrt{x^2-4x}}=\lim_{x\to\infty}\frac{4}{1+\sqrt{1-4/x}}=\frac{4}{2}=2. \bullet$$

In Problems 5 through 23 compute the indicated limits:

5. (1) $\lim\limits_{x\to 3}(x^3+x-5)$; (2) $\lim\limits_{x\to -1}(x^3-x^2+1)$.

6. (1) $\lim\limits_{x\to -1}(2x^3-5x^2+x-4)$; (2) $\lim\limits_{x\to 0}(3x^3+x^2-8x+10)$.

7. (1) $\lim\limits_{x\to 1}[(7x+2)(4x-3)(5x+1)]$;

(2) $\lim\limits_{x\to 2}[(x^2-1)(x-3)(x-5)]$; (3) $\lim\limits_{x\to 0}[(2x-4)(x-1)(x+2)]$.

8. (1) $\lim\limits_{x\to -1}\dfrac{(x+3)(x-2)}{x+2}$; (2) $\lim\limits_{x\to 4}\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$.

9. (1) $\lim\limits_{x\to 3}\dfrac{3}{2x-6}$; (2) $\lim\limits_{x\to 0}\dfrac{4}{3x^2+2x}$.

10. (1) $\lim\limits_{x\to 0}\dfrac{2x^3-2x^2}{5x^3-4x^2}$; (2) $\lim\limits_{x\to 0}\dfrac{3x^3+x}{x}$.

11. (1) $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$; (2) $\lim\limits_{x\to -3/2}\dfrac{4x^2-9}{2x+3}$.

12. (1) $\lim\limits_{x\to 5}\dfrac{x^2-8x+15}{x^2-25}$; (2) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$.

13. (1) $\lim\limits_{x\to 2}\dfrac{3x^2-8x+4}{5x^2-14x+8}$; (2) $\lim\limits_{x\to 5}\dfrac{x^2-7x+10}{x^2-9x+20}$.

14. (1) $\lim\limits_{x\to -3}\dfrac{2x^2+x-15}{3x^2+7x-6}$; (2) $\lim\limits_{x\to -2/3}\dfrac{3x^2+5x+2}{3x^2+8x+4}$.

15. $\lim\limits_{x\to 6}\dfrac{x-6}{\sqrt{x+3}-3}$; (2) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$.

16. (1) $\lim\limits_{x\to 9}\dfrac{3-\sqrt{x}}{4-\sqrt{2x-2}}$; (2) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.

17. (1) $\lim\limits_{x\to 3}\left(\frac{6}{x^2-9}-\frac{1}{x-3}\right)$; (2) $\lim\limits_{x\to -1}\left(\frac{3}{x^3+1}-\frac{1}{x+1}\right)$.

18. (1) $\lim\limits_{x\to\infty}(x^2-5x+6)$; (2) $\lim\limits_{x\to-\infty}(x^3+3x^2)$.

19. (1) $\lim\limits_{x\to\infty}\frac{2}{x^2+3x}$; (2) $\lim\limits_{x\to\infty}\left(5+\frac{2}{x}-\frac{3}{x^2}\right)$.

20. (1) $\lim\limits_{x\to\infty}\frac{3x}{x-2}$; (2) $\lim\limits_{x\to\infty}\frac{x-8}{2x-2}$.

21. (1) $\lim\limits_{x\to\infty}\frac{2x^3-3x^2+1}{x^3+4x^2+2x}$; (2) $\lim\limits_{x\to\infty}\frac{3x^2-5x+4}{x^2+2x+3}$; (3) $\lim\limits_{x\to\infty}\frac{4x^3-x^2}{x^3+3x^2-1}$.

22. (1) $\lim\limits_{x\to\infty}\frac{x^5+x^6}{x^3+x^4}$; (2) $\lim\limits_{x\to\infty}\frac{x^4-x^3+1}{x^3+2x^2+x}$.

23. (1) $\lim\limits_{x\to\infty}(\sqrt{x^2-x}-x)$; (2) $\lim\limits_{x\to\infty}(\sqrt{x^2+5x}-x)$.

Sec. 6.2. The Number e. Natural Logarithms

The following relationship takes place:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}=e. \tag{6.1}$$

The number e is an irrational number ($e\approx 2.718...$, a more precise value: $e\approx 2.7182818$). This number is called the natural base; logarithms based on e are called natural logarithms. For the natural logarithm of a number, the notation ln is used. Then we need not show the base.

Common and natural logarithms are related as follows:

$$\log N = M\ln N = 0.4343\ln N; \tag{6.2}$$

$$\ln N=\frac{1}{M}\log N=2.303\log N, \tag{6.3}$$

where M is the modulus of converting natural logarithms to common logarithms.

24. Find the natural logarithm of each of the numbers: (1) 7; (2) 0.12.

○ By Formula (6.3), we get:

(1) $\ln 7 = 2.303\cdot\log 7 = 2.303\cdot 0.8451 = 1.946$;

(2) $\ln 0.12 = 2.303\cdot\log 0.12 = 2.303\cdot\bar{1}.0792 = 2.303\times(-0.9208) = -2.121$. ●

25. Find the common logarithm of each of the numbers given their natural logarithms: (1) 0.2624; (2) 2.1401.

○ By Formula (6.2), we find:

(1) $\log N = 0.4343\cdot 0.2624 = 0.1140$; (2) $\log N = 0.4343\times 2.1401 = 0.9294$. ●

26. Compute with the aid of the tables of common logarithms: (1) e^3; (2) $\sqrt{e}$; (3) e^{-3}.

○ (1) $\log e^3 = 3 \log e = 3 \cdot 0.4343 = 1.3029$; $e^3 = 20.08$;
(2) $\log \sqrt{e} = 0.5 \log e = 0.5 \cdot 0.4343 = 0.2171$; $\sqrt{e} = 1.648$;
(3) $\log e^{-3} = -3 \log e = -3 \cdot 0.4343 = -1.3029 = \bar{2}.6971$;
$e^{-3} = 0.04978$. ●

27. Compute without using the tables: (1) ln 100; (2) ln 0.001; (3) ln $\sqrt{10}$.

○ (1) $\ln 100 = \ln 10^2 = 2 \ln 10 = 2 \cdot 2.303 = 4.606$;
(2) $\ln 0.001 = \ln 10^{-3} = -3 \ln 10 = -3 \cdot 2.303 = -6.909$;
(3) $\ln \sqrt{10} = 0.5 \ln 10 = 0.5 \cdot 2.303 = 1.151$. ●

Obviously, the procedure is summarized in the following rules:

Rule 1. To change a common logarithm to a natural logarithm, multiply the common loragithm by 2.303 because the logarithm must be made larger.

Rule 2. To change a natural logarithm to a common logarithm, multiply the natural logarithm by 0.4343 because the logarithm must be made smaller.

28. Compute the following limits:

$$(1)\ \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x; \quad (2)\ \lim_{x\to 0}(1+2x)^{5/x}; \quad (3)\ \lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x.$$

○ Performing all necessary transformations and using Formula (6.1), we find:

$$(1)\ \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x/3}\right)^{(x/3)\cdot 3} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x/3}\right)^{x/3}\right]^3$$
$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x/3}\right)^{x/3}\right]^3 = e^3;$$

$$(2)\ \lim_{x\to 0}(1+2x)^{5/x} = \lim_{x\to 0}\left(1+\frac{1}{1/(2x)}\right)^{5\cdot(1/x)\cdot(2/2)}$$
$$= \lim_{1/(2x)\to\infty}\left[\left(1+\frac{1}{1/(2x)}\right)^{1/(2x)}\right]^{10} = \left[\lim_{1/(2x)\to\infty}\left(1+\frac{1}{1/(2x)}\right)^{1/(2x)}\right]^{10} = e^{10};$$

$$(3)\ \lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{-x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^{-1}$$
$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1} = \frac{1}{e}.$$ ●

29. Find the natural logarithm of each of the following numbers. (1) 3; (2) 4; (3) 5; (4) 10; (5) 0.3; (6) 0.8; (7) 5.8; (8) 0.24; (9) 15.6:

30. Find the common logarithms of the numbers given their natural logarithms: (1) 2.0794; (2) 3.6889; (3) 1.959.

31. Compute with the aid of the tables of common logarithms: (1) e^5; (2) $\sqrt[3]{e}$; (3) e^{-2}.

32. Compute without using the tables: (1) ln 1000; (2) ln 0.01; (3) ln $\sqrt[3]{100}$.

In Problems 33 through 35 compute the indicated limits:

33. (1) $\lim\limits_{x\to\infty}\left(1+\frac{2}{3x}\right)^{x}$; (2) $\lim\limits_{x\to\infty}\left(1-\frac{5}{4x}\right)^{x}$.

34. (1) $\lim\limits_{z\to 0}(1+4z)^{3/5z}$; (2) $\lim\limits_{z\to\infty}\left(1+\frac{1}{z}\right)^{-z}$.

35. (1) $\lim\limits_{x\to\infty}\left(\frac{2x}{2x+1}\right)^{x}$; (2) $\lim\limits_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{x+0.5}$.

Sec. 6.3. Miscellaneous Problems

In Problems 36 through 42 compute the given limits:

36. (1) $\lim\limits_{x\to 2}\frac{4x^2-7x-2}{5x^2-11x+2}$; (2) $\lim\limits_{x\to 3}\frac{x^2+2x-15}{x^2-9}$.

37. (1) $\lim\limits_{z\to 0}\frac{2z}{\sqrt{4+z}-\sqrt{4-z}}$; (2) $\lim\limits_{x\to 4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}$.

38. (1) $\lim\limits_{x\to 8}\frac{\sqrt[3]{x-2}}{\sqrt{x}-2\sqrt{2}}$; (2) $\lim\limits_{x\to -27}\frac{x+27}{\sqrt[3]{x}+3}$.

39. (1) $\lim\limits_{n\to\infty}\frac{n^2+a^2}{n^3+a^3}$; (2) $\lim\limits_{n\to\infty}\frac{4n+1}{\sqrt{4n^2+1}}$;

(3) $\lim\limits_{n\to\infty}\frac{1+2+3+\ldots+n}{1+3+5+\ldots+(2n-1)}$.

40. (1) $\lim\limits_{x\to\infty}\frac{x^4-x^2+2}{x^3-x+1}$; (2) $\lim\limits_{x\to\infty}(\sqrt{x^2+x}-x)$.

41. (1) $\lim\limits_{x\to 0}\left(1+\frac{2x}{3}\right)^{2/(3x)}$; (2) $\lim\limits_{x\to 0}\left(1+\frac{5}{3x}\right)^{2x}$.

42. (1) $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}$; (2) $\lim\limits_{x\to\infty}\left(1-\frac{3}{x}\right)^{x}$.

QUIZ No. 1

Form I	Form II
Compute the indicated limits	
(1) $\lim\limits_{x\to 5}\frac{3x^2-17x+10}{3x^2-16x+5}$;	(1) $\lim\limits_{x\to 1}\frac{4x^2-7x+3}{3x^2-2x-1}$;
(2) $\lim\limits_{x\to 5}\frac{5-x}{3-\sqrt{2x-1}}$;	(2) $\lim\limits_{x\to 0}\frac{x}{\sqrt{3+x}-\sqrt{3-x}}$;
(3) $\lim\limits_{z\to\sqrt{3}}\frac{\sqrt{1+z^2}-1}{3z^2}$;	(3) $\lim\limits_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}$;
(4) $\lim\limits_{x\to\infty}\frac{2x^3+x+1}{3x^3+x^2+1}$;	(4) $\lim\limits_{x\to\infty}\frac{5x^4-x^3+2x}{x^4-8x^3+1}$;
(5) $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^{x}$.	(5) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{-x}$.

Sec. 6.4. The Increment of Argument and the Increment of Function

For a function $y = f(x)$ the difference between the two values of the argument x_1 and x_2 from $D(f)$ is called the *increment of the argument* and is denoted by the symbol Δx, that is, $x_2 - x_1 = \Delta x$.

The difference between the two values of the function $y_1 = f(x_1)$ and $y_2 = f(x_2)$ from $E(f)$ corresponding to the values of the argument x_1 and x_2 is called the *increment of the function* and is denoted by the symbol Δy, that is, $\Delta y = f(x_2) - f(x_1) = y_2 - y_1$.

If $x_2 > x_1$, then $\Delta x > 0$; and if $x_2 < x_1$, then $\Delta x < 0$. Correspondingly, the increment of the function $\Delta y > 0$ if $y_2 > y_1$, and $\Delta y < 0$ if $y_2 < y_1$.

The increment of the function $y = f(x)$ is found according to the following scheme. Let the argument x receive an increment Δx, then the new (incremented) value of the argument is equal to $x + \Delta x$, and the corresponding value of the function is equal to $y + \Delta y = f(x + \Delta x)$. To find the increment in the function, it is necessary to subtract the initial value of the function from its new value:

$$\begin{array}{r} y + \Delta y = f(x + \Delta x) \\ \underline{y = f(x)} \\ \Delta y = f(x + \Delta x) - f(x). \end{array}$$

43. Given the function $y = x^2 + x + 1$. Find the increment in the argument and the increment in the function if the argument x changed its value from $x_1 = 2$ to $x_2 = 2.5$.

○ We find the increment in the argument: $\Delta x = x_2 - x_1 = 2.5 - 2 = 0.5$.

Now, we compute the values of the function corresponding to the values of the argument $x_1 = 2$ and $x_2 = 2.5$:

$$y_1 = f(x_1) = f(2) = 2^2 + 2 + 1 = 7; \quad y_2 = f(x_2) = f(2.5) = (2.5)^2 + 2.5 + 1 = 9.75.$$

We then find the increment in the function:

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) = 9.75 - 7 = 2.75. \quad \bullet$$

44. Given the function $y = x^2 + 2x - 4$. Find the increment Δy for $x = 2$ and $\Delta x = 0.5$.

○ Find the incremented value of the function corresponding to the increment Δx:

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 4 = x^2 + 2x\,\Delta x + (\Delta x)^2 + 2x + 2\Delta x - 4.$$

Find the increment in the function:

$$\begin{array}{r} y + \Delta y = x^2 + 2x\,\Delta x + (\Delta x)^2 + 2x + 2\,\Delta x - 4 \\ \underline{y = x^2 + 2x - 4} \\ \Delta y = 2x\,\Delta x + 2\,\Delta x + (\Delta x)^2 = 2 \cdot 2 \cdot 0.5 + 2 \cdot 0.5 + (0.5)^2 = 3.25. \end{array} \quad \bullet$$

45. Given the function $y = x^2 - 2x + 4$. Find the increment in the function if the argument x changed its value from $x_1 = 3$ to $x_2 = 3.5$.

46. Given two functions: (1) $y = x^2 + 2x$; (2) $y = x^3 - 1$. Find the increment Δy for $x = 3$ and $\Delta x = 0.1$ for each function.

47. Given the function $y = 1/x$. Find the increment Δy for $x = 1$ and $\Delta x = 0.2$.

48. Given the function $y = \sqrt{x}$. Find the increment Δy for $x = 1$ and $\Delta x = 0.1$.

49. Given two functions (1) $y = \sqrt{2x}$; (2) $y = \sqrt[3]{x}$. Find the increment Δy for $x = 1$ and $\Delta x = 0.2$.

Sec. 6.5. Continuity of a Function

Definition 1. A function $f(x)$ is said to be *continuous at a point* $x = a$ if the limit of the function, as $x \to a$, is equal to the value of the function for $x = a$, that is, $\lim\limits_{x \to a} f(x) = f(a)$.

Definition 2. A function $y = f(x)$ is said to be *continuous at a point* $x = a$ if it is defined at this point and to an infinitesimal increment of the argument there corresponds an infinitesimal increment of the function, that is, $\lim\limits_{\Delta x \to 0} \Delta y = 0$.

If, for a function $y = f(x)$, the condition of continuity at a point $x = a$ is violated, then such point is called a *point of discontinuity* of the function.

The following statements hold true for elementary functions:

(1) the domain of continuity of an elementary function coincides with its domain of definition, that is, an elementary function is continuous in the whole domain of definition;

(2) an elementary function can have a discontinuity only at separate points of a certain interval but not at all of its points;

(3) an elementary function can be discontinuous only at a point at which it is not defined.

A function is said to be *continuous in an interval* (whether closed or open) if it is continuous at every point of this interval.

50. Investigate for continuity the functions: (1) $y = 3x$; (2) $y = 3x^2 - 2x$.

○ (1) The function $y = 3x$ is defined for all real values of the argument x, that is, the domain of its definition is the whole number line $(-\infty, +\infty)$. The domain of continuity coincides with the domain of its definition which can be readily shown by using Definition 2.

Let the argument x receive an increment Δx and let us find the corresponding increment of the function Δy:

$$\begin{array}{r} y + \Delta y = 3(x + \Delta x) = 3x + 3\Delta x \\ y = 3x \\ \hline \Delta y = 3\Delta x \end{array}$$

We then find the limit Δy as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} 3\Delta x = 3 \lim_{\Delta x \to 0} \Delta x = 3 \cdot 0 = 0.$$

The equality $\lim\limits_{\Delta x\to 0}\Delta y=0$ holds true for any finite value of x, therefore the function $y=3x$ is continuous for any value of x.

(2) The given function is defined in the interval $(-\infty, +\infty)$ and so it is continuous in the same interval. We have

$$\begin{aligned} y+\Delta y &= 3(x+\Delta x)^2-2(x+\Delta x)\\ &= 3x^2+6x\Delta x+3(\Delta x)^2-2x-2\Delta x\\ y &= 3x^2-2x\\ \hline \Delta y &= 6x\Delta x+3(\Delta x)^2-2\Delta x. \end{aligned}$$

Consequently,

$$\lim_{\Delta x\to 0}\Delta y=[6x\lim\Delta x+3(\lim\Delta x)^2-2\lim\Delta x]_{\Delta x\to 0}$$
$$=6x\cdot 0+3\cdot 0^2-2\cdot 0=0.$$

According to Definition 2, the given function is continuous for any finite value of x. ●

51. Determine whether the function $y=x^2-2$ is continuous for $x=3$.

○ In this case, we use Definition 1:

$$\lim_{x\to 3}(x^2-2)=\left(\lim_{x\to 3}x\right)^2-2=3^2-2=7;\quad f(3)=3^2-2=7,$$

that is, $\lim\limits_{x\to 3}(x^2-2)=f(3)$. As $x\to 3$, the limit of the function is equal to the value of the function for $x=3$. Consequently, the function $y=x^2-2$ is continuous at the point $x=3$. ●

In Problems 52 through 54 test the indicated functions for continuity:

52. (1) $y=-5x$; (2) $y=4x-3$.

53. (1) $v=2t^2$; (2) $y=x^2+2$; (3) $s=t^2-t$;
(4) $y=x-3x^2$; (5) $y=x^3$; (6) $y=-x^3-1$; (7) $y=2x^3$.

54. (1) $y=x^2+4x+3$ at the point $x=2$; (2) $y=x^3-5$ at the point $x=1$.

Sec. 6.6. Points of Discontinuity of a Function

If a function $y=f(x)$ has a discontinuity for $x=a$, then, to clear up the character of discontinuity, it is necessary to find the left-hand and the right-hand limits of the function $f(x)$ as $x\to a$.

Depending on the behaviour of a function in the vicinity of the point of discontinuity, we distinguish between two basic kinds of discontinuity: (1) *a discontinuity of the first kind*; in this case, there exist the limits $\lim\limits_{x\to a-0}f(x)$ and $\lim\limits_{x\to a+0}f(x)$; (2) *a discontinuity of the second kind*; in this case, at least one of the limits $\lim\limits_{x\to a-0}f(x)$ and $\lim\limits_{x\to a+0}f(x)$ does not exist or is infinite.

55. For the given functions, find the points of discontinuity and investigate their character:

(1) $y=\frac{x}{x-3}$; (2) $y=\frac{1}{x^2-6x+8}$; (3) $y=3^{1/x}$; (4) $y=\frac{1}{1+5^{1/x}}$.

○ (1) The given function is defined for all values of x, except $x=3$. Since this function is elementary, it is continuous at each point of its domain of definition. Thus, the only point of discontinuity is the point $x=3$. To investigate the character of discontinuity, we find the left-hand and right-hand limits of the function, as $x \to 3$:

$$\lim_{x\to 3-0} \frac{x}{x-3} = -\infty, \quad \lim_{x\to 3+0} \frac{x}{x-3} = +\infty.$$

Consequently, the function $\frac{x}{x-3}$ has an infinite discontinuity at the point $x=3$, that is, $x=3$ is a point of discontinuity of the second kind.

(2) Reasoning in a similar way, we arrive at the conclusion that $x=2$ and $x=4$ are the points of discontinuity of the given function at which the denominator of the fraction vanishes. It is obvious that at these points the function has an infinite discontinuity, that is, $x=2$ and $x=4$ are points of discontinuity of the second kind.

(3) Here, the function is defined for all values of x, except $x=0$. Let us find the left-hand and right-hand limits of the function, as $x \to 0$

$$\lim_{x\to -0} 3^{1/x} = 0, \quad \lim_{x\to +0} 3^{1/x} = +\infty.$$

Since for x tending to zero from the right the function has an infinite limit, $x=0$ is a point of discontinuity of the second kind.

(4) In this case, the point $x=0$ is also the only point of discontinuity. Let us compute the one-sided limits of the function for $x \to 0$:

$$\lim_{x\to -0} \frac{1}{1+5^{1/x}} = 1, \quad \lim_{x\to +0} \frac{1}{1+5^{1/x}} = 0.$$

Since the left-hand and right-hand limits of the function are finite for $x=0$, the latter is a point of discontinuity of the first kind. ●

56. Find the points of discontinuity and investigate their character for the following functions:

(1) $y=\frac{5}{2x-1}$; (2) $y=\frac{1}{x^2}$; (3) $y=\frac{1}{1-x^2}$; (4) $y=\frac{3}{x^2-2x+1}$;

(5) $y=\frac{x-1}{x^2-3x-10}$; (6) $y=1+2^{1/(x-2)}$.

Sec. 6.7. Asymptotes

An *asymptote* to a curve is defined as a straight line which has the property that the distance from a point on the curve to the line tends to zero as the distance from this point to the origin increases without bound.

There are vertical, horizontal, and oblique asymptotes.

Vertical asymptotes. The graph of the function $y = f(x)$ for $x \to a$ has a vertical asymptote if $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$; $x = a$ being a point of discontinuity of the second kind. The equation of the vertical asymptote has the form $x = a$ (Fig. 11(*a*), (*b*)).

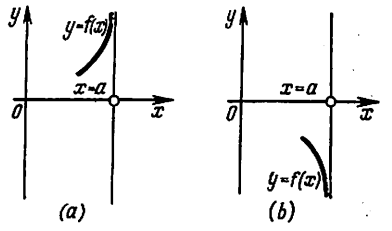


Fig. 11

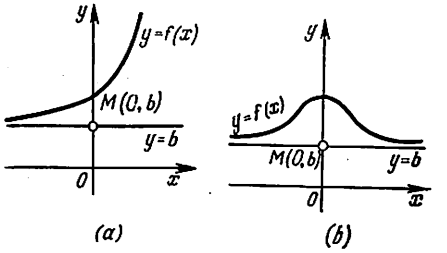


Fig. 12

Horizontal asymptotes. The graph of the function $y = f(x)$ for $x \to +\infty$ or for $x \to -\infty$ has a horizontal asymptote if $\lim\limits_{x \to +\infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b_1$. It may turn out that either only one or none of these limits is finite, then the graph has either one or no horizontal asymptote. The equation of the horizontal asymptote has the form $y = b$ (Fig. 12(*a*), (*b*)).

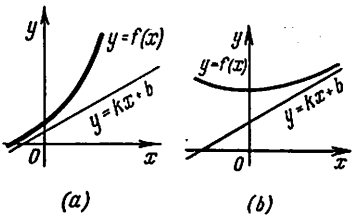


Fig 13

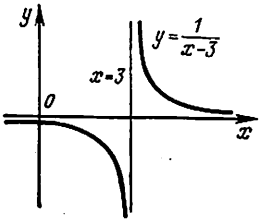


Fig. 14

Oblique asymptotes. Let the graph of the function $y = f(x)$ has an oblique asymptote $y = kx + b$ (Fig. 13(*a*), (*b*)). In this case, the following equality holds true:

$$\lim_{x \to \pm\infty} [f(x) - kx - b] = 0.$$

Factoring out x, we get

$$\lim_{x \to \pm\infty} x \left[\frac{f(x)}{x} - k - \frac{b}{x} \right] = 0.$$

Since $\lim\limits_{x \to \pm\infty} \frac{b}{x} = 0$, we get the formulas for computing the parameters k and b:

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = k, \quad \lim_{x \to \pm\infty} [f(x) - kx] = b.$$

The cases $x \to +\infty$ and $x \to -\infty$ should be considered separately.

57. Find the asymptotes to the following curves:

(1) $y=\frac{1}{x-3}$; (2) $y=\frac{x}{x-1}$; (3) $y=e^{1/x}$; (4) $y=\frac{x}{\sqrt{x^2+1}}$.

○ (1) Since $\lim\limits_{x\to\infty}\frac{1}{x-3}=0$, the curve $y=\frac{1}{x-3}$ has a horizontal asymptote $y=0$. Further, we find $\lim\limits_{x\to 3-0}\frac{1}{x-3}=-\infty$, $\lim\limits_{x\to 3+0}\frac{1}{-3}=+\infty$; consequently, the curve has a vertical asymptote: $x=3$ (Fig. 14).

(2) We have $\lim\limits_{x\to 1-0}\frac{x}{x-1}=-\infty$, $\lim\limits_{x\to 1+0}\frac{x}{x-1}=+\infty$. Hence, $x=1$ is a point of discontinuity of the second kind and, consequently, the curve has a vertical asymptote $x=1$.

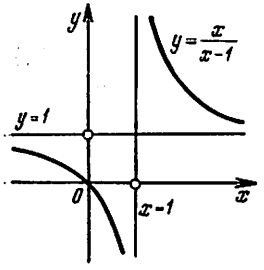


Fig. 15

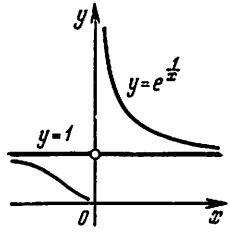


Fig. 16

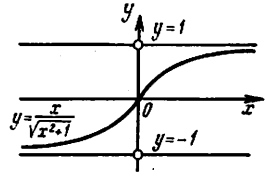


Fig. 17

Let us find the horizontal asymptote:

$$\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{x-1+1}{x-1}=\lim_{x\to\infty}\left(1+\frac{1}{x-1}\right)=1,$$

that is, $y=1$ is the horizontal asymptote of the graph (Fig. 15).

(3) Since $\lim\limits_{x\to\pm\infty}e^{1/x}=1$, the straight line $y=1$ serves as a horizontal asymptote. We then find the vertical asymptote:

$$\lim_{x\to+0}e^{1/x}=+\infty,\quad \lim_{x\to-0}e^{1/x}=0;$$

consequently, $x=0$ is a vertical asymptote (Fig. 16).

(4) Let us find the horizontal asymptote:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+1/x}}=\frac{1}{\sqrt{1+0}}=1,\quad \lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=-1.$$

For $x\to+\infty$ the asymptote is represented by the straight line $y=1$, and for $x\to-\infty$ by the straight line $y=-1$ (Fig. 17). ●

58. Find the asymptotes to the curve $y=\frac{x^2}{x-1}$.

○ We find the oblique asymptote:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x^2}{(x-1)x}=\lim_{x\to\pm\infty}\frac{x}{x-1}=\lim_{x\to\pm\infty}\frac{x-1+1}{x-1}$$

$$=\lim_{x\to\pm\infty}\left(1+\frac{1}{x-1}\right)=1,$$

$$b=\lim_{x\to\pm\infty}[f(x)-kx]=\lim_{x\to\pm\infty}\left[\frac{x^2}{x-1}-x\right]=\lim_{x\to\pm\infty}\frac{x}{x-1}=1.$$

Thus, $k=1$ and $b=1$; consequently, for $x\to+\infty$ and for $x\to-\infty$ the graph of the function has an oblique asymptote:

$$y=x+1.$$

If $x\to1$, then $y\to\pm\infty$, hence the straight line $x=1$ is a vertical asymptote (Fig. 18). ●

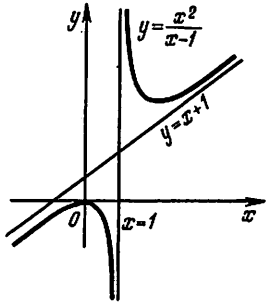


Fig. 18

59. Find the asymptotes to the following curves:

(1) $y=\frac{x^3}{x-1}$; (2) $y=\sqrt{\frac{1}{x}-1}$;

(3) $y=\frac{x^2-1}{x^2+1}$; (4) $y=\frac{1}{1-e^x}$.

60. Find the oblique asymptotes to the following curves:

(1) $y=x+e^{-x}$; (2) $y=\sqrt{x^2-1}$; (3) $y=\frac{x^3}{x^2+1}$.

61. Find the asymptotes to the following curves:

(1) $y=\frac{2}{x+2}$; (2) $y=\frac{5}{x^2-25}$; (3) $y=\frac{x^2+6x-5}{x}$;

(4) $y=\frac{x^2}{\sqrt{x^2-1}}$; (5) $y=\frac{x^2}{x+3}$; (6) $y=\frac{3x^2}{x^2+5}$;

(7) $y=\frac{x^2+1}{x}$; (8) $y=\frac{x^2-5x+4}{x-4}$.

Sec. 6.8. Solving Fractional Rational Inequalities by the Method of Intervals

Solving inequalities by the method of intervals is based on the following property of a continuous function: *if a continuous function vanishes at points x_1 and x_2 ($x_1<x_2$) and has no other roots between these points, then in the interval (x_1, x_2) the function retains its sign.*

Therefore, to find the intervals of constant signs of the function $y=f(x)$, we proceed as follows. On the number line we mark all the points at which the

function $f(x)$ vanishes or undergoes a discontinuity. These points separate the number line into several intervals inside each of which the function $f(x)$ is continuous and does not vanish, that is, preserves its sign. In order to determine this sign, it is sufficient to find the sign of the function at some point of the interval of the x-axis under consideration.

62. Solve the following inequalities:

(1) $(x+3)(x-2)(x-5)<0$;

(2) $(x-3)(x-5)(x^2+2x+5)\leqslant 0$.

○ (1) The polynomial $P(x)=(x+3)(x-2)(x-5)$ vanishes at the points $x_1=-3$, $x_2=2$, and $x_3=5$. These points divide the whole number line into four intervals $(-\infty, -3)$; $(-3, 2)$, $(2, 5)$,



Fig. 19



Fig. 20

$(5, +\infty)$ (Fig. 19) inside each of which the function $P(x)$ retains sign. Let us take an arbitrary point from each interval: $P(-4)<0$; $P(-2)>0$; $P(3)<0$; $P(6)>0$. Consequently, $P(x)<0$ for $x\in(-\infty, -3)$ and for $x\in(2, 5)$. The union of the intervals $(-\infty, -3)\cup(2, 5)$ is the solution of the given inequality.



Fig. 21

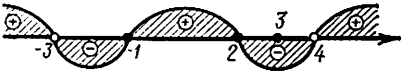


Fig. 22

(2) The trinomial x^2+2x+5 takes on positive values for all $x\in\mathbf{R}$ (since $D=2^2-4\cdot 5<0$). Therefore the given inequality is equivalent to the inequality $(x-3)(x-5)\leqslant 0$. Figure 20 shows the intervals of constant signs of the left-hand side of the given inequality. All the points of the interval $[3, 5]$ are the solutions of the inequality. ●

63. Solve the following inequalities:

(1) $\dfrac{(x-5)(x-3)(x+2)}{(x-4)(x+4)}\leqslant 0$; (2) $\dfrac{(x-3)^2(x-2)(x+1)}{(x-4)(x+3)}\leqslant 0$.

○ (1) Multiplying both sides of the given inequality by the squared denominator $(x-4)^2(x+4)^2$, we get the inequality

$$(x-5)(x-4)(x-3)(x+2)(x+4)\leqslant 0.$$

This inequality has two extraneous solutions: -4 and 4 which should be eliminated from the set of its solutions. The left-hand side of the last inequality vanishes at the points -4, -2, 3, 4, 5. Thus, we obtain the following intervals of constant signs $(-\infty, -4)$, $(-4, -2]$, $[-2, 3]$, $[3, 4)$, $(4, 5]$, $[5, +\infty)$ (Fig. 21). The solution

of the given inequality is represented by the union of the intervals: $(-\infty, -4) \cup [-2, 3] \cup (4, 5]$.

(2) Multiplying both sides of the given inequality by $(x-4)^2 \times (x+3)^2$, we get the inequality

$$(x-3)^2 (x-2)(x+1)(x-4)(x+3) \leqslant 0.$$

This inequality has two extraneous roots -3 and 4 which should be eliminated from the set of solutions. The left-hand side of the last inequality vanishes at the points -3, -1, 2, 3, and 4. Thus, we obtain six intervals of constant signs $(-\infty, -3)$, $(-3, -1]$, $[-1, 2]$, $[2, 3]$, $[3, 4)$, and $(4, +\infty)$ (Fig. 22).

The solution of the given inequality is represented by the union of the intervals: $(-3, -1] \cup [2, 4)$ (since the value $x = 3$ is a solution of the inequality and belongs to the interval $[2, 4)$ it should not be given separately in the answer). ●

64. Solve the inequality $\frac{x^2-2x+3}{x^2-4x+3} > -3$.

○ We have

$$\left(\frac{x^2-2x+3}{x^2-4x+3} > -3\right) \Leftrightarrow \left(\frac{x^2-2x+3}{x^2-4x+3} + 3 > 0\right) \Leftrightarrow \left(\frac{4x^2-14x+12}{x^2-4x+3} > 0\right)$$

$$\Leftrightarrow \left(\frac{2x^2-7x+6}{x^2-4x+3} > 0\right) \Leftrightarrow \left(\frac{2(x-3/2)(x-2)}{(x-1)(x-3)} > 0\right).$$

Multiplying both sides of the last inequality by $(x-1)^2 (x-3)^2$, we get

$$(x-1)(x-3/2)(x-2)(x-3) > 0.$$

Using the method of intervals, we find the solution of the given inequality; $x \in (-\infty, 1) \cup (3/2, 2) \cup (3, +\infty)$. ●

In Problems 65 through 68 solve the indicated inequalities:

65. (1) $(x+2)(x-4)(x-5) < 0$; (2) $(x+1)(x-2)(x-3) > 0$;
(3) $(x+4)(x-1)(x-5) \leqslant 0$.

66. (1) $(x-2)(x-3)(x^2+x+3) \leqslant 0$;
(2) $(x+3)(x+4)(x^2+2x+5) > 0$.

67. (1) $\frac{(x-1)(x-2)(x-3)}{(x+2)(x+1)} \leqslant 0$; (2) $\frac{(x-2)(x-3)(x-4)}{(x+3)(x+2)} > 0$;
(3) $\frac{(x+1)^2(x+2)(x+1)}{(x-1)(x-3)} \leqslant 0$.

68. (1) $\frac{x^2+2x-3}{x^2-2x+8} > 0$; (2) $\frac{x-1}{x-2} > \frac{x-2}{x-1}$;
(3) $\frac{x^2-3x+2}{x-3} > 0$; (4) $\frac{x^2-2x-3}{x^2+2x+8} < 0$.

QUIZ No. 2

Form I

In Problems (1) to (3) solve the given inequalities using the method of intervals:

(1) $(x+2)(x-3)(x-5)>0$;

(2) $\dfrac{(x+3)(x-2)(x-4)}{(x-1)(x+5)} \geqslant 0$;

(3) $\dfrac{3x^2-17x+18}{x^2-5x+4} < 2$.

(4) Given the function $y=1/x^2$. Compute the increment Δy for $x=1$ and $\Delta x=0.1$.

(5) Find the asymptotes to the curve $y=\dfrac{x}{x-2}$.

Form II

(1) $(x+5)(x-4)(x-6)<0$;

(2) $\dfrac{(x-1)(x-3)(x+2)}{(x-4)(x+5)} \leqslant 0$;

(3) $\dfrac{3x^2-14x+14}{x^2-4x+3} > 2$.

(4) Given the function $y=\sqrt{x}$. Compute the increment Δy for $x=1$ and $\Delta x=0.02$.

(5) Find the asymptotes to the curve $y=\dfrac{1}{x+2}$.

Chapter 7
DERIVATIVES

Sec. 7.1. The Rate of Change of a Function

The rate of change of a function $y = f(x)$ is computed according to the following general rule:

I. If the argument x is given an increment Δx, then the function y receives the corresponding increment Δy, that is,

$$y + \Delta y = f(x + \Delta x).$$

II. The increment of the function Δy corresponding to the increment of the argument Δx is found:

$$\begin{array}{r} y+\Delta y = f(x+\Delta x) \\ y = f(x) \\ \hline \Delta y = f(x+\Delta x) - f(x). \end{array}$$

III. The average (or mean) rate of change of the function y for the interval of values of the argument from x to $x + \Delta x$ is expressed by the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The ratio $\frac{\Delta y}{\Delta x}$ indicates the number of units of the increment of a function per unit of the increment of the argument.

IV. The instantaneous (or true) rate of change of the function for a given value of x is the limit to which the average rate $\frac{\Delta y}{\Delta x}$ tends as $\Delta x \to 0$ in the interval of change of the argument from x to $x + \Delta x$, that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

For a linear function $y = kx + b$, the average rate $\frac{\Delta y}{\Delta x} = k$ and the true rate $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = k$ coincide in magnitude, and the numerical value of the true rate is equal to the coefficient k.

1. Find the average rate of change of the function $y = 3x^2 - 6$ as x changes from $x_1 = 3$ to $x_2 = 3.5$.

○ First method. Find the increment in the argument: $\Delta x = x_2 - x_1 = 3.5 - 3 = 0.5$.

Find the values of the function for x_1 and x_2: $y_1 = 3 \cdot 3^2 - 6 = 21$, $y_2 = 3 \cdot (3.5)^2 - 6 = 30.75$.

Compute the increment in the function: $\Delta y = y_2 - y_1 = 30.75 - 21 = 9.75$.

Find the average rate of change of the given function:

$$\frac{\Delta y}{\Delta x} = \frac{9.75}{0.5} = 19.5.$$

Second method. Compute the average rate of change of the function for any value of the argument applying the general rule:

$$\begin{array}{r} y+\Delta y = 3(x+\Delta x)^2 - 6 = 3x^2 + 6x\,\Delta x + 3(\Delta x)^2 - 6 \\ y = 3x^2 - 6 \end{array}$$

$$\Delta y = 6x\,\Delta x + 3(\Delta x)^2;$$

$$\frac{\Delta y}{\Delta x} = \frac{6x\,\Delta x + 3(\Delta x)^2}{\Delta x} = 6x + 3\Delta x.$$

Find the increment in the argument: $\Delta x = x_2 - x_1 = 3.5 - 3 = 0.5$. Compute $\frac{\Delta y}{\Delta x}$ for $x = 3$ and $\Delta x = 0.5$; $\frac{\Delta y}{\Delta x} = 6 \cdot 3 + 3 \cdot 0.5 = 19.5$. ●

2. The motion of a point in a straight line is given by the equation $s = 3t^2 - 2t + 5$ (t is expressed in seconds, and s in metres). Find the velocity of the point at the instant $t = 5$.

○ Find the average velocity of the moving point:

$$\begin{array}{r} s+\Delta s = 3(t+\Delta t)^2 - 2(t+\Delta t) + 5 \\ = 3t^2 + 6t\,\Delta t + 3(\Delta t)^2 - 2t - 2\,\Delta t + 5 \\ s = 3t^2 - 2t + 5 \end{array}$$

$$\Delta s = 6t\,\Delta t + 3(\Delta t)^2 - 2\,\Delta t;$$

$$\frac{\Delta s}{\Delta t} = \frac{6t\,\Delta t + 3(\Delta t)^2 - 2\Delta t}{\Delta t} = 6t + 3\Delta t - 2.$$

Find the true velocity of the moving point at the instant t:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} (6t + 3\Delta t - 2) = 6t - 2.$$

Find the velocity of the moving point after the expiry of the fifth second: $v(5) = 6 \cdot 5 - 2 = 28$ (m/s). ●

3. (1) Find the average rate of change of the function $y = 2x^2 + 5x$ as x changes from $x_1 = 2$ to $x_2 = 3$.

(2) The law of motion of a point is given by the formula $s = 4t^2 - 2$ (t is expressed in seconds, s in metres). Find the average velocity of the moving point during the interval from $t_1 = 4$ to $t_2 = 6$.

4. (1) The motion of a point in a straight line is given by the equation $s = 5t^2$ (t in seconds, s in metres). Find the velocity of the moving point after the expiry of the 10th second.

(2) The motion of a point in a straight line is given by the equation $s = 2t^2 - 8t - 10$ (t in seconds, s in metres). Find the velocity of the point after the expiry of the 8th second.

Sec. 7.2. The Derivative

The *derivative* of a function $f(x)$ at a point x_0 is defined as the limit of the ratio of the increment Δf in the function at this point to the increment Δx in the argument as the latter tends to zero:

$$\lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

A function $f(x)$ having a derivative at each point of a certain interval is said to be *differentiable* in that interval.

For the derivative of a function $y = f(x)$ the following symbols are used: y', y'_x, $\frac{dy}{dx}$ or f', $f'(x)$, $\frac{df(x)}{dx}$. The process of finding the derivative of a function is referred to as the differentiation of the function.

The derivative of a function $y = f(x)$ is computed according to the general differentiation rule:

I. Assigning an increment Δx to the argument x and substituting the new value $x + \Delta x$ for x into the expression of the function, we find the corresponding new value of the function: $y + \Delta y = f(x + \Delta x)$.

II. Subtracting its initial value from the new value of the function, we find the increment in the function: $\Delta y = f(x + \Delta x) - f(x)$.

III. Divide the increment in the function Δy by the increment in the argument Δx, that is, form the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

IV. Find the limit of this ratio for $\Delta x \to 0$, i.e. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ This limit is just the derivative of the function $y = f(x)$.

5. Find: (1) $y'(3)$ if $y = 2x^2 - 3x$; (2) $y'(4)$ if $y = \sqrt{x}$.

○ (1) We find the derivative using the general rule:

$$\begin{aligned} y+\Delta y &= 2(x+\Delta x)^2 - 3(x+\Delta x) \\ &= 2x^2 + 4x\Delta x + 2(\Delta x)^2 - 3x - 3\Delta x \\ y &= 2x^2 - 3x \\ \hline \Delta y &= 4x\Delta x + 2(\Delta x)^2 - 3\Delta x; \end{aligned}$$

$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x - 3$; $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x - 3) = 4x - 3$;

$y' = 4x - 3$.

We then find the value of the derivative for $x = 3$: $y'(3) = 4 \cdot 3 - 3 = 9$.

(2) We find the derivative applying the general rule:

$$\begin{aligned} y+\Delta y &= \sqrt{x+\Delta x} \\ y &= \sqrt{x} \\ \hline \Delta y &= \sqrt{x+\Delta x} - \sqrt{x}; \end{aligned}$$

$$\frac{\Delta y}{\Delta x}=\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x};\quad y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}};\quad y'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}.$$ ●

6. Find: (1) $y'(0)$ if $y=x^2-x$; (2) $y'(1)$ if $y=x^2-5x+4$; (3) $s'(2)$ if $s=t^3$.

7. Find: (1) $y'(3)$ if $y=-3/x$; (2) $y'(-1)$ if $y=1/x^2$.

8. Find: (1) $y'(5)$ if $y=\sqrt{x-1}$; (2) $y'(4)$ if $y=1/\sqrt{x}$; (3) $y'(2\sqrt{2})$ if $y=\sqrt[3]{x}$.

Sec. 7.3. Basic Differentiation Rules. The Derivatives of a Power and a Root

Notation: C—constant; x—argument; u, v, w—functions of x having derivatives.

Basic Differentiation Rules

The derivative of an algebraic sum of functions

$$(u+v-w)'=u'+v'-w'. \tag{7.1}$$

The derivative of a product of two functions

$$(uv)'=u'v+v'u. \tag{7.2}$$

The derivative of a product of three functions

$$(uvw)'=u'vw+v'uw+w'uv. \tag{7.3}$$

The derivative of a product of a constant by a function

$$(Cu)'=Cu'. \tag{7.4}$$

The derivative of a quotient (fraction)

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}. \tag{7.5}$$

Particular cases of Formula (7.5):

$$\left(\frac{u}{C}\right)'=\frac{1}{C}u'; \tag{7.6}$$

$$\left(\frac{C}{v}\right)'=-\frac{C}{v^2}v'. \tag{7.7}$$

If y is a function of u: $y=f(u)$, where u, in turn, is a function of the argument x: $u=\varphi(x)$, that is, if y depends on x via an intermediate argument u, then y is called a *composite function of x (a function of a function)*: $y=f[\varphi(x)]$.

The derivative of a composite function is equal to the product of its derivative with respect to the intermediate argument by the derivative of this argu-

ment with respect to the independent variable:

$$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}, \quad \text{or} \quad y'(x)=y'(u)\cdot u'(x).$$

Proceeding from this relationship, we can obtain the formulas for differentiating composite functions.

When computing derivatives, it is necessary to remember that (by definition)

$$a^0=1\ (a\neq 0);\ a^{-n}=1/a^n\ (a\neq 0);\quad \sqrt[n]{a^m}=a^{m/n}\ (a>0),$$

and to know the following rules of operations with powers and roots:

$$a^n a^m=a^{n+m};\ a^n/a^m=a^{n-m};\ (a^n)^m=a^{nm};$$

$$\sqrt[n]{ab}=\sqrt[n]{a}\,\sqrt[n]{b}=a^{1/n}b^{1/n}\ (a>0,\ b>0);$$

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{a^{1/n}}{b^{1/n}}\ (a>0,\ b>0).$$

Here, m and n are any rational numbers.

Differentiation Formulas

If $u=\varphi(x)$	Formula No.	If $u=x$	Formula No.
		$C'=0$	(7.8)
		$x'=1$	(7.9)
$(u^n)'=nu^{n-1}u'$, where n is any real number	(7.10)	$(x^n)'=nx^{n-1}$, where n is any real number	(7.10a)
$\left(\frac{1}{u}\right)'=-\frac{1}{u^2}u'$	(7.11)	$\left(\frac{1}{x}\right)'=-\frac{1}{x^2}$	(7.11a)
$(\sqrt{u})'=\frac{1}{2\sqrt{u}}u'$	(7.12)	$(\sqrt{x})'=\frac{1}{2\sqrt{x}}$	(7.12a)

In Problems 9 through 14 we shall find the derivatives of the given functions:

9. (1) $y=3x^4$; (2) $y=2x^{-5}$; (3) $y=4x^{1/3}$;
(4) $y=5x^{-2/5}$; (5) $y=5\sqrt[5]{x^3}$.

○ (1) Using Formula (7.4), we factor out the constant multiplier in front of the derivative symbol and then apply Formula (7.10a):

$$y'=3(x^4)'=3\cdot 4x^{4-1}=12x^3.$$

Similarly, using Formulas (7.4) and (7.10a), we get:

(2) $y' = 2 \cdot (-5)\, x^{-5-1} = -10x^{-6} = -\frac{10}{x^6}$;

(3) $y' = 4 \cdot \frac{1}{3}\, x^{1/3-1} = \frac{4}{3}\, x^{-2/3} = \frac{4}{3x^{2/3}} = \frac{4}{3\sqrt[3]{x^2}}$;

(4) $y' = 5 \cdot \left(-\frac{2}{5}\right) x^{-2/5-1} = -2x^{-7/5}$;

(5) $y' = 5 \cdot (x^{3/5})' = 5 \cdot \frac{3}{5}\, x^{3/5-1} = 3x^{-2/5}$. ●

10. (1) $y = \frac{1}{2x^{2/3}}$; (2) $y = 3x^2 \sqrt[3]{x}$; (3) $y = \frac{2x^2}{\sqrt[3]{x}}$.

○ First, transform each of the given functions to the form $y = x^n$, and then take advantage of Formulas (7.4) and (7.10a):

(1) $y = \frac{1}{2x^{2/3}} = \frac{1}{2}\, x^{-2/3}$;

$$y' = \frac{1}{2}\,(x^{-2/3})' = \frac{1}{2} \cdot \left(-\frac{2}{3}\right) x^{-2/3-1} = -\frac{1}{3}\, x^{-5/3}$$

$$= -\frac{1}{3x^{5/3}} = -\frac{1}{3x\sqrt[3]{x^2}};$$

(2) $y = 3x^2 \sqrt[3]{x} = 3x^2 x^{1/3} = 3x^{7/3}$; $y' = 3\,(x^{7/3})' = 3 \cdot \frac{7}{3}\, x^{7/3-1}$

$$= 7x^{4/3} = 7x\sqrt[3]{x};$$

(3) $y = \frac{2x^2}{\sqrt[3]{x}} = \frac{2x^2}{x^{1/3}} = 2x^{5/3}$;

$$y' = 2\,(x^{5/3})' = 2 \cdot \frac{5}{3}\, x^{5/3-1} = \frac{10}{3}\, x^{2/3} = \frac{10}{3}\sqrt[3]{x^2}.$$ ●

11. $f(x) = 1/x^4$; compute $f'(-1)$ and $f'(2)$.

○ We have $f(x) = 1/x^4 = x^{-4}$. Consequently,

$$f'(x) = -4x^{-4-1} = -4x^{-5} = -4/x^5.$$

To compute $f'(-1)$ and $f'(2)$, substitute into the expression of the derivative the values -1 and 2 for x:

$$f'(-1) = -4/(-1)^5 = -4/(-1) = 4;$$

$$f'(2) = -4/2^5 = -4/32 = -1/8.$$ ●

12. $y = 4x^3 - 2x^2 + x - 5$.

○ Applying in succession Formulas (7.1), (7.4), (7.10a), (7.9), and (7.8), we have

$$y' (4x^3)' - (2x^2)' + x' - 5' = 4\,(x^3)' - 2\,(x^2)' + x' - 5'$$

$$= 4 \cdot 3x^2 - 2 \cdot 2x + 1 = 12x^2 - 4x + 1.$$

When the habit of differentiation has been acquired, the intermediate operations can be done mentally, and therefore, when solving such problems, the final result of the differentiation can be written at once. ●

13. $f(x) = (x^3 - 1)(x^2 + x + 1)$.

Using Formulas (7.2), (7.1), (7.10a), (7.8), and (7.9), we find

$$f'(x) = (x^3 - 1)'(x^2 + x + 1) + (x^2 + x + 1)'(x^3 - 1) =$$
$$3x^2(x^2 + x + 1) + (2x + 1)(x^3 - 1) =$$
$$3x^2(x^2 + x + 1) + (2x + 1)(x - 1)(x^2 + x + 1) =$$
$$(x^2 + x + 1)[3x^2 + (2x + 1)(x - 1)] =$$
$$(x^2 + x + 1)(3x^2 + 2x^2 - 2x + x - 1) =$$
$$(x^2 + x + 1)(5x^2 - x - 1). \bullet$$

14. $y = \frac{x^2+1}{x^2-1}$.

○ Using Formulas (7.5), (7.1), (7.10a), and (7.8), we get

$$y' = \frac{(x^2+1)'(x^2-1) - (x^2-1)'(x^2+1)}{(x^2-1)^2} = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2}$$
$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{2x(-2)}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2}. \bullet$$

In Problems 15 through 22 find the derivative of each of the listed functions:

15. (1) $y = x^4$; (2) $y = 2x^3$; (3) $y = 3x^{-5}$; (4) $y = -3x^{-2}$; (5) $y = x^{7/5}$; (6) $y = 4x^{3/2}$; (7) $y = 5x^{-3/5}$; (8) $y = 2\sqrt{x^3}$; (9) $y = \sqrt[4]{x^{-3}}$; (10) $y = \sqrt[3]{x^{-2}}$.

16. (1) $y = -\frac{1}{x^2}$; (2) $y = \frac{3}{x^{4/3}}$; (3) $y = \frac{3}{\sqrt{x^2}}$;

(4) $y = 2x^3\sqrt[3]{x}$; (5) $y = \frac{x^4}{\sqrt{x}}$; (6) $y = \frac{2x^3}{\sqrt{x^3}}$; (7) $y = \frac{2\sqrt{x}}{x^3}$;

(8) $y = \frac{6\sqrt[3]{x}}{\sqrt{x}}$; (9) $y = \sqrt[4]{\frac{1}{x^3}}$; (10) $y = \sqrt[3]{\frac{1}{x^2}}$;

(11) $f(x) = x^{-2}\sqrt{x}\sqrt[3]{x}$; (12) $s = \frac{\sqrt{t^2}\sqrt[3]{t^2}}{t\sqrt{t}}$.

17. (1) $f(x) = 1/x^3$; compute $f'(1/2)$; (2) $f(x) = \sqrt[3]{x^4}$; compute $f'(-8)$; (3) $y = x\sqrt{x}\sqrt[3]{x}$; compute $y'(1)$.

18. (1) $f(x) = -x^3 + 9x^2 + x - 1$; compute $f'(-1)$; (2) $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - 1$; compute $f'(3)$; (3) $f(t) = 0.5\,t^3 + 0.6\,t^2 + 0.8\,t + 8$; compute $f'(1)$.

19. (1) $y = -3x^{-5} + 15x^{-4} - 2x^{-3} + x^{-1} + 2$; (2) $y = 4x^{3/4} + 3x^{2/3} + 4x^{1/2} + 3x$.

20. (1) $y = \sqrt[4]{x^3} - \frac{3}{\sqrt[3]{x}} + \frac{2}{x^2} + \frac{1}{x} + 8$; (2) $y = 2\sqrt{x} + \frac{3}{2\sqrt[3]{x^2}} - \frac{2}{\sqrt{x}} - \frac{1}{x} + 1$.

21. (1) $f(x) = (2x+1)(x^2+3x-1)$; (2) $f(x) = (3x^2+1)(2x^2+3)$; (3) $f(x) = (x^3+x^2+x+1)(x-1)$.

22. (1) $y = \frac{x-a}{x+a}$; (2) $y = \frac{x^2}{2-x^2}$; (3) $y = \frac{x^2-x+1}{x^2+x+1}$; (4) $y = \frac{x^2-x+1}{x^2+1}$.

Sec. 7.4. The Derivative of a Composite Function

In Problems 23 through 27 we shall find the derivative of each function:

23. $y = (x^2 - 5x + 8)^6$.

○ Setting $u = x^2 - 5x + 8$, we get $y = u^6$. By Formula (7.10), we find

$$y' = 6u^5u' = 6(x^2 - 5x + 8)^5 (x^2 - 5x + 8)' = 6(x^2 - 5x + 8)^5 (2x - 5).$$

Such a detailed notation is written only in the process of learning the technique of differentiation. With the practical habit acquired, intermediate computations are carried out mentally. ●

24. $y = \frac{1}{(x^2-1)^4}$.

○ First method. Apply in succession Formulas (7.11) and (7.10):

$$y' = -\frac{1}{[(x^2-1)^4]^2}[(x^2-1)^4]' = -\frac{1}{(x^2-1)^8} \cdot 4(x^2-1)^3 (x^2-1)'$$

$$= -\frac{1}{(x^2-1)^8} \cdot 4(x^2-1)^3 \cdot 2x = -\frac{8x(x^2-1)^3}{(x^2-1)^8} = -\frac{8x}{(x^2-1)^5}.$$

Second method. Introduce a negative exponent and make use of Formula (7.10):

$$y = (x^2-1)^{-4}; \quad y' = -4(x^2-1)^{-4-1}(x^2-1)'$$

$$= -4(x^2-1)^{-5} \cdot 2x = -\frac{8x}{(x^2-1)^5}. \quad ●$$

25. $f(x) = \sqrt{4-x^2}$.

○ Setting $u=4-x^2$, we get $f(x)=\sqrt{u}$. By Formula $(7.12)_2$ we find

$$f'(x)=\frac{1}{2\sqrt{4-x^2}}(4-x^2)'=\frac{-2x}{2\sqrt{4-x^2}}=-\frac{x}{\sqrt{4-x^2}}.$$ ●

26. $y=(x^2+6)\sqrt{x^2-3}$.

○ By the formula for the derivative of a product, we get

$$y'=(x^2+6)'\sqrt{x^2-3}+(\sqrt{x^2-3})'(x^2+6).$$

Find the derivatives in each of the terms and carry out all possible transformations:

$$y'=2x\sqrt{x^2-3}+\frac{2x}{2\sqrt{x^2-3}}(x^2+6)=2x\sqrt{x^2-3}+\frac{x^3+6x}{\sqrt{x^2-3}}$$

$$=\frac{2x(\sqrt{x^2-3})^2+x^3+6x}{\sqrt{x^2-3}}=\frac{2x(x^2-3)+x^3+6x}{\sqrt{x^2-3}}=\frac{2x^3-6x+x^3+6x}{\sqrt{x^2-3}}$$

$$=\frac{3x^3}{\sqrt{x^2-3}}.$$ ●

27. $y=\sqrt[3]{(x^3+1)^2}$.

○ Replace the cube root by a fractional exponent and using Formula (7.10), find the derivative of the power thus obtained:

$$y=\sqrt[3]{(x^3+1)^2}=(x^3+1)^{2/3};\quad y'=\frac{2}{3}(x^3+1)^{-1/3}(x^3+1)'$$

$$=\frac{2}{3}(x^3+1)^{-1/3}\cdot 3x^2=\frac{2x^2}{(x^3+1)^{1/3}}=\frac{2x^2}{\sqrt[3]{x^3+1}}.$$ ●

In Problems 28 through 36 find the derivatives of the indicated functions:

28. $y=(x^3-2x^2+5)^3$; (2) $f(x)=(x^3-1)^6$.

29. (1) $f(x)=(ax^2+bx+c)^n$; (2) $y=(r^2-x^2)^4$.

30. (1) $y=\frac{1}{(1-x^3)^5}$; (2) $y=\frac{1}{(ax+b)^n}$.

31. (1) $y=\frac{(x^4+1)^3}{(x^3+1)^2}$; (2) $y=\left(\frac{a+x}{a-x}\right)^n$.

32. (1) $f(x)=\sqrt{x^2-4x+6}$; (2) $f(t)=\sqrt{t^2-t+1}$; compute $f'(2)$; (3) $y=\sqrt{r^2-x^2}$; (4) $y=\frac{b}{a}\sqrt{a^2-x^2}$; (5) $y=\frac{b}{a}\sqrt{x^2-a^2}$; (6) $y=\sqrt{2px}$.

33. (1) $y=x\sqrt{x^2-1}$; (2) $s=t^2\sqrt{2t-1}$; (3) $s=(t^2+1)\sqrt{t^2-1}$; (4) $y=(2x-1)^2\sqrt{1-2x}$.

34. (1) $y=\frac{1}{\sqrt{ax+b}}$; (2) $y=\frac{1}{\sqrt{x^2+1}}$; (3) $y=\frac{1}{\sqrt{1-x^2}}$;

(4) $y=\frac{1}{\sqrt{x^4-1}}$.

35. (1) $y=\frac{1+2x}{\sqrt{1-2x}}$; (2) $y=\frac{3x}{\sqrt{x^2-1}}$; (3) $y=\frac{x}{\sqrt{x^2+4}}$;

(4) $y=\frac{\sqrt{9+x^2}}{x}$.

36. (1) $y=\sqrt[3]{x^3-1}$; (2) $y=\sqrt[4]{(ax+b)^3}$; (3) $y=\sqrt{(2x-1)^3}$;

(4) $f(t)=\sqrt[3]{t^2+t-1}$; compute $f'(1)$.

Sec. 7.5. Physical Applications of the Derivative

If a point moves in a straight line, the velocity v at a given instant of time $t=t_0$ (called the instantaneous velocity) is defined as the derivative $\frac{ds}{dt}$ of the path s with respect to time t evaluated for $t=t_0$.

The acceleration a at a given instant of time $t=t_0$ is the derivative $\frac{dv}{dt}$ of the velocity v with respect to time t calculated for $t=t_0$.

In the problems considered in this section, the path s is expressed in metres (m), time t in seconds (s), velocity v in metres per second (m/s), and acceleration a in metres per second per second (m/s^2) which is read "metres per second squared".

37. A point moves in a straight line according to the law $s=2t^3+t^2-4$. Find its velocity and acceleration at the instant of time $t=4$.

○ Find the velocity of the point at any instant of time t: $v=\frac{ds}{dt}=6t^2+2t$. Compute the velocity of the point at the instant $t=4$; $v(4)=6\cdot 4^2+2\cdot 4=104$ (m/s).

Find the acceleration of the moving point at any instant of time t: $a=\frac{dv}{dt}=12t+2$. Evaluate the acceleration at the instant of time $t=4$; $a(4)=12\cdot 4+2=50$ (m/s^2). ●

38. A point moves in a straight line as given by the equation $s=6t-t^2$. At what instant of time will the velocity of the point be equal to zero?

○ Determine the velocity of the moving point at any instant of time t: $v=\frac{ds}{dt}=6-2t$. Setting $v=0$, we get $6-2t=0$, whence $t=3$. Thus, the velocity of the point is equal to zero after the expiry of the third second. ●

39. The law of change of the temperature T of a body with time

is given by the equation $T = 0.2t^2$. At what rate does this body get warm at the instant of time $t = 10$?

○ When the body is heated, its temperature T changes with time t, that is, T is a function of time: $T = f(t)$. The rate of heating the body is the derivative of temperature with respect to time: $\frac{dT}{dt} = 0.4t$; $\left(\frac{dT}{dt}\right)_{t=10} = 0.4 \cdot 10 = 4$. Thus, at the instant of time $t = 10$ the body gets warm at a rate of 4 deg/s. ●

40. A body of mass 10 kg moves in a straight line according to the law: $s = 3t^2 + t + 4$. Find the kinetic energy of the body $(mv^2/2)$ four seconds after it started.

○ Find the velocity of the moving body at the instant of time t: $v = \frac{ds}{dt} = 6t + 1$. Evaluate the velocity of the body at the instant of time $t = 4$: $v(4) = 6 \cdot 4 + 1 = 25$ (m/s). Determine the kinetic energy of the body at $t = 4$: $mv^2/2 = 10 \cdot 25^2/2 = 3125$ (J). ●

41. The strength of current I changes with time t according to the law: $I = 0.4t^2$ (I is expressed in amperes, t in seconds). Find the rate of change in the strength of current after the expiry of the 8th second.

○ The rate of change in the strength of current is the derivative of the strength of current with respect to time:

$$\frac{dI}{dt} = 0.8t; \quad \left(\frac{dI}{dt}\right)_{t=8} = 0.8 \cdot 8 = 6.4 \text{ (A/s)}. \quad ●$$

42. Find the velocity and acceleration at the indicated instants of time for a point moving in a straight line if its motion is described by the following equation: (1) $s = t^3 + 5t^2 + 4$, $t = 2$; (2) $s = \sqrt{t}$, $t = 1$; (3) $s = t^2 + 11t + 30$, $t = 3$.

43. Find the acceleration of a moving point at the indicated instants of time if the velocity of the point moving in a straight line is given by the following equation: (1) $v = t^2 + t - 1$, $t = 3$; (2) $v = t^2 + 5t + 1$, $t = 3$.

44. A point moves in a straight line according to the law $s = t^2 - 8t + 4$. At what instant of time will the velocity of the point turn out to be equal to zero?

45. The temperature T of a body changes with time t obeying the law $T = 0.5t^2 - 2t$. At what rate does this body get warm at the instant of time $t = 5$?

46. A body of mass 100 kg moves in a straight line according to the law $s = 5t^2 - 2$. Find the kinetic energy of the body in two seconds after the beginning of motion.

47. The change in the strength of current I with time t is given by the equation $I = 2t^2 - 5t$ (I in amperes, t in seconds). Find the rate of change in the strength of current after the expiry of the 10th second.

Sec. 7.6. The Derivatives of Logarithmic Functions

Differentiation Formulas

If $u = \varphi(x)$	Formula No.	If $u = x$	Formula No.
$(\ln u)' = \frac{1}{u} u'$	(7.13)	$(\ln x)' = \frac{1}{x}$	(7.13a)
$(\log u)' = \frac{0.4343}{u} u'$	(7.14)	$(\log x)' = \frac{0.4343}{x}$	(7.14a)

In Problems 48 and 49 we shall find the derivative of each of the indicated functions:

48. (1) $y = x + \ln x$; (2) $y = 5 \log x$; (3) $y = \ln (ax^2 + b)$.

○ (1) By Formulas (7.1), (7.9), and (7.13a), we get

$$y' = 1 + \frac{1}{x} = \frac{x+1}{x}.$$

(2) We differentiate by Formulas (7.4) and (7.14a):

$$y' = 5 \cdot \frac{0.4343}{x} = \frac{2.1715}{x}.$$

(3) By Formula (7.13), we get

$$y' = \frac{1}{ax^2+b} (ax^2+b)' = \frac{1}{ax^2+b} \cdot 2ax = \frac{2ax}{ax^2+b}. \quad \bullet$$

49. (1) $f(x) = \ln \frac{a-x}{a+x}$; compute $f'(2a)$; (2) $y = \ln \sqrt{2x}$.

○ (1) To simplify finding the derivative, we first take logarithms of the fraction: $f(x) = \ln (a - x) - \ln (a + x)$. Then, by Formulas (7.1), (7.13), (7.8), and (7.9), we get

$$f'(x) = \frac{1}{a-x} (a - x') - \frac{1}{a+x} (a+x)'$$

$$= -\frac{1}{a-x} - \frac{1}{a+x} = \frac{2a}{x^2-a^2}, \quad f'(2a) = \frac{2}{3a}.$$

(2) We take the logarithms of the square root: $y = \frac{1}{2} \ln (2x) = \frac{1}{2} \ln 2 + \frac{1}{2} \ln x$. By Formulas (7.1), (7.8), and (7.13a), we get

$$y' = \frac{1}{2} \cdot \frac{1}{x} = \frac{1}{2x}. \quad \bullet$$

In Problems 50 through 56 find the derivatives of the indicated functions:

50. (1) $f(x)=3\ln x-x^2$; find $f'(1)$; (2) $f(x)=\log x+x^3$; evaluate $f'(-1)$; (3) $y=x^2\ln x$; (4) $y=(1-\ln x)\,x$; (5) $f(z)=z^3-3\ln z$; compute $f'(3)$.

51. (1) $y=\dfrac{\ln x-2}{\ln x}$; (2) $y=\dfrac{\ln x+1}{\ln x}$.

52. (1) $y=\ln 3x$; (2) $y=\ln(2x^2-3)$.

53. (1) $y=\ln\dfrac{x+1}{x-1}$; (2) $y=\ln\dfrac{2}{2+x}$.

54. (1) $y=\log 10x$; (2) $y=\log(2x+1)$.

55. (1) $y=\ln\sqrt{2x-1}$; (2) $y=\ln\sqrt{x^2-a^2}$; (3) $y=\log\sqrt{x^2+4}$; (4) $y=\ln\sqrt{\dfrac{1-x}{1+x}}$.

56. (1) $y=\ln^3 3x$; (2) $y=\ln^2(2x+1)$; (3) $y=\ln^2(x^2-1)$.

Sec. 7.7. The Derivatives of Exponential Functions

Differentiation Formulas

If $u=\varphi(x)$	Formula No.	If $u=x$	Formula No.
$(a^u)'=a^u\ln a\cdot u'$	(7.15)	$(a^x)'=a^x\ln a$	(7.15a)
$(e^u)'=e^u u'$	(7.16)	$(e^x)'=e^x$	(7.16a)

In Problems 57 through 59 we shall find the derivatives of the given functions:

57. $y=2\cdot 5^x+3e^x$.

○ By Formulas (7.1), (7.15a), (7.16a), and (7.4), we get

$$y'=2\cdot 5^x\ln 5+3e^x=2\ln 5\cdot 5^x+3e^x.$$ ●

58. $f(x)=\dfrac{e^x+1}{e^x-1}$; evaluate $f'(-1)$.

○ First method. Applying Formulas (7.5), (7.1), (7.16a), and (7.9), we get

$$f'(x)=\frac{(e^x+1)'(e^x-1)-(e^x-1)'(e^x+1)}{(e^x-1)^2}$$

$$=\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}=\frac{2e^x}{(e^x-1)^2};$$

$$f'(-1)=-\frac{2e^{-1}}{(e^{-1}-1)^2}=-\frac{2e}{(1-e)^2}.$$

Second method. Taking the logarithms of the function, we find the derivative of the logarithm:

$$\ln f(x) = \ln(e^x+1) - \ln(e^x-1);$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{-2e^x}{(e^x+1)(e^x-1)}$$

Consequently,

$$f'(x) = f(x) \cdot \frac{-2e^x}{(e^x+1)(e^x-1)}$$

$$= \frac{e^x+1}{e^x-1} \cdot \frac{-2e^x}{(e^x+1)(e^x-1)} = -\frac{2e^x}{(e^x-1)^2}. \bullet$$

59. (1) $y = 3^{2x^2}$; (2) $y = e^{-2x}$.

○ (1) By Formula (7.15), we get

$$y' = 3^{2x^2} \ln 3 \cdot (2x^2)' = 3^{2x^2} \ln 3 \cdot 4x = 4x \cdot 3^{2x^2} \ln 3.$$

(2) By Formula (7.16), we find

$$y' = e^{-2x}(-2x)' = e^{-2x} \cdot (-2) = -2e^{2x}. \bullet$$

In Problems 60 through 64 find the derivatives of the given functions:

60. $f(x) = \ln x \cdot e^x$; (2) $f(x) = x^2 e^x$; (3) $f(x) = e^x - xe^x$; (4) $y = 3^x e^x$; (5) $y = e^x/2^x$; (6) $f(x) = 5 \ln x + e^x$; evaluate $f'(1)$.

61. (1) $y = \frac{5-e^x}{e^x+2}$; (2) $y = \frac{1-e^x}{e^x}$.

62. (1) $y = 5x^3$; (2) $y = 2^{\sqrt{x}}$; (3) $y = 3^{\ln x}$.

63. (1) $y = e^{-x^2}$; (2) $y = e^{\sqrt{x}}$; (3) $y = e^{\ln x}$.

64. (1) $y = 3(e^{x/3} - e^{-x/3})$; (2) $y = \frac{e^{-x}+e^x}{e^{-x}-e^x}$.

Sec. 7.8. Miscellaneous Problems

In Problems 65 through 71 find the derivative of each of the indicated functions:

65. (1) $y = \frac{x+1}{x-1}$; (2) $f(x) = \frac{x^3+1}{x}$; evaluate $f'(1)$; (3) $y = \frac{x^3+1}{x^3+1}$; (4) $f(u) = \frac{1}{u^2+3x+2}$; evaluate $f'(0)$.

66. (1) $s = \sqrt{t} + \sqrt[3]{t}$; (2) $y = \frac{2}{\sqrt{x}} - \frac{3}{\sqrt[3]{x}}$; (3) $f(x) = \frac{\sqrt{x}+1}{\sqrt{x}}$; evaluate $f'(4)$; (4) $f(x) = \frac{4+\sqrt{x}}{4-\sqrt{x}}$; evaluate $f'(1)$.

67. (1) $f(t) = (t^m + t^n)^3$; evaluate $f'(1)$; (2) $y = \left(\frac{x-1}{x+1}\right)^2$; (3) $y = \frac{3x^3}{(3x-1)^3}$; (4) $f(x) = (x^2-1)^2\sqrt{x^2+1}$; evaluate $f'(\sqrt{3})$.

68. (1) $f(u) = \sqrt{2+\sqrt{2u}}$; compute $f'(2)$; (2) $f(x) = \sqrt{5x^2+2x+1}$; compute $f'(-1)$; (3) $y = \sqrt{\frac{1+ax}{1-ax}}$; (4) $f(z) = \frac{\sqrt{4+z^2}}{z}$; compute $f'(\sqrt{5})$.

69. (1) $f(x) = \frac{x^3}{\sqrt{8+x^3}}$; compute $f'(1)$; (2) $f(x) = \sqrt{\frac{\sqrt{x-1}}{\sqrt{x+1}}}$; evaluate $f'(4)$; (3) $f(x) = \frac{x}{x+\sqrt{1+x^2}}$; evaluate $f'(\sqrt{3})$.

70. (1) $y = \ln\sqrt{\frac{1-4x}{1+4x}}$; (2) $y = \ln(x-\sqrt{1+x^2})$; (3) $y = \ln(x-\sqrt{x^2-1})$; (4) $y = \ln\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$; (5) $y = \ln\sqrt{\frac{1+ax}{1-ax}}$; (6) $y = \ln\sqrt{x}\cdot\ln x^2$.

71. (1) $y = \sqrt{e^{3x}}$; (2) $y = xe^{2x}$.

QUIZ No. 1

Form I

(1) $f(x) = \frac{2}{x} - \frac{8}{\sqrt{x}} + \frac{6}{\sqrt[3]{x^2}} + 2x + 6x^2\sqrt{x}$; find $f'(1)$.

(2) $f(x) = (x^2-2)\sqrt{x^2+1}$; find $f'(\sqrt{3})$.

(3) $f(z) = \frac{9z}{\sqrt{z^2+1}}$; find $f'(2\sqrt{2})$.

(4) $f(x) = e^{2x}\cdot\ln x^2$; find $f'(1)$.

(5) A point moves in a straight line according to the law $s = 2t^3 - 2t^2 - 4$ (s in metres, t in seconds). Find the acceleration of the point after the expiry of the 2nd second.

Form II

(1) $f(x) = \frac{1}{x^2} + \frac{3}{2\sqrt[3]{x^2}} - \frac{4}{\sqrt{x}} + 3x - 2x^2\sqrt{x}$; find $f'(1)$.

(2) $f(u) = (u^2+3)\sqrt{u^2-1}$; find $f'(\sqrt{2})$.

(3) $f(x) = \frac{x}{1-\sqrt{x^2+1}}$; find $f'(\sqrt{3})$.

(4) $f(x) = \sqrt{e^x}\ln x^2$; find $f'(1)$.

(5) A point moves in a straight line according to the law $s = 2t^3 - 3t^2 + 4$ (s in metres, t in seconds). Find the acceleration of the point after the expiry of the 3rd second.

Chapter 8

APPLICATIONS OF THE DERIVATIVE TO INVESTIGATING FUNCTIONS

Sec. 8.1. Increase and Decrease of a Function

A function $y = f(x)$ is said to be *increasing* in the interval (a, b) if for any x_1 and x_2 belonging to this interval and such that $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds.

A function $y = f(x)$ is said to be *decreasing* in the interval (a, b) if for any x_1 and x_2 belonging to this interval and such that $x_1 < x_2$ the inequality $f(x_1) > f(x_2)$ holds.

Both increasing and decreasing functions are called *monotone* functions, and intervals in which a function increases or decreases are called *intervals of monotonicity* of the function.

An increase and a decrease of a function $y = f(x)$ is characterized by the sign of its derivative: *if in some interval* $f'(x) > 0$, *then the function increases in this interval; and if* $f'(x) < 0$, *then the function decreases in this interval.*

In Problems 1 and 2 we shall find the intervals of monotonicity of the indicated functions:

1. (1) $f(x) = x^2 - 8x + 12$; (2) $f(x) = x^3 - 6x^2 + 4$.

○ (1) Find the derivative: $f'(x) = 2x - 8$; we have $(2x - 8 = 0) \Leftrightarrow (x = 4)$. Further reasoning is represented in the table:

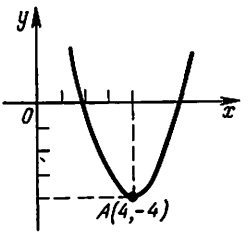


Fig. 23

x	$(-\infty, 4)$	4	$(4, +\infty)$
$f'(x)$	−	0	+
$f(x)$	↘		↗

Thus, the given function decreases in the interval $(-\infty, 4)$ and increases in the interval $(4, +\infty)$ (Fig. 23).

(2) We have $f'(x) = 3x^2 - 12x$, $(3x^2 - 12x = 0) \Leftrightarrow \left[\begin{array}{l} x=0, \\ x=4. \end{array}\right.$ Let us compile a table:

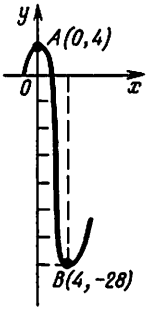


Fig. 24

x	$(-\infty, 0)$	0	$(0, 4)$	4	$(4, +\infty)$
$f'(x)$	+	0	−	0	+
$f(x)$	↗		↘		↗

Thus, the given function increases in the intervals $(-\infty, 0)$ and $(4, +\infty)$, and decreases in the interval $(0, 4)$ (Fig. 24). ●

2. (1) $y = \frac{1}{2x}$; (2) $y = \ln x$; (3) $y = \sqrt{x - x^2}$.

○ (1) The domain of definition of the given function is the entire number line, except for the point $x = 0$. We find $y' = -1/(2x^2)$. It is obvious that $y' < 0$ for all x's from the domain of definition of the function, that is, the function $y = 1/(2x)$ decreases in the intervals $(-\infty, 0)$ and $(0, +\infty)$ (Fig. 25).

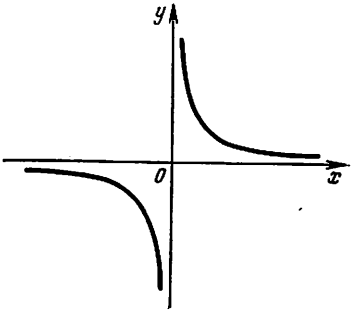


Fig. 25

(2) The domain of definition of the given function is the interval $(0, +\infty)$. It is obvious that the derivative $y' = 1/x$ is positive in this interval. Consequently, the function $y = \ln x$ increases in the interval $(0, +\infty)$.

(3) To find the domain of definition of the given function, we solve the inequality $x - x^2 \geqslant 0$, whence we get $0 \leqslant x \leqslant 1$. Thus, the given function is defined in the interval $[0, 1]$.

Find the derivative $y' = \frac{1-2x}{2\sqrt{x-x^2}}$. Since the denominator of the fraction is positive, the sign of this fraction coincides with the sign of its numerator. Taking into consideration that the function is defined for $0 \leqslant x \leqslant 1$, we get: $y' > 0$ for $1 - 2x > 0$, that is, for $0 < x < 1/2$; $y' < 0$ for $1 - 2x < 0$, that is, for $1/2 < x < 1$. Consequently, the given function increases in the interval $[0, 1/2)$ and decreases in the interval $(1/2, 1)$. ●

In Problems 3 through 11 find the intervals of monotonicity of the given functions:

3. (1) $f(x) = x^2 - 6x + 5$; (2) $f(x) = 2x^2 - 4x + 5$; (3) $f(x) = -x^2 + 4x + 1$.

4. (1) $f(x) = x^3 - 3x^2 + 1$; (2) $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2$.

5. (1) $f(x) = x^4 - 4x + 3$; (2) $f(x) = x^4 - 32x + 40$; (3) $f(x) = -\frac{1}{4}x^4 - x + 1$.

6. (1) $f(x) = 2x^3 - 9x^2 + 12x - 15$; (2) $f(x) = -2x^3 + 15x^2 - 36x + 20$.

7. (1) $f(x) = -\frac{1}{x}$; (2) $f(x) = \frac{1}{3x-2}$.

8. (1) $y = \ln x^2$; (2) $y = \ln \frac{1}{x}$.

9. (1) $y=\frac{1}{2}x^2-\ln x$; (2) $y=\ln x-\frac{1}{3}x^3$.

10. (1) $y=e^{-x}$; (2) $y=e^{x^2}$; (3) $y=e^{1/x}$.

11. (1) $y=\sqrt{x-2x^2}$; (2) $y=\sqrt{x^2-3x}$.

Sec. 8.2. Investigating a Function for an Extremum with the Aid of Its First Derivative

A point x_0 from the domain of definition of a function $f(x)$ is called a *point of minimum* of this function if there exists a δ-neighbourhood $(x_0-\delta, x_0+\delta)$ of the point x_0 such that for all $x \neq x_0$ from this neighbourhood the inequality $f(x) \geqslant f(x_0)$ is fulfilled.

A point x_0 from the domain of definition of a function $f(x)$ is called a *point of maximum* if there exists a δ-neighbourhood $(x_0-\delta, x_0+\delta)$ of the point x_0 such that for all $x \neq x_0$ from this neighbourhood the inequality $f(x) \leqslant f(x_0)$ is fulfilled.

The points of minimum and maximum are called *extreme points* (or *points of extremum*) of a given function, and the values of the function at these points are called a *minimum* or a *maximum* (or, generally, *extrema*) of the function.

Points of extremum may be represented only by *stationary* points, that is, by the points belonging to the domain of definition of a function at which the derivative $f'(x)$ vanishes or suffers a discontinuity, that is, does not exist.

If, as x passes through the stationary point x_0, the derivative $f'(x)$ changes sign, then the function $f(x)$ has an extremum at the point x_0: a minimum if the derivative changes sign from minus to plus, and a maximum if it changes sign from plus to minus. And if the derivative $f'(x)$ does not change sign, as x passes through the stationary point x_0, then the function $f(x)$ has no extremum at the point x_0.

The Rule for Finding Extrema of a Function $y=f(x)$ with the Aid of the First Derivative

I. Find the derivative $f'(x)$.

II. Find the stationary points of the function $y=f(x)$, that is, the points at which the derivative is equal to zero or does not exist.

III. Investigate the sign of the derivative $f'(x)$ in the subintervals into which the domain of definition of the function $f(x)$ is divided by means of the found stationary points. The stationary point x_0 is a point of minimum if it separates the subinterval in which $f'(x)<0$ from the subinterval in which $f'(x)>0$, otherwise it is a point of maximum. And if the sign of the derivative does not change in the adjoining subintervals separated by the stationary point x_0, then the function has no extremum at this point.

IV. Evaluate the function at the points of extremum.

In Problems 12 through 14 we shall investigate the given functions for extremum:

12. (1) $f(x)=x^2-4x$; (2) $f(x)=-x^2+5x+6$.

○ (1) We find $f'(x) = 2x - 4$. Setting $f'(x) = 0$, we get the only stationary point $x = 2$. Further reasoning is represented by the table

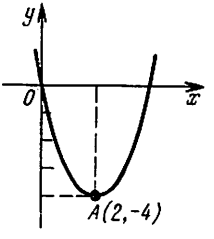


Fig. 26

x	$(-\infty, 2)$	2	$(2, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	↘	Minimum $f_{\min} = f(2) = -4$	↗

The graph of the function $f(x) = x^2 - 4x$ is a parabola shown in Fig. 26. The point of minimum $(2; -4)$ is the vertex of the parabola.

(2) We find $f'(x) = -2x + 5$; $(-2x + 5 = 0) \Leftrightarrow (x = 5/2)$ and then compile a table:

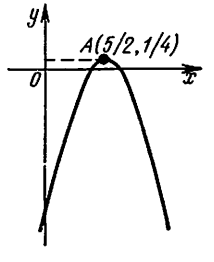


Fig. 27

x	$(-\infty, 5/2)$	5/2	$(5/2, +\infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	↗	Maximum $f_{\max} = f(5/2) = 1/4$	↘

The graph of the function $f(x) = -x^2 + 5x - 6$ is represented by the parabola shown in Fig. 27. ●

13. (1) $f(x) = \frac{1}{2}x^4$; (2) $f(x) = x^3 - 3x^2$.

○ (1) Find $f'(x) = 2x^3$: $(2x^3 = 0) \Leftrightarrow (x = 0)$. Compile a table:

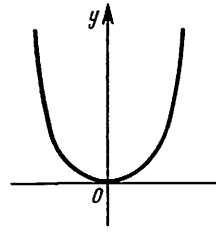


Fig. 28

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	↗	Minimum $f_{\min} = f(0) = 0$	↗

The graph of the function $f(x) = \frac{1}{2}x^4$ is shown in Fig. 28.

(2) We have $f'(x) = 3x^2 - 6x$; $(3x^2 - 6x = 0) \Leftrightarrow \left[\begin{array}{l} x=0, \\ x=2. \end{array}\right.$ Compile a table:

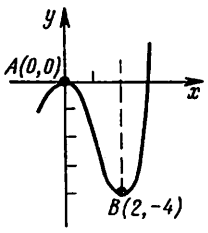


Fig. 29

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
$f'(x)$	+	0	−	0	+
$f(x)$	↗	Maximum $f_{\max} = f(0) = 0$	↘	Minimum $f_{\min} = f(2) = -4$	↗

The graph of the function $f(x) = x^3 - 3x^2$ is represented in Fig. 29.

14. $f(x) = \sqrt[3]{x^2}(x-5)$.

⊙ We find $f'(x) = \frac{2}{3\sqrt[3]{x}}(x-5) + \sqrt[3]{x^2} = \frac{2(x-5)+3x}{3\sqrt[3]{x}} = \frac{5}{3} \cdot \frac{x-2}{\sqrt[3]{x}}$.

In this case, we have two stationary points: $x = 0$ (there is no derivative at this point) and $x = 2$ (the derivative is equal to zero at this point). We then compile the following table:

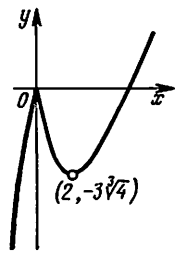


Fig. 30

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
$f'(x)$	+	Non-existent	−	0	+
$f(x)$	↗	Maximum $f_{\max} = f(0) = 0$	↘	Minimum $f_{\min} = f(2) = -3\sqrt[3]{4} \approx -4.8$	↖

The graph of the function $f(x) = \sqrt[3]{x^2}(x-5)$ is represented in Fig. 30. ●

In Problems 15 through 25 investigate the given functions for extremum:

15. (1) $f(x) = x^2 - x$; (2) $f(x) = x^2 + 3x$.

16. (1) $f(x) = -x^2 + 2x$; (2) $f(x) = -x^2 - x$.

17. (1) $f(x) = x^2 - 8x + 12$; (2) $f(x) = x^2 - 4x + 3$; (3) $f(x) = x^2 - 10x + 9$.

18. (1) $f(x) = -x^2 + 2x + 3$; (2) $f(x) = -x^2 - x + 6$; (3) $f(x) = -2x^2 + x + 1$.

19. (1) $f(x) = 2x^4 - x$; (2) $f(x) = -\frac{1}{4}x^4 + 8x$.

20. (1) $f(x) = \frac{1}{3}x^3 - 4x$; (2) $f(x) = \frac{1}{3}x^3 - x^2$.

21. (1) $f(x) = 2x^3 - 9x^2 + 12x - 8$; (2) $f(x) = 2x^3 - 3x^2 - 12x + 8$; (3) $f(x) = 2x^3 + 9x^2 + 12x - 2$.

22. (1) $f(x) = 5 - 2\sqrt[3]{x^2}$; (2) $f(x) = 3\sqrt[3]{x^2} - x$.

23. (1) $f(x) = 6\sqrt[3]{x^2}(x + 1)$; (2) $f(x) = \sqrt[3]{x^2}(10 - x)$.

24. (1) $f(x) = e^x + e^{-x}$; (2) $f(x) = x^2e^{-x}$.

25. (1) $f(x) = x - 2\ln x$; (2) $f(x) = x\ln x$.

Sec. 8.3. Investigating a Function for Extremum with the Aid of the Second Derivative

If y' is the derivative of the function $y = f(x)$, then the derivative of y' with respect to x (provided that it exists) is called the *second derivative* (or a *derivative of the second order*). The following symbols are used to denote the second derivative: y'', y''_x, $\frac{d^2y}{dx^2}$ or $f''(x)$, $\frac{d^2f(x)}{dx^2}$.

The Rule for Finding the Extrema of a Function $y = f(x)$ with the Aid of the Second Derivative

I. Find the derivative $f'(x)$.

II. Find the stationary points of the given function at which $f'(x) = 0$.

III. Find the second derivative $f''(x)$.

IV. Investigate the sign of the second derivative at each of the stationary points. If the second derivative turns out to be negative, then at such a point the function has a maximum, and if it turns out to be positive, then it has a minimum. If the second derivative is equal to zero, then the point of extremum should be found with the aid of the first derivative.

V. Compute the values of the function at the points of extremum.

26. Investigate the following functions for extremum with the aid of the second derivative:

(1) $f(x) = x^2 - 2x - 3$; (2) $f(x) = x^3 - 9x^2 + 24x = 12$.

○ (1) We find the first derivative: $f'(x) = 2x - 2$. Solving the equation $f'(x) = 0$, we obtain a stationary point $x = 1$. We now find the second derivative; $f''(x) = 2$. Since the second derivative is positive at a stationary point, for $x = 1$ the function has a minimum: $f_{\min} = f(1) = -4$.

(2) We find $f'(x) = 3x^2 - 18x + 24$; $(3x^2 - 18x + 24 = 0) \Leftrightarrow (x^2 - 6x + 8 = 0) \Leftrightarrow \left[\begin{array}{l} x = 2, \\ x = 4. \end{array}\right.$ Let us now find the second de-

rivative $f''(x) = 6x - 18$ and determine its sign at the stationary points. Since $f''(2) = 6 \cdot 2 - 18 < 0$, for $x = 2$ the function has a maximum; since $f''(4) = 6 \cdot 4 - 18 > 0$, for $x = 4$ the function has a minimum. We finally evaluate the function at the points of extremum: $f_{max} = f(2) = 2^3 - 9 \cdot 2^2 + 24 \cdot 2 - 12 = 8$, $f_{min} = f(4) = 4^3 - 9 \cdot 4^2 + 24 \cdot 4 - 12 = 4$. ●

In Problems 27 through 29 investigate the given functions for extremum by means of the second derivative:

27. (1) $f(x) = 2x^2 - 3$; (2) $f(x) = x^2 - 2x$; (3) $f(x) = 2x^2 - 5x + 2$; (4) $f(x) = -x^2 + 4x$; (5) $f(x) = -x^2 + x + 6$.

28. (1) $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 4$; (2) $f(x) = \frac{1}{3}x^3 - 3x^2 + 5x + 5$; (3) $f(x) = x^3 - \frac{9}{2}x^2 + 6x - 2$; (4) $f(x) = x^4 + 3x^2 - 4$.

29. (1) $f(x) = \frac{x^2+1}{x}$; (2) $f(x) = \frac{3x}{x^2+1}$.

Sec. 8.4. The Largest and Smallest Values of a Function

To find the smallest and the largest value of a function continuous in some interval, it is necessary:

(1) to find the stationary points belonging to the given interval and to compute the values of the function at these points;

(2) to evaluate the function at the end points of the interval;

(3) to compare the obtained values; then the smallest (or the least) and the largest of them are just the smallest and the largest value, respectively, of the function in the interval under consideration.

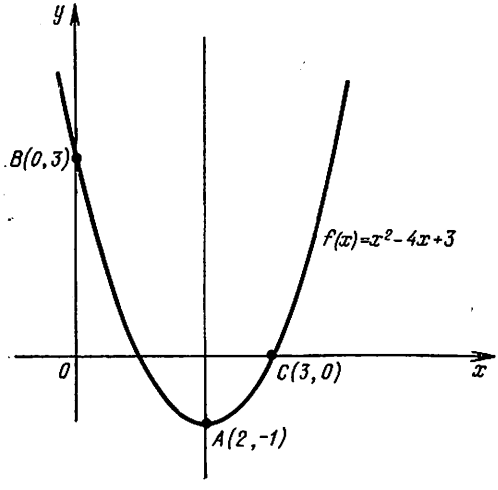


Fig. 31

30. Find the smallest and largest values of the function $f(x) = x^2 - 4x + 3$ in the interval $[0, 3]$.

○ We have $f'(x) = 2x - 4$; $2x - 4 = 0$, that is, $x = 2$ is a stationary point. We find $f(2) = -1$ and then compute the values of the function at the end points of the given interval: $f(0) = 3$, $f(3) = 0$.

Thus, the smallest value of the function is equal to -1 and is attained by the function at an inner point of the interval considered,

and the largest value of the function is equal to 3 and is taken on at the left-hand endpoint of the interval (Fig. 31.) ●

In Problems 31 through 33 find the smallest and largest values of the given functions in the indicated intervals:

31. (1) $f(x) = x^2 - 6x + 13$, $x \in [0, 6]$; (2) $f(x) = 8 - 0.5x^2$, $x \in [-2, 2]$.

32. (1) $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$; $x \in [1, 3]$; (2) $f(x) = 6x^2 - x^3$, $x \in [-1, 6]$.

33. (1) $f(x) = x^3 - 3x^2 - 9x + 35$, $x \in [-4, 4]$; (2) $f(x) = -x^3 + 9x^2 - 24x + 10$, $x \in [0, 3]$.

Sec. 8.5. Problems on Finding the Smallest and Largest Values of Quantities

34. Of all rectangles of a given perimeter, find the one having the largest area.

○ Let the perimeter of a rectangle be equal to p. We denote the length of one of the sides of the rectangle by x, then the length of the other side will be equal to $\frac{p-2x}{p} = \frac{p}{2} - x$. Denoting the area of the rectangle by y, we have

$$y = x\left(\frac{p}{2} - x\right) = \frac{p}{2}x - x^2 \left(0 < x < \frac{p}{2}\right).$$

We now investigate the function for maximum and minimum with the aid of the second derivative:

$$y' = \frac{p}{2} - 2x; \quad \frac{p}{2} - 2x = 0; \quad x = \frac{p}{4}; \quad y'' = -2.$$

The second derivative is negative, consequently, the function has a maximum for $x = p/4$. Thus, of all the rectangles of a given perimeter, a square has the largest area. ●

35. At what height h should a lamp be hung up above the centre of a circular ground of radius a to provide maximum illumination of its boundary?

○ It is known from physics that illumination E is inversely proportional to the square of the distance from a light source and directly proportional to the cosine of the angle of incidence (the angle formed by the normal to the surface and the direction of light flux), that is,

$$E = k \cdot \frac{\cos \alpha}{r^2},$$

where k depends on the intensity of the light source placed at the point A (Fig. 32). From the triangle OAB, we have $\cos\alpha = h/r$, and $r = \sqrt{h^2 + a^2}$. Taking h for the independent variable, we get

$$E = k \cdot \frac{h}{\sqrt{h^2+a^2}\,(h^2+a^2)} = k \cdot \frac{h}{(h^2+a^2)^{3/2}} \quad (h > 0).$$

We then investigate the function for extremum with the aid of the first derivative:

$$E' = k \cdot \frac{(h^2+a^2)^{3/2} - \frac{3}{2}(h^2+a^2)^{1/2} \cdot 2h \cdot h}{(h^2+a^2)^3} = k \cdot \frac{(h^2+a^2)^{1/2}(h^2+a^2-3h^2)}{(h^2+a^2)^3}$$

$$= k \cdot \frac{a^2-2h^2}{(h^2+a^2)^{5/2}} = k \cdot \frac{\left(\frac{a}{\sqrt{2}}-h\right)\left(\frac{a}{\sqrt{2}}+h\right)}{(h^2+a^2)^{5/2}}; \quad E' = 0 \text{ for } h = \frac{a}{\sqrt{2}}.$$

Since $E' > 0$ in the interval $(0, a/\sqrt{2})$ and $E' < 0$ in the interval $(a/\sqrt{2}, +\infty)$, for $h = a/\sqrt{2}$ the function has a maximum, that is, for the value $h = a/\sqrt{2} \approx 0.7a$ the illumination at the point B is the greatest. ●

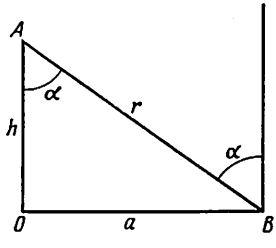


Fig. 32

36. The motion of a body travelling in a straight line is described by the equation $s = -t^3 + 9t^2 - 24t - 8$. Find the maximum velocity of the moving body (s in metres, t in seconds).

○ The velocity of the body is the first derivative of the path with respect to time: $v = s' = -3t^2 + 18t - 24$. We then investigate this function for a maximum and a minimum with the aid of the second derivative:

$$v'' = -6t + 18; \quad -6t + 18 = 0; \quad t = 3; \quad v'' = -6.$$

The second derivative is negative, consequently, the velocity is the greatest for $t = 3$. We now find the value of the velocity at the instant of time $t = 3$:

$$v(3) = -3 \cdot 3^2 + 18 \cdot 3 - 24 = 3 \text{ (m/s)}. \quad ●$$

37. The sum of two positive numbers is equal to a. What are these numbers if the sum of their cubes is the smallest?

38. The product of two positive numbers is equal to a. What are the numbers if their sum is the smallest?

39. A rectangle can be bent from a piece of wire 50 cm long. What must this rectangle be to have the greatest area?

40. Of all the rectangles of a given perimeter $2p$ find the one having the smallest diagonal.

41. Of all the rectangles inscribed in a circle of radius R find the one having the greatest area.

42. In a semicircle of radius R inscribe a rectangle having the greatest area.

43. In a semicircle of radius R inscribe a rectangle having the greatest perimeter.

44. Of all the triangles in which the sum of the base and altitude is equal to a, find the one having the greatest area.

45. An isosceles triangle is inscribed in a circle of radius a. For what proportion of the sides will the triangle have the greatest area?

46. In a triangle with base a and altitude h a rectangle having the greatest possible area is inscribed (the base of the inscribed rectangle lies on the base of the triangle). Find the sides of the rectangle.

47. In a right triangle whose legs are a and b, a rectangle having the greatest area is inscribed so that one of its sides lies on the hypotenuse. Find the sides of the rectangle.

48. In an equilateral triangle with perimeter $3m$, a rectangle having the greatest possible area is inscribed. Find the sides of the rectangle.

49. The motion of a body travelling in a straight line is described by the equations $s = -t^3 + 3t^2 + 9t + 3$. Find the maximum velocity of the moving body (s is expressed in metres, t in seconds).

50. The motion of a body thrown vertically upward is given by the equation $s = v_0 t - 0.5gt^2$. Find the greatest height reached by the body.

51. The motion of a body thrown vertically upward is given by the equation $s = 19.6t - 4.9t^2$. Find the greatest height reached by the body (s in metres, t in seconds).

Sec. 8.6. Direction of Convexity of the Graph of a Function

A curve $y = f(x)$ is said to be *convex downward* in the interval (a, b) if it lies above the tangent at any point of this interval (Fig. 33(a)).

A curve $y = f(x)$ is said to be *convex upward* in the interval (a, b) if it lies below the tangent at any point of this interval (Fig. 33(b)).

Intervals in which the graph of a function is convex either upward or downward are called *intervals of convexity of the graph of a function.*

The direction of convexity of the curve which is the graph of the function $y = f(x)$ is characterized by the sign of its second derivative; *if in some interval* $f''(x) > 0$, *then the curve is convex downward in this interval; and if* $f''(x) < 0$, *then the curve is convex upward in this interval.*

52. Test the curve $f(x) = 1/x$ for the direction of convexity at the points $x_1 = -2$ and $x_2 = 1$.

○ We find $f'(x) = -1/x^2$, $f''(x) = 2/x^3$. Substituting the values $x_1 = -2$ and $x_2 = 1$ into the expression for the second derivative, we get: $f''(-2) = 2/(-2)^3 < 0$, $f''(1) = 2/1 > 0$. Thus, at the point $x = -2$ the curve is convex upward and at the point $x = 1$ it is convex downward. ●

53. Find the intervals of convexity of the following curves: (1) $f(x) = x^3$; (2) $f(x) = x^4 - 2x^3 + 6x - 4$.

○ (1) First of all, we find $f'(x) = 3x^2$, $f''(x) = 6x$. In the interval $(-\infty, 0)$ we have $f''(x) < 0$, that is, in this interval the curve

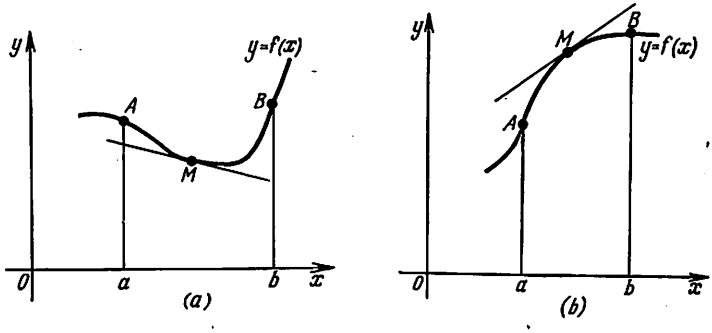


Fig. 33

is convex upward, in the interval $(0, +\infty)$, we have $f''(x) > 0$, that is, in this interval the curve is convex downward (Fig. 34).

(2) We find $f'(x) = 4x^3 - 6x^2 + 6$, $f''(x) = 12x^2 - 12x = 12x(x - 1)$. It is obvious that in the intervals $(-\infty, 0)$ and $(1, +\infty)$ the inequality $f''(x) > 0$ is fulfilled, that is, in these intervals the curve is convex downward, while in the interval $(0, 1)$ the inequality $f''(x) < 0$ occurs, that is, in this interval the curve is convex upward. ●

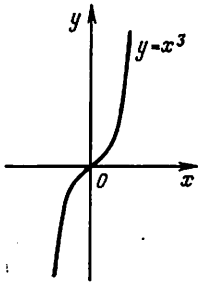


Fig. 34

54. Test the following curves for convexity at the indicated points: (1) $y = -1/x$ at the points $x_1 = -1$ and $x_2 = 1$; (2) $y = 1/x^2$ at the points $x_1 = -2$ and $x_2 = 1$.

In Problems 55 and 56 find the intervals of convexity for each of the given curves:

55. (1) $y = 2x^3$; (2) $y = x^2$; (3) $y = -x^2 - 1$; (4) $y = x^2 + 3x - 1$.

56. (1) $y = x^3 - 6x^2 + 2x - 6$; (2) (2) $y = x^4 - 2x^3 - 12x^2 + 24x + 8$.

Sec. 8.7. Points of Inflection

The point of the graph of a function $y = f(x)$ separating the intervals of convexity of opposite directions of this graph is called a *point of inflection*.

Points of inflection can be represented only by the *stationary points* belonging to the domain of definition of the function $y = f(x)$ at which the second derivative $f''(x)$ is equal to zero (vanishes) or does not exist (suffers a discontinuity).

If, when passing through a stationary point x_0, the second derivative changes sign, then the graph of the function has a point of inflection $(x_0, f(x_0))$.

The Rule for Finding Inflection Points of the Graph of a Function $y = f(x)$

I. Find the second derivative $f''(x)$.

II. Find the stationary points of the function $y = f(x)$ at which $f''(x)$ is equal to zero or does not exist.

III. Investigate the sign of the second derivative $f''(x)$ in the subintervals into which the domain of definition of the function $f(x)$ is divided by found stationary points. And if the stationary point x_0 separates the intervals of convexity of opposite directions, then x_0 is the abscissa of the point of inflection.

IV. Evaluate the function at the inflection points.

57. Find the points of inflection of the following curves:

(1) $f(x) = 6x^2 - x^3$; (2) $f(x) = x + \sqrt[3]{x^5} - 2$.

○ (1) We find $f'(x) = 12x - 3x^2$, $f''(x) = 12 - 6x$. Setting $f''(x) = 0$, we obtain the only stationary point $x = 2$. Since in the interval $(-\infty, 2)$ we have $f''(x) > 0$ and in the interval $(2, +\infty)$ we have $f''(x) < 0$, for $x = 2$ the curve has a point of inflection. We now find the ordinate of this point: $f(2) = 16$. Thus, $(2, 16)$ is a point of inflection.

(2) We find $f'(x) = 1 + \frac{5}{3}\sqrt[3]{x^2}$, $f''(x) = \frac{10}{9} \cdot \frac{1}{\sqrt[3]{x}}$. Here, $x = 0$ is a stationary point. At this point the second derivative suffers a discontinuity (is nonexistent). It is obvious that $f''(x) < 0$ in the interval $(-\infty, 0)$ and $f''(x) > 0$ in the interval $(0, +\infty)$, that is, for $x = 0$ the curve has a point of inflection $(0, -2)$. ●

In Problems 58 through 60 find the points of inflection of the indicated curves:

58. (1) $f(x) = x^3 - x$; (2) $y = \frac{1}{3}x^3 - 3x^2 + 8x - 4$.

59. (1) $f(x) = x^4 - 10x^3 + 36x^2 - 100$;
(2) $f(x) = x^4 - 8x^3 + 18x^2 - 48x + 31$;
(3) $f(x) = x^4 - 6x^3 + 12x^2 - 10$.

60. (1) $f(x) = xe^{-x}$; (2) $f(x) = e^{-x^2}$.

Sec. 8.8. Constructing the Graphs of Functions

The General Scheme for Constructing Graphs of Functions

I. Find the domain of definition of a given function.

II. Find out whether the function is even, odd, or periodic*.

III. Find the points at which the graph intersects the coordinate axes (if it involves no difficulties).

IV. Find the asymptotes to the graph of the function.

V. Find the intervals of monotonicity of the function and its extrema.

VI. Find the intervals of convexity of the graph and the points of inflection.

VII. Construct the desired graph using the results obtained.

* See Sections 9.4 and 9.6.

61. Construct the graph of the function $y = x^3 - 6x^2 + 9x - 3$.

⊙ 1. The function is defined throughout the number line, that is, $D(y) = \mathbf{R}$.

2. The given function is neither even nor odd; besides, it is not periodic.

3. We find the point of intersection of the graph with the y-axis; setting $x = 0$, we get $y = -3$. In this case, it is difficult to find the points at which the graph intersects the x-axis.

4. Obviously, the graph of the given function has no asymptotes.

5. We then find the derivative: $y' = 3x^2 - 12x + 9$. Further, we have $(3x^2 - 12x + 9 = 0) \Leftrightarrow (x^2 - 4x + 3 = 0) \Leftrightarrow \left[\begin{array}{l} x=1, \\ x=3. \end{array}\right.$ The points $x = 1$ and $x = 3$ divide the domain of definition of the given function into three subintervals: $(-\infty, 1)$ $(1, 3)$, and $(3, +\infty)$. In the subintervals $(-\infty, 1)$ and $(3, +\infty)$ we have $y' > 0$, that is, the function increases, while in the subinterval $(1, 3)$ the first derivative $y' < 0$, that is, the function decreases. When passing through the point $x = 1$, the derivative changes sign from plus to minus, and when passing through the point $x = 3$ from minus to plus. Hence, $y_{\max} = y(1) = 1$, $y_{\min} = y(3) = -3$.

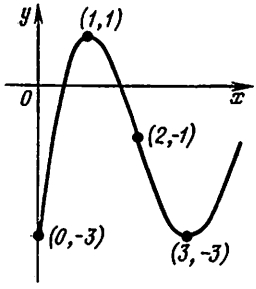


Fig. 35

6. We now find the second derivative: $y'' = 6x - 12$; $6x - 12 = 0$, $x = 2$. The point $x = 2$ separates the domain of definition of the given function into two subintervals $(-\infty, 2)$ and $(2, +\infty)$. In the first of them $y'' < 0$, and in the second $y'' > 0$, that is, in the subinterval $(-\infty, 2)$ the curve is convex upward, while in the subinterval $(2, +\infty)$ it is convex downward. Thus, we get a point of inflection: $(2, -1)$.

7. Using the data thus obtained, we construct the desired graph (Fig. 35). ●

62. Construct the graph of the function $y = \dfrac{x^2}{x-3}$.

○ 1. Find the domain of definition of the given function: $D(y) = (-\infty, 3) \cup (3, +\infty)$.

2. The given function is neither even nor odd nor periodic.

3. For $x = 0$ we get $y = 0$, that is, the graph passes through the origin.

4. Since $\lim\limits_{x \to 3 \pm 0} f(x) = \pm\infty$, the straight line $x = 3$ serves as a vertical asymptote to the graph.

Further, we find:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^2}{x(x-3)} = 1,$$

$$b=\lim_{x\to\pm\infty}[f(x)-kx]=\lim_{x\to\pm\infty}\left[\frac{x^2}{x-3}-x\right]=\lim_{x\to\pm\infty}\frac{3x}{x-3}=3.$$

Consequently, the straight line $y = x + 3$ is an oblique asymptote to the graph.

5. We find

$$y'=\frac{2x(x-3)-x^2}{(x-3)^2}=\frac{x^2-6x}{(x-3)^2}=\frac{x(x-6)}{(x-3)^2}.$$

The derivative y' vanishes at the points $x = 0$ and $x = 6$ and suffers a discontinuity (i.e. does not exist) for $x = 3$. These points divide the number line into four subintervals: $(-\infty, 0)$, $(0, 3)$, $(3, 6)$, and $(6, +\infty)$. Let us investigate the sign of y' in each of them; obviously, $y' > 0$ in the subintervals $(-\infty, 0)$ and $(6, +\infty)$ (in these subintervals the function increases) and $y' < 0$ in the subintervals $(0, 3)$ and $(3, 6)$ (in these subintervals the function decreases). When passing through the point $x = 0$, the derivative changes sign from plus to minus, that is, this is a point of maximum, and when passing through the point $x = 6$, it changes sign from minus to plus, that is, this is a point of minimum. We find $y_{max} = y(0) = 0$, $y_{min} = y(6) = 12$.

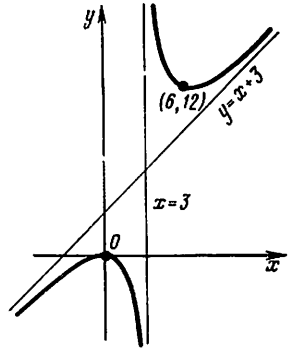


Fig. 36

6. We find

$$y''=\frac{(2x-6)(x-3)^2-2(x-3)(x^2-6x)}{(x-3)^4}=\frac{18}{(x-3)^3}.$$

The second derivative vanishes nowhere, but does not exist for $x = 3$. In the subinterval $(-\infty, 3)$ we have $y'' < 0$, that is, in this subinterval the curve is convex upward; in the subinterval $(3, +\infty)$ we have $y'' > 0$, that is, in this subinterval the curve is convex downward. There is no point of inflection.

7. Using the data thus obtained, we construct the graph of the given function (Fig. 36). ●

In Problems 63 through 75 investigate the given functions and construct their graphs:

63. (1) $y = 2x^2 - 8x$; (2) $y = -3x^2 + 12x$; (3) $y = x^2 + 5x + 4$; (4) $y = -x^2 + 2x + 15$.

64. (1) $y=\frac{1}{3}x^3-9$; (2) $y=x^3-3x$; (3) $y=3x^3-x$; (4) $y=-x^3+x$.

65. (1) $y=\frac{1}{4}x^4$; (2) $y=\frac{1}{5}x^5$.

66. (1) $y = x^3 + 6x^2 + 9x + 8$; (2) $y = 2x^3 - 3x^2 - 12x - 1$;

(3) $y = x^3 - 6x^2 + 16$; (4) $y = 2x^3 + 3x^2 - 12x - 10$.

67. (1) $y = x^4 - 5x^2 + 4$; (2) $y = -x^4 + 8x^2 + 9$.

68. (1) $y = \frac{1}{x^2+1}$; (2) $y = \frac{1}{1-x^2}$.

69. (1) $y = \frac{x}{x^2-4}$; (2) $y = \frac{x^2-4}{x}$.

70. (1) $y = \frac{1}{x^2-7x+12}$; (2) $y = \frac{(x+1)(x+8)}{x}$.

71. (1) $y = \frac{x^3}{x^2-1}$; (2) $y = \frac{6-x^3}{x^2}$.

72. (1) $y = x - \sqrt{x}$; (2) $y = x^2\sqrt{x-3}$.

73. (1) $y = \ln(x^2+1)$; (2) $y = x \ln x$.

74. (1) $y = 3^{1/x}$; (2) $y = e^{-x^2}$.

75. (1) $y = (x-1)e^x$; (2) $y = x^2 e^{-x}$.

QUIZ No. 1

Form I

(1) Find the intervals of monotonicity of the function

$$y = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 1.$$

(2) Find the smallest and largest values of the function

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x - \frac{1}{3}$$

in the interval $[-2, 2]$.

(3) Find the intervals of convexity and inflection points of the following curves:

(a) $y = x^3 + 3x^2$;

(b) $y = \frac{1}{3}x^3 - 4x$.

(4) Given the motion of a point in a straight line $s = -\frac{1}{6}t^3 + \frac{1}{2}t^2 + \frac{1}{2}t + 1$ (t in seconds, s in metres), find the maximum velocity.

Form II

(1) Find the intervals of monotonicity of the function

$$y = x^4 - 4x + 4.$$

(2) Find the smallest and largest values of the function

$$y = \frac{1}{3}x^3 + x^2 - 3x - 4$$

in the interval $[-4, 2]$.

(3) Find the intervals of convexity and inflection points of the following curves:

(a) $y = x^3 - 12x^2 + 145$;

(b) $y = \frac{1}{3}x^3 + x^2 + \frac{1}{3}$.

(4) Given the motion of a point in a straight line $s = -\frac{1}{3}t^3 + 3t^2 + 5t + 3$ (t in seconds, s in metres), find the maximum velocity.

Chapter 9
TRIGONOMETRIC FUNCTIONS

Sec. 9.1. Radian Measure of Arcs and Angles

1. Basic Formulas for Radian Measure of Arcs. When measuring arcs (and the corresponding central angles) in radians, the unit of measurements is the radian which is defined as a central angle subtended in a circle by an arc whose length is equal to the radius of this circle. Thus, the radian (or circular) measure of an angle is the ratio of the arc it subtends to the radius of the circle in which it is the central angle (a constant ratio for all such circles).

The radian measure of an arc is computed by the formula

$$a = l/R, \tag{9.1}$$

where a is the radian measure of the arc; l, the length of the circular arc; R, the radius of this arc.

The formula for converting the degree measure to radian measure has the form

$$a = (\pi/180^\circ) \cdot \alpha, \tag{9.2}$$

where α is the degree measure of an arc (angle).

The radian measure of 1° is equal to 0.0175 radian.

The formula for changing radians to degrees has the form

$$\alpha = (180^\circ/\pi) \cdot \alpha. \tag{9.3}$$

The degree measure of 1 radian is equal to $57^\circ\ 17'\ 44.8'' \approx 57.3^\circ$.

The length of a circular arc is equal to the radian measure of the arc multiplied by the radius of this arc:

$$l = aR. \tag{9.4}$$

The area of a sector of a circle is equal to one-half the radian measure of the arc of the sector multiplied by the squared radius of the circle:

$$S_{\text{sect}} = aR^2/2. \tag{9.5}$$

2. Rotary Motion of a Point. Basic Notions. When considering rotary motion of a rigid body about a fixed axis, we distinguish between linear and angular velocity.

The velocity of any point of a rigid body in rotary motion is called the *linear velocity.*

The linear velocity v of a point performing uniform motion over a circle of radius R is computed by the formula

$$v = 2\pi R/T, \tag{9.6}$$

where T is the period of rotation, that is, the time (in seconds) during which the point completes one revolution.

The angle through which the radius of any point of a uniformly rotating rigid body is rotated per second is called the *angular velocity.* The angular velocity is expressed in radians per second (rad/s).

The relation between the angular velocity ω and the period of rotation T is expressed by the formula

$$\omega = 2\pi/T. \tag{9.7}$$

The linear velocity v of a point situated at a distance R from the axis of rotation and its angular velocity are related as follows

$$v = \omega R. \tag{9.8}$$

In a nonuniform rotation of a rigid body, its angular velocity ω is defined as the rate of change of the angle φ during time t. The angular velocity (in rad/s) in this case is the derivative of the angle of rotation φ with respect to

time t:

$$\omega = \frac{d\varphi}{dt}. \tag{9.9}$$

The angular acceleration ε (in rad/s^2) is the derivative of the angular velocity ω with respect to time t:

$$\varepsilon = \frac{d\omega}{dt}. \tag{9.10}$$

1. What is the exact radian measure of the following arcs: (1) 240°; (2) 300°?

○ By Formula (9.2), we get

(1) $a = (\pi/180°)\cdot 240° = 4\pi/3$; (2) $a = (\pi/180°)\cdot 300° = 5\pi/3$. ●

2. What is the exact degree measure of the indicated arcs: (1) $7\pi/6$; (2) $5\pi/4$?

○ By Formula (9.3), we obtain

(1) $\alpha = (180°/\pi)\cdot(7\pi/6) = 210°$;

(2) $\alpha = (180°/\pi)\cdot(5\pi/4) = 225°$. ●

3. A wheel whose radius is equal to 0.65 m has rotated through an angle of 1.4 radians. Find the path covered by a point on the rim.

○ By Formula (9.4), we find: $l = 1.4\cdot 0.65 = 0.91$ (m). ●

4. The arc of a sector of a circle measures 0.94 radian. Compute the area of the sector if the radius of the circle is equal to 0.65 m.

○ By Formula (9.5), we find: $S_{\text{sect}} = 0.5\cdot 0.94\cdot 0.65^2 \approx 0.20$ (m^2). ●

5. A point of a wheel situated at a distance of 0.56 m from its centre rotates uniformly with linear velocity 4.6 m/s. Find the period of rotation of the wheel.

○ From Formula (9.6) we find T and substitute the numerical values of R and v into the expression found for T:

$$T = 2\pi R/v = 2\pi\cdot 0.56/4.6 \approx 0.77 \text{ (s)}. \quad ●$$

6. The linear velocity at the rim of a uniformly rotating flywheel, whose radius is 0.64 m, is equal to 256 m/s. Find the angular velocity of the flywheel.

○ From the formula (9.8) we find the angular velocity ω and then substitute the numerical values of v and R into the obtained expression:

$$\omega = v/R = 256/0.64 = 400 \text{ (rad/s)}. \quad ●$$

7. In the process of braking, during t seconds a flywheel rotates through an angle $\varphi = 3 + 8t - t^2$. Find: (1) the angular velocity of the flywheel rotation for $t = 3$ seconds; (2) the angular acceleration at the instant of time t; (3) the instant at which the flywheel will cease rotating.

○ (1) The angular velocity is the derivative of the angle of rotation φ with respect to time t, that is, $\omega = \frac{d\varphi}{dt} = 8 - 2t$. We find the angular velocity at the instant of time $t = 3$ s: $\omega(3) = 8 - 2\cdot 3 = 2$ (rad/s).

(2) The angular acceleration ε is the derivative of the angular velocity ω with respect to time t, that is, $\varepsilon = \frac{d\omega}{dt} = -2$ (rad/s^2).

(3) Setting $\omega = 0$, we find t: $8 - 2t = 0$, $t = 4$ s. ●

8. What is the exact radian measure of the following arcs (orally): (1) 30°; (2) 45°; (3) 60°; (4) 90°; (5) 120°; (6) 135°; (7) 150°; (8) 180°; (9) 210°; (10) 225°; (11) 270°; (12) 330°?

9. Using the tables, find the radian measure of the following angles: (1) 15° 18′; (2) 71°10′; (3) 15°17′; (4) 115° 44′; (5) 215°10′; (6) 312°19′; (7) 57°25′; (8) 87°30′.

10. Using a slide rule, find the radian measure of the following arcs: (1) 14.5°; (2) 27.3°; (3) 75°; (4) 130°; (5) 38.7°; (6) 86°.

11. Find the degree measure of the following arcs: (1) $5\pi/36$; (2) $7\pi/12$; (3) $11\pi/18$; (4) $5\pi/9$; (5) $11\pi/20$; (6) $13\pi/30$; (7) $11\pi/6$; (8) $4\pi/3$.

12. Using the tables, find the degree measure of the following arcs: (1) 0.3008; (2) 0.5728; (3) 1.0472; (4) 1.3454; (5) 1.4850; (6) 1.7453.

13. Compute the radius of a circle if its arc of length 0.84 m measures 1.5 radians.

14. Compute the perimeter of a sector of a circle whose arc measures 0.85 radian, and the radius of the circle is equal to 0.38 m.

15. The radius of a circle is equal to 0.56 m, and the area of its sector amounts to 0.72 m^2. Find the arc of the sector in radians.

16. A circular sector, whose area is 0.39 m^2, is subtended by an arc measuring 1.4 radians. Find the radius of the circle.

17. A point of the rim of a uniformly rotating flywheel has a linear velocity 1.6 m/s. The period of rotation of the flywheel is equal to $\pi/4$ s. Find the radius of the flywheel.

18. A point 0.12 m from the axis of rotation of a uniformly rotating gear has a linear velocity 0.48 m/s. Find the period of rotation of the gear.

19. A wheel, whose radius is 0.125 m, rotates uniformly with an angular velocity of 24 rad/s. Compute the linear velocity at the point lying on the rim of the wheel.

20. A rigid body rotates about an axis according to the law $\varphi = 10t - t^2$. Find: (1) the angular velocity of rotation at the instant of time $t = 2$ s; (2) the angular acceleration at the instant t; (3) the instant when the body will cease rotating.

21. The angles of a triangle are in the proportion 1 : 3 : 5. Compute their size in radian measure.

22. Four points divide the circumference of a circle in the ratio 3 : 4 : 5 : 6. Compute the corresponding arcs in radians.

23. The radius of a circular arc is $R = 0.12$ m, its radian measure is equal to 2.5. Find the length of this arc.

24. Compute in radians the size of an inscribed angle subtended by an arc whose radian measure is equal to $4\pi/15$.

25. The radian measure of an arc is equal to $5\pi/9$. At what angle is the chord subtending this arc seen from the points of the arc?

26. Compute (in rad/s) the angular velocity of the hour-hand, minute-hand, and second-hand of a watch.

Sec. 9.2. Unit Number Circle

A circle with centre at the origin and radius equal to 1 is called the *unit circle*. The equation of the unit circle has the form $x^2 + y^2 = 1$ (Fig. 37).

Let us take the point A (1, 0) of the unit circle for the reference point of arc measurement, and the positive semiaxis OX for the initial side of the central angle formed by the radius vector $\overrightarrow{OM}$ and the x-axis (the point M lies on the unit circle).

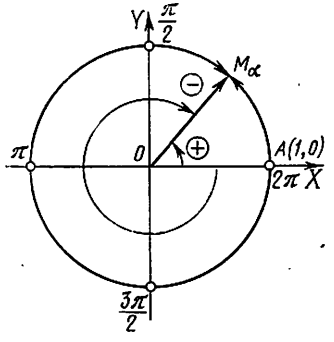


Fig. 37

The rotation of the radius vector $\overrightarrow{OM}$ from the positive semiaxis OX anticlockwise will be called positive, and the arc $\overset{\smile}{AM}$ and the central angle AOM will be termed positive; in opposite rotation (that is, clockwise), they will be regarded as negative.

The arcs of Quadrant I are contained within the interval $(0, \pi/2)$, those of Quadrant II in the interval $(\pi/2, \pi)$, of Quadrant III in the interval $(\pi, 3\pi/2)$, and the arcs of Quadrant IV in the interval $(3\pi/2, 2\pi)$.

There is an infinite set of arcs having a given origin A and a given end M (given initial and terminal sides of an angle). The set of these arcs (angles), both positive and negative, is expressed by the formula

$$\alpha = 2\pi k + \alpha_1 \text{ or } \alpha = 360°k + \alpha_1, \tag{9.11}$$

where $0 \leqslant \alpha_1 < 2\pi$ $(0 \leqslant \alpha < 360°)$ and $k \in \mathbf{Z}$.

If to each real number α on the unit circle there corresponds a point M_α, i.e. the terminal point of the arc AM for which the arc AM has the magnitude α, then such a unit circle is called the *unit number circle*.

The length of the entire unit number circle is equal to 2π. If two numbers differ from each other by a number multiple of 2π, then they are represented on the unit number circle by one and the same point. If two numbers correspond to one and the same point of the unit number circle, then their difference is multiple of 2π.

27. Indicate on the unit circle: (1) the point M of the arc $\overset{\smile}{AM} = 17\pi$; (2) the point M of the arc $\overset{\smile}{AM} = 1950°$; (3) the point M of the arc $\overset{\smile}{AM} = -1380°$.

○ (1) By Formula (9.11), we get $17\pi = 16\pi + \pi = 2\pi \cdot 8 + \pi$, $k = 8$. The arc $\overset{\smile}{AM}$ terminates at the point M $(-1; 0)$.

(2) We find $\overset{\smile}{AM} = 1950° = 360° \cdot 5 + 150°$, $k = 5$. The arc $\overset{\smile}{AM}$ terminates in Quadrant II.

(3) We have $\overset{\smile}{AM} = -1380° = 360° \cdot (-4) + 60°$, $k = -4$. The arc $\overset{\smile}{AM}$ terminates in Quadrant I. Here, the arc α_1 is represented by the least (in the absolute value) number ($\alpha_1 = 60°$). In order for α_1 to be the least (in the absolute value) number, it should be taken in the interval $0 \leqslant \alpha_1 < 180°$ or $-180° < \alpha_1 \leqslant 0$. ●

28. Write in the general form the terminal points of the arcs of the unit circle: (1) whose abscissas are equal to zero; (2) whose ordinates are equal to zero.

○ (1) The terminal points of the arcs $\pi/2$ and $3\pi/2$ ($-\pi/2$) have abscissas equal to zero; consequently, the set of the terminal points of the arcs with abscissas equal to zero is written in the form $\pi/2 + \pi k$, $k \in \mathbf{Z}$.

(2) The terminal points of the arcs 0 (2π) and π have ordinates equal to zero; consequently, the set of the terminal points of arcs with ordinates equal to zero is written in the form πk, $k \in \mathbf{Z}$. ●

29. Indicate in which quadrant of the unit circle the following arcs terminate: (1) 120°; (2) 315°; (3) —220°; (4) 850°; (5) 500°; (6)—120°; (7) $\pi/4$; (8) $7\pi/6$; (9)—$5\pi/3$; (10) $4\pi/15$; (11) $8\pi/3$; (12)—$5\pi/4$; (13) 0.76; (14) 5.8; (15) —7.2; (16) —3.7; (17) 15; (18) 3.3.

30. Represent the angle α by Formula (9.11), taking the least (in the absolute value) angle α_1 if: (1) $\alpha = 2200°$; (2) $\alpha = -550°$; (3) $\alpha = -740°$; (4) $\alpha = 1170°$; (5) $\alpha = -1450°$, (6) $\alpha = 520°$.

31. Give the general notation of the arcs terminating in the following points (1) A (1, 0), (2) B (0, 1), (3) C (—1, 0); (4) D (0,—1).

32. The unit circle is divided into twelve equal parts beginning with the point A (1, 0). Write the sets of terminal points of the arcs ending in the division points.

33. Find the values of the following arcs: (1) πk; (2) $\pi/4 + 2\pi k$; (3) $\pi/3 + 2\pi k$; (4) $(-1)^k \pi/6 + \pi k$ for $k = -2$; —1; 0; 1; 2.

34. On the unit circle, mark the arcs α satisfying the following conditions: (1) $0 < \alpha < \pi/4$; (2) $\pi/4 < \alpha < \pi/3$; (3) $-\pi/6 < \alpha < \pi/6$; (4) $-2\pi/3 < \alpha < 0$; (5) $|\alpha| \leqslant \pi/3$; (6) $|\alpha| \leqslant \pi/2$.

35. On the unit circle: (1) construct the points corresponding to the following numbers: $\pi/6$; $\pi/4$; $\pi/3$; $2\pi/3$; $-2\pi/3$; $-3\pi/4$;

(2) construct approximately the points corresponding to the following numbers: 1.5; 3.5; 5; 6; —2; —4.

36. A point on a unit circle lying at the terminal point of the arc $\pi/6$ begins moving uniformly in the circle in the positive direction with the velocity $\omega = 4\pi/3$ rad/s. In which quadrant will the point appear after: (1) two seconds; (2) three seconds; (3) four seconds; (4) ten seconds?

37. Write the general form of the arcs α if:
(1) $2\alpha = \pi/4 + 2\pi k$; (2) $3\alpha - \pi/6 = 5\pi/6 + 2\pi k$; (3) $4\alpha + \pi/3 = 2\pi/3 + \pi k$; (4) $10° + \alpha = 50° + 180° k$.

Sec. 9.3. Trigonometric Functions of Numerical Argument

The abscissa X of the point M_α of the numerical unit circle (Fig. 38) is called the cosine of the number α:

$$X = \cos \alpha. \tag{9.12}$$

The ordinate Y of the point M_α of the numerical unit circle is called the *sine* of the number α:

$$Y = \sin \alpha. \tag{9.13}$$

The domain of definition of cosine and sine is the set of all real numbers, that is, $D(\cos \alpha) = \mathbf{R}$, $D(\sin \alpha) = \mathbf{R}$.

The ratio of the sine of the number α to its cosine is termed the *tangent* of the number α:

$$\tan \alpha = \sin \alpha / \cos \alpha. \tag{9.14}$$

The domain of definition of tangent is the set of all real numbers, except numbers of the form $\pi/2 + \pi k$, $k \in \mathbf{Z}$.

The ratio of the cosine of the number α to its sine is called the *cotangent* of the number α:

$$\cot \alpha = \cos \alpha / \sin \alpha. \tag{9.15}$$

The domain of definition of cotangent is the set of all real numbers, except numbers of the form πk, $k \in \mathbf{Z}$.

The reciprocal of the cosine of the number α is called the *secant* of the number α:

$$\sec \alpha = 1/\cos \alpha. \tag{9.16}$$

The domain of definition of secant is the set of all real numbers, except numbers of the form $\pi/2 + \pi k$, $k \in \mathbf{Z}$.

The reciprocal of the sine of the number α is called the *cosecant* of the number α:

$$\csc \alpha = 1/\sin \alpha. \tag{9.17}$$

The domain of definition of cosecant is the set of all real numbers, except numbers of the form πk, $k \in \mathbf{Z}$.

When solving problems, trigonometric functions are regarded either as functions of a number or as functions of an arc (angle) defined on the set of all real numbers or on the set of all arcs (angles). In this case, to every real number or every arc (angle) there corresponds the only value of a trigonometric function.

The functions $\cos \alpha$ and $\sin \alpha$ are bounded, since $E(\cos \alpha) = [-1, 1]$, $E(\sin \alpha) = [-1, 1]$.

The functions $\tan \alpha$ and $\cot \alpha$ are not bounded, since each of them can attain any real value, that is, $E(\tan \alpha) = \mathbf{R}$ and $E(\cot \alpha) = \mathbf{R}$.

38. Can the sine and cosine be equal to: (1) 0.85; (2) 7/6; (3) $-\sqrt[3]{3}/\sqrt{3}$; (4) -1.01; (5) $3/\sqrt{10}$; (6) $\sqrt{10}/\pi$; (7) $a/\sqrt{a^2 + b^2}$, $a > 0$, $b > 0$; (8) $a/\sqrt{a^2 - b^2}$, $a > 0$, $|b| < a$?

39. Can sine, cosine, tangent, and cotangent take on a value equal to: (1) 1/2; (2) $\sqrt{2}$; (3) -5.1; (4) $\sqrt{0.5}$; (5) 2.2; (6) $\sqrt{3}/2$?

40. Can the functions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, and $\cot \alpha$ take on negative values if α expresses the size of an angle of a triangle?

41. What is the difference between the domains of definition of $\sin \alpha$ and $\tan \alpha$?

42. Compare the absolute value of the tangent of any number with that of the sine of this number.

43. What sets of values have the functions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, and $\cot \alpha$?

44. Find the values of α in the interval $[0, 2\pi]$ for which the indicated equality is fulfilled: (1) $\sin \alpha = 0$; (2) $\sin \alpha = 1$; (3) $\sin \alpha = -1$; (4) $\cos \alpha = 0$; (5) $\cos \alpha = 1$; (6) $\cos \alpha = -1$.

45. Find the values of α in the interval $[0, 2\pi]$ for which the indicated equality is fulfilled: (1) $\tan \alpha = 0$; (2) $\tan \alpha = 1$; (3) $\tan \alpha = -1$; (4) $\cot \alpha = 0$; (5) $\cot \alpha = 1$; (6) $\cot \alpha = -1$.

Sec. 9.4. Signs, Numerical Values, and Properties of Evenness and Oddness of Trigonometric Functions

A function $y = f(x)$ is said to be *even* if for all values of x from the domain of definition of this function $f(-x) = f(x)$.

A function $y = f(x)$ is said to be *odd* if for all values of x from the domain of definition of this function $f(-x) = -f(x)$.

The properties of evenness and oddness of trigonometric functions are expressed by the following formulas:

$$\sin(-\alpha) = -\sin \alpha; \tag{9.18}$$

$$\cos(-\alpha) = \cos \alpha; \tag{9.19}$$

$$\tan(-\alpha) = -\tan \alpha; \tag{9.20}$$

$$\cot(-\alpha) = -\cot \alpha. \tag{9.21}$$

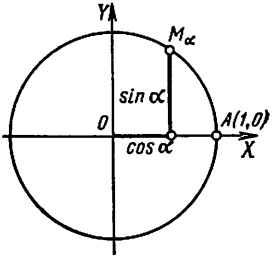


Fig. 38

The signs of trigonometric functions for each quadrant are represented in Table 3.

The values of trigonometric functions of certain arcs (angles) are given in Table 4.

Table 3

Function	Quadrant			
	I $0 < \alpha < \pi/2$	II $\pi/2 < \alpha < \pi$	III $\pi < \alpha < 3\pi/2$	IV $3\pi/2 < \alpha < 2\pi$
$\sin \alpha$	+	+	−	−
$\cos \alpha$	+	−	−	+
$\tan \alpha$	+	−	+	−
$\cot \alpha$	+	−	+	−

Table 4

	Quadrant I					Quadrant II				Quadrant III				Quadrant IV			
	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Non-exist.	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Non-exist.	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0
$\cot\alpha$	Non-exist.	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	−1	$-\sqrt{3}$	Non-exist.	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	−1	$-\sqrt{3}$	Non. exist.

46. What signs have the following values: (1) $\cos 150°$; (2) $\sin 320°$; (3) $\tan 220°$; (4) $\cot 400°$?

○ (1) $90° < 150° < 180°$ (Quadrant II); $\cos 150° < 0$.

(2) $270° < 320° < 360°$ (Quadrant IV); $\sin 320° < 0$.

(3) $180° < 220° < 270°$ (Quadrant III); $\tan 220° > 0$.

(4) $360° < 400° < 360° + 90°$ (Quadrant I); $\cot 400° > 0$. ●

47. Using the unit circle, determine the signs of the following differences: (1) $\sin(2\pi/3) - \sin(3\pi/4)$; (2) $\cos(2\pi/3) - \cos(3\pi/4)$; (3) $\tan(2\pi/3) - \tan(3\pi/4)$; (4) $\cot(2\pi/3) - \cot(3\pi/4)$.

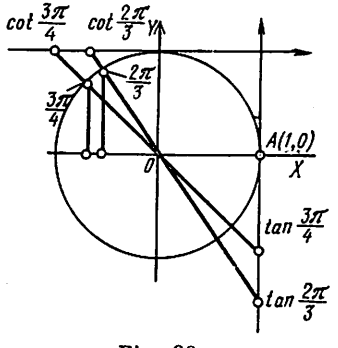


Fig. 39

○ (1) Comparing the ordinates of the terminal points of the arcs $2\pi/3$ and $3\pi/4$, we get $\sin(2\pi/3) - \sin(3\pi/4) > 0$ (Fig. 39).

(2) Comparing the abscissas of the terminal points of the arcs $2\pi/3$ and $3\pi/4$, we get $\cos(2\pi/3) - \cos(3\pi/4) > 0$.

(3) Comparing the ordinates of the points on the axis of tangents, we have $\tan(2\pi/3) - \tan(3\pi/4) < 0$.

(4) Comparing the abscissas of the points on the axis of cotangents, we obtain $\cot(2\pi/3) - \cot(3\pi/4) > 0$. ●

48. Simplify the following:

(1) $\sin^2(-\alpha) - \cos(-\alpha) + \tan(-\alpha)$;

(2) $\sin(-3\pi/2) + \cos(-\pi) + \tan(-2\pi)$.

○ Using the Formulas (9.18), (9.19), and (9.20), we get

(1) $\sin^2(-\alpha) - \cos(-\alpha) + \tan(-\alpha) = (-\sin\alpha)^2 - \cos\alpha - \tan\alpha = \sin^2\alpha - \cos\alpha - \tan\alpha$;

(2) $\sin(-3\pi/2) + \cos(-\pi) + \tan(-2\pi) = -\sin(3\pi/2) + \cos\pi - \tan 2\pi = 1 + (-1) - 0 = 0$. ●

49. In which quadrant can the arc α terminate if:

(1) $|\tan(-\alpha)| = -\tan\alpha$;

(2) $|\cot(-\alpha)| = -\cot\alpha$; (3) $\sin(-\alpha) > 0$?

○ (1) $|\tan(-\alpha)| = -\tan\alpha$; $-\tan\alpha \geqslant 0$, $\tan\alpha \leqslant 0$; α terminates in Quadrant II or IV;

(2) $|\cot(-\alpha)| = -\cot\alpha$; $-\cot\alpha \geqslant 0$, $\cot\alpha \leqslant 0$; α terminates in Quadrant II or IV;

(3) $\sin(-\alpha) > 0$; $-\sin\alpha > 0$; $\sin\alpha < 0$; α terminates in Quadrant III or IV. ●

50. Compute:

$$\tan^2(\pi/3) + \cot(\pi/6) - 2\sin(\pi/3) + \sin\pi + 4\cos(3\pi/2) - 2\cos(\pi/3).$$

○ $\tan^2(\pi/3) + \cot(\pi/6) - 2\sin(\pi/3) + \sin\pi + 4\cos(3\pi/2) - 2\cos(\pi/3) = (\sqrt{3})^2 + \sqrt{3} - 2\cdot(\sqrt{3}/2) + 0 + 4\cdot 0 - 2\cdot(1/2) = 2$. ●

51. What signs do the following values have:
(1) $\sin 170°$; (2) $\cos 300°$; (3) $\tan 160°$; (4) $\cot 315°$;
(5) $\tan 450°$; (6) $\sin 400°$; (7) $\sin (7\pi/3)$;
(8) $\cos (4\pi/3)$; (9) $\sin (5\pi/4)$; (10) $\cos (7\pi/5)$;
(11) $\tan (8\pi/3)$; (12) $\cot (9\pi/4)$?

52. Using the unit circle, determine the signs of the following differences: (1) $\sin 130° - \sin 140°$;
(2) $\cos 50° - \cos 70°$; (3) $\tan 220° - \tan 210°$;
(4) $\cot 220° - \cot 210°$; (5) $\sin 50° - \tan 50°$;
(6) $\cos 50° - \cot 50°$;
(7) $\cot 300° - \cot 315°$; (8) $\sin 70° - \cos 70°$.

53. Using the unit circle, determine the signs of the following products: (1) $\sin 100° \cdot \sin 120°$;
(2) $\cos 210° \cdot \sin 210°$; (3) $\cos 200° \cdot \sin 110°$;
(4) $\tan 140° \cdot \tan 220°$; (5) $\cos 315° \cdot \tan 215°$;
(6) $\sin 150° \cdot \cos 150 \cdot \tan 150°$; (7) $\sin 320° \cdot \cos 125 \cdot \tan 250°$;
(8) $\sin 230° \cdot \tan 160 \cdot \cot 340°$.

54. Compute: (1) $\cos (-\pi) \cdot \sin (-\pi/2) \cdot \sin (-3\pi/2)$;
(2) $2 \cos (-\pi) \cdot \cos (-2\pi) \cdot \sin (-3\pi/2)$;
(3) $\sin (-\pi) + \cos (-\pi) + \tan (-\pi)$.

55. In which quadrant can the arc α terminate if:
(1) $\tan (-\alpha) > 0$; (2) $|\sin (-\alpha)| = -\sin \alpha$;
(3) $|\cos (-\alpha)| = \cos \alpha$; (4) $\sin (-\alpha) < 0$?

56. Compute the following:
(1) $\cot (\pi/2) + \tan \pi - \sin (3\pi/2) - \cos (-\pi/2) + \sin \pi$;
(2) $\sin (\pi/2) - \cos (3\pi/2) + \cos \pi - \tan 0 + \cot (3\pi/2)$;
(3) $\sin 180° + \sin 270° - \cot 90° + \tan 180° - \cos 90°$;
(4) $4 \sin \pi \cdot \cos 2\pi + 5 \tan \pi - \cot (3\pi/2)$;
(5) $2 \sin 2\pi + 5 \cos (3\pi/2) + 3 \tan \pi - 2 \cot (\pi/2)$;
(6) $2 \sin (\pi/3) + 2 \cos (\pi/4) - 3 \tan (\pi/3) + \cot (\pi/2)$;
(7) $\sin^2 (\pi/4) - 2 \cos^2 (\pi/3) - 5 \tan^2 (\pi/4)$;

(8) $\dfrac{5 + \cot^4 (\pi/6) - \tan^2 (\pi/4)}{\cot (\pi/4) - 4 \cos^2 (\pi/3) - 8 \sin^3 (\pi/6)}$;

(9) $4 \sin^2 (-\pi/3) + 9 \tan^2 (-\pi/6) - 4 \cos^2 (-\pi/3) + 16 \cot^2 (-\pi/4)$;

(10) $\dfrac{(a \tan (\pi/4))^2 - (b \sin (\pi/2))^2}{2a^2 \sin (\pi/6) + 2ab \cos 0 + (b \cot (\pi/4))^2}$;

(11) $\dfrac{(2a \sin (\pi/6))^3 - (b \tan (\pi/4))^3 - (2ab \cos (\pi/2))^2}{(a \cos 0)^2 + 2ab \cos (\pi/3) + 2b^2 \cos^2 (\pi/4)}$;

(12) $\sin (-\pi/6) - 2 \tan (-\pi/4) + \cos (-\pi/3) - \cot (-\pi/2)$;
(13) $\cos^3 (-\pi/3) - \cot^3 (-\pi/6) + \sin^3 (-\pi/6)$.

57. (1) $f(x) = 4 \sin 3x + 5 \cos 3x - 2 \sin x$; compute $f(0)$, $f(\pi/6)$; $f(\pi/3)$; $f(\pi)$;
(2) $f(x) = \sin x + \sin 2x + \sin 3x$; compute $f(\pi/6)$;

(3) $f(x) = \sin\left(\frac{\pi}{4} + x\right) + 2\cos(\pi + 2x) + 3\tan\left(\frac{\pi}{2} - x\right)$; compute $f(\pi/4)$;

(4) $\frac{\sin\alpha\cos\beta}{\sin(\alpha+\beta)+\cos(\alpha+\beta)}$; compute for $\alpha = \pi/3$ and $\beta = \pi/6$.

58. Determine the signs of the following expressions if $0 < \alpha < \pi/2$: (1) $\sin(\pi/2 + \alpha)$; (2) $\cos(3\pi/2 - \alpha)$; (3) $\sin(\pi + \alpha)$; (4) $\cos(3\pi/2 - \alpha)$; (5) $\sin(3\pi/2 + \alpha)$; (6) $\sin(3\pi + \alpha)$; (7) $\tan(\pi + \alpha)$; (8) $\cot(\pi - \alpha)$; (9) $\sin(2\pi - \alpha)$; (10) $\cos(2\pi + \alpha)$; (11) $\tan(3\pi/2 + \alpha)$; (12) $\tan(1.57 - \alpha)$.

59. Determine the signs of the following expressions:

(1) $\sin(7\pi/6)\cdot\cos(3\pi/5)$; (2) $\sin^2(4\pi/3)\cdot\sin(5\pi/4)$; (3) $\sin(2\pi/3)\times\cos(2\pi/3)\cdot\cos(\pi/4)$; (4) $\sin 2.3\cdot\cos(-1.7)\cdot\sin(-1.5)$.

60. To which quadrant does the arc α belong if: (1) $\sin\alpha = \cos\alpha$; (2) $\sin\alpha = \cos^2\alpha$; (3) $\sin\alpha = \cos^3\alpha$; (4) $\sin\alpha = \cos^4\alpha$; (5) $\cos\alpha = \sin^2\alpha$; (6) $\cos\alpha = \sin^3\alpha$; (7) $\tan\alpha = \cot\alpha$; (8) $\tan\alpha = \cot^2\alpha$?

61. Determine the signs of the following expressions if: $0 < \alpha < \pi/2$:

(1) $\cos(\pi/2 - \alpha)\cdot\tan(2\pi - \alpha)\cdot\cot(\pi/2 + \alpha)$;

(2) $\sin(3\pi/2 + \alpha)\cos(\pi + \alpha)\cdot\tan(\pi + \alpha)$;

(3) $\cos^2(\pi + \alpha)\cdot\tan^2(\pi/2 + \alpha)\cdot\cot^3(\pi - \alpha)$.

62. Compute:

(1) $\sin\pi + \cot(-\pi/2) + \cos(-3\pi/2) + \tan\pi$;

(2) $\cos(-\pi) + \sin(-\pi) + \tan(-\pi/4) + \cot(-\pi/4)$;

(3) $a^2\cos 0 + 2ab\cos\pi + b^2\sin(\pi/2)$;

(4) $\sin(\pi/6)\cdot\cos(\pi/3)\cdot\tan(\pi/4)\cdot\cot(\pi/6)$;

(5) $5\tan^2(\pi/4) - 8\sin^2(\pi/6) + 4\cos^2(\pi/3)$;

(6) $\sin^3(\pi/6) - \cos^3(\pi/4) + \tan^3(\pi/4) - \cos^3(\pi/3)$;

(7) $\sin^2(\pi/4)\cdot\cos^2(-\pi/4)\cdot\tan^2(-\pi/3)$.

63. Simplify:

(1) $\frac{(a\cos 0)^2 - (b\cot(\pi/4))^2}{2a^2\cos(\pi/3) + 2ab\cos\pi + b^2\tan(\pi/4)}$;

(2) $\frac{(a\sin(\pi/2))^4 - (b\tan)\pi/4))^4}{(a\cos 2\pi)^2 - (b\sin(-\pi/2))^2}$;

(3) $(-a\sin(-3\pi/2))^3 + (ab\tan 2\pi)^3 + (b\cos 0)^3$;

(4) $\frac{a\cos 0 - ab\sin 0 + b\tan(\pi/4)}{a\cos 2\pi - b\sin 2\pi}$.

64. Check the following equalities:

(1) $\sin^2(\pi/6) + \cos^2(\pi/4) + \cos\pi + \tan^2(\pi/3) - \cot(3\pi/2) - \sin^2(\pi/3) = 2$;

(2) $\sin\pi + 3\cos(3\pi/2) - \tan^2(\pi/3) + \cot^2(\pi/6) = 0$;

(3) $2\sin^2(\pi/4) + 4\sin^2(\pi/3) + \cos^2(\pi/4) - \sin(\pi/6) = 4$;

(4) $\sin^3(\pi/4) + \cos^3(\pi/4) - \sin(\pi/4) = 0$;

(5) $\frac{2 - \sin^2(-\pi/6) - \cos^2(-\pi/6)}{1 - 2\cos^2(-\pi/6)} = -2$;

(6) $2\sin(-\pi/6) + 3\cos(-\pi/2) - 3\cot(-\pi/4) + 4\tan 0 = 2$.

Sec. 9.5. Fundamental Trigonometric Identities

If two functions of the same arguments have one and the same domain of definition and take on equal values for all permissible values of the arguments, then they are said to be *identically equal*.

A statement of equality which is true for all permissible values of the arguments is called the *identity*.

If an identity contains trigonometric functions, then such an identity is termed a *trigonometric identity*.

A passage from a given function to an identically equal function is called an *identical transformation of the function*.

When proving trigonometric identities, the following techniques are usually used: (1) we transform any member of an equality (usually the one which represents a more complicated expression) so that the identical transformations performed result in an expression standing in the other side of the equality; (2) we transform simultaneously both sides of the identity to be proved until it is obvious that the expressions obtained in both sides are identically equal; (3) using the property of the proportion that the product of the means equals the product of the extremes, we check to make sure that these products are equal to each other.

The Fundamental Trigonometric Identities

$$\sin^2\alpha + \cos^2\alpha = 1; \tag{9.22}$$

$$\tan\alpha \cot\alpha = 1,\ \alpha \neq \pi k/2,\ k \in \mathbf{Z}; \tag{9.23}$$

$$1 + \tan^2\alpha = 1/\cos^2\alpha,\ \alpha \neq \pi/2 + \pi k,\ k \in \mathbf{Z}; \tag{9.24}$$

$$1 + \cot^2\alpha = 1/\sin^2\alpha,\ \alpha \neq \pi k,\ k \in \mathbf{Z}. \tag{9.25}$$

Using the fundamental trigonometric identities, we can express in terms of a given trigonometric function all the remaining functions. These expressions are tabulated below.

65. Given: $\sin\alpha = 3/5$, $\alpha \in (\pi/2, \pi)$. Find: (1) $\cos\alpha$; (2) $\tan\alpha$; (3) $\cot\alpha$.

○ (1) $\cos\alpha = -\sqrt{1 - (3/5)^2} = -4/5$ (the radical is negative, since in Quadrant II $\cos\alpha < 0$); (2) $\tan\alpha = (3/5) \div (-4/5) = -3/4$; (3) $\cot\alpha = -4/3$. ●

66. Given: $\cos\alpha = -12/13$, $\alpha \in (\pi, 3\pi/2)$. Compute: (1) $\sin\alpha$; (2) $\tan\alpha$; (3) $\cot\alpha$.

○ (1) $\sin\alpha = -\sqrt{1 - (-12/13)^2} = -5/13$; (2) $\tan\alpha = (-5/13) \div (-12/13) = 5/12$; (3) $\cot\alpha = 12/5$. ●

67. Given: $\tan\alpha = -3/4$, $\alpha \in (\pi/2, \pi)$. Compute: (1) $\cot\alpha$; (2) $\cos\alpha$; (3) $\sin\alpha$.

○ (1) $\cot\alpha = -4/3$; (2) by Formula (9.24), we get $\cos^2\alpha = \frac{1}{1+\tan^2\alpha}$; $\cos\alpha = -\frac{1}{\sqrt{1+(-3/4)^2}} = -\frac{4}{5}$;

Table 5

Given function	Desired function			
	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	$\sin\alpha$	$\pm\sqrt{1-\sin^2\alpha}$	$\dfrac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}}$	$\dfrac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
$\cos\alpha$	$\pm\sqrt{1-\cos^2\alpha}$	$\cos\alpha$	$\dfrac{\pm\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\dfrac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}}$
$\tan\alpha$	$\dfrac{\tan\alpha}{\pm\sqrt{1+\tan^2\alpha}}$	$\dfrac{1}{\pm\sqrt{1+\tan^2\alpha}}$	$\tan\alpha$	$\dfrac{1}{\tan\alpha}$
$\cot\alpha$	$\dfrac{1}{\pm\sqrt{1+\cot^2\alpha}}$	$\dfrac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}$	$\dfrac{1}{\cot\alpha}$	$\cot\alpha$

(3) $\sin\alpha=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\frac{3}{5}$. ●

68. Given: $\cot\alpha = 8/15$, $\alpha\in(0,\ \pi/2)$. Compute: (1) $\tan\alpha$; (2) $\sin\alpha$; (3) $\cos\alpha$.

○ (1) $\tan\alpha = 15/8$; (2) by Formula (9.25), we get $\sin^2\alpha=\frac{1}{1+\cot^2\alpha}$; $\sin\alpha=\frac{1}{\sqrt{1+(8/15)^2}}=\frac{15}{17}$; (3) $\cos\alpha=\sqrt{1-(15/17)^2}=\frac{8}{17}$. ●

69. Simplify: (1) $\frac{\cos\alpha}{1+\sin\alpha}+\tan\alpha$; (2) $\sin^4\alpha+\cos^4\alpha+2\sin^2\alpha\cos^2\alpha$.

○ (1) $\frac{\cos\alpha}{1+\sin\alpha}+\tan\alpha=\frac{\cos\alpha}{1+\sin\alpha}+\frac{\sin\alpha}{\cos\alpha}=\frac{\cos^2\alpha+\sin\alpha+\sin^2\alpha}{(1+\sin\alpha)\cos\alpha}$

$=\frac{1+\sin\alpha}{(1+\sin\alpha)\cos\alpha}=\frac{1}{\cos\alpha}=\sec\alpha\left(\alpha\neq\frac{\pi}{2}+\pi k,\ k\in\mathbf{Z}\right)$;

(2) $\sin^4\alpha+\cos^4\alpha+2\sin^2\alpha\cos^2\alpha=(\sin^2\alpha+\cos^2\alpha)^2=1^2=1$. ●

70. Prove the following identities: (1) $\frac{\sin\alpha}{1+\cos\alpha}+\frac{1+\cos\alpha}{\sin\alpha}=\frac{2}{\sin\alpha}$; (2) $1+\sin\alpha+\cos\alpha+\tan\alpha=(1+\cos\alpha)(1+\tan\alpha)$;

(3) $\frac{\sin\alpha}{1-\cos\alpha}=\frac{1+\cos\alpha}{\sin\alpha}$.

○ (1) $\left(\frac{\sin\alpha}{1+\cos\alpha}+\frac{1+\cos\alpha}{\sin\alpha}=\frac{2}{\sin\alpha}\right)\Leftrightarrow\left(\frac{\sin^2\alpha+1+2\cos\alpha+\cos^2\alpha}{(1+\cos\alpha)\sin\alpha}=\frac{2}{\sin\alpha}\right)\Leftrightarrow\left(\frac{1+1+2\cos\alpha}{(1+\cos\alpha)\sin\alpha}=\frac{2}{\sin\alpha}\right)\Leftrightarrow\left(\frac{2(1+\cos\alpha)}{(1+\cos\alpha)\sin\alpha}=\frac{2}{\sin\alpha}\right)$

$\Leftrightarrow\left(\frac{2}{\sin\alpha}=\frac{2}{\sin\alpha}\right)$, $\alpha\neq\pi k$, $k\in\mathbf{Z}$.

(2) Transform the left-hand side:

$$1+\sin\alpha+\cos\alpha+\tan\alpha=1+\sin\alpha+\cos\alpha+\frac{\sin\alpha}{\cos\alpha}$$

$$=\frac{\cos\alpha+\sin\alpha\cos\alpha+\cos^2\alpha+\sin\alpha}{\cos\alpha}=\frac{(\sin\alpha+\cos\alpha)+\cos\alpha(\sin\alpha+\cos\alpha)}{\cos\alpha}$$

$$=\frac{(\sin\alpha+\cos\alpha)(1+\cos\alpha)}{\cos\alpha},\quad \alpha\neq\frac{\pi}{2}+\pi k,\quad k\in\mathbf{Z}.$$

Transform the right-hand side:

$$(1+\cos\alpha)(1+\tan\alpha)=(1+\cos\alpha)\left(1+\frac{\sin\alpha}{\cos\alpha}\right)$$

$$=(1+\cos\alpha)\frac{\sin\alpha+\cos\alpha}{\cos\alpha}=\frac{(\sin\alpha+\cos\alpha)(1+\cos\alpha)}{\cos\alpha},$$

$$\alpha\neq\frac{\pi}{2}+\pi k,\quad k\in\mathbf{Z}.$$

Since the left and right members are equal to each other, the desired identity has been proved.

(3) On the basis of the property of any proportion, we get

$$(\sin^2\alpha=(1-\cos\alpha)(1+\cos\alpha))\Leftrightarrow(\sin^2\alpha=1-\cos^2\alpha)$$

$$\Leftrightarrow(\sin^2\alpha=\sin^2\alpha),\ \alpha\neq\pi k,\ k\in\mathbf{Z}.\ \bullet$$

71. Evaluate the given function:

(1) $y=\frac{\sin^3 x+\cos^3 x}{\sin^3 x-\cos^3 x}$, if $\tan x=2$,

(2) $y=\frac{\sin^2 x+\sin x\cos x+2}{3\sin x\cos x+\cos^2 x-4}$, if $\tan x=3$.

○ (1) Since both the numerator and denominator are homogeneous polynomials of the same power of $\sin x$ and $\cos x$, they can be divided by $\cos^2 x$; then we get

$$y=\frac{\tan^3 x+1}{\tan^3 x-1}=\frac{2^3+1}{2^3-1}=\frac{9}{7}.$$

(2) Multiplying 2 in the numerator and 4 in the denominator by the trigonometric unity $(\sin^2 x+\cos^2 x)$, and dividing then both the numerator and denominator by $\cos^2 x$, we get

$$y=\frac{\sin^2 x+\sin x\cos x+2(\sin^2 x+\cos^2 x)}{3\sin x\cos x+\cos^2 x-4(\sin^2 x+\cos^2 x)}=\frac{3\sin^2 x+\sin x\cos x+2\cos^2 x}{3\sin x\cos x-4\sin^2 x-3\cos^2 x}$$

$$=\frac{3\tan^2 x+\tan x+2}{3\tan x-4\tan^2 x-3}=\frac{3\cdot 3^2+3+2}{3\cdot 3-4\cdot 3^2-3}=-\frac{16}{15}.\ \bullet$$

72. Compute the values of the three remaining trigonometric functions if: (1) $\sin\alpha = -5/13$ and $\alpha \in (3\pi/2, 2\pi)$ (2) $\cos\alpha = -8/17$ and $\alpha \in (\pi/2, \pi)$; (3) $\tan\alpha = 8/15$ and $\alpha \in (\pi, 3\pi/2)$; (4) $\cot\alpha = -7/24$ and $\alpha \in (3\pi/2, 2\pi)$.

73. Simplify the following expressions:

(1) $\sin^2\alpha + \tan^2\alpha + \cos^2\alpha$; (2) $\sin^4\alpha - \cos^4\alpha + \cos^2\alpha$;

(3) $\dfrac{\tan\alpha}{1-\tan^2\alpha} + \dfrac{\cot\alpha}{1-\cot^2\alpha}$; (4) $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{\sin\alpha}{1-\cos\alpha}$;

(5) $\tan^2\alpha\cos^2\alpha + \cot^2\alpha\sin^2\alpha$; (6) $\cos^4 x + \sin^2 x\cos^2 x + \sin^2 x$;

(7) $\dfrac{\cos^3\alpha - \sin^3\alpha}{1+\sin\alpha\cos\alpha}$; (8) $\sin\alpha\cos\alpha(\tan\alpha + \cot\alpha)$.

74. Prove the following identities:

(1) $\dfrac{\sin^2 x}{\sin x - \cos x} + \dfrac{\sin x + \cos x}{1-\tan^2 x} = \sin x + \cos x$;

(2) $\tan^2\alpha - \sin^2\alpha = \sin^2\alpha\tan^2\alpha$;

(3) $\sin^3\alpha(1+\cot\alpha) + \cos^3\alpha(1+\tan\alpha) = \sin\alpha + \cos\alpha$;

(4) $\dfrac{1}{1+\tan^2\alpha} + \dfrac{1}{1+\cot^2\alpha} = 1$; (5) $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} = \tan\alpha\tan\beta$;

(6) $\dfrac{\sin x}{1+\cot x} + \dfrac{\cos x}{1+\tan x} = \dfrac{1}{\sin x+\cos x}$; (7) $\dfrac{\tan^2\alpha}{1+\tan^2\alpha}\cdot\dfrac{1+\cot^2\alpha}{\cot^2\alpha}$ $= \tan^2\alpha$; (8) $\dfrac{1-(\sin\alpha+\cos\alpha)^2}{\sin\alpha\cos\alpha-\cot\alpha} = 2\tan^2\alpha$; (9) $\dfrac{\tan\alpha}{\tan\alpha+\cot\alpha} = \sin^2\alpha$;

(10) $1 - \sin^6 z - \cos^6 z = 3\sin^2 z\cos^2 z$;

(11) $\cos^2\alpha(1-\tan\alpha)(1+\tan\alpha) = \cos^4\alpha - \sin^4\alpha$.

75. Compute:

(1) $y = \dfrac{\sin^2 x - 3\cos^2 x}{2\sin^2 x + \cos^2 x}$, if $\tan x = 3$;

(2) $y = \dfrac{3\sin^2 x + 2\cos^2 x - 1}{\sin^2 x - \sin x\cos x + 2}$, if $\tan x = 1$.

76. Simplify:

(1) $(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2$; (2) $\sin^2\alpha + \cos^4\alpha - \sin^4\alpha$; (3) $\sin^4\alpha + \sin^2\alpha\cos^2\alpha + \cos^2\alpha$; (4) $\cot^2\alpha - \cos^2\alpha\cot^2\alpha - \cos^2\alpha$;

(5) $\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}} - \sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$;

(6) $(1+\sin\alpha)(\tan\alpha+\cot\alpha)(1-\sin\alpha)$;

(7) $(1+\tan\alpha)^2 + (1-\tan\alpha)^2$; (8) $\tan\alpha - \sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}$.

77. Prove the following identities:

(1) $(\cot\alpha + 1)^2 + (\cot\alpha - 1)^2 = 2/\sin^2\alpha$;

(2) $\cos\alpha + \sin\alpha \tan\alpha - 1/\cos\alpha = 0$,

(3) $\tan^2\alpha\,(1 + \tan\alpha)\,(1 + \cot^2\alpha) - (1 - \tan^2\alpha)^2 = 4\tan^2\alpha$;

(4) $\cos\alpha\,(\sin\alpha + \cos\alpha)\,(1 - \tan\alpha) = \cos^4\alpha - \sin^4\alpha$;

(5) $\dfrac{\cos\alpha}{\tan\alpha} + \dfrac{\sin\alpha}{\cot\alpha} = (\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha - 1)$;

(6) $2\sin^2\alpha - \cos^2\alpha\,(\tan^2\alpha + \cot^2\alpha) + (\tan\alpha - \cot\alpha)^2 + 1 = \tan^2\alpha$;

(7) $(1 + 2\sin\alpha\cos\alpha)(\tan\alpha - 1) = (\sin^2\alpha - \cos^2\alpha)(\tan\alpha + 1)$;

(8) $(\tan\alpha + \cot\alpha)(1 + \cos\alpha)(1 - \cos\alpha) = \tan\alpha$;

(9) $2(\cos^6\alpha + \sin^6\alpha) - 3(\cos^4\alpha + \sin^4\alpha) = -1$;

(10) $\dfrac{\tan^2\alpha}{\sin^2\alpha\cos^2\alpha} = (1 + \tan^2\alpha)^2$;

(11) $\dfrac{\cos^2\alpha - \cos^2\alpha\sin^2\beta}{\sin^2\alpha\sin^2\beta} = \cot^2\alpha\cot^2\beta$;

(12) $\sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta + \cos^2\alpha = 1$.

78. Evaluate the given expressions if $\tan z = 2$:

(1) $\sin^4 z + \cos^4 z$; (2) $\sin^6 z + \cos^6 z$;

(3) $\dfrac{\sin^4 z - \cos^4 z}{\sin^6 z - \cos^6 z}$; (4) $\dfrac{\sin^3 z - 2\cos^3 z + 3\cos z}{3\sin z + 2\cos z}$.

79. Given: $\tan\alpha + \cot\alpha = 3$. Find:

(1) $\tan\alpha - \cot\alpha$; (2) $\tan^2\alpha - \cot^2\alpha$; (3) $\tan^2\alpha + \cot^2\alpha$;

(4) $\tan^3\alpha + \cot^3\alpha$.

QUIZ No. 1

Form I

(1) Compute:

$$\frac{(\sin(\pi/4) + \cos(3\pi/2))\cdot\tan(\pi/3)}{\cot(\pi/6) - \cot(\pi/2)}.$$

(2) Determine the sign of the expression

$$\frac{\cos 100^\circ \tan 250^\circ}{\sin 300^\circ \cot 100^\circ}.$$

(3) Evaluate the remaining trigonometric functions of the angle α if $\sin\alpha = -3/5$ and $\alpha \in (3\pi/2, 2\pi)$.

(4) Evaluate the remaining trigonometric functions of the angle α if $\cot\alpha = \sqrt{3}/3$ and $\alpha \in (\pi, 3\pi/2)$.

Form II

(1) Compute:

$$\frac{2\tan(\pi/4)\,(\tan(\pi/3) + \cos(\pi/6))}{\cos\pi - 2\sin(3\pi/2)}.$$

(2) Determine the sign of the expression

$$\frac{\tan 150^\circ \sin 200^\circ}{\cos 320^\circ \cot 140^\circ}.$$

(3) Evaluate the remaining trigonometric functions of the angle α if $\cos\alpha = -4/5$ and $\alpha \in (\pi/2, \pi)$.

(4) Evaluate the remaining trigonometric functions of the angle α if $\tan\alpha = \sqrt{3}$ and $\alpha \in (\pi, 3\pi/2)$.

(5) Prove the identity $\frac{\tan\alpha}{1-\tan^2\alpha}\times\frac{\cot^2\alpha-1}{\cot\alpha}=1$ and indicate the permissible values for α.

(5) Prove the identity $\frac{\tan\alpha}{1+\cot\alpha}+\frac{\cot\alpha}{1+\tan\alpha}=\tan\alpha+\cot\alpha-1$ and indicate the permissible values for α.

Sec. 9.6. Periodicity of Trigonometric Functions

A function f is said to be *periodic* if there is a number $\lambda\neq 0$ such that for any α from the domain of definition of the function the numbers $\alpha-\lambda$ and $\alpha+\lambda$ also belong to this domain and the following equality is fulfilled:

$$f(\alpha-\lambda)=f(\alpha)=f(\alpha+\lambda).$$

In this case, the number λ is called the *period* of the function f, the numbers of the form $n\lambda$, $n\in\mathbf{Z}$, $n\neq 0$ are also its periods. The smallest positive period for sine and cosine is equal to 2π and for tangent and cotangent it is equal to π.

The periodicity property of trigonometric functions can be expressed by the following identities:

$$\sin\alpha=\sin(\alpha+2\pi k),\ k\in\mathbf{Z}, \tag{9.26}$$

$$\cos\alpha=\cos(\alpha+2\pi k),\ k\in\mathbf{Z}, \tag{9.27}$$

$$\tan\alpha=\tan(\alpha+\pi k),\ k\in\mathbf{Z}, \tag{9.28}$$

$$\cot\alpha=\cot(\alpha+\pi k),\ k\in\mathbf{Z}. \tag{9.29}$$

80. Compute:

(1) $\cos 3660°$, (2) $2\cos 4.5\pi+\sin(19\pi/3)$, (3) $\sin(-300°)-\tan(-150°)$.

○ On the basis of the property of periodicity of cosine and sine we get:

(1) $\cos 3660°=\cos(360°\cdot 10+60°)=\cos 60°=1/2$,

(2) $2\cos 4.5\pi+\sin(19\pi/3)\times 2\cos(2\pi\cdot 2+0.5\pi)+\sin(2\pi\cdot 3+\pi/3)=2\cos 0.5\pi+\sin(\pi/3)=0+\sqrt{3}/2=\sqrt{3}/2$.

(3) Adding one period to each of the arguments, we obtain

$$\sin(-300°)-\tan(-150°)=\sin(-300°+360°)-\tan(-150°+180°)$$
$$=\sin 60°-\tan 30°=\sqrt{3}/2-\sqrt{3}/3=\sqrt{3}/6.$$ ●

81. Find the periods of the functions: (1) $y=\sin 3x$, (2) $y=\cos(x/2)$.

○ (1) Denoting the desired period by λ we get

$$\sin 3(x+\lambda)=\sin 3x \text{ or } \sin(3x+3\lambda)=\sin 3x.$$

Hence we conclude that $3\lambda=2\pi$, i.e. $\lambda=2\pi/3$.

(2) Similarly, we have

$$\cos\left(\frac{x+\lambda}{2}\right)=\cos\frac{x}{2} \quad\text{or}\quad \cos\left(\frac{x}{2}+\frac{\lambda}{2}\right)=\cos\frac{x}{2},$$

whence $\lambda/2=2\pi$, i.e. $\lambda=4\pi$. ●

82. Find the periods of the indicated functions:
(1) $y = \sin 2x + \cos 3x$, (2) $y = \sin(3x/2) + \sin(2x/3)$.

○ (1) We find the period of each of the addends:

$$\sin 2(x + \lambda_1) = \sin 2x, \quad \sin(2x + 2\lambda_1) = \sin 2x,$$
$$2\lambda_1 = 2\pi, \quad \lambda_1 = \pi;$$
$$\cos 3(x + \lambda_2) = \cos 3x, \quad \cos(3x + 3\lambda_2) = \cos 3x,$$
$$3\lambda_2 = 2\pi, \quad \lambda_2 = 2\pi/3.$$

Every number multiple of the period is itself a period, therefore the common multiple of the numbers λ_1 and λ_2 is the period of the function y. The least common multiple of the numbers π and $2\pi/3$ equal to the least common multiple of the numerators of the periods λ_1 and λ_2 is 2π.

(2) We have

$$\sin\frac{3(x+\lambda_1)}{2} = \sin\frac{3x}{2}, \quad \sin\left(\frac{3x}{2} + \frac{3\lambda_1}{2}\right) = \sin\frac{3x}{2},$$
$$3\lambda_1/2 = 2\pi, \quad \lambda_1 = 4\pi/3;$$
$$\sin\frac{2(x+\lambda_2)}{3} = \sin\frac{2x}{3}, \quad \sin\left(\frac{2x}{3} + \frac{2\lambda_2}{3}\right) = \sin\frac{2x}{3};$$
$$2\lambda_2/3 = 2\pi, \quad \lambda_2 = 3\pi.$$

The least common multiple of the numerators of the periods λ_1 and λ_2 is equal to 12π; consequently, the period of the given function is equal to 12π. ●

83. Compute: (1) $\cos 7230°$; (2) $\sin 900°$; (3) $\tan 585°$; (4) $\cot 750°$; (5) $\sin 1843°$.

84. Find: (1) $\sin 6.2\pi + \cos 4.1\pi$; (2) $\tan(13\pi/4) + \cot(21\pi/4)$; (3) $\sin(19\pi/3) - \cos(19\pi/3)$; (4) $\sin(82\pi - 0.192) + \cos(22\pi + 1.501)$; (5) $\sin 7.854 \cdot \tan 3.927$.

85. Find the periods of the following functions: (1) $y = \cos 3x$; (2) $y = \cos(x/4)$; (3) $y = \tan 2x$; (4) $y = \cot(x/5)$.

86. Find the periods of the following functions: (1) $y = \sin 5x - \cos 4x + 1$; (2) $y = 2\sin(x/4) - 3\sin(x/3)$; (3) $y = \tan(2x/3) - 4\cot(3x/2) - 2$; (4) $y = \sin(3x/4) - 3\cos(5x/8) + \cos 5x$.

87. Compute: (1) $2\sin 750° - 3\cos 900° + \tan 405°$; (2) $\tan^2 600° + \cot^2 585° + 3$; (3) $\sin(-330°) + \sin(-690°)$; (4) $\tan(-135°) - \tan 225°$; (5) $\cos(-3\pi) - \sin(-7\pi)$; (6) $\dfrac{\sin(-11\pi/2) + \tan(-5\pi)}{\cos(-5\pi) + \cot(-21\pi/4)}$; (7) $\sin(-5\pi/2)\cos 8\pi$.

88. Simplify:

(1) $\sin^2(6\pi - \alpha) + \sin^2(10\pi + \alpha)$;

(2) $\sin^2\left(\frac{\pi}{3} - 4\pi\right) + \cos^2\left(8\pi - \frac{\pi}{3}\right) + 2$;

(3) $\cos(\alpha - 6\pi) + \cos(12\pi + \alpha)$;

(4) $\sin^2(2\pi + \alpha) + \cos^2(6\pi - \alpha) + 1$.

89. Prove the following identities and indicate the domains of their definition:

(1) $\dfrac{\cos^2(4\pi-\alpha)}{\tan^2(9\pi+\alpha)+1}=\cos^4\alpha$; (2) $\dfrac{\cos(6\pi+\alpha)\tan(3\pi-\alpha)}{\sin(4\pi-\alpha)\cot(5\pi+\alpha)}=\tan\alpha$;

(3) $\dfrac{\cot(13\pi-x)+\tan(4\pi+x)}{\tan(5\pi+x)-\cot(7\pi+x)}=1$;

(4) $\sin(6\pi-x)\cos(8\pi-x)\tan(9\pi-x)\cot(10\pi-x)=-\sin x\cos x$.

Sec. 9.7. Inverse Trigonometric Functions

The function $y=\sin x$ is reversible on the interval $[-\pi/2,\ \pi/2]$ that is, it has an inverse function which is called *arcsine* and is denoted as $y=\arcsin x$:

$$D(\arcsin x)=[-1,\ 1],\ E(\arcsin x)=[-\pi/2,\ \pi/2];$$

$$\sin(\arcsin x)=x,\ \text{where } x\in[-1,\ 1];\ \arcsin(-x)=-\arcsin x.$$

The function $y=\cos x$ is reversible on the interval $[0,\ \pi]$, that is, it has an inverse function which is called *arccosine* and is denoted as $y=\arccos x$:

$$D(\arccos x)=[-1,\ 1],\ E(\arccos x)=[0,\ \pi];$$

$$\cos(\arccos x)=x,\ \text{where } x\in[-1,\ 1];\ \arccos(-x)=\pi-\arccos x.$$

The function $y=\tan x$ is reversible on the interval $(-\pi/2,\ \pi/2)$, that is, it has an inverse function which is called *arctangent* and is denoted as $y=\arctan x$:

$$D(\arctan x)=\mathbf{R},\ E(\arctan x)=(-\pi/2,\ \pi/2);$$

$$\tan(\arctan x)=x,\ \text{where } x\in\mathbf{R},\ \arctan(-x)=-\arctan x.$$

The function $y=\cot x$ is reversible on the interval $(0,\ \pi)$ that is, it has an inverse function which is called *arccotangent* and is denoted as $y=\operatorname{arccot} x$:

$$D(\operatorname{arccot} x)=\mathbf{R},\ E(\operatorname{arccot} x)=(0,\ \pi);$$

$$\cot(\operatorname{arccot} x)=x,\ \text{where } x\in\mathbf{R},\ \operatorname{arccot}(-x)=\pi-\operatorname{arccot} x.$$

90. Check whether the given equalities are valid: (1) $\arcsin(1/2)=\pi/6$; (2) $\arccos(-\sqrt{3}/2)=5\pi/6$; (3) $\arctan\sqrt{3}=-\pi/3$.

○ (1) $\arcsin(1/2)=\pi/6$, since $\sin(\pi/6)=1/2$ and $\pi/6\in[-\pi/2,\ \pi/2]$; (2) $\arccos(-\sqrt{3}/2)=5\pi/6$, since $\cos(5\pi/6)=-\sqrt{3}/2$ and $5\pi/6\in[0,\ \pi]$ (3) $\arctan\sqrt{3}=-\pi/3$, since $\tan(-\pi/3)=-\sqrt{3}$ and $-\pi/3\in(-\pi/2,\ \pi/2)$. ●

91. Compute the following: (1) $\arcsin 0.7880$; (2) $\arccos 0.9063$; (3) $\arctan 2.145$; (4) $\operatorname{arccot} 0.9657$.

○ Using four-place tables, we find:

(1) $\arcsin 0.7880=\alpha$, $\sin\alpha=0.7880$, $\alpha=52^\circ\approx 0.9076$, $\arcsin 0.7880\approx 0.9076$;

(2) $\arccos 0.9063=\alpha$, $\cos\alpha=0.9063$, $\alpha=25^\circ\approx 0.4363$, $\arccos 0.9063\approx 0.4363$; (3) $\arctan 2.145=\alpha$, $\tan\alpha=2.145$, $\alpha=65^\circ\approx 1.1345$, $\arctan 2.145\approx 1.1345$;

(4) $\operatorname{arccot} 0.9657 = \alpha$, $\cot \alpha = 0.9657$, $\alpha = 46° \approx 0.8029$, $\operatorname{arccot} 0.9657 \approx 0.8029$. ●

92. Check whether the given equalities are true: (1) $\arcsin (\sqrt{2}/2) = \pi/4$; (2) $\arcsin (-1/2) = -\pi/6$; (3) $\arccos (\sqrt{3}/2) = \pi/6$; (4) $\arccos (\sqrt{2}/2) = 3\pi/4$; (5) $\arctan 1 = \pi/4$; (6) $\arctan \sqrt{3} = \pi/3$; (7) $\arctan (-\sqrt{3}/3) = \pi/6$; (8) $\operatorname{arccot} \sqrt{3} = -2\pi/3$.

93. Compute: (1) $\arcsin 0.4067$; (2) $\arcsin 0.9962$; (3) $\arccos 0.9848$; (4) $\arccos 0.1736$; (5) $\arctan 0.2679$; (6) $\arctan 2.747$; (7) $\operatorname{arccot} 2.145$; (8) $\operatorname{arccot} 0.1944$.

94. Compute: (1) $\arcsin (-\sqrt{3}/2)$; (2) $\arccos (-1/2)$; (3) $\arctan (-\sqrt{3}/3)$; (4) $\operatorname{arccot} (-1)$; (5) $\operatorname{arccot} (-\sqrt{3})$; (6) $\arcsin (-0.9033)$; (7) $\arccos (-0.8965)$; (8) $\arctan (-1.4659)$; (9) $\operatorname{arccot} (-1.3663)$; (10) $\operatorname{arccot} (-0.3096)$.

95. Prove that the following inequalities are true: (1) $\arcsin (1/2) < \arccos (1/2)$; (2) $\arccos 0 > \arcsin 0$; (3) $\arcsin (1/4) > \arcsin (1/6)$; (4) $\arctan \sqrt{3} > \operatorname{arccot} \sqrt{3}$.

96. Compute: (1) $\arcsin (\sqrt{2}/2) + \arccos (\sqrt{2}/2)$, (2) $\arcsin (-1/2) + \arccos (-1/2)$; (3) $\arctan (-1) + \operatorname{arccot} (-1)$; (4) $\arcsin 1 + \arccos 1 + \arctan 1 + \operatorname{arccot} 1$; (5) $\arcsin (-1) + \arccos (-1)$; (6) $\arcsin 0 + \arccos 0$; (7) $\arcsin 0 + \arcsin 1 + \arcsin (-1)$; (8) $\arccos 0 + \arccos 1 + \arccos (-1)$.

Sec. 9.8. Constructing the Arc (Angle) Given the Value of a Trigonometric Function

97. Find the set of arcs α whose sine is equal to a.

○ On the axis OY of the unit circle construct a point $N(0, a)$ and draw through it a straight line parallel to the axis OX. Consider two cases.

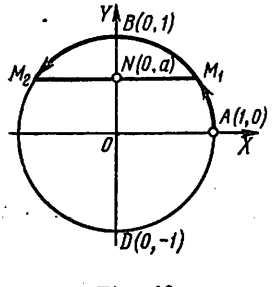


Fig. 40

Case I. Let $|a| < 1$, then the straight line $y = a$ will intersect the unit circle at points M_1 and M_2 (Fig. 40) which are symmetric with respect to the axis OY. To the point M_1 there corresponds the arc $\overset{\frown}{A\ }_1 = \arcsin a$, and to the point M_2 the arc $\pi - \arcsin a$. Either of these arcs has a sine equal to a. The set of arcs terminating in the point M_1 and having a sine equal to a is expressed by the formula

$$\alpha = \arcsin a + 2\pi k \quad (k \in \mathbf{Z}),$$

and the set of arcs terminating in point M_2 and having a sine also equal to a is expressed by the formula

$$\alpha = \pi - \arcsin a + 2\pi k,$$

or

$$\alpha = -\arcsin a + \pi (2k + 1) \ (k \in \mathbf{Z}).$$

Since $(-1)^n = 1$ for $n = 2k$ (i.e. if n is even) and $(-1)^n = -1$ for $n = 2k + 1$ (i.e. if n is odd), these two formulas can be united into one:

$$\alpha = (-1)^n \arcsin a + \pi n \ (n \in \mathbf{Z}).$$

Case II. Let $a = \pm 1$, then the point $N\ (0, a)$ coincides with the point $B\ (0, 1)$ if $a = 1$ and with the point $D\ (0, -1)$ if $a = -1$

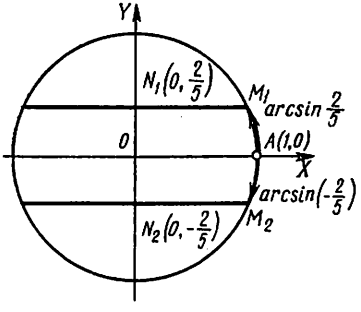


Fig. 41

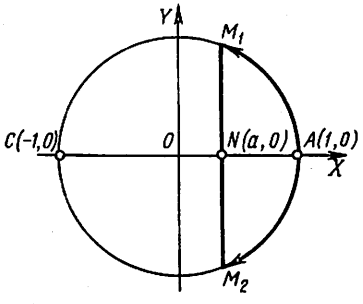


Fig. 42

(Fig. 40). The set of arcs terminating in the point $B\ (0, 1)$ (for $a = 1$) is expressed by the formula

$$\alpha = \pi/2 + 2\pi k \ (k \in \mathbf{Z}),$$

and the set of arcs terminating in the point $D\ (0, -1)$ (for $a = 1$), by the formula

$$\alpha = -\pi/2 + 2\pi k \ (k \in \mathbf{Z}).$$

Of all the arcs (angles) whose sine is equal to a where $|a| \leqslant 1$, the arc $\arcsin a$ from the interval $[-\pi/2, \pi/2]$ is regarded to be the principal one. ●

98. Write the principal arcs whose sine is equal to: (1) 0; (2) —1; (3) 1; (4) $\sqrt{3}/2$; (5) —1/2.

○ (1) $\alpha = \arcsin 0 = 0$; (2) $\alpha = \arcsin(-1) = -\arcsin 1 = -\pi/2$; (3) $\alpha = \arcsin 1 = \pi/2$; (4) $\alpha = \arcsin(\sqrt{3}/2) = \pi/3$; (5) $\alpha = \arcsin(-1/2) = -\arcsin(1/2) = -\pi/6$. ●

99. Construct the principal arcs $\arcsin(2/5)$ and $\arcsin(-2/5)$.

○ The construction is carried out in Fig. 41:
$\overset{\smile}{AM}_1 = \arcsin(2/5)$, $\overset{\smile}{AM}_2 = \arcsin(-2/5) = -\arcsin(2/5)$. ●

100. Write the set or arcs whose sine is equal to 1/2.

○ There are two points serving as the terminal points of the arcs α_1 and α_2 whose sine is equal to 1/2:

$$\alpha_1 = \arcsin(1/2) = \pi/6 \quad \text{and} \quad \alpha_2 = \pi - \arcsin(1/2) = \pi - \pi/6.$$

Consequently, the desired set of arcs is expressed by the formulas $\alpha = \pi/6 + 2\pi k$ and $\alpha = \pi - \pi/6 + 2\pi k = -\pi/6 + \pi(2k + 1)$, or

$$\alpha = (-1)^n \pi/6 + \pi n \ (n \in \mathbf{Z}). \quad \bullet$$

101. Find the set of arcs α whose cosine is equal to a.

○ On the axis OX of the unit circle we construct the point $N(a, 0)$ and draw through it a straight line parallel to the axis OY. Consider two cases.

Case I. Let $|a| < 1$, then the straight line $x = a$ will intersect the unit circle at the points M_1 and M_2 which are symmetric with respect to the axis OX (Fig. 42). To the point M_1 there corresponds the arc $\overset{\smile}{AM}_1 = \arccos a$, and to the point M_2 the arc $\overset{\smile}{AM}_2 = -\arccos a$. Each of the arcs has a cosine equal to a. The set of arcs terminating in the point M_1 and having the cosine equal to a is expressed by the formula

$$\alpha = \arccos a + 2\pi k \ (k \in \mathbf{Z}),$$

and the set of arcs terminating in the point M_2 and having the cosine also equal to a, by the formula

$$\alpha = -\arccos a + 2\pi k \ (k \in \mathbf{Z}).$$

These two formulas can be combined into one:

$$\alpha = \pm \arccos \alpha + 2\pi k \ (k \in \mathbf{Z}).$$

Case II. Let $a = \pm 1$; then the point $N(a, 0)$ coincides with the point $A(1, 0)$ if $a = 1$, and with the point $C(-1, 0)$ if $a = -1$ (Fig. 42). The set of arcs terminating in the point $A(1, 0)$ (for $a = 1$) is expressed by the formula

$$\alpha = 2\pi k \ (k \in \mathbf{Z}),$$

and the set of arcs terminating in the point $C(-1, 0)$ (for $a = -1$) by the formula

$$\alpha = \pi + 2\pi k = \pi(2k + 1) \ (k \in \mathbf{Z}).$$

Of all the arcs (angles) whose cosine is equal to a, where $|a| \leqslant 1$, the arc $\arccos a$ from the interval $[0, \pi]$ is regarded to be the principal one. ●

102. Write the principal arcs whose cosine is equal to: (1) 0; (2) 1; (3) -1; (4) 1/2; (5) $-\sqrt{2}/2$; (6) $-3/4$.

○ (1) $\alpha = \arccos 0 = \pi/2$, (2) $\alpha = \arccos 1 = 0$, (3) $\alpha = \arccos(-1) = \pi$, (4) $\alpha = \arccos(1/2) = \pi/3$, (5) $\alpha = \arccos(-\sqrt{2}/2) = \pi - \arccos(\sqrt{2}/2) = \pi - \pi/4 = 3\pi/4$, (6) $\alpha = \arccos(-3/4) = \pi - \arccos 3/4$). ●

103. Construct the principal arcs arccos (2/3) and arccos (—2/3).

○ The construction is shown in Fig. 43: $\overset{\smile}{AM_1}$ = arccos (2/3) and AM_2 = arccos (—2/3) = π — arccos (2/3). ●

104. Write the set of arcs whose cosine is equal to 1/2.

○ There are two points serving as the terminal points of the arcs α_1 and α_2 whose cosine is equal to 1/2: α_1 = arccos (1/2) = $\pi/3$ and α_2 = —arccos (1/2) = $-\pi/3$. Consequently, the desired set of arcs is expressed by the formula

$$\alpha = \pm\pi/3 + 2\pi k \ (k \in Z). \bullet$$

105. Find the set of arcs α whose tangent is equal to a.

○ On the axis of tangents (Fig. 44), we construct the point N (1, a). Through this point and the origin draw a straight line to intersect

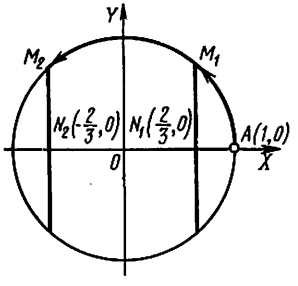


Fig. 43

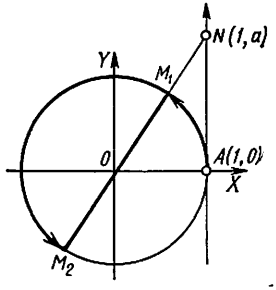


Fig. 44

the unit circle at the points M_1 and M_2. The tangent of the arcs AM_1 and AM_2 is equal to the ordinate a of the point N which is the point of intersection of the extension of the radius OM_1 and the axis of tangents. To the point M_1 there corresponds the arc $\overset{\smile}{AM_1}$ = arctan a, and to the point M_2 the arc $\overset{\smile}{AM_2}$ = arctan $a + \pi$. Each of these arcs has the tangent equal to a. The set of arcs terminating at the points M_1 and M_2 is represented by the formula

$$\alpha = \arctan a + \pi k \ (k \in Z).$$

Of all the arcs (angles) having the given tangent a, the arc arctan a from the interval $(-\pi/2, \pi/2)$ is regarded to be the principal one. ●

106. Write the principal arcs whose tangent is equal to: (1) 0; (2) $\sqrt{3}$; (3) $-\sqrt{3}/3$; (4) 1; (5) —1.

○ (1) α = arctan 0 = 0; (2) α = arctan $\sqrt{3} = \pi/3$; (3) α = arctan $(-\sqrt{3}/3) = -\arctan(\sqrt{3}/3) = -\pi/6$; (4) α = arctan 1 = $\pi/4$; (5) α = arctan (—1) = — arctan 1 = $-\pi/4$. ●

107. Construct the principal arcs arctan (4/3) and arctan (—4/3).

○ The construction is represented in Fig. 45: $\overset{\smile}{AM_1}$ = arctan (4/3); $\overset{\smile}{AM_2}$ = arctan (—4/3) = —arctan (4/3). ●

108. Write the set of arcs whose tangent is equal to $\sqrt{3}$.

○ On the circle, there are two points serving as terminal points of the arcs α_1 and α_2 whose tangent is equal to $\sqrt{3}$: $\alpha_1 = \arctan \sqrt{3} = \pi/3$ and $\alpha_2 = \arctan \sqrt{3} + \pi = \pi/3 + \pi$. Consequently, the desired set of arcs is expressed by the formula

$$\alpha = \pi/3 + \pi k \ (k \in \mathbf{Z}). \bullet$$

109. Find the set of arcs α whose cotangent is equal to a.

○ On the axis of cotangents (Fig. 46), we construct the point $N\,(a, 1)$. Through this point and the origin, draw a straight line to intersect the unit circle at the points M_1 and M_2. The cotangent of

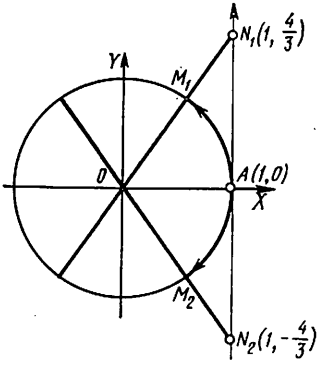


Fig. 45

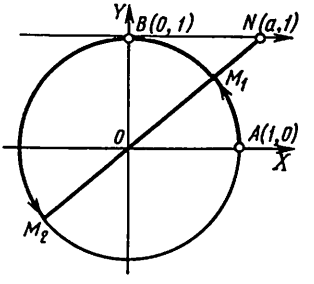


Fig. 46

the arcs AM_1 and AM_2 is equal to the abscissa a of the point N which is the point of intersection of the extension of the radius OM_1 and the axis of cotangents. To the point M_1 there corresponds the arc $\overset{\smile}{AM}_1 = \operatorname{arccot} a$, and to the point M_2, the arc $\overset{\smile}{AM}_2 = \operatorname{arccot} a + \pi$. Each of the arcs has the cotangent equal to a. The set of arcs terminating in the points M_1 and M_2 is written by one formula

$$\alpha = \operatorname{arccot} a + \pi k \ (k \in \mathbf{Z}).$$

Of all the arcs (angles) having the given cotangent a, the arc $\operatorname{arccot} a$ from the interval $(0, \pi)$ is regarded to be the principal one. ●

110. Write the principal arcs whose cotangent is equal to: (1) $\sqrt{3}/3$; (2) -1; (3) $\sqrt{3}$; (4) $-\sqrt{3}$.

○ (1) $\alpha = \operatorname{arccot}(\sqrt{3}/3) = \pi/3$; (2) $\alpha = \pi - \operatorname{arccot} 1 = \pi - \pi/4 = 3\pi/4$; (3) $\alpha = \operatorname{arccot} \sqrt{3} = \pi/6$; (4) $\alpha = \pi - \operatorname{arccot} \sqrt{3} = \pi - \pi/6 = 5\pi/6$. ●

111. Construct the principal arcs $\operatorname{arccot} 1$ and $\operatorname{arccot}(-1)$.

○ The construction is fulfilled in Fig. 47: $\overset{\smile}{AM}_1 = \operatorname{arccot} 1 = \pi/4$; $\overset{\smile}{AM}_2 = \pi - \operatorname{arccot} 1 = \pi - \pi/4 = 3\pi/4$. ●

112. Write the set of arcs whose tangent is equal to $\sqrt{3}$.

○ On the circle, there are two points serving as the terminal points of the arcs α_1 and α_2 whose cotangent is equal to $\sqrt{3}$: $\alpha_1 = \operatorname{arccot} \sqrt{3} = \pi/6$ and $\alpha_2 = \operatorname{arccot} \sqrt{3} + \pi = \pi/6 + \pi$. Consequently, the desired set of arcs is expressed by the formula

$$\alpha = \pi/6 + \pi k \ (k \in Z). \quad \bullet$$

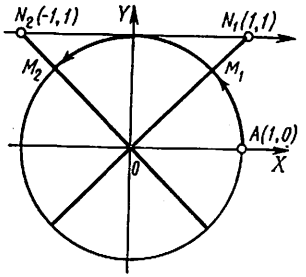


Fig. 47

113. Write the principal arcs whose sine is equal to: (1) 1/2; (2) $\sqrt{2}/2$; (3) $-\sqrt{2}/2$; (4) $-\sqrt{3}/2$; (5) 3/4.

114. Write the set of arcs whose sine is equal to: (1) $\sqrt{3}/2$; (2) $-1/2$; (3) 1; (4) $-\sqrt{2}/2$.

115. Construct the arcs whose sine is equal to: (1) 1/2; (2) $-2/3$; (3) 0.6. Write the set of arcs corresponding to these values of sine.

116. Write the set of arcs whose cosine is equal to: (1) $-1/2$; (2) $\sqrt{3}/2$; (3) $-\sqrt{2}/2$; (4) $-\sqrt{3}/2$.

117. Construct the arcs whose cosine is equal to: (1) 4/5; (2) $-4/5$; (3) 0.6. Write the set of arcs corresponding to these values of cosine.

118. Write the principal arcs whose tangent is equal to: (1) $\sqrt{3}/3$; (2) $-\sqrt{3}$; (3) 1/2; (4) -0.7.

119. Write the set of arcs whose tangent is equal to: (1) 0; (2) $\sqrt{3}/3$; (3) -1; (4) $-\sqrt{3}$; (5) $\sqrt{2}$.

120. Construct the arcs whose tangent is equal to: (1) $-2/3$; (2) 2; (3) -1.5. Write the set of arcs corresponding to these values of the tangent.

121. Write the set of arcs whose cotangent is equal to: (1) $-\sqrt{3}/3$; (2) $-\sqrt{3}$; (3) 1; (4) 1/2.

122. Construct the arcs whose cotangent is equal to: (1) -2; (2) 0.8; (3) $-2/3$. Write the set of arcs corresponding to these values of cotangent.

Sec. 9.9. Trigonometric Equations

Simplest trigonometric equations are defined as equations of the form

$$\sin x = m, \ \cos x = m, \ \tan x = m, \ \cot x = m,$$

where m is a given number.

To solve a simplest trigonometric equation means to find the set of all values of the argument (arcs or angles) for which a given trigonometric function takes on a given value m.

In Problems 123 through 131 we shall solve the indicated equations:

123. (1) $\sin x = m$; (2) $\sin x = 1/2$.

○ (1) If $|m| \leqslant 1$, then, on the unit circle, there are two arcs: $\arcsin m$ and $\pi - \arcsin m$ whose sine is equal to m, and whose terminal points are symmetric with respect to the axis OY (Fig. 48).

The arc $\arcsin m$ from the interval $[-\pi/2, \pi/2]$ whose sine is equal to m is called the principal solution of the equation $\sin x = m$. The

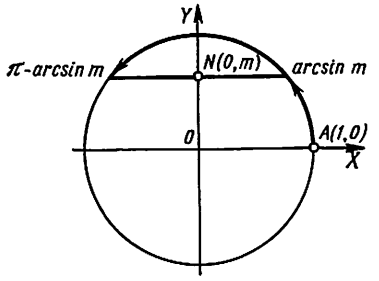


Fig. 48

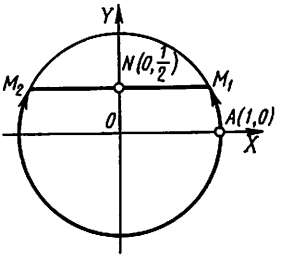


Fig. 49

set of all desired arcs satisfying the equation $\sin x = m$ are found by adding any integral number of periods of sine to the two arcs:

$$x = \begin{cases} \arcsin m + 2\pi k, \\ \pi - \arcsin m + 2\pi k \end{cases} \quad \text{or} \quad x = \begin{cases} \arcsin m + 2\pi k, \\ -\arcsin m + \pi\,(2k+1). \end{cases}$$

The set of roots of the equation can be expressed by one formula (see Problem 97):

$$x = (-1)^n \arcsin m + \pi n \ (n \in \mathbf{Z}).$$

If $|m| > 1$, then the equation has no solutions.

Particular cases:

$$\sin x = -1, \quad x = -\pi/2 + 2\pi k, \quad k \in \mathbf{Z};$$

$$\sin x = 0, \quad x = \pi k, \quad k \in \mathbf{Z};$$

$$\sin x = 1, \quad x = \pi/2 + 2\pi k, \quad k \in \mathbf{Z}.$$

(2) The principal solution is represented by the arc $\overset{\smile}{AM}_1 = \pi/6$ from the interval $[-\pi/2, \pi/2]$, whose sine is $1/2$ (Fig. 49). The set of the equation roots has the form $\{(-1)^n \pi/6 + \pi n \mid n \in \mathbf{Z}\}$. ●

124. (1) $\cos x = m$; (2) $\cos x = -1/2$.

○ (1) If $|m| \leqslant 1$, then on the unit circle there are two arcs symmetric with respect to the axis OX: $\overset{\smile}{AM}_1 = \arccos m$ and $\overset{\smile}{AM}_2 = -\arccos m$, whose cosine is equal to m (Fig. 50).

The arc $\arccos m$ from the interval $[0, \pi]$ whose cosine is equal to m is called the principal solution of the equation $\cos x = m$. The set of all desired arcs satisfying the equation $\cos x = m$ is found by

adding any integral number of periods of cosine to the two found arcs:

$$x = \pm \arccos m + 2\pi k, \quad k \in \mathbf{Z}.$$

If $|m| > 1$, then the equation has no solutions.

Particular cases:

$$\cos x = -1, \quad x = \pm\pi + 2\pi k, \text{ or } x = \pi(2k+1), \quad k \in \mathbf{Z};$$

$$\cos x = 0, \quad x = \pi/2 + \pi k, \quad k \in \mathbf{Z};$$

$$\cos x = 1, \quad x = 2\pi k, \quad k \in \mathbf{Z}.$$

(2) The principal solution is represented by the arc $\overset{\smile}{AM_1} = \pi - \pi/3 = 2\pi/3$ from the interval $[0, \pi]$ whose cosine is equal to

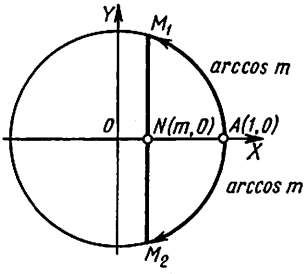


Fig. 50

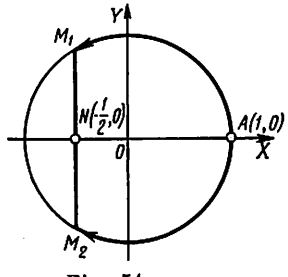


Fig. 51

$-1/2$ (Fig. 51). The set of roots of the equation has the form $\{\pm 2\pi/3 + 2\pi k, \ k \in \mathbf{Z}\}$. ●

125. (1) $\tan x = m$; $\tan x = \sqrt{3}$.

○ (1) The arc $\arctan m$ from the interval $(-\pi/2, \pi/2)$ whose tangent is equal to m is termed the principal solution of the equation $\tan x = m$. The set of all desired arcs satisfying the equation $\tan x = m$ is found by adding any integral number of periods of tangent:

$$x = \arctan m + \pi k, \quad k \in \mathbf{Z}.$$

A particular case: $\tan x = 0$, $x = \pi k$, $k \in \mathbf{Z}$.

(2) The arc $\pi/3$ from the interval $(-\pi/2, \pi/2)$ whose tangent is equal to $\sqrt{3}$ is the principal solution (Fig. 52). The set of the equation roots has the form $\{\pi/3 + \pi k \mid k \in \mathbf{Z}\}$. ●

126. (1) $\cot x = m$; (2) $\cot x = -1$.

○ (1) The arc $\operatorname{arccot} m$ from the interval $(0, \pi)$ whose cotangent is equal to m is called the principal solution of the equation $\cot x = m$. The set of all desired arcs satisfying the equation $\cot x = m$ is found by adding any integral number of periods of cotangent:

$$x = \operatorname{arccot} m + \pi k, \quad k \in \mathbf{Z}.$$

A particular case: $\cot x = 0$, $x = \pi/2 + \pi k$, $k \in \mathbf{Z}$.

(2) The principal solution is represented by the arc $\smile AM = \pi - \pi/4 = 3\pi/4$ from the interval $(0, \pi)$ whose cotangent is equal to -1 (Fig. 53). The set of the equation roots has the form $\{3\pi/4 + \pi k \mid k \in Z\}$. ●

127. (1) $\sin 2x = 1/2$; (2) $\tan(3x + 2) = -1$; (3) $\cos(\cos x) = 1/2$.

○ (1) $2x = (-1)^k \pi/6 + \pi k$; the set of the roots of the equation has the form

$$\left\{(-1)^k \frac{\pi}{12} + \frac{\pi k}{2} \,\middle|\, k \in \mathbf{Z}\right\};$$

(2) $3x + 2 = -\pi/4 + \pi k$; $3x = -\pi/4 - 2 + \pi k$; the set of the roots of the equation has the form

$$\left\{-\frac{\pi}{12} - \frac{2}{3} + \frac{\pi k}{3} \,\middle|\, k \in \mathbf{Z}\right\};$$

(3) $\cos x = \pm\pi/3 + 2\pi k$; the last equation has no roots, since for any $k \in \mathbf{Z}$ its right-hand side exceeds unity in absolute value. ●

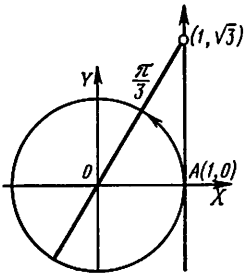


Fig. 52

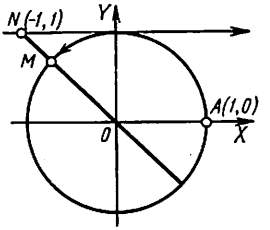


Fig. 53

128. (1) $\sin^2 x = m$; (2) $\cos^2 x = m$ $(0 \leqslant m \leqslant 1)$.

○ (1) $\sin^2 x = m \Leftrightarrow \left[\begin{array}{l} \sin x = \sqrt{m}, \\ \sin x = -\sqrt{m} \end{array}\right.$

$$\Leftrightarrow \left[\begin{array}{l} x = \pi k + (-1)^k \arcsin \sqrt{m}, \\ x = \pi k - (-1)^k \arcsin \sqrt{m}, \quad k \in \mathbf{Z}. \end{array}\right.$$

In this notation of the solution the factor $(-1)^k$, controlling the sign of the second terms, is unnecessary. If for some integral k in front of $\arcsin \sqrt{m}$ the plus sign is taken in the first formula, then for the same k the minus sign is taken in the second formula and vice versa (depending on whether k is even or odd). Therefore the two

formulas can be combined into a simpler one:

$$x = \pi n \pm \arcsin \sqrt{m}, \quad n \in \mathbf{Z}.$$

(2) $\cos^2 x = m \Leftrightarrow \left[\begin{array}{l} \cos x = \sqrt{m}, \\ \cos x = -\sqrt{m} \end{array}\right.$

$\Leftrightarrow \left[\begin{array}{l} x = 2\pi k \pm \arccos \sqrt{m}, \\ x = 2\pi k \pm (\pi - \arccos \sqrt{m}, \quad k \in \mathbf{Z} \end{array}\right.$

$\Leftrightarrow \left[\begin{array}{l} x = 2\pi k \pm \arccos \sqrt{m}, \\ x = \pi(2k \pm 1) \pm \arccos \sqrt{m}, \quad k \in \mathbf{Z}. \end{array}\right.$

Combining both formulas, we get

$$x = \pi n \pm \arccos \sqrt{m}, \; n \in \mathbf{Z}. \bullet$$

129. (1) $\tan^2 x = m$; (2) $\cot^2 x = m$ $(m > 0)$.

○ (1) $\tan^2 x = m \Leftrightarrow \left[\begin{array}{l} \tan x = \sqrt{m}, \\ \tan x = -\sqrt{m} \end{array}\right.$

$\Leftrightarrow \left[\begin{array}{l} x = \pi k + \arctan \sqrt{m}, \\ x = \pi k - \arctan \sqrt{m}, \quad k \in \mathbf{Z} \end{array}\right. \Leftrightarrow x = \pi k \pm \arctan \sqrt{m}, \quad k \in \mathbf{Z}.$

(2) Analogously, we find $x = \pi k \pm \operatorname{arccot} \sqrt{m}$, $k \in \mathbf{Z}$. ●

130. (1) $2 \sin^2 x - 7 \sin x + 3 = 0$;

(2) $4 \cos^2 x + \sin x - 1 = 0$; (3) $2 \cot 3x + \tan 3x + 3 = 0$;

(4) $\tan x \cos x + \tan x - \cos x - 1 = 0$; (5) $\tan^3 x = \tan x$.

○ (1) We solve this equation with respect to $\sin x$:

$2 \sin^2 x - 7 \sin x + 3 = 0 \Leftrightarrow \left[\begin{array}{l} \sin x = 1/2, \\ \sin x = 3 \end{array}\right.$

$\Leftrightarrow \left[\begin{array}{l} x = (-1)^k \pi/6 + \pi k, \quad k \in \mathbf{Z}, \\ \varnothing. \end{array}\right.$

Answer: $\{(-1)^k \pi/6 + \pi k \mid k \in \mathbf{Z}\}$.

(2) We have

$4(1 - \sin^2 x) + \sin x - 1 = 0 \Leftrightarrow 4 \sin^2 x - \sin x - 3 = 0$

$\Leftrightarrow \left[\begin{array}{l} \sin x = -3/4, \\ \sin x = 1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = (-1)^{k+1} \arcsin (3/4) + \pi k, \\ x = \pi/2 + 2\pi k, \quad k \in \mathbf{Z}. \end{array}\right.$

Answer: $\{(-1)^{k+1} \arcsin (3/4) + \pi k,\ \pi/2 + 2\pi k \mid k \in \mathbf{Z}\}$.

(3) $2 \cot 3x + \tan 3x + 3 = 0 \Leftrightarrow \begin{cases} 2/\tan 3x + \tan 3x + 3 = 0, \\ \tan 3x \neq 0, \\ 3x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$

$\Leftrightarrow \begin{cases} \tan^2 3x + 3 \tan 3x + 2 = 0, \\ 3x \neq \pi k, \\ 3x \neq \pi/2 + \pi k, \ k \in \mathbf{Z} \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} \tan 3x = -2, \\ \tan 3x = -1 \end{array}\right. \\ 3x \neq \pi k, \\ 3x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$

$\Leftrightarrow \begin{cases} \left[\begin{array}{l} 3x = -\arctan 2 + \pi k, \\ 3x = -\pi/4 + \pi k \end{array}\right. \\ 3x \neq \pi k, \\ 3x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow \begin{cases} x = -(1/3) \arctan 2 + \pi k/3, \\ x = -\pi/12 + \pi k/3, \quad k \in \mathbf{Z}. \end{cases}$

Answer: $\{-(1/3) \arctan 2 + \pi k/3,\ -\pi/12 + \pi k/3 \mid k \in \mathbf{Z}\}$.

(4) $\tan x \cos x + \tan x - \cos x - 1 = 0$

$\Leftrightarrow \begin{cases} (\cos x + 1)(\tan x - 1) = 0, \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} \cos x = -1, \\ \tan x = 1 \end{array}\right. \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$

$\Leftrightarrow \begin{cases} \left[\begin{array}{l} x = \pi(2k+1), \\ x = \pi/4 + \pi k \end{array}\right. \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow \left[\begin{array}{l} x = \pi(2k+1), \\ x = \pi/4 + \pi k, \quad k \in \mathbf{Z}. \end{array}\right.$

Answer: $\{\pi(2k+1),\ \pi/4 + \pi k \mid k \in \mathbf{Z}\}$.

(5) $\tan^3 x - \tan x = 0 \Leftrightarrow \begin{cases} \tan x (\tan^2 x - 1) = 0, \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$

$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \tan x = 0, \\ \tan^2 x = 1, \end{array}\right. \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x = \pi k, \\ x = \pm\pi/4 + \pi k \end{array}\right. \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$

$\Leftrightarrow \left[\begin{array}{l} x = \pi k, \\ x = \pm\pi/4 + \pi k, \quad k \in \mathbf{Z}. \end{array}\right.$

Answer: $\{\pi k,\ \pm\pi/4 + \pi k \mid k \in \mathbf{Z}\}$. ●

131. (1) $\sin x - \cos x = 0$; (2) $\sin^2 x - 4 \sin x \cos x + 3 \cos^2 x = 0$; (3) $2 \sin^2 x + 5 \sin x \cos x + \cos^2 x - 4 = 0$.

○ (1) $\sin x - \cos x = 0 \Leftrightarrow \begin{cases} \sin x/\cos x = 1, \\ \cos x \neq 0 \end{cases}$

$\Leftrightarrow \begin{cases} \tan x = 1, \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow x = \pi/4 + \pi k, \quad k \in \mathbf{Z}.$

(2) We solve the homogeneous equation:

$$\sin^2 x - 4\sin x\cos x + 3\cos^2 x = 0 \Leftrightarrow$$

$$\begin{cases} \dfrac{\sin^2 x}{\cos^2 x} - \dfrac{4\sin x\cos x}{\cos^2 x} + \dfrac{3\cos^2 x}{\cos^2 x} = 0, \\ \cos x \neq 0 \end{cases} \Leftrightarrow \begin{cases} \tan^2 x - 4\tan x + 3 = 0, \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \tan x = 1, \\ \tan x = 3 \end{array}\right. \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases} \Leftrightarrow \begin{cases} x = \pi/4 + \pi k, \\ x = \arctan 3 + \pi k \\ x \neq \pi/2 + \pi k, \quad k \in \mathbf{Z} \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} x = \pi/4 + \pi k, \\ x = \arctan 3 + \pi k, \quad k \in \mathbf{Z}. \end{array}\right.$$

Answer: $\{\pi/4 + \pi k,\ \arctan 3 + \pi k \mid k \in \mathbf{Z}\}$.

(3) Multiplying the constant term by $\sin^2 x + \cos^2 x$, we get the homogeneous equation:

$$2\sin^2 x + 5\sin x\cos x + \cos^2 x - 4(\sin^2 x + \cos^2 x) = 0$$

or

$$2\sin^2 x - 5\sin x\cos x + 3\cos^2 x = 0.$$

Dividing all the terms by $\cos^2 x$ ($x \neq \pi/2 + \pi k$), we get

$$2\tan^2 x - 5\tan x + 3 = 0 \Leftrightarrow \left[\begin{array}{l} \tan x = 1, \\ \tan x = 3/2 \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x = \pi/4 + \pi k, \\ x = \arctan(3/2) + \pi k, \quad k \in \mathbf{Z}. \end{array}\right.$$

Answer: $\{\pi/4 + \pi k,\ \arctan(3/2) + \pi k \mid k \in \mathbf{Z}\}$. ●

In Problems 132 through 143 solve the indicated equations:

132. (1) $\sin x = \sqrt{2}/2$; (2) $\sin x = -\sqrt{2}/2$; (3) $\sin x = -\sqrt{3}/2$; (4) $\sin x = \sqrt{3}/2$; (5) $\sin x = 4/5$.

133. (1) $\cos x = 1/2$; (2) $\cos x = -\sqrt{2}/2$; (3) $\cos x = -\sqrt{3}/2$; (4) $\cos x = \sqrt{3}/2$; (5) $\cos x = -0.3$.

134. (1) $\tan x = -\sqrt{3}/3$; (2) $\tan x = 1$; (3) $\tan x = 1.327$.

135. (1) $\cot x = 1$; (2) $\cot x = -\sqrt{3}$; (3) $\cot x = -\sqrt{3}/3$; (4) $\cot x = 2.05$.

136. (1) $\sin(x/2 + \pi/6) = 1/4$; (2) $\tan(3x + 1) = 1$; (3) $\tan 3x = \sqrt{3}/3$; (4) $\sin \pi x = \sqrt{2}/2$; (5) $\cos x^2 = 1$.

137. (1) $\sin^2 x = 1/2$; (2) $\sin^2 x = 1$; (3) $\sin^2 x = 3/4$; (4) $\sin^2 x = 0$.

138. (1) $\cos^2 x = 1$; (2) $\cos^2 x = 1/2$; (3) $\cos^2 x = 1/9$; (4) $\cos^2 x = 0$.

139. (1) $\tan^2 x = 1$; (2) $\cot^2 x = 3$; (3) $\tan^2 x = 1/3$.

140. (1) $2\sin^2 x + 3\cos x - 3 = 0$;
(2) $\cos^2 x - \cos x - 2 = 0$; (3) $5\cot^2 x - 8\cot x + 3 = 0$;
(4) $3\sin^2 x + \cos^2 x - 2 = 0$; (5) $7\sin^2 x - 5\cos^2 x + 2 = 0$;
(6) $\tan x + \cot x = 0$; (7) $\sin^2 x - \cos^2 x = \cos x$.

141. (1) $\cos 2x = 1$; (2) $\tan (x - \pi/2) = 1$;
(3) $\tan (2x + \pi/2) = -1$; (4) $\tan x (\sin x + \cos x) = 0$;
(5) $\cos x (\tan x - 1) = 0$; (6) $\tan (x/2) (1 + \cos x) = 0$.

142. (1) $\sin 2x/\cos x = 0$; (2) $\cos^2 x + \sin x \cos x - 1 = 0$;
(3) $\sin x = 1/\cos x$; (4) $\tan^3 x + \tan^2 x - 3 \tan x - 3 = 0$;
(5) $2 \sin x - 3 \cos x = 0$.

143. (1) $\sin^2 x - 10 \sin x \cos x + 21 \cos^2 x = 0$;
(2) $8 \sin^2 x + \sin x \cos x + \cos^2 x - 4 = 0$;
(3) $\sin^2 x - 6 \sin x \cos x + 5 \cos^2 x = 0$;
(4) $9 \sin^2 x + 25 \cos^2 x + 32 \sin x \cos x = 25$.

Sec. 9.10. Trigonometric Inequalities

Simplest trigonometric inequalities are represented by the inequalities

$$\sin x < m, \ \sin x > m, \ \cos x < m, \ \cos x > m,$$
$$\tan x < m, \ \tan x > m, \ \cot x < m, \ \cot x > m,$$

where m is a given number.

To solve a simplest trigonometric inequality means to find the set of all values of the argument (arcs or angles) which turn the given inequality into a true numerical inequality.

In Problems 144 through 147 we shall solve the given inequalities:

144. (1) $\sin x < 1/2$; (2) $|\sin x| > 1/2$.

○ (1) Taking into consideration the boundedness property of sine, we may rewrite the given inequality as follows $-1 \leqslant \sin x < 1/2$.

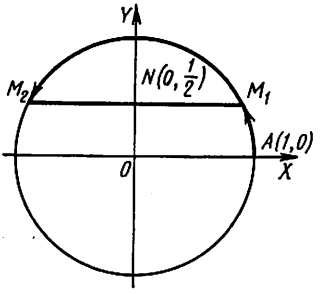


Fig. 54

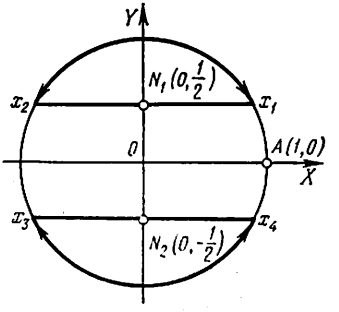


Fig. 55

We then have $\overset{\smile}{AM}_1 = \pi/6$, $\overset{\smile}{AM}_2 = -\pi - \pi/6 = -7\pi/6$ (Fig. 54). The inequality $\sin x < 1/2$ is satisfied by the arcs from the interval $(-7\pi/6, \pi/6)$. By virtue of the periodicity of sine, the general solution is represented by the set of arcs of the form $(-7\pi/6 + 2\pi k, \pi/6 + 2\pi k)$, $k \in \mathbf{Z}$.

(2) This inequality is fulfilled for all arcs $x_1 < x < x_2$ and $x_3 < x < x_4$, where $x_1 = \pi/6$, $x_2 = \pi - \pi/6 = 5\pi/6$, $x_3 = x_1 + \pi =$

$\pi/6 + \pi$ and $x_4 = x_2 + \pi = 5\pi/6 + \pi$, that is, for $\pi/6 < x < 5\pi/6$ and $\pi/6 + \pi < x < 5\pi/6 + \pi$ (Fig. 55). The general solution is the set of arcs of the form $(\pi/6 + \pi k, 5\pi/6 + \pi k)$, $k \in \mathbf{Z}$. ●

145. (1) $\cos x > -1/2$; (2) $\cos x < -1/2$; (3) $|\cos x| > \sqrt{2}/2$.

○ (1) Let us rewrite the given inequality in the following way: $-1/2 < \cos x \leqslant 1$. The inequality $\cos x > -1/2$ is satisfied by the

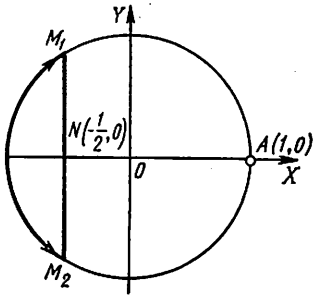


Fig. 56

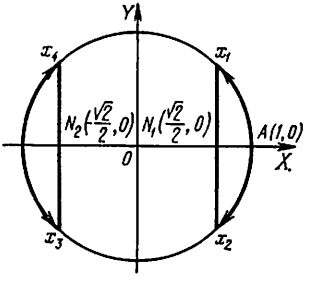


Fig. 57

arcs from the interval $(-2\pi/3, 2\pi/3)$ (see Fig. 51). The general solution is represented by the set of arcs of the form $(-2\pi/3 + 2\pi k, 2\pi/3 + 2\pi k)$, $k \in \mathbf{Z}$.

(2) Taking into account the boundedness property of cosine, we may rewrite the inequality as follows: $-1 \leqslant \cos x < -1/2$. We have $\overset{\smile}{AM}_1 = \pi - \pi/3 = 2\pi/3$, $\overset{\smile}{AM}_2 = \pi + \pi/3 = 4\pi/3$ (Fig. 56). The inequality $\cos x > -1/2$ is satisfied by the arcs from the interval $(2\pi/3, 4\pi/3)$. By virtue of the periodicity of cosine, the general solution is represented by the set of arcs of the form $(2\pi/3 + 2\pi k, 4\pi/3 + 2\pi k)$, $k \in \mathbf{Z}$.

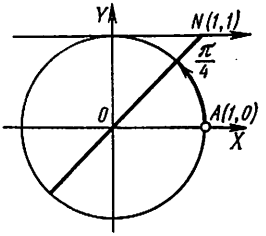


Fig. 58

(3) This inequality is fulfilled for all arcs $x_2 < x < x_1$ and $x_4 < x < x_3$, where $x_1 = \pi/4$, $x_2 = -\pi/4$, $x_3 = x_1 + \pi = \pi/4 + \pi$ and $x_4 = x_2 - \pi = -\pi/4 - \pi$, that is, for $-\pi/4 < x < \pi/4$ and $-\pi/4 - \pi < x < \pi/4 + \pi$. The general solution is represented by the set of arcs of the form $(-\pi/4 + \pi k, \pi/4 + \pi k)$, $k \in \mathbf{Z}$ (Fig. 57). ●

146. (1) $\tan x > \sqrt{3}$; (2) $\cot x > 1$.

○ (1) Since tangent is unbounded, we have $\sqrt{3} < \tan x < +\infty$. The inequality $\tan x > \sqrt{3}$ is satisfied by the arcs from the interval $(\pi/3, \pi/2)$. By virtue of the periodicity of tangent, the general solution is represented by the set of arcs of the form $(\pi/3 + \pi k, \pi/2 + \pi k)$, $k \in \mathbf{Z}$.

(2) Since cotangent is unbounded, we have $1 < \cot x < +\infty$. The inequality $\cot x > 1$ is satisfied by the arcs from the interval

$(0, \pi/4)$ (Fig. 58). The general solution is represented by the set of arcs of the form $(\pi k, \pi/4 + \pi k)$, $k \in \mathbf{Z}$. ●

147. (1) $\sin(x/2) > 1/2$; (2) $\tan 2x < -1$.

○ (1) $\pi/6 + 2\pi k < x/2 < 5\pi/6 + 2\pi k$, $\pi/3 + 4\pi k < x < 5\pi/3 + 4\pi k$ or $x \in [\pi/3 + 4\pi k, 5\pi/3 + 4\pi k)$, $k \in \mathbf{Z}$.

(2) $-\pi/2 + \pi k < 2x < -\pi/4 + \pi k$, $-\pi/4 + \pi k/2 < x < \pi/8 + \pi k/2$, or $x \in (-\pi/4 + \pi k/2, -\pi/8 + \pi k/2)$, $k \in \mathbf{Z}$. ●

In Problems 148 through 152 solve the given inequalities:

148. (1) $\sin x < 0$; (2) $\sin x > 0$; (3) $\sin x < 1$; (4) $\sin x < -1/2$; (5) $\sin x > -1/2$; (6) $\sin x > -\sqrt{3}/2$; (7) $|\sin x| < 1/2$; (8) $\sin x > 1/2$.

149. (1) $\cos x < 0$; (2) $\cos x > 0$, (3) $\cos x > -1$; (4) $\cos x < 1$; (5) $\cos x < 1/2$; (6) $\cos x > 1/2$; (7) $|\cos x| < 1/2$.

150. (1) $\tan x < -\sqrt{3}$; (2) $\tan x > -\sqrt{3}$; (3) $|\tan x| < \sqrt{3}$.

151. (1) $\cot x < -1$; (2) $\cot x > -1$; (3) $\cot x > \sqrt{3}$; (4) $|\cot x| < 1$.

152. (1) $\sin 2x < -1/2$; (2) $\cos(x/2) > -1/2$; (3) $\cot(x/3) > 1$; (4) $\tan 3x > -1$; (5) $\sin x > -1$; (6) $\sin x < -\sqrt{2}/2$; (7) $\sin x > -\sqrt{2}/2$; (8) $\cos x < -\sqrt{3}/2$; (9) $\tan x < \sqrt{3}/3$; (10) $\cot x < 1$.

QUIZ No. 2

Form I	Form II
Solve the following:	Solve the following:
(1) $\sin(x/2) < 0$;	(1) $\cot x > -\sqrt{3}$;
(2) $\cos(x/2) = 0$;	(2) $3 \sin(x/2 + \pi/6) = 3/2$;
(3) $\sin 3x = 0.5$;	(3) $\cos 2x \tan x = 0$;
(4) $2 \cos x + \tan x - 2 \tan x \cos x - 1 = 0$;	(4) $4 \sin^3 x + 4 \sin^2 x - 3 \sin x - 3 = 0$;
(5) $2 \sin^2 x - 3 \cos^2 x + \sin x \cos x = 0$.	(5) $3 \sin^2 x + 2 \sin x \cos x - 5 \cos^2 x = 0$.

Sec. 9.11. The Property of the Half-period of Sine and Cosine

When the argument is increased or decreased by π the sine and cosine functions change only sign:

$$\sin \alpha = -\sin(\alpha \pm \pi); \tag{9.30}$$

$$\cos \alpha = -\cos(\alpha \pm \pi). \tag{9.31}$$

Adding to the argument π multiplied by any odd number, we obtain the following formulas

$$\sin \alpha = -\sin[\alpha + \pi(2k + 1)], \tag{9.32}$$

$$\cos \alpha = -\cos[\alpha + \pi(2k + 1)], \tag{9.33}$$

that is, as the argument is changed by $\pi(2k + 1)$, the sine and cosine functions remain unchanged except for a reversal of sign.

153. Compute: (1) $\sin 150°$; (2) $\sin(-120°)$; (3) $\cos 225°$C; (4) $\cos(-240°)$; (5) $\sin 570°$; (6) $\sin(-585°)$; (7) $\cos 600°$.

○ In (1)-(4) use Formulas (9.30) and (9.31), and also the evenness and oddness of trigonometric functions:

(1) $\sin 150° = -\sin(150° - 180°) = -\sin(-30°) = \sin 30° = 1/2$;

(2) $\sin(-120°) = -\sin(-120° + 180°) = -\sin 60° = -\sqrt{3}/2$;

(3) $\cos 225° = -\cos(225° - 180°) = -\cos 45° = -\sqrt{2}/2$;

(4) $\cos(-240°) = -\cos(-240° + 180°) = -\cos(-60°) = -\cos 60° = -1/2$.

In (5)-(7) use Formulas (9.32) and (9.33):

(5) $\sin 570° = -\sin(570° - 3\cdot 180°) = -\sin 30° = -1/2$;

(6) $\sin(-585°) = -\sin(-585° + 3\cdot 180°) = -\sin(-45°) = \sin 45° = \sqrt{2}/2$;

(7) $\cos 600° = -\cos(600° - 3\cdot 180°) = -\cos 60° = -1/2$. ●

154. Compute: (1) $\sin(-7\pi/6)$; (2) $\cos(-2\pi/3)$;

(3) $\sin 7\frac{1}{3}\pi$.

○ (1) $\sin\left(-\frac{7\pi}{6}\right) = -\sin\left(-1\frac{1}{6}\pi + \pi\right) = -\sin\left(-\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$;

(2) $\cos\left(-\frac{2\pi}{3}\right) = -\cos\left(-\frac{2\pi}{3} + \pi\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$;

(3) $\sin 7\frac{1}{3}\pi = -\sin\left(7\frac{1}{3}\pi - 7\pi\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$. ●

In Problems 155 through 157 compute the indicated sines and cosines:

155. (1) $\sin(-225°)$; (2) $\cos(-150°)$; (3) $\cos 135°$; (4) $\sin 135°$; (5) $\sin 210°$; (6) $\cos 570°$.

156. (1) $\sin(2\pi/3)$; (2) $\cos(-3\pi/4)$; (3) $\sin(5\pi/4)$; (4) $\sin(-5\pi/4)$; (5) $\sin(31\pi/6)$; (6) $\cos(5\pi/4)$.

157. (1) $\sin(-2.76)$; (2) $\cos(-3.87)$; (3) $\cos 4.33$; (4) $\sin 4.26$.

Sec. 9.12. Reduction Formulas

Reduction formulas make it possible to express trigonometric functions of the angles $\pi/2 \pm \alpha$, $\pi \pm \alpha$, $3\pi/2 \pm \alpha$, $2\pi \pm \alpha$ in terms of trigonometric functions of the angle α (see Table 6).

When using the reduction formulas, the reader is recommended to apply the following rules:

1. If α is laid off from the axis OX, then the name of the function to be reduced, that is, of the function of the argument $-\alpha$, $\pi \pm \alpha$, $2\pi \pm \alpha$, remains unchanged. And if α is laid off from the axis OY, then the name of the function being reduced, that is, of a function of the argument $\pi/2 \pm \alpha$, $3\pi/2 \pm \alpha$, is replaced by a similar name (sine by cosine, tangent by cotangent, and vice versa).

Table 6

	Argument	Function			
		sin	cos	tan	cot
1	$-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
2	$\pi/2-\alpha$ $(90°-\alpha)$	$\cos\alpha$	$\sin\alpha$	$\cot\alpha$	$\tan\alpha$
3	$\pi/2+\alpha$ $(90°+\alpha)$	$\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$
4	$\pi-\alpha$ $(180°-\alpha)$	$\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
5	$\pi+\alpha$ $(180°+\alpha)$	$-\sin\alpha$	$-\cos\alpha$	$\tan\alpha$	$\cot\alpha$
6	$3\pi/2-\alpha$ $(270°-\alpha)$	$-\cos\alpha$	$-\sin\alpha$	$\cot\alpha$	$\tan\alpha$
7	$3\pi/2+\alpha$ $(270°+\alpha)$	$-\cos\alpha$	$\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$
8	$2\pi-\alpha$ $(360°-\alpha)$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
9	$2\pi+\alpha$ $(360°+\alpha)$	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$

2. The sign of the trigonometric function in the right-hand side is found by the sign of the left-hand side in the supposition that $0 < \alpha < \pi/2$.

158. Write the reduction formula for $\tan(3\pi/2 + \alpha)$.

○ Since α is laid off from the axis OY, tangent should be replaced by cotangent. The formula is true for all permissible values of the argument α, consequently, it is also true for $0 < \alpha < \pi/2$; but in this case the arc $3\pi/2 + \alpha$ terminates in Quadrant IV in which tangent is negative. Hence, $\tan(3\pi/2 + \alpha) = -\cot\alpha$. ●

In Problems 159 through 161 find indicated values:

159. (1) $\sin 135°$; (2) $\cot 150°$; (3) $\cos 70°$; (4) $\cos 240°$; (5) $\sin 310°$; (6) $\tan 320°$.

○ (1) $\sin 135° = \sin(90° + 45°) = \cos 45° = \sqrt{2}/2$;
(2) $\cot 150° = \cot(90° + 60°) = -\tan 60° = -\sqrt{3}$;
(3) $\cos 70° = \cos(90° - 20°) = \sin 20° = 0.3420$;
(4) $\cos 240° = \cos(270° - 30°) = -\sin 30° = -1/2$;
(5) $\sin 310° = \sin(270° + 40°) = -\cos 40° = -0.7660$;
(6) $\tan 320° = \tan(270° + 50°) = -\cot 50° = -0.8391$. ●

160. (1) $\sin 225°$; (2) $\cos 150°$; (3) $\tan 210°$; (4) $\sin 330°$; (5) $\tan 315°$; (6) $\cos 240°$; (7) $\sin 200°$; (8) $\cot 140°$.

○ (1) $\sin 225° = \sin(180° + 45°) = -\sin 45° = -\sqrt{2}/2$;
(2) $\cos 150° = \cos(180° - 30°) = -\cos 30° = -\sqrt{3}/2$;
(3) $\tan 210° = \tan(180° + 30°) = \tan 30° = \sqrt{3}/3$;
(4) $\sin 330° = \sin(360° - 30°) = -\sin 30° = -1/2$;

(5) $\tan 315° = \tan (360° - 45°) = -\tan 45° = -1$;
(6) $\cos 240° = \cos (180° + 60°) = -\cos 60° = -1/2$;
(7) $\sin 200° = \sin (180° + 20°) = -\sin 20° = -0.3420$;
(8) $\cot 140° = \cot (180° - 40°) = -\cot 40° = -1.1918$. ●

161. (1) $\sin 2.15$; (2) $\tan 4.85$.

○ (1) $\sin 2.15 = \sin (3.14 - 0.99) = \sin 0.99 = \sin 57° = 0.84$;
(2) $\tan 4.85 = \tan (4.85 - 3.14) = \tan 1.71 = \tan (1.57 + 0.14) = -\cot 0.14 = -\cot 8° = -7.1$. ●

In Problems 162 through 164 find the indicated values:

162. (1) $\cos 150°$; (2) $\tan 135°$; (3) $\sin 120°$; (4) $\cot 130°$; (5) $\cos 210°$; (6) $\sin 260°$; (7) $\tan 220°$; (8) $\cot 200°$; (9) $\sin 210°$; (10) $\sin 350°$; (11) $\cos 280°$; (12) $\tan 340°$; (13) $\cot 325°$; (14) $\sin 345°$; (15) $\cos 295°$; (16) $\tan 335°$.

163. (1) $\cos 225°$; (2) $\sin 150°$; (3) $\cot 210°$; (4) $\tan 225°$; (5) $\cos 315°$; (6) $\tan 120°$; (7) $\cot 150°$; (8) $\sin 220°$; (9) $\cos 230°$; (10) $\tan 250°$; (11) $\sin 315°$; (12) $\cos 340°$.

164. (1) $\sin 3.52$; (2) $\cos 3.68$; (3) $\cot 5.11$.

Sec. 9.13. Miscellaneous Problems

165. Compute:
(1) $\sin 9135° + \cos (-585°) + \tan 1395° + \cot (-630°)$;
(2) $\sin (-810°) + \cos (-900°) + \tan (-395°) \cot 575°$;
(3) $\sin (-2383°) - \sin (-2023°) + \cos (-485°) - \cos (-125°)$;
(4) $3 \tan 930° + \sin 1200° - \cos 1410°$;
(5) $\cos 510° - \sin 480° + \cos 840° + \sin 1230°$;
(6) $\sin (-13\pi/6) + \cos (17\pi/3) + \tan (22\pi/3) - \cot (37\pi/4)$;
(7) $\sin (-47\pi/3) - \tan (21\pi/4) + \tan (-23\pi/4) - \cot (19\pi/6)$.

166. Simplify:
(1) $\sin (\alpha - 3\pi/2) \cos (2\pi - \alpha) - \sin (\pi - \alpha) \sin (\pi + \alpha)$;
(2) $\dfrac{\tan (\alpha - \pi/2) - \cot (\pi - \alpha) + \cos (\alpha - 3\pi/2)}{\sin (\pi + \alpha)}$.

167. Prove the following identities:
(1) $\dfrac{\cos^2 (3\pi/2 - \alpha)}{\tan^2 (\alpha - 2\pi)} + \dfrac{\cos^2 (-\alpha)}{\tan^2 (\alpha - 3\pi/2)} = 1$;
(2) $\dfrac{\sin (\pi + \alpha)}{\cot (\pi/2 - \alpha)} \cdot \dfrac{\tan (\alpha - \pi)}{\cot (\pi + \alpha)} \cdot \dfrac{\cos (2\pi - \alpha)}{\cos (3\pi/2 - \alpha)} = \sin \alpha$;
(3) $\dfrac{1 - \cot^2 (\alpha - 3\pi/2)}{\cot (\alpha + \pi/2)} \cdot \dfrac{\tan (\alpha - \pi/2)}{1 - \cot^2 (\alpha - 2\pi)} = -1$.

168. Solve the following equations:
(1) $\cos^2 (\pi - x) + 8 \cos (\pi + x) + 7 = 0$;
(2) $2 \cos^2 (x - \pi) + 3 \sin (\pi + x) = 0$;
(3) $2 \sin^2 x + 5 \sin (3\pi/2 - x) - 2 = 0$;
4) $5 \cos^2 (x - 3\pi/2) - 2 \cos (x - \pi/2) = 0$;

(5) $3\sin^2(x-3\pi/2)-\cos(x+4\pi)=0$;
(6) $5\tan^2(x-\pi)+12\tan(\pi-x)=0$;
(7) $2\tan^2(3\pi/2+x)+3\tan(\pi/2+x)=0$.

QUIZ No. 3

Form I

(1) Simplify: $\cos(3\pi/2+\alpha)\times\tan(\pi/2-\alpha)-\sin(\pi/2-\alpha)+\cot(3\pi/2-\alpha)-\cot(\pi/2-\alpha)$.
(2) Compute: $\cot 225^\circ-\cot 675^\circ-\cos 495^\circ+\cos 765^\circ$.

(3) Prove the identity:
$$\frac{\sin(\pi-\alpha)\ \cot(\pi/2-\alpha)}{\tan(\pi+\alpha)\ \tan(\pi/2+\alpha)}\times\frac{\cos(2\pi-\alpha)}{\sin(-\alpha)}=\sin\alpha.$$

(4) Solve the equation:
$\sin^2(x-\pi)+6\cos(x-\pi/2)\times\cos(x+\pi)+8\sin^2(x-3\pi/2)=0$.

(5) Solve the equation
$\cot^2(x-\pi/2)-\cot(x-3\pi/2)-2=0$.

Form II

(1) Simplify: $\sin(\alpha-2\pi)\times\cos(3\pi/2-\alpha)+\tan(\pi-\alpha)\times\tan(3\pi/2+\alpha)+\cos^2(\pi/2-\alpha)$.
(2) Compute:
$$\frac{\cos^2 336^\circ-\cos^2 156^\circ+\tan 100^\circ\tan 350^\circ}{\tan^2 72^\circ+\cot 162^\circ}-\frac{\tan^2 18^\circ}{2}.$$

(3) Prove the identity:
$$\frac{\sin^2(3\pi/2+\alpha)}{\cot^2(\alpha-2\pi)}+\frac{\sin^2(-\alpha)}{\cot^2(\alpha-3\pi/2)}=1.$$

(4) Solve the equation:
$\sin^2(x+\pi)-10\sin(x+\pi)\times\cos(x-\pi)+21\sin^2(3\pi/2+x)=0$.

(5) Solve the equation
$\cot^2(x-3\pi/2)-4\tan(x-\pi)+3=0$.

Sec. 9.14. Trigonometric Functions of an Algebraic Sum of Two Arguments (Addition Formulas)

To find trigonometric functions of a sum and a difference of two arguments, the following formulas are used:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta; \tag{9.34}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta; \tag{9.35}$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta; \tag{9.36}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta; \tag{9.37}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta},\quad \alpha\neq\frac{\pi}{2}(2k+1),$$

$$\beta\neq\frac{\pi}{2}(2k+1),\quad \tan\alpha\tan\beta\neq 1; \tag{9.38}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta},\quad \alpha\neq\frac{\pi}{2}(2k+1),$$

$$\beta\neq\frac{\pi}{2}(2k+1),\quad \tan\alpha\tan\beta\neq -1; \tag{9.39}$$

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta},\quad \alpha\neq\pi k,\quad \beta\neq\pi k,\quad \alpha\neq-\beta+\pi k; \tag{9.40}$$

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha},\quad \alpha\neq\pi k,\quad \beta\neq\pi k,\quad \alpha\neq\beta+\pi k. \tag{9.41}$$

In Problems 169 through 173 we shall find the indicated values:

169. (1) $\sin(\alpha+\beta)$ if $\sin\alpha=3/5$, $\cos\beta=-5/13$, $\alpha\in(\pi/2,\pi)$, $\beta\in(\pi,3\pi/2)$; (2) $\sin(\alpha-\beta)$ if $\cos\alpha=-4/5$, $\sin\beta=-24/25$, $\alpha\in(\pi,3\pi/2)$, $\beta\in(3\pi/2,2\pi)$; (3) $\cos(\alpha+\beta)$ if $\tan\alpha=-24/7$, $\tan\beta=15/8$, $\alpha\in(\pi/2,\pi)$, $\beta\in(\pi,3\pi/2)$.

○ (1) Find $\cos\alpha$ and $\sin\beta$ if $\alpha\in(\pi/2,\pi)$, $\beta\in(\pi,3\pi/2)$:

$$\cos\alpha=-\sqrt{1-(3/5)^2}=-4/5,\quad \sin\beta=-\sqrt{1-(-5/13)^2}=-12/13.$$

By Formula (9.34), we get

$$\sin(\alpha+\beta)=(3/5)\cdot(-5/13)+(-4/5)\cdot(-12/13)=33/65.$$

(2) Find $\sin\alpha=-\sqrt{1-(4/5)^2}=-3/5$; $\cos\beta=\sqrt{1-(-24/25)^2}=7/25$. By Formula (9.35), we get

$$\sin(\alpha-\beta)=(-3/5)\cdot(7/25)-(-4/5)\cdot(-24/25)=-117/125.$$

(3) From the formula $1+\tan^2\alpha=1/\cos^2\alpha$ we have $\cos\alpha=\pm1/\sqrt{1+\tan^2\alpha}$. Taking into account that $\alpha\in(\pi/2,\pi)$, we find $\cos\alpha=-1/\sqrt{1+(-24/7)^2}=-7/25$, $\sin\alpha=\sqrt{1-(-7/25)^2}=24/25$. Similarly, we find $\cos\beta=-8/17$ and $\sin\beta=-15/17$. By Formula (9.36), we get

$$\cos(\alpha+\beta)=(-7/25)\cdot(-8/17)-(24/25)\cdot(-15/17)=416/425.\ \bullet$$

170. (1) $\sin(\arcsin(3/5)+\arcsin(4/5))$;
(2) $\cos(\arccos(3/5)+\arcsin(8/17))$.

○ (1) Setting $\arcsin(3/5)=\alpha$ and $\arcsin(4/5)=\beta$, we have $\sin\alpha=3/5$, $\alpha\in[-\pi/2,\pi/2]$ and $\sin\beta=4/5$, $\beta\in[-\pi/2,\pi/2]$. We find $\cos\alpha=\sqrt{1-(3/5)^2}=4/5$ and $\cos\beta=\sqrt{1-(4/5)^2}=3/5$. Consequently,

$$\begin{aligned}\sin(\arcsin(3/5)+\arcsin(4/5))&=\sin(\alpha+\beta)\\&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\&=(3/5)\cdot(3/5)+(4/5)\cdot(4/5)=1.\end{aligned}$$

(2) Setting $\arccos(3/5)=\alpha$ and $\arcsin(8/17)=\beta$, we have $\cos\alpha=3/5$, $\alpha\in[0,\pi]$ and $\sin\beta=8/17$, $\beta\in[-\pi/2,\pi/2]$. We find $\sin\alpha=\sqrt{1-(3/5)^2}=4/5$ and $\cos\beta=\sqrt{1-(8/17)^2}=15/17$. Thus

$$\begin{aligned}\cos(\arccos(3/5)+\arcsin(8/17))&=\cos(\alpha+\beta)\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\&=(3/5)\cdot(15/17)-(4/5)\cdot(8/17)=13/85.\ \bullet\end{aligned}$$

171. (1) $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$;
(2) $\cos 47^\circ \cos 17^\circ + \sin 47^\circ \sin 17^\circ$.

○ (1) $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ = \sin (20^\circ + 40^\circ) = \sin 60^\circ = \sqrt{3}/2$;

(2) $\cos 47^\circ \cos 17^\circ + \sin 47^\circ \sin 17^\circ = \cos (47^\circ - 17^\circ) = \cos 30^\circ = \sqrt{3}/2$.

172. (1) $\tan (\alpha + \beta)$ if $\tan \alpha = 1/5$, $\tan \beta = 2/3$; (2) $\cot (\alpha - \beta)$ if $\tan \alpha = 3/2$, $\tan \beta = 5/2$; (3) $\tan (\pi/4 + \alpha)$ if $\sin \alpha = 12/13$, $\alpha \in (\pi/2, \pi)$.

○ (1) By Formula (9.38), we get

$$\tan (\alpha+\beta) = \frac{1/5+2/3}{1-(1/5)\cdot(2/3)} = 1.$$

(2) We find $\cot \alpha = 1/\tan \alpha = 2/3$, $\cot \beta = 1/\tan \beta = 2/5$. By formula (9.41), we get

$$\cot (\alpha-\beta) = \frac{(2/3)\cdot(2/5)+1}{2/5-2/3} = -\frac{19}{4}.$$

(3) We have

$$\tan \left(\frac{\pi}{4}+\alpha\right) = \frac{\tan (\pi/4)+\tan \alpha}{1-\tan (\pi/4) \tan \alpha} = \frac{1+\tan \alpha}{1-\tan \alpha}.$$

By the formula $\tan \alpha = \dfrac{\sin \alpha}{\pm \sqrt{1-\sin^2 \alpha}}$, we find $\tan \alpha = -\dfrac{12/13}{\sqrt{1-(12/13)^2}} = -\dfrac{12}{5}$.

Substituting the found value of the tangent, we get

$$\tan \left(\frac{\pi}{4}+\alpha\right) = \frac{1-12/5}{1+12/5} = -\frac{7}{17}. \quad \bullet$$

173. (1) $\tan (\arctan (1/2) + \arctan (3/2))$;
(2) $\cot (\arcsin (4/5) + \arctan 3)$.

○ (1) By Formula (9.38), we get

$$\tan (\arctan (1/2)+\arctan (3/2)) = \frac{\tan \arctan (1/2)+\tan \arctan (3/2)}{1-\tan \arctan (1/2) \tan \arctan (3/2)} = \frac{1/2+3/2}{1-(1/2)\cdot(3/2)} = 8.$$

(2) Denoting $\arcsin (4/5) = \alpha$ and $\arctan 3 = \beta$, we get $\sin \alpha = 4/5$, $\alpha \in [-\pi/2, \pi/2]$ and $\tan \beta = 3$, $\beta \in (-\pi/2, \pi/2)$. Further, we find $\cot \alpha = 3/4$; $\cot \beta = 1/3$. By Formula (9.40), we get

$$\cot (\arcsin (4/5)+\arctan 3) = \cot (\alpha+\beta) = \frac{(3/4)(1/3)-1}{3/4+1/3} = -\frac{9}{13}. \quad \bullet$$

174. Simplify: (1) $\sin \alpha \cos 2\alpha + \cos \alpha \sin 2\alpha$;

(2) $\dfrac{\sin (\alpha+\beta)-\sin \beta \cos \alpha}{\sin (\alpha-\beta)+\sin \beta \cos \alpha}$; (3) $\dfrac{\tan (\pi/4+\alpha)-\tan \alpha}{1+\tan (\pi/4+\alpha) \tan \alpha}$.

○ (1) Using Formula (9.34), we get

$$\sin\alpha\cos 2\alpha+\cos\alpha\sin 2\alpha=\sin(\alpha+2\alpha)=\sin 3\alpha.$$

(2) We have

$$\frac{\sin(\alpha+\beta)-\sin\beta\cos\alpha}{\sin(\alpha-\beta)+\sin\beta\cos\alpha}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\beta\cos\alpha}{\sin\alpha\cos\beta-\cos\alpha\sin\beta+\sin\beta\cos\alpha}$$

$$=\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}=1.$$

(3) Using Formula (9.39), we get

$$\frac{\tan(\pi/4+\alpha)-\tan\alpha}{1+\tan(\pi/4+\alpha)\tan\alpha}=\tan(\pi/4+\alpha-\alpha)=\tan(\pi/4)=1.\ \bullet$$

175. Prove the indicated identities:

(1) $\dfrac{\sqrt{2}\cos\alpha-2\cos\left(\frac{\pi}{4}+\alpha\right)}{2\sin\left(\frac{\pi}{4}+\alpha\right)-\sqrt{2}\sin\alpha}=\tan\alpha;$

(2) $\dfrac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}=\tan(\alpha+\beta)\tan(\alpha-\beta).$

○ (1) Simplifying the left-hand side of the equality, we get

$$\frac{\sqrt{2}\cos\alpha-2\cos\left(\frac{\pi}{4}+\alpha\right)}{2\sin\left(\frac{\pi}{4}+\alpha\right)-\sqrt{2}\sin\alpha}=\frac{2\left(\frac{\sqrt{2}}{2}\cos\alpha-\cos\left(\frac{\pi}{4}+\alpha\right)\right)}{2\left(\sin\left(\frac{\pi}{4}+\alpha\right)-\frac{\sqrt{2}}{2}\sin\alpha\right)}$$

$$=\frac{\cos(\pi/4)\cos\alpha-\cos(\pi/4)\cos\alpha+\sin(\pi/4)\sin\alpha}{\sin(\pi/4)\cos\alpha+\cos(\pi/4)\sin\alpha-\cos(\pi/4)\sin\alpha}=\frac{\sin(\pi/4)\sin\alpha}{\sin(\pi/4)\cos\alpha}=\tan\alpha,$$

that is, the identity has been proved.

(2) Simplifying the right-hand side of the equality, we get:

$$\tan(\alpha+\beta)\tan(\alpha-\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\cdot\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}.\ \bullet$$

176. Solve the following equations:

(1) $\sin 2x\cos x+\cos 2x\sin x=0;$

(2) $\sin(\pi/4+x)+\cos(x-\pi/4)=0;$ (3) $\dfrac{\tan x+1}{1-\tan x}=1.$

○ (1) Using Formula (9.34), we get

$$(\sin 2x\cos x+\cos 2x\sin x=0)\Leftrightarrow(\sin(2x+x)=0)$$

$$\Leftrightarrow(\sin 3x+0)\Leftrightarrow(3x=\pi k)\Leftrightarrow x=\pi k/3,\ k\in Z.$$

(2) By Formulas (9.34) and (9.37), we get

$$\sin(\pi/4)\cos x+\cos(\pi/4)\sin x+\cos x\cos(\pi/4)$$

$$+\sin x\sin(\pi/4)=0.$$

Reducing by $\sin(\pi/4)$ [$\sin(\pi/4) = \cos(\pi/4)$] and combining like terms, we arrive at the equation

$$2\sin x + 2\cos x = 0, \text{ i.e. } \sin x + \cos x = 0.$$

Under the condition that $\cos x \neq 0$, we have $\tan x = -1$; $x = -\pi/4 + \pi k$, $k \in \mathbf{Z}$.

(3) We have

$$\left(\frac{\tan x + 1}{1 - \tan x} = 1\right) \Leftrightarrow \left(\frac{\tan x + \tan(\pi/4)}{1 - \tan x \tan(\pi/4)} = 1\right) \Leftrightarrow$$

$$(\tan(x + \pi/4) = 1) \Leftrightarrow (x + \pi/4 = \pi/4 + \pi k) \Leftrightarrow x = \pi k, \quad k \in \mathbf{Z}.$$

177. Without using the tables, compute the values of the sine and cosine of the arcs $\pi/12$, $5\pi/12$ and $7\pi/12$ ($\pi/12 = \pi/3 - \pi/4$; $5\pi/12 = \pi/6 + \pi/4$, $7\pi/12 = \pi/4 + \pi/3$).

In Problems 178 through 180 compute the indicated values:

178. (1) $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$, if $\cos\alpha = 4/5$, $\sin\beta = -3/5$, $\alpha \in (3\pi/2, 2\pi)$, $\beta \in (\pi, 3\pi/2)$; (2) $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$, if $\sin\alpha = 8/17$ and $\cos\beta = 3/5$, $\alpha \in (\pi/2, \pi)$ and $\beta \in (3\pi/2, 2\pi)$; (3) $\sin(\pi/4 + \alpha)$ and $\cos(\pi/4 + \alpha)$, if $\tan\alpha = -3/4$, $\alpha \in (\pi/2, \pi)$.

179. $\cos(\arccos(1/7) - \arccos(11/14))$;
(2) $\sin(\arcsin(5/13) + \arcsin(12/13))$; (3) $\tan(\arctan(2/3) - \arctan(1/3))$; (4) $\cot(\operatorname{arccot} 5 - \operatorname{arccot}(1/5))$.

180. (1) $\tan(\pi/12)$; (2) $\tan(5\pi/12)$; (3) $\tan(7\pi/12)$;

(4) $\dfrac{\tan(\pi/15) + \tan(4\pi/15)}{1 - \tan(\pi/15)\tan(4\pi/15)}$, (5) $\cot(\pi/4 + \alpha)$, if $\sin\alpha = -1/2$ and $\alpha \in (\pi, 3\pi/2)$.

181. Simplify the following expressions:

(1) $\cos(\pi/4)\cos(\pi/6) - \sin(\pi/4)\sin(\pi/6)$;

(2) $\sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4)$;

(3) $\sin(\alpha + \beta) - \sin(\alpha - \beta)$; (4) $\cos(\pi/3 + \alpha)\cos(\pi/3 - \alpha) - \cos^2\alpha$;

(5) $2\sin(\pi/4 + \alpha)\sin(\pi/4 - \alpha)\sin^2\alpha$;

(6) $\sin 2\alpha - \cos 2\alpha \tan\alpha$;

(7) $\dfrac{1 + \tan(\pi/4 - \alpha)}{1 - \tan(\pi/4 - \alpha)}$.

182. Prove the given identities:

(1) $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$;

(2) $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$;

(3) $\cos\left(\dfrac{\pi}{3} - \alpha\right) = \dfrac{1}{2}(\cos\alpha + \sqrt{3}\sin\alpha)$;

(4) $\sin\left(\dfrac{\pi}{4} + \alpha\right)\dfrac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha)$.

183. Solve the following equations: (1) $\sin 2x \cos x = \cos 2x \sin x$;
(2) $8\sin(\pi/6 - x) - 3\cos x = 0$;
(3) $5\sin(\pi/3 + x) + 7\sin(\pi/3 - x) = 0$.

Sec. 9.15. Miscellaneous Problems

In Problems 184 and 185 compute the indicated values:

184. (1) $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$, if $\cos\alpha = 3/5$ and $\cos\beta = 7/25$, $\alpha \in (0, \pi/2)$ and $\beta \in (0, \pi/2)$; (2) $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$, if $\sin\alpha = -4/5$ and $\cos\beta = -24/25$, $\alpha \in (\pi, 3\pi/2)$, $\beta \in (\pi, 3\pi/2)$.

185. (1) $\tan(\alpha-45°)$, if $\cot\alpha = 2/3$; (2) $\cot(\alpha-45°)$, if $\tan\alpha = 3/2$.

186. Simplify: (1) $\sin(\pi/12)+\cos(\pi/12)$; (2) $\sin^2(\alpha-\beta)+\sin^2\beta+2\sin(\alpha-\beta)\cos\alpha\sin\beta$.

In Problems 187 through 190 prove the indicated identities:

187. (1) $\sin(\pi/6+\alpha)+\sin(\pi/6-\alpha)=\cos\alpha$;

(2) $\cos^2\alpha+\cos^2(\pi/3+\alpha)+\cos^2(\pi/3-\alpha)=3/2$;

(3) $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\tan(\pi/4+\alpha)$;

(4) $\dfrac{\tan(\pi/4+\alpha)-\tan(\pi/4-\alpha)}{\tan(\pi/4+\alpha)+\tan(\pi/4-\alpha)}=2\sin\alpha\cos\alpha$;

(5) $\tan(\pi/4+\alpha)\tan(\pi/4-\alpha)+\sin(\pi/6+\alpha)+\sin(\pi/6-\alpha)=1+\cos\alpha$.

188. (1) $\dfrac{\sin^2(\alpha+\beta)+\sin^2(\alpha-\beta)}{2\cos^2\alpha\cos^2\beta}=\tan^2\alpha+\tan^2\beta$;

(2) $\dfrac{\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)}{2\sin^2\alpha\sin^2\beta}-1=\cot^2\alpha\cot^2\beta$;

(3) $\dfrac{\tan(\alpha-\beta)+\tan\beta}{\tan(\alpha+\beta)-\tan\beta}=\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}$;

(4) $(\tan\alpha+\tan\beta)\cot(\alpha+\beta)+(\tan\alpha-\tan\beta)\cot(\alpha-\beta)=2$;

(5) $\dfrac{\tan\alpha+\tan\beta}{\tan(\alpha+\beta)}+\dfrac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)}=2$.

189. (1) $\sin(\alpha+\beta+\gamma)=\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma$;

(2) $\cos(\alpha+\beta+\gamma)=\cos(\alpha+\beta)\cos\gamma-\sin(\alpha+\beta)\sin\gamma$;

(3) $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}+\dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma}+\dfrac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha}=0$.

190. (1) $\cos 15° + \sqrt{3}\sin 15° = \sqrt{2}$;

(2) $\cos 59°\cos 79° + \cos 31°\cos 11° + \cos 20° = 2\cos 20°$;

(3) $\dfrac{\sin 40°\cos 15°-\cos 40°\sin 15°}{\cos 15°\cos 10°-\sin 15°\sin 10°}=\tan 25°$;

(4) $\dfrac{\cos 115°\sin 305°+\sin 35°\cos 25°}{\cos 160°\sin 230°-\cos 40°\sin 70°}=\sqrt{3}$.

191. Solve the following equations: (1) $\cos 2x\cos x=\sin 2x\sin x$;

(2) $\cos\left(\dfrac{\pi}{4}+2x\right)\cos\left(\dfrac{\pi}{4}-2x\right)=\dfrac{1}{2}\cos^2 2x$; (3) $\cos\dfrac{3x}{2}=\cos\dfrac{x}{2}\times\cos x$.

QUIZ No. 4

Form I

(1) Compute: $\cos(\alpha+\beta)$ if $\sin\alpha=-15/17$, $\cos\beta=8/17$, $\alpha\in(\pi, 3\pi/2)$ and $\beta\in(3\pi/2, 2\pi)$.

(2) Compute: $\sin(\arcsin(12/13)-\arccos(15/17))$.

(3) Prove the identity:

$$\frac{\sin(\alpha-\beta)\cos\beta+\cos(\alpha-\beta)\sin\beta}{\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta}=\tan\alpha.$$

(4) Prove the identity:

$\tan\alpha\tan\beta+(\tan\alpha+\tan\beta)\times\cot(\alpha+\beta)=1$.

(5) Solve the equation:

$$2\sin\left(\frac{4\pi}{3}-x\right)-\sin\left(\frac{4\pi}{3}+x\right)=0.$$

Form II

(1) Compute: $\sin(\alpha-\beta)$ if $\cos\alpha=3/5$, $\cos\beta=-7/25$, $\alpha\in(3\pi/2, 2\pi)$ and $\beta\in(\pi, 3\pi/2)$.

(2) Compute: $\cos(\arcsin(15/17)+\arccos(-12/13))$.

(3) Prove the identity:

$$\frac{\sin(\pi/4+\alpha)-\cos(\pi/4+\alpha)}{\sin(\pi/4+\alpha)+\cos(\pi/4+\alpha)}=\tan\alpha.$$

(4) Prove the identity:

$$\tan(\pi/3+\alpha)-\tan(\pi/3-\alpha)=\frac{8\tan\alpha}{1-3\tan^2\alpha}.$$

(5) Solve the equation:

$$\sin x\sin 3x+\cos 4x=0.$$

Sec. 9.16. Trigonometric Functions of Double Angle

The double-angle formulas (or indentities) make it possible to express the sine, cosine, tangent, etc., of twice an angle in terms of functions of the angle:

$$\sin 2\alpha=2\sin\alpha\cos\alpha; \tag{9.42}$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha; \tag{9.43}$$

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}, \quad \alpha\neq\frac{\pi}{2}+\pi k, \quad \alpha\neq\frac{\pi}{4}+\frac{\pi k}{2}; \tag{9.44}$$

$$\cot 2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha}, \quad \alpha\neq\frac{\pi k}{2}. \tag{9.45}$$

From Formula (9.43) the following relationships are readily obtained:

$$\cos 2\alpha=2\cos^2\alpha-1; \tag{9.43a}$$

$$\cos 2\alpha=1-2\sin^2\alpha. \tag{9.43b}$$

192. Compute: (1) $\sin 2\alpha$, if $\cos\alpha=4/5$ and $3\pi/2<\alpha<2\pi$; (2) $\cos 2\alpha$, if $\cos\alpha=-0.2$ and $\pi/2<\alpha<\pi$; (3) $\tan 2\alpha$, if $\tan\alpha=3/4$ and $\pi<\alpha<3\pi/2$.

○ (1) Find $\sin\alpha=-\sqrt{1-(4/5)^2}=-3/5$. By Formula (9.42), we get

$$\sin 2\alpha=2\cdot(-3/5)\cdot(4/5)=-24/25.$$

(2) By Formula (9.43a), we find $\cos 2\alpha=2(-0.2)^2-1=-0.92$.

(3) By Formula (9.44), we find $\tan 2\alpha=\frac{2\cdot(3/4)}{1-(3/4)^2}=3\frac{3}{7}$. ●

193. Express: (1) $\sin 3\alpha$ in terms of $\sin\alpha$; (2) $\cos 3\alpha$ in terms of $\cos\alpha$.

○ (1) $\sin 3\alpha = \sin(2\alpha + \alpha) = \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha = 2\sin\alpha\cos\alpha\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha = 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha = 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha = 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha$;

(2) $\cos 3\alpha = \cos(2\alpha + \alpha) = \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha = (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha = 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha = 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha = 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha = 4\cos^3\alpha - 3\cos\alpha$. ●

In Problems 194 and 195 find the indicated values:

194. $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ if: (1) $\sin\alpha = -3/5$ and $\pi < \alpha < 3\pi/2$; (2) $\cos\alpha = 5/13$ and $3\pi/2 < \alpha < 2\pi$; (3) $\tan\alpha = -3/4$ and $\pi/2 < \alpha < \pi$.

195. (1) $\cot z$, if $\tan(z/2) = 5/3$; (2) $\sin 3\alpha$, $\cos 3\alpha$ and $\tan 3\alpha$, if $\sin(3\alpha/2) = -5/13$ and $\pi < \alpha < 3\pi/2$; (3) $\cos 4x$ and $\tan 4x$, if $\tan x = 1/5$ and $\pi < x < 3\pi/2$.

196. Find the numerical values of the expressions: (1) $\sin 2\alpha/(2\cos\alpha)$, if $\cos\alpha = -4/5$ and $\pi < \alpha < 3\pi/2$; (2) $\cos 2\alpha/\sin\alpha$, if $\sin\alpha = -3/5$ and $3\pi/2 < \alpha < 2\pi$.

197. Express: (1) $\sin 4\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$; (2) $\cos 4\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$; (3) $\tan 3\alpha$ in terms of $\tan\alpha$; (4) $\cos 4\alpha$ in terms of $\cos\alpha$; (5) $\sin 5\alpha$ in terms of $\sin\alpha$.

198. Simplify: (1) $1 - 2\cos^2\left(\frac{\pi}{4} - \frac{4x}{3}\right)$; (2) $2\cos^2\left(\frac{\pi}{4} + \frac{3x}{2}\right) - 1$;

(3) $1 - 2\sin^2\left(\frac{\pi}{4} - \frac{5x}{2}\right)$; (4) $2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right) - 1$.

199. Prove the following identities:

(1) $2\sin^2\alpha + \cos 2\alpha = 1$; (2) $1 + \cos 2\alpha = 2\cos^2\alpha$;

(3) $\frac{1+\cos 2\alpha}{1-\cos 2\alpha} = \cot^2\alpha$; (4) $\frac{\sin 2\alpha - \sin\alpha}{1 - \cos\alpha + \cos 2\alpha} = \tan\alpha$;

(5) $\cos^4\alpha + \sin^4\alpha = 1 - 0.5\sin^2 2\alpha$;

(6) $\cos^6\alpha + \sin^6\alpha = 1 - 0.75\sin^2 2\alpha$;

(7) $\frac{1+\sin 2\alpha}{\cos 2\alpha} = \frac{\sin\alpha + \cos\alpha}{\cos\alpha - \sin\alpha}$; (8) $\frac{\sin^3\alpha + \sin 3\alpha}{\cos^3\alpha - \cos 3\alpha} = \cot\alpha$;

(9) $\frac{1 - \cos 2\alpha + \sin 2\alpha}{1 + \cos 2\alpha + \sin 2\alpha} = \tan\alpha$; (10) $\frac{2 - \sin 4\alpha\cot 2\alpha}{\sin 4\alpha} = \tan 2\alpha$;

(11) $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} = 2$; (12) $\frac{1+\cos 2\alpha}{\cos 2\alpha} \cdot \frac{1+\cos 4\alpha}{\sin 4\alpha} = \cot\alpha$;

(13) $\cos 4\alpha + 4\cos 2\alpha + 3 = 8\cos^4\alpha$.

200. Solve the following equations:

(1) $\sin 2x - \sin x = 0$; (2) $\sin x \cdot \cos x = 1/2$; (3) $\cos^2 x - \sin^2 x = 1$.

Sec. 9.17. Trigonometric Functions of Half-angle

The half-angle formulas (identities) are formulas expressing trigonometric functions of half an angle in terms of functions of the angle

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}; \tag{9.46}$$

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}; \tag{9.47}$$

$$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},\quad \alpha\neq\pi(2k+1); \tag{9.48}$$

$$\cot\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}},\quad \alpha\neq 2\pi k; \tag{9.49}$$

$$\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha},\quad \alpha\neq\pi(2k+1); \tag{9.50}$$

$$\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha},\quad \begin{array}{l}\text{in the left-hand side } \alpha\neq\pi(2k+1),\\ \text{in the right-hand side } \alpha\neq\pi k;\end{array} \tag{9.51}$$

$$\cot\frac{\alpha}{2}=\frac{1+\cos\alpha}{\sin\alpha}\quad \begin{array}{l}\text{in the left-hand side } \alpha\neq 2\pi k,\\ \text{in the right-hand side } \alpha\neq\pi k;\end{array} \tag{9.52}$$

$$\cot\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha},\quad \alpha\neq 2\pi k. \tag{9.53}$$

In Formulas (9.46) and (9.47), the sign in front of the radical is determined by the sign of the quadrant to which the arc $\alpha/2$ belongs.

In Formulas (9.48) and (9.49), the sign in front of the radical is taken so as to coincide with the sign of $\tan(\alpha/2)$, that is, the plus sign if the arc $\alpha/2$ belongs to Quadrants I and III, and the minus sign if $\alpha/2$ is an arc belonging to Quadrants II or IV.

Instead of Formulas (9.48) and (9.49), it is possible to make use of Formulas (9.50) to (9.53) which yield a rational expression of $\tan(\alpha/2)$ in terms of trigonometric functions of the argument α.

In the equalities (9.51) and (9.52), the left-hand and right-hand sides have different domains of definition. The domain of definition of the left-hand side of the equality $\alpha\neq 2\pi$ $(k+1)$, while that of the right-hand side is $\alpha\neq\pi k$. In the equality (9.52), the left-hand member is defined for $\alpha\neq 2\pi k$, while the right-hand member for $\alpha\neq\pi k$. Applying formulas (9.51) and (9.52) for solving trigonometric equations, one should bear in mind that the domains of definition of these formulas do not coincide.

All trigonometric functions of any angle can be expressed in terms of the tangent of half this angle by the formulas:

$$\sin\alpha=\frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)},\quad \alpha\neq\pi(2k+1); \tag{9.54}$$

$$\cos\alpha=\frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)},\quad \alpha\neq\pi(2k+1); \tag{9.55}$$

$$\tan\alpha=\frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)},\quad \alpha\neq\pi(2k+1),\quad \alpha\neq\frac{\pi}{2}+\pi k; \tag{9.56}$$

$$\cot\alpha=\frac{1-\tan^2(\alpha/2)}{2\tan(\alpha/2)},\quad \alpha\neq\pi k. \tag{9.57}$$

201. Given: $\cos\alpha = 1/2$ and $\pi/2 < \alpha < \pi$. Compute: $\sin(\alpha/2)$, $\cos(\alpha/2)$, and $\tan(\alpha/2)$.

○ By Formulas (9.46) and (9.47), we find

$$\sin\frac{\alpha}{2}=\sqrt{\frac{1-1/2}{2}}=\frac{1}{2}\,;\quad \cos\frac{\alpha}{2}=\sqrt{\frac{1+1/2}{2}}=\frac{\sqrt{3}}{2}$$

(in both cases, we take the radical with the plus sign, since it follows from the hypothesis that $\pi/4 < \alpha/2 < \pi/2$). Further, we have: $\tan(\alpha/2) = (1/2):(\sqrt{3}/2) = 1/\sqrt{3} = \sqrt{3}/3$. ●

202. Find $\tan(\alpha/2)$ if (1) $\sin\alpha = 4/5$ and $\pi/2 < \alpha < \pi$; (2) $\cos\alpha = -4/5$ and $\pi < \alpha < 3\pi/2$; (3) $\tan\alpha = 2\sqrt{2}$ and $\pi < \alpha < 3\pi/2$.

○ (1) Find $\cos\alpha = -\sqrt{1-(4/5)^2} = -3/5$. By Formula (9.50), we get

$$\tan\frac{\alpha}{2}=\frac{4}{5}\div\left(1-\frac{3}{5}\right)=2.$$

(2) Find $\sin\alpha = -\sqrt{1-(-4/5)^2} = -3/5$. By Formula (9.50), we get $\tan\frac{\alpha}{2}=\left(-\frac{3}{5}\right)\div\left(1-\frac{4}{5}\right)=-3$.

(3) Compute $\sin\alpha$ and $\cos\alpha$:

$$\sin\alpha=-\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}=-\frac{2\sqrt{2}}{\sqrt{1+(2\sqrt{2})^2}}=-\frac{2\sqrt{2}}{3}.$$

$$\cos\alpha=-\frac{1}{\sqrt{1+(2\sqrt{2})^2}}=-\frac{1}{3}.$$

By Formula (9.50), we get $\tan\frac{\alpha}{2}=-\frac{2\sqrt{2}}{3}\div\left(1-\frac{1}{3}\right)=-\sqrt{2}$. ●

203. Simplify: (1) $\dfrac{\sin\alpha}{1-\cos\alpha}$; (2) $\dfrac{1+\cos 2\alpha}{\sin 2\alpha}$.

○ (1) $\dfrac{\sin\alpha}{1-\cos\alpha}=\dfrac{2\sin(\alpha/2)\cos(\alpha/2)}{2\sin^2(\alpha/2)}=\dfrac{\cos(\alpha/2)}{\sin(\alpha/2)}=\cot(\alpha/2)$.

(2) $\dfrac{1+\cos 2\alpha}{\sin 2\alpha}=\dfrac{2\cos^2\alpha}{2\sin\alpha\cos\alpha}=\dfrac{\cos\alpha}{\sin\alpha}=\cot\alpha$. ●

204. Prove the following identities:

(1) $1 + \cos\alpha = 2\cos^2(\alpha/2)$; (2) $1 - \cos\alpha = 2\sin^2(\alpha/2)$.

○ (1) Replacing α by $\alpha/2$ in the formula $\cos 2\alpha = 2\cos^2\alpha - 1$, we get

$$\cos\alpha = 2\cos^2(\alpha/2) - 1 \text{ or } 1 + \cos\alpha = 2\cos^2(\alpha/2).$$

(2) Analogously, substituting $\alpha/2$ for α in the formula $\cos 2\alpha = 1 - 2\sin^2\alpha$, we obtain

$$\cos\alpha = 1 - 2\sin^2(\alpha/2) \text{ or } 1 - \cos\alpha = 2\sin^2(\alpha/2).$$ ●

205. Compute: (1) $\sin\alpha$, if $\tan(\alpha/2)=2$; (2) $\cos\alpha$, if $\tan(\alpha/2)=3$; (3) $\tan\alpha$, if $\tan(\alpha/2)=\sqrt{3}$; (4) $\cot\alpha$, if $\tan(\alpha/2)=-\sqrt{2}$; (5) $\dfrac{5\cos\alpha-3}{10\sin\alpha+1}$, if $\tan(\alpha/2)=3$.

○ Using Formulas (9.54)-(9.57), we get:

(1) $\sin\alpha=\dfrac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}=\dfrac{2\cdot 2}{1+2^2}=\dfrac{4}{5}$;

(2) $\cos\alpha=\dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}=\dfrac{1-3^2}{1+3^2}=-\dfrac{4}{5}$;

(3) $\tan\alpha=\dfrac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)}=\dfrac{2\sqrt{3}}{1-(-\sqrt{3})^2}=-\sqrt{3}$;

(4) $\cot\alpha=\dfrac{1-\tan^2(\alpha/2)}{2\tan(\alpha/2)}=\dfrac{1-(-\sqrt{2})^2}{2\cdot(-\sqrt{2})}=\dfrac{\sqrt{2}}{4}$;

(5) $\cos\alpha=\dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}=\dfrac{1-9}{1+9}=-\dfrac{4}{5}$, $\sin\alpha=\dfrac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}=\dfrac{6}{1+9}=\dfrac{3}{5}$,

$\dfrac{5\cos\alpha-3}{10\sin\alpha+1}=\dfrac{5\cdot(-4/5)-3}{10\cdot(3/5)+1}=-1$. ●

206. Solve the following equations:

(1) $\sin\left(\frac{\pi}{2}+\frac{x}{2}\right)-\cos(\pi+x)+1=0$;

(2) $\sin x+\cos x=1$; (3) $3\sin x+4\cos x=4$.

○ (1) $\left(\sin\left(\frac{\pi}{2}+\frac{x}{2}\right)-\cos(\pi+x)+1=0\right)$

$\Leftrightarrow\left(\cos\frac{x}{2}+\cos x+1=0\right)\Leftrightarrow\left(\cos\frac{x}{2}+2\cos^2\frac{x}{2}=0\right)$

$\Leftrightarrow\left(\cos\frac{x}{2}\left(1+2\cos\frac{x}{2}\right)=0\right)\Leftrightarrow\left[\begin{array}{l}\cos\frac{x}{2}=0,\\ 1+2\cos\frac{x}{2}=0\end{array}\right.$

$\Leftrightarrow\left[\begin{array}{l}\cos\frac{x}{2}=0,\\ \cos\frac{x}{2}=-\frac{1}{2}\end{array}\right.\Leftrightarrow\left[\begin{array}{l}\frac{x}{2}=\frac{\pi}{2}+\pi k,\\ \frac{x}{2}=\pm\frac{2\pi}{3}+2\pi k,\quad k\in\mathbf{Z}\end{array}\right.$

$\Leftrightarrow\left[\begin{array}{l}x=\pi(2k+1),\\ x=\pm\frac{4\pi}{3}+4\pi k,\quad k\in\mathbf{Z}.\end{array}\right.$

Answer: $\left\{\pi(2k+1),\ \frac{4\pi}{3}(3k\pm 1) \mid k\in\mathbf{Z}\right\}$.

(2) $(\sin x+\cos x=1) \Leftrightarrow (\sin x=1-\cos x) \Leftrightarrow \left(2\sin\frac{x}{2}\cos\frac{x}{2}=2\sin^2\frac{x}{2}\right) \Leftrightarrow \left(2\sin\frac{x}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)=0\right)$

$$\Leftrightarrow \left[\begin{array}{l}\sin\frac{x}{2}=0,\\ \cos\frac{x}{2}-\sin\frac{x}{2}=0\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\sin\frac{x}{2}=0,\\ \tan\frac{x}{2}=1\end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l}\frac{x}{2}=\pi k,\\ \frac{x}{2}=\frac{\pi}{4}+\pi k,\quad k\in\mathbf{Z}\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x=2\pi k,\\ x=\frac{\pi}{2}+2\pi k,\quad k\in\mathbf{Z}.\end{array}\right.$$

Answer: $\left\{2\pi k,\ \frac{\pi}{2}+2\pi k \mid k\in\mathbf{Z}\right\}$.

(3) Let us express $\sin x$ and $\cos x$ in terms of $z=\tan(x/2)$; we have $\sin x=\frac{2z}{1+z^2}$, $\cos x=\frac{1-z^2}{1+z^2}$. Then

$$\frac{6z}{1+z^2}+\frac{4-4z^2}{1+z^2}=4 \Leftrightarrow 4z^2-3z=0 \Leftrightarrow \left[\begin{array}{l}z=0,\\ z=3/4.\end{array}\right.$$

The given equation is equivalent to the combination of equations: $\tan(x/2)=0$ and $\tan(x/2)=3/4$, whence $x/2=\pi k$ and $x/2=\arctan(3/4)+\pi k$. *Answer*: $\{2\pi k,\ 2\arctan(3/4)+2\pi k \mid k\in Z\}$. ●

In Problems 207 through 209 find the indicated values:

207. $\sin(\alpha/2)$, $\cos(\alpha/2)$ and $\tan(\alpha/2)$ if: (1) $\cos\alpha=-7/25$ and $\pi/2<\alpha<\pi$; (2) $\sin\alpha=-15/17$ and $3\pi/2<\alpha<2\pi$; (3) $\tan\alpha=4/3$ and $0<\alpha<\pi/2$.

208. (1) $\cos(\alpha/2)$ if $\tan\alpha=-12/5$ and $5\pi/2<\alpha<3\pi$; (2) $\sin(\alpha/2)$ if $\cot\alpha=5/12$ and $3\pi<\alpha<7\pi/2$.

209. (1) $\sin\alpha$ if $\cot(\alpha/2)=1/3$; (2) $\cos\alpha$ if $\cot(\alpha/2)=1/2$; (3) $\tan\alpha$ if $\cot(\alpha/2)=\sqrt{3}/3$; (4) $\frac{5\cos\alpha+4}{10\sin\alpha-1}$ if $\tan(\alpha/2)=2$.

210. Solve the following equations: (1) $1-\cos x=\sin(x/2)$; (2) $1+\cos x=\cos(x/2)$; (3) $1+\cos x=\sin x$.

Sec. 9.18. Miscellaneous Problems

In Problems 211 through 213 compute the indicated values:

211. (1) $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ if $\sin\alpha=4/5$ and $\alpha\in(0,\pi/2)$; (2) $\sin(\alpha/2)$, $\cos(\alpha/2)$, and $\tan(\alpha/2)$ if $\cos\alpha=1/2$ and $\alpha\in(0,\pi/2)$; (3) $\sin(\alpha/4)$, $\cos(\alpha/4)$, and $\tan(\alpha/4)$ if $\sin\alpha=-24/25$ and $\alpha\in(3\pi/2,2\pi)$.

212. (1) $\frac{\tan(\pi/8)}{1-\tan^2(\pi/8)}$; (2) $\frac{1-\tan^2(\pi/12)}{1+\tan^2(\pi/12)}$; (3) $\cos 20^\circ \cos 40^\circ \cos 80^\circ$.

213. (1) $\sin(2 \arcsin (40/41))$; (2) $\cos(2 \arccos (2/3))$; (3) $\tan(2 \arctan (7/25))$.

In Problems 214 and 215 simplify the given expressions:

214. (1) $1-\cos 40^\circ$; (2) $\frac{1-\cos\alpha}{1+\cos\alpha}$; (3) $\frac{1-\sin\alpha}{1+\sin\alpha}$;

(4) $\cot\alpha(1-\cos 2\alpha)$; (5) $\tan\alpha(1+\cos 2\alpha)$;

(6) $\left(\tan\alpha+\frac{1}{\cos\alpha}\right)\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$.

215. (1) $\frac{\sin\alpha\cos\alpha}{\cos^2\alpha-\sin^2\alpha}$; (2) $\frac{2\sin^2\alpha-1}{1-2\cos^2\alpha}$; (3) $\frac{1-\tan^2\alpha}{1+\tan^2\alpha}$;

(4) $\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}$; (5) $\frac{1+\cos 2\alpha}{1-\cos 2\alpha}$; (6) $2\cos^2\left(\frac{\pi}{4}-\alpha\right)-\sin 2\alpha$.

In Problems 216 through 218 prove the indicated identities:

216. (1) $\frac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=\tan^2\frac{x}{2}$; (2) $\frac{\sin 2\alpha}{1+\cos 2\alpha}\frac{\cos\alpha}{1+\cos\alpha}=\tan\frac{\alpha}{2}$;

(3) $\frac{\cos\alpha-\cos 2\alpha-1}{\sin\alpha-\sin 2\alpha}=\cot\alpha$; (4) $\frac{\cos 2\alpha}{1+\sin 2\alpha}=\cot\left(\frac{\pi}{4}+\alpha\right)$.

217. (1) $\frac{1+\sin 2\alpha-\cos 2\alpha}{1+\sin 2\alpha+\cos 2\alpha}=\tan\alpha$;

(2) $\frac{1+\cos\alpha+\cos 2\alpha+\cos 3\alpha}{\sin 2\alpha+2\sin\alpha\cos 2\alpha}=\cot\alpha$;

(3) $\frac{\sin^3\alpha+\sin 3\alpha}{\sin\alpha}+\frac{\cos^3\alpha-\cos 3\alpha}{\cos\alpha}=3$;

(4) $\cos^4\alpha-\sin^4\alpha=\cos 2\alpha$;

(5) $\tan(\pi/12)+\cot(\pi/12)=4$;

(6) $\cos^3\alpha\cos 3\alpha+\sin^3\alpha\sin 3\alpha=\cos^3 2\alpha$.

218. (1) $\tan(\pi/4+\alpha)+\tan(\alpha-\pi/4)=2\tan 2\alpha$;

(2) $(\sin\alpha-\sin\beta)^2+(\cos\alpha-\cos\beta)^2=4\sin^2\frac{\alpha-\beta}{2}$;

(3) $\sin\alpha(\sin\alpha-\sin\beta)+\cos(\cos\alpha-\cos\beta)=2\sin^2\frac{\alpha-\beta}{2}$.

219. Solve the following equations:

(1) $\sin x\cos x=1/4$; (2) $\cos^2 x-\sin^2 x=-1/2$;

(3) $\cot x-\cot 2x=2$; (4) $\tan x\tan(\pi/3+x)\tan(\pi/3-x)=1$.

(5) $\tan x-\tan 2x=0$; (6) $1+\cos x=2\cos(x/2)$;

(7) $1-\cos x=2\sin(x/2)$; (8) $\sqrt{3}\sin x-\cos x=1$.

QUIZ No. 5

Form I

(1) Compute $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ if $\sin\alpha = 1/2$ and $\alpha \in (\pi/2, \pi)$.

(2) Compute $\sin(\alpha/2)$, $\cos(\alpha/2)$, and $\tan(\alpha/2)$ if $\cos\alpha = 3/5$ and $\alpha \in (3\pi/2, 2\pi)$.

(3) Prove the identity:

$$\frac{2\sin\alpha + \sin 2\alpha}{2\sin\alpha - \sin 2\alpha} = \cot^2\frac{\alpha}{2}.$$

(4) Prove the identity:

$4\sin 20^\circ \sin 50^\circ \sin 70^\circ = \sin 80^\circ$.

(5) Solve the equation:

$\sqrt{3}\sin x + \cos x = 1$.

Form II

(1) Compute $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ if $\cos\alpha = 3/5$ and $\alpha \in (0, \pi/2)$.

(2) Compute $\sin(\alpha/2)$, $\cos(\alpha/2)$ and $\tan(\alpha/2)$ if $\sin\alpha = \sqrt{3}/2$ and $\alpha \in (0, \pi/2)$.

(3) Prove the identity:

$$\frac{1 + \sin 2\alpha}{\cos 2\alpha} = \tan\left(\frac{\pi}{4} + \alpha\right).$$

(4) Prove the identity:

$8\sin 10^\circ \sin 50^\circ \sin 70^\circ = 1$.

(5) Solve the equation:

$1 - \cos x = \sin x \sin(x/2)$.

Sec. 9.19. Transforming a Product of Trigonometric Functions into an Algebraic Sum

To transform a product of trigonometric functions into a sum, the following formulas are used:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]; \tag{9.58}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]; \tag{9.59}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]. \tag{9.60}$$

220. Transform into an algebraic sum: (1) $\sin 5x \sin 3x$;

(2) $\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4}$.

○ (1) By Formula (9.60), we obtain

$$\sin 5x \sin 3x = \frac{1}{2}(\cos(5x - 3x) - \cos(5x + 3x)) = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 8x.$$

(2) Using Formula (9.59) twice, we get

$$\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4} = \left(\cos\frac{x}{2}\cos\frac{x}{4}\right)\cos\frac{x}{3}$$

$$= \frac{1}{2}\left(\cos\frac{3x}{4} + \cos\frac{x}{4}\right)\cos\frac{x}{3} = \frac{1}{2}\left(\cos\frac{3x}{4}\cos\frac{x}{3} + \cos\frac{x}{4}\cos\frac{x}{3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(\cos\frac{13x}{12} + \cos\frac{5x}{12}\right) + \frac{1}{2}\left(\cos\frac{7x}{12} + \cos\frac{x}{12}\right)\right)$$

$$= \frac{1}{4}\cos\frac{13x}{12} + \frac{1}{4}\cos\frac{5x}{12} + \frac{1}{4}\cos\frac{7x}{12} + \frac{1}{4}\cos\frac{x}{12}. \quad ●$$

221. Represent the following trigonometric functions in the form of sums of first powers: (1) $\sin^2 x$; (2) $\cos^2 x$; (3) $\sin^3 x$.

○ (1) $\sin^2 x = \sin x \cdot \sin x = \frac{1}{2}(\cos 0 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$;

(2) $\cos^2 x = \cos x \cdot \cos x = \frac{1}{2}(\cos 0 + \cos 2x) = \frac{1}{2} + \frac{1}{2}\cos 2x$;

(3) $\sin^3 x = \sin^2 x \cdot \sin x = \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\sin x$

$= \frac{1}{2}\sin x - \frac{1}{2}\sin x \cos 2x = \frac{1}{2}\sin x - \frac{1}{4}(\sin 3x - \sin x)$

$= \frac{1}{2}\sin x - \frac{1}{4}\sin 3x + \frac{1}{4}\sin x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

Hence, we get the formula for the sine of a triple argument: $\sin 3x = 3\sin x - 4\sin^3 x$. ●

In Problems 222 and 223 transform each of the given products into an algebraic sum:

222. (1) $\cos 7x \cos 5x$; (2) $\sin 11x \sin x$; (3) $\sin 5x \cos 2x$; (4) $\sin(\alpha - \beta)\cos(\alpha + \beta)$; (5) $\cos(\alpha + \beta)\cos(2\alpha + \beta)$; (6) $\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{x+y}{2}$.

223. (1) $\sin\left(\frac{\pi}{4} + x\right)\sin\left(\frac{\pi}{4} - x\right)$;

(2) $4\cos\left(\frac{\pi}{12} - x\right)\cos\left(\frac{\pi}{12} + x\right)$;

(3) $4\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{12} - x\right)$;

(4) $4\cos\left(\frac{\pi}{6} + x\right)\sin\left(\frac{\pi}{3} - x\right)$.

224. Represent in the form of a sum of first powers: (1) $\cos^3 x$; (2) $\sin^4 x$; $\cos^4 x$; (4) $\sin^5 x$; (5) $\cos^5 x$.

Sec. 9.20. Transforming an Algebraic Sum of Trigonometric Functions into a Product

1. **Basic Formulas.** To transform algebraic sums of trigonometric function into a product (that is, to reduce an algebraic sum to the form convenient for taking logarithms), the following formulas are used:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{9.61}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}; \tag{9.62}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{9.63}$$

$$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}; \tag{9.64}$$

$$\tan\alpha+\tan\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}, \quad \alpha\neq\frac{\pi}{2}+\pi k, \quad \beta\neq\frac{\pi}{2}+\pi k; \tag{9.65}$$

$$\tan\alpha-\tan\beta=\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}, \quad \alpha\neq\frac{\pi}{2}+\pi k, \quad \beta\neq\frac{\pi}{2}+\pi k. \tag{9.66}$$

The following formulas are also frequently used:

$$1+\cos\alpha=2\cos^2\frac{\alpha}{2}; \tag{9.67}$$

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2}; \tag{9.68}$$

$$1+\sin\alpha=2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right); \tag{9.69}$$

$$1-\sin\alpha=2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right). \tag{9.70}$$

2. The Conditions of Equality of Trigonometric Functions Having the Same Name. For the sines of two different arguments to be equal to each other, it is necessary and sufficient that the following conditions be fulfilled:

$$(\sin x=\sin y)\Leftrightarrow\left[\begin{array}{l}x+y=(2k+1)\pi,\\ x-y=2\pi k,\ k\in Z.\end{array}\right.$$

For the cosines of two different arguments to be equal to each other, it is necessary and sufficient that the following conditions be fulfilled:

$$(\cos x=\cos y)\Leftrightarrow\left[\begin{array}{l}x+y=2\pi k,\\ x-y=2\pi k,\ k\in Z.\end{array}\right.$$

For the tangents of two different arguments to be equal to each other, it is necessary and sufficient that the following conditions be fulfilled:

$$(\tan x=\tan y)\Leftrightarrow\left\{\begin{array}{l}x\neq\pi/2+\pi k,\ y\neq\pi/2+\pi k,\\ x-y=\pi k,\ k\in Z.\end{array}\right.$$

In Problems 225 and 226 transform each of the given algebraic sums into a product:

225. (1) $\sin 40^\circ+\sin 20^\circ$; (2) $\cos(\pi/5)-\cos(7\pi/10)$; (3) $\cot(2\pi/7)-\cot(\pi/7)$; (4) $\cos^2\alpha-\sin^2\beta$; (5) $\sin 2\alpha\cos 3\alpha-2\sin^2\alpha\sin 3\alpha$.

○ (1) By Formula (9.61), we get

$$\sin 40^\circ+\sin 20^\circ=2\sin\frac{40^\circ+20^\circ}{2}\cos\frac{40^\circ-20^\circ}{2}$$

$$=2\sin 30^\circ\cos 10^\circ=2\cdot\frac{1}{2}\cos 10^\circ=\cos 10^\circ.$$

(2) By Formula (9.64), we get

$$\cos(\pi/5)-\cos(7\pi/10)=2\sin(9\pi/20)\sin(\pi/4)=\sqrt{2}\sin(9\pi/20).$$

(3) Expressing the cotangents in terms of tangents, we use Formula (9.66):

$$\cot(2\pi/7) - \cot(\pi/7) = \tan(\pi/2 - 2\pi/7) - \tan(\pi/2 - \pi/7)$$
$$= \tan(3\pi/14) - \tan(5\pi/14) = -\frac{\sin(\pi/7)}{\cos(3\pi/14)\cos(5\pi/14)}.$$

(4) Using Formulas (9.67), (9.68), and (9.63), we get

$$\cos^2\alpha - \sin^2\beta = \left(\frac{1+\cos 2\alpha}{2}\right) - \left(\frac{1-\cos 2\beta}{2}\right) = \frac{1}{2}(\cos 2\alpha + \cos 2\beta)$$
$$= \frac{1}{2}\cdot 2\cos\frac{2(\alpha+\beta)}{2}\cos\frac{2(\alpha-\beta)}{2} = \cos(\alpha+\beta)\cos(\alpha-\beta).$$

(5) $\sin 2\alpha \cos 3\alpha - 2\sin^2\alpha \sin 3\alpha = 2\sin\alpha\cos\alpha\cos 3\alpha - 2\sin^2\alpha\sin 3\alpha = 2\sin\alpha(\cos\alpha\cos 3\alpha - \sin\alpha\sin 3\alpha) = 2\sin\alpha\cos 4\alpha$. ●

226. (1) $1 + \sin\alpha + \cos\alpha$; (2) $\sin x + \sin 2x + \sin 3x$; (3) $\sin A + \sin B + \sin C$, if $A + B + C = \pi$; (4) $\sin 20° + \sin 34° + \sin 24° + \sin 30°$.

○ (1) Transforming the expression $1 + \cos\alpha$ into a product by Formula (9.67) and using the double-angle formula for $\sin\alpha$, we get

$$1 + \cos\alpha + \sin\alpha = 2\cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$
$$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right) = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right)$$
$$= 4\cos\frac{\alpha}{2}\cos\frac{\pi}{4}\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = 2\sqrt{2}\cos\frac{\alpha}{2}\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

(2) We then transform the sum of the first two terms (or of any two terms) into a product, and apply the double-angle formula to the third term

$$\sin x + \sin 2x + \sin 3x = 2\sin\frac{3x}{2}\cos\frac{x}{2} + 2\sin\frac{3x}{2}\cos\frac{3x}{2}$$
$$= 2\sin\frac{3x}{2}\left(\cos\frac{x}{2} + \cos\frac{3x}{2}\right) = 2\sin\frac{3x}{2}\cdot 2\cos x\cos\frac{x}{2}$$
$$= 4\sin\frac{3x}{2}\cos x\cos\frac{x}{2}.$$

(3) Since $C = \pi - (A + B)$, we have $\sin C = \sin(\pi - (A + B)) = \sin(A + B)$.

Therefore

$$\sin A + \sin B + \sin C = \sin A + \sin B + \sin(A + B)$$
$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cdot 2\sin\frac{A+B}{2}\cos\frac{A+B}{2}$$
$$= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$
$$= 4\sin\frac{A+B}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$$

But $A+B+C=\pi$, i.e. $\frac{A+B}{2}=\frac{\pi-C}{2}=\frac{\pi}{2}-\frac{C}{2}$. Hence, $\sin\frac{A+B}{2}=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\frac{C}{2}$, whence

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(4) Noting that $20^\circ+34^\circ=24^\circ+30^\circ$, we perform successively the transformations by Formulas (9.61) and (9.63):

$$\sin 20^\circ+\sin 34^\circ+\sin 24^\circ+\sin 30^\circ=2\sin 27^\circ\cos 7^\circ$$
$$+2\sin 27^\circ\cos 3^\circ=2\sin 27^\circ(\cos 7^\circ+\cos 3^\circ)$$
$$=4\sin 27^\circ\cos 5^\circ\cos 2^\circ. \bullet$$

227. Solve the following equations: (1) $\sin 4x=\sin 3x$; (2) $\cos 5x=\cos\left(x-\frac{\pi}{4}\right)$; (3) $\cos x=\sin 3x$; (4) $\tan 5x=\tan x$.

○ Let us take advantage of the conditions for equality of two trigonometric functions having the same name:

$$(1)\ (\sin 4x=\sin 3x)\Leftrightarrow\left[\begin{array}{l}4x+3x=(2k+1)\pi,\\4x-3x=2\pi k,\ k\in\mathbf{Z}.\end{array}\right.$$

$$\Leftrightarrow\left[\begin{array}{l}7x=(2k+1)\pi,\\x=2\pi k,\ k\in\mathbf{Z}.\end{array}\right.\quad \textit{Answer:}\ \left\{(2k+1)\frac{\pi}{7},\ 2k\pi\,|\,k\in\mathbf{Z}\right\}.$$

$$(2)\ \left(\cos 5x=\cos\left(x-\frac{\pi}{4}\right)\right)\Leftrightarrow\left[\begin{array}{l}5x+x-\frac{\pi}{4}=2\pi k,\\5x-x+\frac{\pi}{4}=2\pi k,\ k\in\mathbf{Z}\end{array}\right.$$

$$\Leftrightarrow\left[\begin{array}{l}6x=\frac{\pi}{4}+2\pi k,\\4x=-\frac{\pi}{4}+2\pi k,\ k\in\mathbf{Z}.\end{array}\right.$$

Answer: $\left\{\frac{\pi}{24}(8k+1),\ \frac{\pi}{16}(8k-1)\,|\,k\in\mathbf{Z}\right\}$.

$$(3)\ (\cos x=\sin 3x)\Leftrightarrow\left(\cos x=\cos\left(\frac{\pi}{2}-3x\right)\right)$$

$$\Leftrightarrow\left[\begin{array}{l}x+\pi/2-3x=2\pi k,\\x-\pi/2+3x=2\pi k,\ k\in\mathbf{Z}\end{array}\right.\Leftrightarrow\left[\begin{array}{l}-2x=2\pi k-\pi/2,\\4x=2\pi k+\pi/2,\ k\in\mathbf{Z}.\end{array}\right.$$

Answer: $\left\{\frac{\pi}{4}(4k+1),\quad\frac{\pi}{8}(4k+1)\,|\,k\in\mathbf{Z}\right\}$.

$$(4)\ (5x-x=\pi k)\Leftrightarrow(4x=\pi k)\Leftrightarrow(x=\pi k/4).$$

From the set of solutions we have to eliminate those values of x for which both sides of the equation do not exist, that is, the values

of the form $\pi/2\,(2k+1)$. In the set $x = \pi k/4 = (\pi/2)\,(k/2)$ such values occur if $k/2$ is an odd number: $k/2 = 2n+1$, that is, $k = 4n+2$. Consequently, the equation is satisfied by the set of roots of the form $\pi k/4$ for $k \neq 4n+2$. ●

In Problems 228 through 230 transform each of the given expressions into a product:

228. (1) $\cos(\pi/3) - \cos(2\pi/3)$; (2) $\cos\beta - \sin\alpha$; (3) $2\cos^2\alpha - \sin 2\alpha$; (4) $\cos 20^\circ + \sin 50^\circ$; (5) $\tan 25^\circ - \cot 75^\circ$; (6) $\sin^2 5\alpha - \sin^2 3\alpha$.

229. (1) $\sin\alpha\cos\beta + 2\sin^2(\alpha/2)\sin\beta$; (2) $\sin 10^\circ + 2\sin 5^\circ\cos 15^\circ + \cos 50^\circ$.

230. (1) $\sin 16^\circ + \sin 26^\circ - \sin 42^\circ$; (2) $\sin A + \sin B + \sin(A+B)/2$; (3) $\sin(\alpha/2) - \sin(3\alpha/2) + \cos\alpha$; (4) $\sin A + \cos B + \cos C$; where $A + B + C = \pi$; (5) $\sin 25^\circ + \sin 37^\circ + \sin 27^\circ + \sin 35^\circ$.

231. Show that for the angles A, B, and C of any triangle the following relationships hold true:

(1) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$;

(2) $1 - \cos A + \cos B + \cos C = 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

(3) $\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot\frac{A}{2}\cot\frac{B}{2}$;

(4) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

232. Prove the indicated identities:

(1) $1 + \sin\alpha + \cos\alpha = 4\cos\frac{\alpha}{2}\sin\frac{\pi}{4}\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$;

(2) $\sin\alpha + \cos\alpha - 1 = 2\sqrt{2}\sin\frac{\alpha}{2}\cos\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$;

(3) $2\cos^2\alpha + \cos\alpha - 1 = 2\cos(3\alpha/2)\cos(\alpha/2)$.

233. Solve the following equations:

(1) $\sin 5x = \cos\left(\frac{\pi}{2} - 7x\right)$; (2) $\cot\left(\frac{\pi}{6} - x\right) = -\tan\left(\frac{\pi}{6} - 2x\right)$;

(3) $\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right) = 0$;

(4) $\cos(x - 70^\circ) - \sin(x + 70^\circ) = 0$.

Sec. 9.21. Transformations with the Aid of an Auxiliary Argument

A given number a may be regarded as the value of a trigonometric function of a certain argument called an *auxiliary angle*, since for any value of a we have the equality $\tan(\arctan a) = a$ and for $|a| \leqslant 1$ the equalities $\sin(\arcsin a) = a$ and $\cos(\arccos a) = a$. This allows us to represent an algebraic sum of any two numbers as an algebraic sum of values of trigonometric functions, and, conse-

quently, to apply the formulas for transforming an algebraic sum of trigonometric functions into a product.

234. Transform into a product by introducing an auxiliary argument: (1) $\sqrt{2}\sin\alpha + 1$; (2) $\sqrt{3} + \tan\alpha$; (3) $4\cos^2\alpha - 1$; (4) $a + b$; (5) $a^2 + b^2$.

○ (1) $\sqrt{2}\sin\alpha + 1 = \sqrt{2}\left(\sin\alpha + \frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\sin\alpha + \sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\sin\left(\frac{\pi}{8} + \frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$;

(2) $\sqrt{3} + \tan\alpha = \tan\frac{\pi}{3} + \tan\alpha = \frac{\sin\left(\frac{\pi}{3}+\alpha\right)}{\cos\frac{\pi}{3}\cos\alpha} = \frac{2\sin\left(\frac{\pi}{3}+\alpha\right)}{\cos\alpha}$;

(3) $4\cos^2\alpha - 1 = 4\left(\cos^2\alpha - \frac{1}{4}\right) = 4\left(\cos\alpha + \frac{1}{2}\right)\left(\cos\alpha - \frac{1}{2}\right)$

$= 4\left(\cos\alpha + \cos\frac{\pi}{3}\right)\left(\cos\alpha - \cos\frac{\pi}{3}\right)$

$= 4\cdot 2\cos\left(\frac{\alpha}{2} + \frac{\pi}{6}\right)\cos\left(\frac{\alpha}{2} - \frac{\pi}{6}\right) 2\sin\left(\frac{\alpha}{2} + \frac{\pi}{6}\right)\sin\left(\frac{\pi}{6} - \frac{\alpha}{2}\right)$

$= 4\sin\left(\alpha + \frac{\pi}{3}\right)\sin\left(\frac{\pi}{3} - \alpha\right)$;

(4) $a + b = a\left(1 + \frac{b}{a}\right) = a(1 + \tan\varphi) = a\left(\tan\frac{\pi}{4} + \tan\varphi\right)$

$= \frac{a\sin\left(\frac{\pi}{4}+\varphi\right)}{\cos\frac{\pi}{4}\cos\varphi} = \frac{a\sqrt{2}\sin\left(\frac{\pi}{4}+\varphi\right)}{\cos\varphi}$, where $\varphi = \arctan\frac{b}{a}$.

(5) $a^2 + b^2 = a^2\left(1 + \frac{b^2}{a^2}\right) = a^2(1 + \tan^2\varphi) = \frac{a^2}{\cos^2\varphi}$, where $\varphi = \arctan\frac{b}{a}$. ●

235. Transform the sum $4\sin\alpha + 3\cos\alpha$ into a product.

○ Let us represent the given expression in the following way

$$4\sin\alpha + 3\cos\alpha = \sqrt{4^2+3^2}\left(\frac{4}{\sqrt{4^2+3^2}}\sin\alpha + \frac{3}{\sqrt{4^2+3^2}}\cos\alpha\right)$$

$$= 5\left((4/5)\sin\alpha + (3/5)\cos\alpha\right).$$

Since $(4/5)^2 + (3/5)^2 = 1$, there is an angle φ such that $4/5 = \cos\varphi$, $3/5 = \sin\varphi$. Then

$4\sin\alpha + 3\cos\alpha = 5(\sin\alpha\cos\varphi + \sin\varphi\cos\alpha) = 5\sin(\alpha + \varphi)$,

where $\tan\varphi = \sin\varphi/\cos\varphi = 3/4$. Thus,

$$4\sin\alpha + 3\cos\alpha = 5\sin(\alpha + \arctan(3/4)).$$ ●

236. Using the method of introducing an auxiliary argument, solve the trigonometric equation $\sin x + \sqrt{3}\cos x = 2$.

○ We transform the left-hand member of the given equation into a product:

$$\sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$$

$$= 2\left(\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right) = 2\sin\left(x + \frac{\pi}{3}\right).$$

Thus,

$$\left(\sin\left(x + \frac{\pi}{3}\right) = 1\right) \Leftrightarrow \left(x + \frac{\pi}{3} = \frac{\pi}{2} + 2\pi k\right)$$

$$\Leftrightarrow \left(x = \frac{\pi}{2} - \frac{\pi}{3} + 2\pi k\right) \Leftrightarrow \left(x = \frac{\pi}{6} + 2\pi k, \quad k \in \mathbf{Z}\right). \bullet$$

In Problems 237 and 238 transform each of the indicated expressions into a product by introducing an auxiliary argument:

237. (1) $0.5 + \cos\alpha$; (2) $1 - 2\cos\alpha$; (3) $1 - 2\sin\alpha$; (4) $1 + 2\sin\alpha$; (5) $1 + \tan\alpha$; (6) $1 - \tan^2\alpha$; (7) $1 - \cot^2\alpha$; (8) $\sqrt{3} + 2\cos\alpha$; (9) $\sin^2\alpha - 0.75$; (10) $1 - \sqrt{2}\sin\alpha$; (11) $(a - b)/(a + b)$.

238. (1) $\sqrt{3}\sin\alpha - \cos\alpha$; (2) $4\sin\alpha - 3\cos\alpha$; (3) $2\sin\alpha - \sqrt{5}\cos\alpha$.

239. Solve the following equations: (1) $\sqrt{3}\sin x - \cos x = 1$; (2) $\sqrt{3}\sin x + \cos x = 2$; (3) $\sin x + \sqrt{3}\cos x = \sqrt{3}$.

Sec. 9.22. Miscellaneous Problems

240. Transform the following products into sums of trigonometric functions of first power:

(1) $4\cos\frac{\alpha}{2}\cos\alpha\sin\frac{3\alpha}{2}$; (2) $4\cos\frac{\alpha}{2}\cos\alpha\sin\frac{5\alpha}{2}$;

(3) $4\cos\frac{\alpha}{2}\cos\alpha\cos\frac{5\alpha}{2}$.

241. Simplify the following products by transforming them into sums: (1) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$; (2) $\tan 20^\circ \tan 40^\circ \times \tan 60^\circ \tan 80^\circ$.

242. Transform into a product:

(1) $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha$;

2) $\sin 3\alpha + \sin 2\alpha + 2\sin\frac{3\alpha}{2}\cos\frac{\alpha}{2}$.

243. Solve the following equations:

(1) $\sin\left(x+\frac{\pi}{3}\right)\cos\left(x-\frac{\pi}{6}\right)=1$;

(2) $\sin\left(x+\frac{\pi}{3}\right)\cos\left(x-\frac{\pi}{3}\right)=-\frac{1}{4}$.

In Problems 244 through 246 prove the indicated identities:

244. (1) $\sin^2\alpha\cos\alpha = (1/4)(\cos\alpha - \cos 3\alpha)$;

(2) $\sin^3\alpha\cos\alpha = (1/8)(2\sin 2\alpha - \sin 4\alpha)$;

(3) $\sin^2\alpha\cos^3\alpha = (1/16)(2\cos\alpha - \cos 3\alpha - \cos 5\alpha)$;

(4) $\sin^3\alpha\cos^3\alpha = (1/32)(3\sin 2\alpha - \sin 6\alpha)$.

245. (1) $4\sin\alpha\sin 2\alpha\sin 3\alpha = \sin 2\alpha + \sin 4\alpha - \sin 6\alpha$;

(2) $4\sin\alpha\sin\beta\cos(\alpha+\beta) = \cos 2\alpha + \cos 2\beta - \cos 2(\alpha+\beta) - 1$;

(3) $\sin\left(\frac{\pi}{4}+\alpha\right)\sin\left(\frac{\pi}{4}-\alpha\right)\cos 2\alpha = \frac{1}{4}+\frac{1}{4}\cos 4\alpha$.

246. (1) $\dfrac{\sin\alpha-\sin 2\alpha-\sin 4\alpha+\sin 5\alpha}{\cos\alpha-\cos 2\alpha-\cos 4\alpha+\cos 5\alpha}=\tan 3\alpha$;

(2) $\dfrac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}=\cot\alpha$;

(3) $\dfrac{\sin 80^\circ-\sin 70^\circ}{\sin 29^\circ-\sin 19^\circ}=\dfrac{\cos 20^\circ-\cos 50^\circ}{\cos 31^\circ+\sin 11^\circ}$;

(4) $\dfrac{(\cos\alpha+\cos(\alpha/2))^2-(\sin\alpha+\sin(\alpha/2))^2}{\sin 2\alpha-\sin\alpha}=\cot(\alpha/4)$;

(5) $\dfrac{1}{\sin^2\alpha}+\dfrac{1}{\sin^2\beta}-\dfrac{2\cos(\alpha-\beta)}{\sin\alpha\sin\beta}=\dfrac{\sin^2(\alpha-\beta)}{\sin^2\alpha\sin^2\beta}$.

QUIZ No 6.

Form I

(1) Simplify:
$\sin 87^\circ-\sin 59^\circ-\sin 93^\circ+\sin 61^\circ$.

(2) Prove the identity:
$\sin 10^\circ\sin 20^\circ\sin 70^\circ\sin 80^\circ = (1/4)\cos 50^\circ\cos 70^\circ$.

(3) Prove the identity:
$1+\sin\alpha+\cos\alpha = 2\sqrt{2}\cos\frac{\alpha}{2}\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$.

(4) Solve the equation:
$\sin 3x+\sin x=2\sin 2x$.

(5) Solve the equation:
$\sin 2x=\cos x-\cos 3x$.

Form II

(1) Simplify:
$\cos 65^\circ+\sqrt{3}\sin 5^\circ+\sin 85^\circ$.

(2) Prove the identity:
$\cos 10^\circ\sin 20^\circ\cos 70^\circ\sin 80^\circ = \cos^2 10^\circ\sin^2 20^\circ$.

(3) Prove the identity:
$1+\sin\alpha-\cos\alpha = 2\sqrt{2}\sin\frac{\alpha}{2}\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$.

(4) Solve the equation:
$\sin^4 x-\cos^4 x=\sin 2x$.

(5) Solve the equation:
$\cos x-\sin x=\sqrt{2}\sin 2x$.

Sec. 9.23. Computing the Limits of Trigonometric Functions. The Limit of the Ratio $\sin x/x$ as $x \to 0$

When computing the limits of trigonometric functions, we often use the limit of the ratio of the sine of an arc to the arc itself:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad \text{or} \quad \lim_{x\to 0}\frac{x}{\sin x}=1. \tag{9.71}$$

In Problems 247 and 248 compute the given limits:

247. (1) $\lim\limits_{x\to -\pi/2}\dfrac{\cos^2 x}{1+\sin^3 x}$; (2) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}$; (3) $\lim\limits_{z\to 0}\dfrac{\sin^3 2z}{z^3}$;

(4) $\lim\limits_{x\to 0}\dfrac{\sin^3 x}{x^2}$; (5) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x}$.

⊙ (1) It is obvious that, as $x \to -\pi/2$, both the numerator and denominator of the fraction tend to zero. Factorizing the numerator and denominator into factors and reducing the fraction by $(1 + \sin x)$, we get

$$\lim_{x\to -\pi/2}\frac{\cos^2 x}{1+\sin^3 x}=\lim_{x\to -\pi/2}\frac{1-\sin^2 x}{(1+\sin x)(1-\sin x+\sin^2 x)}$$

$$=\lim_{x\to -\pi/2}\frac{1-\sin x}{1-\sin x+\sin^2 x}=\frac{1-\sin(-\pi/2)}{1-\sin(-\pi/2)+\sin^2(-\pi/2)}$$

$$=\frac{1+1}{1+1+1}=\frac{2}{3}.$$

(2) Transforming the given expression and using (9.71), we get

$$\lim_{x\to 0}\frac{\sin 4x}{3x}=\lim_{x\to 0}\frac{4\sin 4x}{4\cdot 3x}=\lim_{x\to 0}\frac{4\sin 4x}{3\cdot 4x}=\frac{4}{3}\lim_{4x\to 0}\frac{\sin 4x}{4x}=\frac{4}{3}\cdot 1=\frac{4}{3}.$$

(3) $$\lim_{z\to 0}\frac{\sin^3 2z}{z^3}=\lim_{z\to 0}\left(\frac{\sin 2z}{z}\right)^3=8\left(\lim_{2z\to 0}\frac{\sin 2z}{2z}\right)^3=8\cdot 1=8.$$

(4) $$\lim_{x\to 0}\frac{\sin^3 x}{x^2}=\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\cdot\sin x\right)=\lim_{x\to 0}\left(\left(\frac{\sin x}{x}\right)^2\sin x\right)$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\sin x=1^2\cdot 0=0.$$

(5) Transforming the difference between cosines, we get

$$\lim_{x\to 0}\frac{\cos x-\cos 3x}{x}=2\lim_{x\to 0}\frac{\sin 2x\sin x}{x}$$

$$=4\lim_{2x\to 0}\frac{\sin 2x}{2x}\lim_{x\to 0}\sin x=4\cdot 1\cdot 0=0. \quad \bullet$$

248. (1) $\lim\limits_{x\to \pi/2}\dfrac{\cot x}{x-\pi/2}$; (2) $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{2x}$.

○ (1) Let us make the substitution $x-\pi/2=y$, then $x=\pi/2+y$; if $x\to\pi/2$, then $y\to 0$. Consequently,

$$\lim_{x\to\pi/2}\frac{\cot x}{x-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\cot(\pi/2+y)}{y}=-\lim_{y\to 0}\frac{\tan y}{y}$$

$$=-\lim_{y\to 0}\frac{\sin y}{y}\cdot\lim_{y\to 0}\frac{1}{\cos y}=-1\cdot 1=-1.$$

(2) Setting $\arcsin 3x=\alpha$, we have $\sin\alpha=3x$. Transforming:

$$\frac{\arcsin 3x}{2x}=\frac{3\arcsin 3x}{2\cdot 3x}=\frac{3}{2}\,\frac{\alpha}{\sin\alpha},$$

we then get

$$\lim_{x\to 0}\frac{\arcsin 3x}{2x}=\lim_{\alpha\to 0}\left(\frac{3}{2}\,\frac{\alpha}{\sin\alpha}\right)=\frac{3}{2}\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=\frac{3}{2}\cdot 1=\frac{3}{2}.$$ ●

In Problems 249 through 256 compute the indicated limits:

249. (1) $\lim\limits_{x\to\pi}\dfrac{\sin^2 x}{1+\cos x}$; (2) $\lim\limits_{x\to\pi/4}\dfrac{\sin x-\cos x}{\tan x-1}$.

250. (1) $\lim\limits_{x\to\pi/2}\dfrac{1-\sqrt{1+\cot x}}{\cot x}$; (2) $\lim\limits_{x\to 0}\dfrac{\tan x}{\sqrt{1-\tan x}-1}$.

251. (1) $\lim\limits_{x\to 0}\dfrac{3x}{\sin 2x}$; (2) $\lim\limits_{\alpha\to 0}\dfrac{5\alpha}{\cos\alpha}$;

(3) $\lim\limits_{x\to 0}\dfrac{\sin^2(x/4)}{x^2}$; (4) $\lim\limits_{z\to 0}\dfrac{\sin^3 3z}{z^3}$.

252. (1) $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x}$; (2) $\lim\limits_{x\to 0}\dfrac{\sin^3 x}{x^4}$.

253. (1) $\lim\limits_{x\to 0}\dfrac{\sin 2x\cos x}{x}$; (2) $\lim\limits_{x\to 0}\dfrac{\sin 2x\sin x}{x^2}$;

(3) $\lim\limits_{x\to 0}\dfrac{\sin 3x+\sin x}{x}$; (4) $\lim\limits_{x\to 0}\dfrac{\sin 2x+\sin 4x}{x}$;

(5) $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos x}{x}$.

254. (1) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{x^2}$; (2) $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x}$.

255. (1) $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{\pi/2-x}$; (2) $\lim\limits_{x\to\pi/2}\dfrac{\pi/2-x}{\cot x}$.

256. (1) $\lim\limits_{z\to 0}\dfrac{\arcsin 2z}{7z}$; (2) $\lim\limits_{x\to 0}\dfrac{\arctan 5x}{3x}$.

Sec. 9.24. The Derivatives of Trigonometric Functions

Differentiation Formulas

If $u=\varphi(x)$	Formula No.	If $u=x$	Formula No.
$(\sin u)'=\cos u\cdot u'$	(9.72)	$(\sin x)'=\cos x$	(9.72a)
$(\cos u)'=-\sin u\cdot u'$	(9.73)	$(\cos x)'=-\sin x$	(9.73a)
$(\tan u)'=\dfrac{1}{\cos^2 u}\,u'$	(9.74)	$(\tan x)'=\dfrac{1}{\cos^2 x}$	(9.74a)
$(\cot u)'=-\dfrac{1}{\sin^2 u}\,u'$	(9.75)	$(\cot x)'=-\dfrac{1}{\sin^2 x}$	(9.75a)

In Problems 257 and 258 we shall find the derivatives of the indicated functions:

257. (1) $f(x)=\dfrac{1-\sin x}{1+\sin x}$; find $f'(\pi/4)$;

(2) $f(x)=\dfrac{\tan x-1}{\tan x}$; find $f'(\pi/3)$.

○ (1) By Formulas (7.5) and (9.72a), we get

$$f'(x)=\frac{(1-\sin x)'(1+\sin x)-(1+\sin x)'(1-\sin x)}{(1+\sin x)^2}$$

$$=\frac{-\cos x(1+\sin x)-\cos x(1-\sin x)}{(1+\sin x)^2}=-\frac{2\cos x}{(1+\sin x)^2};$$

$$f'(\pi/4)=-\frac{2\cos(\pi/4)}{(1+\sin(\pi/4))^2}=-\frac{\sqrt{2}}{(1+\sqrt{2}/2)^2}=8-6\sqrt{2}.$$

(2) By Formulas (7.5) and (9.74a), we get

$$f'(x)=\frac{\dfrac{1}{\cos^2 x}\tan x-\dfrac{1}{\cos^2 x}(\tan x-1)}{\tan^2 x}$$

$$=\frac{\dfrac{1}{\cos^2 x}(\tan x-\tan x+1)}{\tan^2 x}=\frac{1}{\sin^2 x};$$

$$f'(\pi/3)=\frac{1}{\sin^2(\pi/3)}=\frac{4}{3}.$$ ●

258. (1) $y=\sin(2x^2+3)$; (2) $y=\sin^3 mx$.

○ (1) Setting $2x^2 + 3 = u$, we get $y = \sin u$. By Formula (9.72), we find

$$y' = \cos u \cdot u' = \cos (2x^2 + 3)\,(2x^2 + 3)' = \cos (2x^2 + 3) \cdot 4x = 4x \cos (2x^2 + 3) \cdot$$

(2) Setting $mx = u$, we get $y = \sin^3 u$. Applying in succession Formulas (7.10) and (9.72), we get

$$y' = 3 \sin^2 u\,(\sin u)' = 3 \sin^2 u \cos u \cdot u' = 3 \sin^2 mx \cos mx\,(mx') = 3 \sin^2 mx \cos mx \cdot m = 3m \sin^2 mx \cos mx. \bullet$$

In Problems 259 through 275 find the derivative for each of the given functions:

259. (1) $f(x) = \dfrac{\sin x - 1}{\sin x}$; compute $f'(\pi/3)$; (2) $y = x^2 + \sin x$; (3) $y = x \sin x$.

260. (1) $y = \sin 3x$; (2) $f(x) = \sin (4x - 1)$; (3) $s = \sin t^2$; (4) $f(\theta) = \sin (\theta/2)$; compute $f'(\pi/2)$.

261. (1) $y = \sin^2 x$; (2) $t = \sin^3 5\varphi^2$; (3) $y = 1/\sin x$; (4) $y = 1/\sin 3x$; (5) $y = 1/\sin^2 x$.

262. (1) $f(t) = \sqrt{\sin t}$; (2) $y = \sqrt{\sin 2x}$; (3) $y = 1/\sqrt{\sin 3x}$; (4) $y = 1/\sqrt{\sin^3 x}$.

263. (1) $f(x) = \dfrac{1 - \sin x}{1 + \cos x}$; compute $f'(\pi/4)$;

(2) $f(x) = \dfrac{\cos x + 1}{\cos x - 1}$; compute $f'(\pi/3)$; (3) $y = 2 \sin x - \cos x + 3$; (4) $f(x) = 2 \sin x - 2 \cos x$; compute $f'(\pi/6)$.

264. (1) $f(t) = \sin t \cos t$; (2) $f(x) = \sin x\,(1 - \cos x)$; (3) $y = x \cos x$; (4) $f(x) = \cos x\,(1 + \sin x)$.

265. (1) $y = \cos x^3$; (2) $y = \cos (1/x^2)$; (3) $y = \cos^3 x$; (4) $y = 1/\cos 2x$; (5) $y = 1/\cos^2 x$.

266. (1) $y = \sqrt{\cos 2x}$; (2) $y = 1/\sqrt{\cos x^2}$.

267. (1) $y = (1 - \cos 2x) \sin 2x$; (2) $y = \dfrac{\cos^2 x}{1 + \cos^2 x}$; (3) $f(x) = \cos^3 x \sin x$; compute $f'(\pi/3)$.

268. (1) $y = \dfrac{\tan x}{1 - \tan x}$; (2) $y = \tan x - x$; (3) $f(u) = u^2 \tan u$; (4) $f(x) = \sin x + \tan x$; compute $f'(\pi)$.

269. (1) $y = \tan (ax + b)$; (2) $y = \tan (x/3)$; (3) $y = \tan x^2$; (4) $y = \tan \sqrt{2x}$; (5) $y = \tan^2 3x$.

270. (1) $y = \tan x \sin^2 x$; (2) $y = 3x - \tan 3x$; (3) $f(x) = \tan^2 x \sin x$; compute $f'(\pi/3)$; (4) $y = \tan \frac{x}{2} + \frac{1}{3} \tan^3 \frac{x}{2}$.

271. $f(x) = \frac{\tan(x/2)}{1+\tan(x/2)}$; compute $f'(\pi/2)$; (2) $y = \frac{1-\tan 2x}{\tan 2x}$.

272. (1) $y = \cot x + x$; (2) $f(x) = \frac{1+\cot x}{\cot x}$;
(3) $f(x) = \cot x - \tan x$; compute $f'(\pi/4)$.

273. (1) $y = \cot x^3$; (2) $y = \cot^3 x$; (3) $y = -\cot \frac{x}{2} - \frac{1}{3} \cot^3 \frac{x}{2}$; (4) $y = 1/\cot^2 2x$.

274. Find the velocity of a point moving in a straight line according to the law $s = 4 \sin 3t$ at the instant of time $t = \pi/9$ (s in metres, t in seconds).

275. Find the velocity of a point moving in a straight line according to the law $s = -2 \cos 2t$ at the instant $t = \pi/6$.

Sec. 9.25. The Derivatives of Inverse Trigonometric Functions

Differentiation Formulas

If $u = \varphi(x)$	Formula No.	If $u = x$	Formula No.
$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} u'$, $\lvert u \rvert < 1$	(9.76)	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $\lvert x \rvert < 1$	(9.76a)
$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} u'$, $\lvert u \rvert < 1$	(9.77)	$(\arccos x')' = -\frac{1}{\sqrt{1-x^2}}$, $\lvert x \rvert < 1$	(9.77a)
$(\arctan u)' = \frac{1}{1+u^2} u'$	(9.78)	$(\arctan x)' = \frac{1}{1+x^2}$	(9.78a)
$(\operatorname{arccot} u)' = -\frac{1}{1+u^2} u'$	(9.79)	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	(9.79a)

In Problems 276 and 277 find the derivative for each of the given functions:

276. (1) $f(x) = 5 \arcsin x - 3 \arccos x$; compute $f'(\sqrt{3}/2)$; (2) $f(x) = 3 \arctan x - 2 \operatorname{arccot} x$; compute $f'(2)$.

○ Using Formulas (9.76a) and (9.77a), we get

$$f'(x)=\frac{5}{\sqrt{1-x^2}}+\frac{3}{\sqrt{1-x^2}}=\frac{8}{\sqrt{1-x^2}}; \qquad f'\left(\frac{\sqrt{3}}{2}\right)=\frac{8}{\sqrt{1-3/4}}=16.$$

(2) By Formulas (7.1), (9.78a), and (9.79a), we find

$$f'(x)=\frac{3}{1+x^2}+\frac{2}{1+x^2}=\frac{5}{1+x^2}; \quad f'(2)=\frac{5}{1+2^2}=1. \bullet$$

277. (1) $y=\arcsin 2x$; (2) $y=\arccos\sqrt{2x}$; (3) $y=\operatorname{arccot} 3x$.
○ (1) By Formula (9.76), we get

$$y'=\frac{1}{\sqrt{1-(2x)^2}}(2x')=\frac{1}{\sqrt{1-4x^2}}\cdot 2=\frac{2}{\sqrt{1-4x^2}}.$$

(2) By Formula (9.77), we get

$$y'=-\frac{1}{\sqrt{1-(\sqrt{2x})^2}}(\sqrt{2x})'=-\frac{1}{\sqrt{1-2x}}\cdot\frac{1}{2\sqrt{2x}}\cdot 2$$

$$=-\frac{1}{\sqrt{1-2x}}\,\frac{1}{\sqrt{2x}}=-\frac{1}{\sqrt{2x(1-2x)}}.$$

(3) By Formula (9.79), we find

$$y'=-\frac{1}{1+(3x)^2}\cdot(3x)'=-\frac{1}{1+9x^2}\cdot 3=-\frac{3}{1+9x^2}. \bullet$$

In Problems 278 through 284 find the derivatives of the given functions:

278. (1) $f(x)=2\arcsin x+\arccos x$; compute $f'(\sqrt{2}/2)$;
(2) $f(x)=5\arcsin x+2\arccos x$; compute $f'(1/2)$;
(3) $y=x(\arcsin x+\arccos x)$.

279. (1) $y=\arcsin 3x$; (2) $y=\arccos(x/a)$;
(3) $y=\arcsin x^2$; (4) $y=\arccos ax$.

280. (1) $y=\arcsin\sqrt{3x}$; (2) $y=\arccos\sqrt{x-1}$;
(3) $y=\arcsin\dfrac{x^2-a^2}{x^2+a^2}$.

281. (1) $f(x)=\arctan x$; compute $f'(\sqrt{3})$;
(2) $y=x(\arctan x+\operatorname{arccot} x)$.

282. (1) $y=\arctan x^2$; (2) $y=\operatorname{arccot} 3x$; (3) $y=\arctan(a/x)$;
(4) $y=\operatorname{arccot}(x/a)$.

283. (1) $y=\arctan\sqrt{x}$; (2) $y=\arctan(1/\sqrt{x})$.

284. (1) $y=\operatorname{arccot}\dfrac{1+x}{1-x}$; (2) $y=\operatorname{arccot}\dfrac{x-1}{x+1}$.

Sec. 9.26. The Second Derivative and Its Applications

285. Find the second derivatives of the following functions: (1) $y = \sin^2 x$; (2) $y = \ln \sin x$.

○ (1) $y' = 2 \sin x \cos x = \sin 2x$; $y'' = \cos 2x \cdot 2 = 2 \cos 2x$;

(2) $y' = \frac{1}{\sin x} \cdot \cos x = \cot x$; $y'' = -\frac{1}{\sin^2 x}$. ●

286. Find the velocity and acceleration of a point moving in a straight line according to the law $s = 2 \sin (\pi t/3)$ at the instant $t = 1$.

○ $v = \frac{ds}{dt} = 2 \cos \frac{\pi t}{3} \frac{\pi}{3} = \frac{2\pi}{3} \cos \frac{\pi t}{3}$;

$v(1) = \frac{2\pi}{3} \cos \frac{\pi}{3} = \frac{2\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{3}$;

$a = \frac{d^2 s}{dt^2} = \frac{dv}{dt} = -\frac{2\pi}{3} \sin \frac{\pi t}{3} \frac{\pi}{3} = -\frac{2\pi^2}{9} \sin \frac{\pi t}{3}$;

$a(1) = -\frac{2\pi^2}{9} \sin \frac{\pi}{3} = -\frac{2\pi^2}{9} \cdot \frac{\sqrt{3}}{2} = -\frac{\pi^2 \sqrt{3}}{9}$. ●

287. Find the second derivatives of the following functions: (1) $y = \cos x$; (2) $y = \tan x$; (3) $y = \ln \cos x$; (4) $y = \ln \tan x$; (5) $s = e^{\cos t}$; (6) $s = e^{-\sin t}$.

288. At the instant $t = 1$, find the velocity and acceleration of a point moving in a straight line according to the law: (1) $s = \sin (\pi t/4)$; (2) $s = -\cos (\pi t/3)$.

289. A material point of mass m moves in a straight line according to the law $s = -\sin 3t$. Find the force F under the action of which the point moves at the instant $t = \pi/6$.

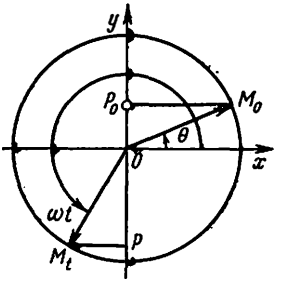


Fig. 59

Sec. 9.27. Harmonic Oscillations

1. **Basic Concepts.** If a point M moves in a circle of radius R (Fig. 59) with a constant angular velocity ω, then the projection P of the point M on the y axis oscillates along this axis according to the law

$$y = R \sin (\omega t + \theta). \qquad (9.80)$$

The motion of the point P is called *simple harmonic oscillation*. The quantity R (amplitude of oscillation) expresses the maximum deviation of the oscillating point P from the origin; ω is the angular velocity of the point M in radians per second; t is the time (in seconds) during which the point M displaces from the position M_0 to the position M_t; θ is the initial phase of oscillation.

At $t = 0$ the radius vector $\overrightarrow{OM_0}$ forms an angle θ with the x-axis. During t seconds the radius vector will rotate through an angle ωt and will form an angle $\omega t + \theta$ with the x-axis. Consequently, the projection of a point moving uniformly in a circle of radius R with angular velocity ω harmonically oscillates with amplitude R and initial phase θ.

The time T during which the point P passes through all of its phases and the point M completes one revolution while moving on the circle is called the *period* of harmonic oscillation, that is, T is the period of the function $y = R \sin(\omega t + \theta)$. Since during time T the point P completes one revolution, that is, describes an arc of 2π radians, during the unit time it describes an angle ω equal to $2\pi/T$ radians; therefore the angular velocity

$$\omega = 2\pi/T. \tag{9.81}$$

Hence it follows that

$$T = 2\pi/\omega. \tag{9.82}$$

The inverse of the period of oscillation T, that is, $1/T = \omega/(2\pi)$, is called the *frequency* of oscillation. The frequency* of oscillation indicates the number of oscillations n completed per second

$$T = 1/n. \tag{9.83}$$

2. Addition of Two Simple Harmonic Oscillations with the Same Period Let there be two harmonic oscillations with one and the same period T:

$$y_1 = R_1 \sin\left(\frac{2\pi}{T}t + \theta_1\right) \quad \text{and} \quad y_2 = R_2 \sin\left(\frac{2\pi}{T}t + \theta_2\right).$$

The addition of the given oscillations yields again a simple harmonic oscillation

$$R_1 \sin\left(\frac{2\pi}{T}t + \theta_1\right) + R_2 \sin\left(\frac{2\pi}{T}t + \theta_2\right) = R \sin\left(\frac{2\pi}{T}t + \theta\right),$$

whose amplitude R and the initial phase θ are related with the amplitudes and initial phases of the given oscillations by the following equalities:

$$R = \sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos(\theta_1 - \theta_2)}, \quad \tan\theta = \frac{R_1\sin\theta_1 + R_2\sin\theta_2}{R_1\cos\theta_2 + R_2\cos\theta_2}.$$

290. Set up the equation of harmonic oscillation if the amplitude is equal to 10, the period to 0.5 s, and the initial phase to 1.5.

○ By Formula (9.81), we find $\omega = 2\pi/0.5 = 4\pi$. Substituting $R = 10$, $\theta = 1.5$, $\omega = 4\pi$ into the equality (9.80), we get $y = 10 \sin(4\pi t + 1.5)$. ●

291. Find the period, amplitude, and initial phase of the following functions:

(1) $y = 3 \sin\left(2x - \frac{\pi}{6}\right)$; (2) $y = -3 \sin\frac{x}{3}$;

(3) $y = \cos\left(\frac{\pi}{4} - 2x\right)$.

○ (1) Here $R = 3$, $\omega = 2$, $\theta = -\pi/6$. The period T is found from the relationship (9.82), i.e. $T = 2\pi/2 = \pi$.

*Sometimes, frequency is defined as the quantity ω. It expreses the number of oscillations performed by the point P during 2π seconds.

(2) Here $R = |-3| = 3$, $\omega = 1/3$, $T = 2\pi$; $(1/3) = 6\pi$. To compute the initial phase, let us write the given function in the form $y = 3 \sin\left(\frac{x}{3} + \pi\right)$, whence $\theta = \pi$.

(3) Transform the given function in the following way:

$$y = \cos\left(\frac{\pi}{4} - 2x\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - 2x\right)\right) = \sin\left(2x + \frac{\pi}{4}\right),$$

whence $\theta = \pi/4$, $R = 1$, $\omega = 2$, $T = 2\pi/2 = \pi$. ●

292. Set up the equation of a harmonic oscillation if its amplitude is equal to 5, the frequency of oscillation to 3, and the initial phase to 0.8.

293. Find the period, amplitude, and initial phase of the following functions: (1) $y = 2 \sin\left(3x + \frac{\pi}{3}\right)$; (2) $y = \frac{1}{3} \sin\left(\pi x - \frac{\pi}{4}\right)$; (3) $y = \sin(x - 5)$.

294. A point mass m is in simple harmonic motion according to the law $s = 5 \sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$. Find the force F that started the mass moving at the moment $t = 0$.

295. Reduce to the form $R \sin(\omega t + \theta)$ the following expressions: (1) $12 \sin 2t + 5 \cos 2t$; (2) $8 \sin(5x + \pi/6) - 15 \cos(5x + \pi/6)$ (see Problem 235).

296. Find the amplitude and initial phase of the sums of the following harmonic oscillations: (1) $y_1 = 3 \sin(t/2)$ and $y_2 = 5 \sin(t/2)$; (2) $y_1 = 2 \sin 2t$ and $y_2 = 2 \sin(2t + \pi/3)$; (3) $y_1 = \sqrt{2} \sin 5t$ and $y_2 = \sqrt{2} \cos 5t$.

Sec. 9.28. Basic Properties of Trigonometric Functions

297. Find the domain of definition of each of the following functions: (1) $y = 1/\sin x$; (2) $y = 1/(1 - \cos x)$; (3) $y = \tan 4x$; (4) $y = 1/\cot x$.

○ (1) $\sin x \neq 0$, $x \neq \pi k$, $k \in \mathbf{Z}$;

(2) $1 - \cos x \neq 0$, $\cos x \neq 1$, $x \neq 2\pi k$, $k \in \mathbf{Z}$;

(3) $4x \neq \pi/2 + \pi k$, $x \neq \pi/8 + \pi k/4$, $k \in \mathbf{Z}$;

(4) $\cot x \neq 0$, $x \neq \pi/2 + \pi k$, besides, $x \neq \pi k$, consequently, $x \neq \pi k/2$, $k \in \mathbf{Z}$. ●

298. For what values of the argument the functions: (1) $y = 2 \sin 3x$; (2) $y = (1/2) \cos 2x$ take on the largest and smallest values?

○ (1) We have $\sin 3x = 1$, $3x = \pi/2 + 2\pi k$, hence, for $x = \pi/6 + 2\pi k/3$, $k \in \mathbf{Z}$, the function takes on the largest value equal to 2.

Analogously, $\sin 3x = -1$, $3x = -\pi/2 + 2\pi k$, that is, for $x = -\pi/6 + 2\pi k/3$, $k \in \mathbf{Z}$, the function attains the smallest value equal to -2.

(2) We have $\cos 2x = 1$, $2x = 2\pi k$, that is, for $x = \pi k$, $k \in \mathbf{Z}$, the function takes on the largest value equal to $1/2$. Similarly, $\cos 2x = -1$, $2x = \pi(2k + 1)$, that is, for $x = (\pi/2)(2k + 1)$, $k \in \mathbf{Z}$, the function attains the smallest value equal to $-1/2$. ●

299. Find the domain of definition of each of the following functions: (1) $y = 1/\sin x^2$; (2) $y = \tan x/\sin x$; (3) $y = \sin x/\tan x$; (4) $y = \cot 3x$; (5) $y = (1 + \sin x)/\cos x$.

300. Find the domain of definition of each of the following functions: (1) $y = \tan x \cot x$; (2) $y = \cos x + \cot x$; (3) $y = 1/(\sin x + \cos x)$; (4) $y = 1/(\sin^2 x - \sin x)$; (5) $y = \cot x/(\sin x - \cos x)$.

301. For what values of the argument do the following functions attain the largest and smallest values: (1) $y = \sin(x - 1)$; (2) $y = \cos(\pi/4 + x)$; (3) $y = 3 \sin 4x$; (4) $y = (1/2) \cos 5x$?

302. Find the set of values of each of the following functions: (1) $y = \sin |x|$; (2) $y = |\sin x|$; (3) $y = \cos |x|$; (4) $y = |\cos x|$.

303. What are the boundaries of variation of each of the following functions: (1) $y = 1 - \sin x$; (2) $y = 3 + \sin x$; (3) $y = 2 - \cos x$; (4) $y = 5 + \cos x$.

304. Do the following functions have the largest and smallest values: (1) $y = \tan x$; (2) $y = |\tan x|$; (3) $y = \tan^2 x$?

305. Find the largest and smallest values of the following functions: (1) $y = 4 + \sin(x - \pi/12)$; (2) $y = 6 - \sin^2 x$; (3) $y = 5 - 3|\sin x|$.

306. Find the range of each of the indicated functions: (1) $y = \sin x + \cos x$; (2) $y = 3 \sin x + \sqrt{3} \cos x$; (3) $y = \sin x - \sqrt{3} \cos x$, (see Problem 235).

307. Investigate with the aid of the derivative and formulate the basic properties of the following functions: $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$.

Sec. 9.29. Constructing Graphs of Trigonometric Functions

308. Construct the graphs of the following functions: (1) $y = A \sin x$; (2) $y = \sin(x + \alpha)$; (3) $y = \sin \omega x$.

○(1) If $A > 0$, then the given function has the same intervals of increase and decrease as $\sin x$; the largest value of the function is equal to A, and the smallest value to $-A$. The graph is obtained by stretching the sine curve $y = \sin x$ A times from the axis of abscissas. Such transformation is called the *transformation of amplitude*. Figure 60 represents the graphs of the functions: $y = \sin x$, $y = 2 \sin x$, and $y = (1/2) \sin x$.

(2) The graph of the given function is obtained from the graph of the function $y = \sin x$ by translating the origin to the point O_1 $(-\alpha, 0)$. Such transformation is called the *phase shift*. Figure 61 shows the graph of the function $y = \sin(x - \pi/4)$.

(3) The largest value of this function is equal to 1, and the smallest value to -1; the period is equal to $2\pi/\omega$. Setting $\omega = 1$, $\omega = 2$ and $\omega = 1/2$, we construct the graphs of the functions $y = \sin x$, $y = \sin 2x$, and $y = \sin(x/2)$.

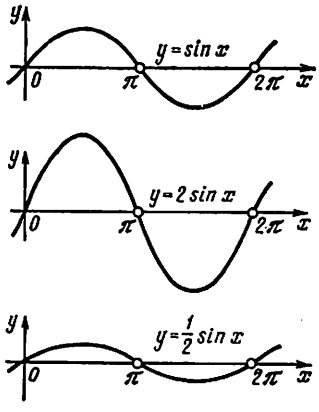


Fig. 60

The graph of the function $y = \sin 2x$ (period $T = \pi$) can be obtained by "compressing" the sine curve $y = \sin x$ twofold along the axis of abscissas. Anal-

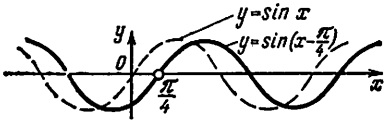


Fig. 61

ogously, the graph of the function $y = \sin(x/2)$ (period $T = 4\pi$) can be obtained by "stretching" the sine curve $y = \sin x$ twofold along

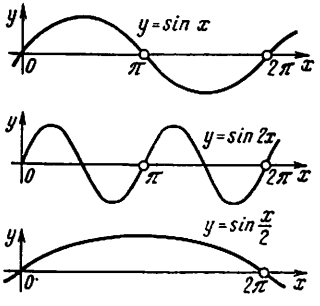


Fig. 62

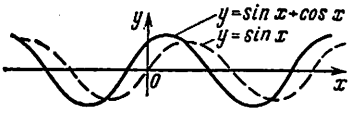


Fig. 63

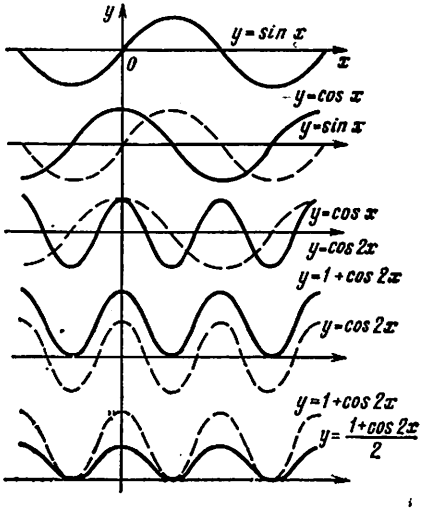


Fig. 64

the axis of abscissas. Such transformation is called the *period transformation*. The relevant graphs are represented in Fig. 62. ●

309. Construct the graphs of the following functions: (1) $y = \sin x + \cos x$; (2) $y = \cos^2 x$.

○(1) We transform the given function in the following way:

$$y = \sin x + \cos x = \sqrt{2} \sin (x + \pi/4).$$

The graph of this function is constructed in Fig. 63.

(2) We write the given function in the form: $y = \cos^2 x = (1/2)(1 + \cos 2x)$. The successive stages of constructing the desired graph are shown in Fig. 64. ●

In Problems 310 through 313 graph the indicated functions:

310. (1) $y = 3 \sin x$; (2) $y = 2 \cos x$; (3) $y = (-1/3) \sin x$.

311. (1) $y = \sin (x - \pi/6)$; (2) $y = \cos (x + \pi/3)$; (3) $y = \tan (x - \pi/4)$.

312. (1) $y = \cos 2x$; (2) $y = \cos (x/3)$; (3) $y = \tan (x/2)$.

313. (1) $y = \sin x - \cos x$; (2) $y = \sin^2 x$.

Sec. 9.30. Miscellaneous Problems

314. The cross-section of an open chute has the shape of an isosceles trapezoid whose base and lateral sides are equal to a (Fig. 65). What must be the size of the angle α at which the walls of the chute are inclined to its altitude drawn from the vertex of the obtuse angle to ensure the greatest chute capacity?

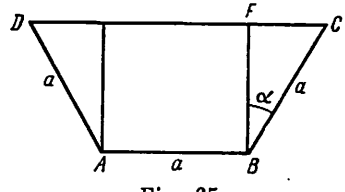


Fig. 65

In Problems 315 through 321 find the derivative for each of the indicated functions:

315. (1) $y = (1/3) \sin^3 x - \sin x$; (2) $y = \cos (x + a) \sin (x - a)$.

316. (1) $y = \dfrac{\tan 3x}{\tan 3x - 1}$; (2) $y = \tan 2x - \cot 2x$; (3) $y = \tan^2 2x + \cot^2 2x$.

317. (1) $y = \ln \cot x$; (2) $f(x) = \ln \sin (x/3)$; compute $f'(\pi/2)$; (3) $y = \ln \cos^2 x$.

318. (1) $y = \ln \sqrt{\dfrac{1 + \cos x}{1 - \cos x}}$; (2) $y = \ln \sin^2 (x - 1)$; (3) $u = \ln \tan^2 z^2$.

319. (1) $s = \ln e^{\sin 2t}$; (2) $y = e^{\sin x} \cos x$; (3) $y = e^{\tan x} \cos^2 x$.

320. (1) $y = \arccos \sqrt{1 - x^2}$; (2) $u = \operatorname{arccot} \dfrac{1+z}{1-z}$; (3) $y = \arctan \sqrt{x} + \operatorname{arccot} \sqrt{x}$.

321. (1) $y = \arccos \sqrt{1 - e^{2x}}$; (2) $y = \arcsin \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$; (3) $y = \arctan \sqrt{e^x - 1}$.

QUIZ No. 7

Form I

Compute the derivatives for the given values of the argument:

(1) $f(x) = \sin^2 \ln e^x$, $f'(0)$;

(2) $f(x) = 3 \ln \sqrt{\cos 2x}$, $f'(\pi/8)$;

(3) $f(x) = \arccos \sqrt{x}$, $f'(1/2)$;

(4) $f(x) = 8 \sin^2 x \cos x$, $f'(\pi/4)$.

(5) A point moves in a straight line according to the law $s = \sin^2 t$. Find the instant of time t at which its acceleration is equal to 1.

Form II

Compute the derivatives for the given values of the argument:

(1) $f(x) = \ln \tan^2 2x$, $f'(\pi/8)$;

(2) $f(x) = 2 \ln \sqrt{\sin 2x}$, $f'(\pi/8)$;

(3) $f(x) = \arctan e^{-x}$, $f'(0)$;

(4) $f(x) = \sin 2x\,(1 + \cos 2x) \times f'(\pi/4)$.

(5) A point moves in a straight line according to the law $s = \sin^2 t$. Find the instant t at which its acceleration is equal to 0.

Chapter 10

DIFFERENTIAL OF A FUNCTION. APPLYING THE DIFFERENTIAL TO APPROXIMATE CALCULATIONS

Sec. 10.1. Computing the Differential of a Function

The *differential* of a function $y = f(x)$ (or the *first differential*) is defined as the product of the derivative of this function $f'(x)$ by an arbitrary increment in the argument Δx:

$$dy = f'(x)\,\Delta x.$$

The differential of the argument is equal to the increment in the argument: $dx = \Delta x$. Therefore, the differential of a function is equal to the product of its derivative by the differential of the argument:

$$dy = f'(x)\,dx. \tag{10.1}$$

The *second differential* is defined as the differential of the first differential:

$$d^2y = f''(x)\,dx^2, \tag{10.2}$$

that is, the second differential of a function $y = f(x)$ is equal to the product of the second derivative of this function and the square of the differential of the argument.

1. Find the first differentials of the following functions:
(1) $y = (x^3 - 2)^4$; (2) $y = \sqrt{x^2 - 1}$; (3) $y = \ln \sin \sqrt{x}$; $y = \arcsin \sqrt{2x}$.

○Take advantage of Relationship (10.1):

(1) $dy = ((x^3 - 2)^4)'\,dx = 4(x^3 - 2)^3 \cdot 3x^2\,dx = 12x^2(x^3 - 2)^3\,dx$;

(2) $dy = (\sqrt{x^2 - 1})'\,dx = \dfrac{2x}{2\sqrt{x^2 - 1}}\,dx = \dfrac{x\,dx}{\sqrt{x^2 - 1}}$;

(3) $dy\,(\ln \sin \sqrt{x})'\,dx = \dfrac{1}{\sin \sqrt{x}} \cdot \cos \sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{\cot \sqrt{x}\,dx}{2\sqrt{x}}$;

(4) $dy = (\arcsin \sqrt{2x})'\,dx = \dfrac{1}{\sqrt{1 - (\sqrt{2x})^2}} \cdot \dfrac{1}{2\sqrt{2x}} \cdot 2\,dx$

$= \dfrac{dx}{\sqrt{2x(1 - 2x)}}$. ●

2. Find the second differentials of the following functions:
(1) $y = \ln \sin^2 2x$; (2) $y = e^{-x}$.

○Take advantage of relationship (10.2):

(1) $y' = \dfrac{1}{\sin^2 2x} \cdot 2 \sin 2x \cos 2x \cdot 2 = 4 \cot 2x$; $y'' = -4 \cdot \dfrac{1}{\sin^2 2x} \cdot 2 =$

$-\dfrac{8}{\sin^2 2x}$; $d^2y = y''\,dx^2 = -\dfrac{8\,dx^2}{\sin^2 2x}$;

(2) $y' = -e^{-x}$; $y'' = e^{-x}$; $d^2y = y''\,dx^2 = e^{-x}\,dx^2$. ●

3. Find the first differentials of the following functions: (1) $y =$

$(1-x^2)^5$; (2) $y=(ax^2+b)^3$; (3) $y=\sqrt{4-2x^2}$; (4) $y=1/\sqrt{2x-1}$; (5) $y=\ln\cos^2 x$; (6) $y=\ln(1/\sqrt{x})$; (7) $y=\arccos x^2$; (8) $y=\operatorname{arccot}(1/x)$.

4. Find the second differentials of the following functions: (1) $y=\ln\cos^2 x$; (2) $y=\ln\tan 2x$; (3) $y=a^{3x}$; (4) $y=\arccos x$; (5) $y=\arctan x^2$.

Sec. 10.2. Absolute and Relative Errors

Consider the function $y=f(x)$. Suppose that the quantity x is obtained by direct measurement or as a result of approximate calculation. Then, in the process of finding the quantity x, we omit unvoluntarily an error Δx.

Let x be an approximate value of the argument (the quantity being measured), Δx, the absolute error of the quantity x, $\Delta x/x$ the relative error of the quantity x, and $x+\Delta x$, the true value of the measured quantity (Δx can be both a positive and a negative number).

Then x defines an approximate value of the function $f(x)$, and $x+\Delta x$ its true value $f(x+\Delta x)$, whence it follows that the absolute error of the function

$$|\Delta y| = |f(x+\Delta x)-f(x)|.$$

For small values of Δx (close to zero) the quantity Δy may be approximately replaced by the differential dy:

$$\Delta y = f(x+\Delta x)-f(x)\approx f'(x)\,dx = dy.$$

The advantage of replacing the function Δy by its differential dy consists in that dy depends on Δx linearly, while Δy usually represents a more complicated dependence on Δx.

Setting $\Delta y\approx dy$, we get the expression for the relative error ε of the quantity y:

$$\varepsilon=\left|\frac{dy}{y}\right|.$$

5. Compare the relative errors in computing the area of a circle of radius $r=125$ cm, regarding that the absolute error is equal to: (1) the increment in the area of the circle; (2) the differential of the area of the circle.

○ (1) We first find the increment ΔS in the area of the circle and the relative error $\Delta S/S$ in computing the area of the circle $S=\pi r^2$. We assume that the error of measuring the radius does not exceed ± 0.5 cm. We have

$$\Delta S=\pi(r+\Delta r)^2-\pi r^2=\pi(2r\,\Delta r+(\Delta r)^2)$$
$$=\pi(2\cdot 125\cdot 0.5+0.25)=125.25\pi;$$
$$\frac{\Delta S}{S}=\frac{125.25}{\pi\cdot 125^2}=0.008016\approx 0.8\%.$$

(2) We now find the differential dS and relative error dS/S in computing the area of the circle:

$$dS=2\pi r\,\Delta r=2\pi\cdot 125\cdot 0.5=125\pi;\ \frac{dS}{S}=\frac{2\pi r\,\Delta r}{\pi r^2}=2\cdot\frac{dr}{r}.$$

Hence, the relative error in computing the area of the circle is equal to twice the relative error obtained when measuring the radius:

$$\frac{dS}{S} = 2 \cdot \frac{dr}{r} = 2 \cdot \frac{0.5}{125} = 0.8\%.$$

Thus, we see that in the second case the computation was considerably simpler and was carried out without loss of its accuracy.

Let us determine the relative approximation error when replacing the increment ΔS by the differential dS:

$$\Delta S - dS = 125.25\pi - 125\pi = 0.25\pi; \frac{\Delta S - dS}{dS} = \frac{0.25\pi}{125\pi} = 0.002 = 0.2\%.$$

The relative approximation error amounts only to 0.2%. ●

6. Find the relative error in measuring the length of the circle if $r = 50$ cm, and $\Delta r = 0.5$ cm.

7. Find the relative error of computing the quantity given by the equation $y = x^3$ if $x = 2$ and $\Delta x = 0.01$.

Sec. 10.3. Computing an Approximate Numerical Value of a Function

Let there be given a function $y = f(x)$; the increment in this function $\Delta y = f(x + \Delta x) - f(x)$, its differential $dy = f'(x)\,dx$. For sufficiently small (close to zero) increments in the argument Δx, we shall assume that $\Delta y \approx dy$, that is, that the increment in the function is approximately equal to its differential.

Replacing the increment in the function by its differential, we get

$$f'(x)\,dx \approx f(x + \Delta x) - f(x),$$

whence

$$f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x. \qquad (10.3)$$

The application of this formula makes the computation of a numerical value of a function considerably simpler; geometrically, this corresponds to the replacement of a portion of the curve by a segment of the tangent line.

8. Find the approximate value of the increment in the function $y = 2x^3 + 5$ for $x = 2$ and $\Delta x = 0.001$.

○ We have $\Delta y \approx dy = 6x^2\,dx = 6 \cdot 2^2 \cdot 0.001 = 0.024$. The exact value of the increment

$$\Delta y = 2(x + \Delta x)^3 + 5 - 2x^3 - 5 = 6x^2\,\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3$$
$$= 6 \cdot 4 \cdot 0.001 + 6 \cdot 2 \cdot 0.000001 + 2 \cdot 0.000000001 = 0.024012002. \bullet$$

9. A ball of radius R is being heated. What will be the increment in its volume if its radius has become longer by ΔR?

○ The volume of the ball is computed by the formula $V = (4/3)\pi R^3$. Assuming the increment ΔR in the argument R to be small, we replace the increment in the volume of the ball by its differential: $\Delta V \approx dV$. Consequently, to compute the increment in

the volume of the ball, it suffices to find the differential of the function $V = (4/3)\,\pi R^3$, i.e. $dV = 4\pi R^2\,dR$. ●

10. Find the approximate value of the function $f(x) = 5x^3 - 2x + 3$ for $x = 2.01$.

○ Setting $x = 2$ and $\Delta x = 0.01$, we get

$$f(x) = f(2) = 5\cdot 2^3 - 2\cdot 2 + 3 = 39;$$

$$f'(x)\,\Delta x = f'(2)\cdot 0.01 = (5x^3 - 2x + 3)'\,\Delta x$$

$$= (15x^2 - 2)\,\Delta x = (15\cdot 2^2 - 2)\cdot 0.01 = 0.58.$$

Using Formula (10.3), we find $f(2.01) = 39 + 0.58 = 39.58$.

Let us find the exact value of the function: $f(2.01) = 5\cdot(2.01)^3 - 2\cdot 2.01 + 3 = 39.583005$. ●

11. Find the approximate value of the increment in each of the given functions: (1) $y = 3x^2 + 5x + 1$ for $x = 3$ and $\Delta x = 0.001$; (2) $y = x^3 + x - 1$ for $x = 2$ and $\Delta x = 0.01$; (3) $y = \ln x$ for $x = 10$ and $\Delta x = 0.01$.

12. What will be the increment in the volume of a cube due to its heating if its edge is 10 cm long, the elongation of the edge being equal to 0.02 cm?

13. Find the approximate values of the following functions: (1) $f(x) = 2x^2 - x + 1$ for $x = 2.01$; (2) $f(x) = x^2 + 3x + 1$ for $x = 3.02$; (3) $f(x) = (1/3)\,x^3 + (1/2)\,x^2 - 2x + 4$ for $x = 1.1$.

Sec. 10.4. Deriving Formulas for Approximate Calculations

Applying Formula (10.3), it is easy to obtain various formulas for finding approximate numerical values. Below, we shall consider certain formulas having practical importance in approximate calculations.

1. Approximate Computation of Powers. Consider the function $f(x) = x^n$. Let the argument x receive a small increment Δx. We now compute the approximate value of the function $f(x + \Delta x) = (x + \Delta x)^n$, applying Formula (10.3). We have

$$f(x + \Delta x) = (x + \Delta x)^n;\ f(x) = x^n;\ f'(x)\,\Delta x = nx^{n-1}\,\Delta x,$$

whence

$$(x + \Delta x)^n \approx x^n + nx^{n-1}\,\Delta x. \tag{10.4}$$

Particular cases of Formula (10.4):

(1) $n = 2$, $(x + \Delta x)^2 \approx x^2 + 2x\,\Delta x$; (2) $n = 3$, $(x + \Delta x)^3 \approx x^3 + 3x^2\,\Delta x$; (3) $x = 1$, $(1 + \Delta x)^n \approx 1 + n\,\Delta x$.

2. Approximate Computation of Roots. Consider the function $f(x) = \sqrt[n]{x}$. Let the argument x receive a small increment Δx. We then compute the approximate value of the function $f(x + \Delta x) = \sqrt[n]{x + \Delta x}$. We have

$$f(x+\Delta x) = \sqrt[n]{x+\Delta x};\quad f(x) = \sqrt[n]{x};\quad f'(x)\,\Delta x = \frac{1}{n}\,\frac{\Delta x}{\sqrt[n]{x^{n-1}}},$$

whence

$$\sqrt[n]{x+\Delta x} \approx \sqrt[n]{x} + \frac{\Delta x}{n\sqrt[n]{x^{n-1}}}. \qquad (10.5)$$

Particular cases of Formula (10.5):

(1) $n=2$, $\sqrt{x+\Delta x} \approx \sqrt{x} + \frac{\Delta x}{2\sqrt{x}}$; (2) $n=3$, $\sqrt[3]{x+\Delta x} \approx \sqrt[3]{x} + \frac{\Delta x}{3\sqrt[3]{x^2}}$;

(3) $x=1$, $\sqrt[n]{1+\Delta x} \approx 1 + \frac{\Delta x}{n}$.

3. Approximate Computation of Reciprocal Quantities. Consider the function $f(x) = 1/x$. Let the argument x receive a small increment Δx. We compute the approximate value of the function $f(x + \Delta x) = \frac{1}{x+\Delta x}$. We have

$$f(x+\Delta x) = \frac{1}{x+\Delta x}; \quad f(x) = \frac{1}{x}; \quad f'(x)\,\Delta x = -\frac{\Delta x}{x^2},$$

whence

$$\frac{1}{x+\Delta x} \approx \frac{1}{x} - \frac{\Delta x}{x^2}. \qquad (10.6)$$

Particular cases of Formula (10.6):

(1) $\Delta x < 0$, $\frac{1}{x-\Delta x} \approx \frac{1}{x} + \frac{\Delta x}{x^2}$; (2) $x=1$, $\frac{1}{1+\Delta x} \approx 1-\Delta x$; (3) $x=1$ and $\Delta x < 0$, $\frac{1}{1-\Delta x} \approx 1+\Delta x$.

4. Approximate Computation of Sines and Tangents of Small Angles. Let for the function $f(x) = \sin x$ the argument $x = 0$ receive a small increment Δx. We then compute the approximate value of the function:

$$f(x + \Delta x) = \sin(x + \Delta x) = \sin(0 + \Delta x) = \sin \Delta x.$$

We have

$$f(x + \Delta x) = \sin \Delta x; \; f(x) = \sin x = \sin 0 = 0;$$
$$f'(x)\,\Delta x = \cos x\,\Delta x = \cos 0 \cdot \Delta x = \Delta x,$$

whence

$$\sin \Delta x \approx 0 + \Delta x; \; \sin \Delta x \approx \Delta x.$$

The sine of a small angle is approximately equal to the angle itself (the angle is taken in radians).

Analogously, we can show that the following approximate equality occurs:

$$\tan \Delta x \approx \Delta x.$$

14. Find the approximate values: (1) $(4.012)^2$; (2) $\sqrt{1.006}$; (3) $1/1.004$.

○ Setting in Relationship (10.4) $x = 4$, $\Delta x = 0.012$, we get $(4.012)^2 = (4 + 0.012)^2 \approx 4^2 + 2 \cdot 4 \cdot 0.012 = 16.096 \approx 16.1$ (the exact answer is 16.096144).

(2) Setting in Relationship (10.5) $x = 1$, $\Delta x = 0.006$, we get $\sqrt{1.006} = \sqrt{1 + 0.006} \approx 1 + 0.006/2 = 1.003$.

(3) Setting $x = 1$, $\Delta x = 0.004$ in Relationship (10.6), we get $1/(1 + 0.004) = 1 - 0.004 = 0.996$. ●

15. Compute $\sin 12'$.

○ Since $12' = 0.0035$ radian, $\sin 0.0035 = 0.0035$. Using the tables of natural values of sine, we find $\sin 12' = 0.0035$. ●

16. Find the approximate values of the following powers: (1) $(9.06)^2$; (2) $(1.012)^3$; (3) $(9.95)^3$; (4) $(1.005)^{10}$; (5) $(0.975)^4$.

17. Find the approximate values of the indicated roots: (1) $\sqrt[3]{1.012}$; (2) $\sqrt{25.16}$; (3) $\sqrt{24.84}$; (4) $\sqrt{101}$; (5) $\sqrt{99.5}$; (6) $\sqrt[10]{1.03}$.

18. Find the approximate values of the following quantities: (1) $1/0.99$; (2) $1/9.93$; (3) $1/(1.004)^2$.

19. Compute: (1) $\sin 42'$; (2) $\sin 2°06'$; (3) $\tan 1°12'$; (4) $\tan 3°18'$.

Sec. 10.5. Computations by the Method of Taking Strict Account of Errors

When carrying out computations, it is often necessary to know the bounds of the errors of intermediate calculations and the final result. Such method of carrying out approximate computations is called the method of taking strict account of the errors. To use this method successfully, it is necessary to know how to find the bounds of relative errors of an algebraic sum, product, power, root, and a quotient.

20. Prove that the relative error of a product does not exceed the sum of the relative errors of its factors.

○ Let there be given a function $y = uv$, where $u = f(x)$ and $v = \varphi(x)$. Let us take the logarithm of this function and find the differential:

$$\ln y = \ln u + \ln v: \quad \frac{dy}{y} = \frac{du}{u} + \frac{dv}{v}.$$

Since the absolute value of a sum does not exceed the sum of the absolute values of the addends:

$$\left|\frac{du}{u} + \frac{dv}{v}\right| \leqslant \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|,$$

we have

$$\left|\frac{dy}{y}\right| = \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right| \quad \text{or} \quad \varepsilon(uv) = \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|. \; ●$$

21. Prove that the relative error of a quotient does not exceed the sum of the relative errors of the dividend and divisor.

○ Let there be given a function $y = u/v$, where $u = f(x)$ and $v = \varphi(x)$. Taking the logarithms and finding the differential of the function $y = u/v$, we get

$$\ln y = \ln u - \ln v; \quad \frac{dy}{y} = \frac{du}{u} - \frac{dv}{v}.$$

Since the absolute value of the difference does not exceed the sum of the absolute values of the minuend and subtrahend, we have

$$\left|\frac{dy}{y}\right| = \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right| \quad \text{or} \quad \varepsilon\left(\frac{u}{v}\right) = \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|. \; ●$$

22. Prove that the relative error of a power is equal to the relative error of the base multiplied by the exponent.

○ Let there be given a function $y = x^n$. Let us take the logarithm of this function and find its differential:

$$\ln y = n \ln x; \quad \frac{dy}{y} = n \frac{dx}{x}.$$

The relative error $\varepsilon(x^n) = n \frac{dx}{x}$.

Particular cases: (1) $n = 2$, $\varepsilon(x^2) = 2 \frac{dx}{x}$; (2) $n = 3$; $\varepsilon(x^3) = 3 \frac{dx}{x}$. ●

23. Prove that the relative error of a root is equal to the relative error of the radicand divided by the index of the root.

○ Let there be given the function $y = \sqrt[n]{x}$. We take the logarithm of it and find the differential

$$\ln y = \frac{1}{n} \ln x; \quad \frac{dy}{y} = \frac{1}{n} \frac{dx}{x}.$$

The relative error $\varepsilon(\sqrt[n]{x}) \frac{1}{n} \frac{dx}{x}$.

Particular cases: (1) $n = 2$, $\varepsilon(\sqrt{x}) = \frac{1}{2} \frac{dx}{x}$;

(2) $n = 3$, $\varepsilon(\sqrt[3]{x}) \frac{1}{3} \frac{dx}{x}$. ●

24. Find the relative error of the number x made when computing this number from its logarithm $y = \log x$.

○ Suppose that $\log x$ was computed with an error Δy, then, when finding from it the number x, an error Δx will be introduced. The relative error of the number x is equal to $\Delta x/x$. Since the absolute error of the logarithm $\Delta y \approx 0.4343 \, \Delta x/x$, we have

$$\frac{\Delta x}{x} \approx \frac{\Delta y}{0.4343}.$$

Thus, the relative error of the number x, computed from its logarithm, is independent of the value of the number x and is dependent only on the error with which the logarithm of the number x was found. ●

25. Find the relative error of the accuracy of reading from a slide rule with a scale 250 mm long.

○ Suppose that the cursor is set to a necessary division and readings are taken from the scale with an error not exceeding 0.1 mm.

Let us find the absolute error of the logarithm of the number. The entire length (250 mm) of the scale corresponds to the number whose logarithm is equal to unity ($\log 10 = 1$). Consequently, the absolute error of the logarithm of a number per 0.1 mm of the scale length

will be $\frac{1}{250}$, that is, $\Delta y = 0.1/250 = 0.004$. Since $\Delta y = 0.4343\,\Delta x/x$, we have

$$\frac{\Delta x}{x} \approx \frac{\Delta y}{0.4343} = \frac{0.004}{0.4343} \approx 0.00092 \approx 0.001 = 0.1\%,$$

that is, the relative error of the accuracy of reading is equal to 0.1% (for any portion of the scale). ●

26. When measuring a rectangular field, its length was found to be equal to $u = 60$ m and its width to $v = 23$ m. The error of measuring the length does not exceed 0.3 m and that of measuring the width 0.2 m. Determine the bounds of the error, taking the area of the field to be equal to $60 \cdot 23 = 1380$ m², and the relative error made in the area computation.

○ We have $|du| < 0.3$, $|dv| < 0.2$. Under the most unfavourable conditions $|du| = 0.3$, $|dv| = 0.2$. Find the absolute error of product

$$d(uv) = v\,du + u\,dv = 23 \cdot 0.3 + 60 \cdot 0.2 = 18.9 \approx 19 \text{ (m}^2\text{)}.$$

This is the greatest value of the absolute error which may be introduced by taking the area of the field to be equal to 1380 m². Rounding off the error with excess and taking it to be equal to 20 m², we find the bounds of the error made in the computation of the area. Thus, the area of the field does not exceed $1380 + 20 = 1400$ (m²) and is not less than $1380 - 20 = 1360$ (m²).

The relative error is computed by the formula: $\varepsilon(uv) = \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|$, i.e.

$$\varepsilon(uv) = \frac{0.3}{60} + \frac{0.2}{23} = \frac{1}{200} + \frac{2}{230} \approx 0.014 = 1.4\%.$$

Thus, the relative error does not exceed 1.4%. ●

27. To find the density of a body, its mass (m_1) and the mass of the water (m_2) displaced by the body were determined: $m_1 = 484$ g and $m_2 = 62$ g. The absolute errors $\Delta m_1 = 0.5$ g and $\Delta m_2 = 0.4$ g. Find the relative error in computing the body density.

○ Since $y = m_1/m_2$, we have

$$\left|\frac{dy}{y}\right| = \left|\frac{dm_1}{m_1}\right| + \left|\frac{dm_2}{m_2}\right| = \frac{0.5}{484} + \frac{0.4}{62} \approx 0.00103$$

$$+ 0.00645 = 0.00748 \approx 0.7\%. \;\bullet$$

28. Find the relative error in measuring the volume of a cube if its edge is equal to 12.5 cm. The absolute error $\Delta x = 0.05$ cm.

○ Setting $dx = 0.05$ cm, we have

$$\varepsilon(x^3) = 3\,\frac{dx}{x} = 3 \cdot \frac{0.05}{12.5} = \frac{15}{1250} \approx 0.012 = 1.2\%. \;\bullet$$

29. Find the relative error in computing the length of the side of a square if the area of the square is equal to 37.7 cm^2. The absolute error $\Delta x = 0.05$ cm.

○ Denoting the length of the side of the square by y and its area by x, we get $y = \sqrt{x} = \sqrt{37.7}$, $dx = 0.05$;

$$\varepsilon(\sqrt{37.7}) = \frac{1}{2} \cdot \frac{0.05}{37.7} = \frac{0.05}{75.4} \approx 0.000664 \approx 0.1\%. \quad \bullet$$

30. When measuring the area of a parallelogram, its base was found to be equal to $a = 70$ cm ($\Delta a = 0.4$ cm) and its altitude to $h = 48$ cm ($\Delta h = 0.3$ cm). Determine the relative error in computing the area of the parallelogram.

31. Given two approximate numbers: 82.6 and 64.8. Find the relative error of their quotient.

32. Find the relative error in measuring the area of a square room if the rounded-off length of its side was taken to be equal to 6.4 m (the absolute error is equal to 0.05 m).

33. Find the relative error of computing the length of the side of a square if the area of the square is equal to 68.5 cm^2.

Sec. 10.6. Miscellaneous Problems

34. Find the first differentials of the following functions:

(1) $y = e^x \sin x$; (2) $y = a^x e^x$; (3) $y = e^x \sqrt{2x}$;

(4) $y = (e^x - e^{-x})^2$; (5) $y = \frac{e^x}{e^x - 1}$; (6) $y = \frac{e^x + 1}{e^x}$.

35. Compute the approximate values of the increments in the indicated functions: (1) $y = \sin 2x$ for $x = \pi/6$ and $\Delta x = 0.02$; (2) $y = \ln x^2$ for $x = 20$ and $\Delta x = 0.01$; (3) $y = \arcsin x$ for $x = \sqrt{3}/2$ and $\Delta x = 0.02$.

36. Compute the approximate values of the given functions: (1) $f(x) = x^3 + x^2 + x + 1$ for $x = 0.001$; (2) $f(x) = x^4 - 1$ for $x = -3.3$; (3) $f(x) = x/\sqrt{x^2 + 3}$ for $x = 1.1$.

37. Find the relative error of computing the quantity given by the equation: (1) $y = x^2$ for $x = 10$ and $\Delta x = 0.01$; (2) $y = x^3$ for $x = 3$ and $\Delta x = 0.02$.

38. Derive the formulas for computing the relative errors of the indicated functions: (1) $y = \sin^2 3x$; (2) $y = \tan^2 x$.

39. Derive the formulas for computing the relative errors of the indicated functions: (1) $y = e^{\sin^2 x}$; (2) $y = 3^{\sqrt{x}}$.

QUIZ No. 1

Form I

(1) Compute the differential of the function $y = \ln \cos^2 x$ for $x = \pi/4$ and $dx = 0.01$.

(2) Compute the relative error of the function $V = (4/3)\pi R^3$ for $R = 300$ and $dR = 0.3$.

(3) Find the approximate value of the increment in the function $y = x^3 - x^2$ for $x = 2$ and $\Delta x = 0.01$.

(4) Find the approximate value of the function $f(x) = x^3 - x^2 + x - 3$ for $x = 3.03$.

(5) Compute the approximate value of the quantity $1/0.998$.

Form II

(1) Compute the differential of the function $y = \ln \tan 2x$ for $x = \pi/8$ and $dx = 0.03$.

(2) Compute the relative error of the function $y = x^3$ for $x = 750$ and $dx = 0.5$.

(3) Find the approximate value of the increment in the function $y = 2\sqrt{x} + 4$ for $x = 25$ and $\Delta x = 0.01$.

(4) Find the approximate value of the function $f(x) = 3x^3 - x^2 + 5x - 1$ for $x = 3.02$.

(5) Compute the approximate value of the quantity $(1.02)^7$.

Chapter 11
INDEFINITE INTEGRAL

Sec. 11.1. Basic Integration Formulas. Direct Integration

1. Basic Integration Formulas. The function $F(x)$ is said to be the *antiderivative* (primitive) of the function $f(x)$ in the interval $[a, b]$ if at any point of this interval its derivative is equal to $f(x)$:

$$F'(x) = f(x) \Rightarrow dF(x) = f(x)\,dx, \quad x \in [a, b].$$

We have seen that the process of finding a derivative or differential is called differentiation. We now come to the inverse process of finding the function itself when the derivative $f(x)$ (or differential $f(x)\,dx$) is given. This inverse process is called integration or antidifferentiation.

The collection of antiderivatives for the function $f(x)$ or for the differential $f(x)\,dx$ is called the *indefinite integral* and is denoted by the symbol $\int f(x)\,dx$. Thus

$$\int f(x)\,dx = F(x) + C \text{ if } d[F(x) + C] = f(x)\,dx.$$

Here, $f(x)$ is the integrand function; $f(x)\,dx$ is the integrand expression; and C is an arbitrary constant.

Let us consider the basic properties of the indefinite integral.

1°. *The indefinite integral of the differential of a function is equal to this function plus an arbitrary constant*:

$$\int dF(x) = F(x) + C.$$

2°. *The differential of an indefinite integral is equal to the integrand expression, and the derivative of an indefinite integral is equal to the integrand function*:

$$d\int f(x)\,dx = f(x)\,dx, \quad \left(\int f(x)\,dx\right)' = f(x)$$

3°. *The indefinite integral of an algebraic sum of functions is equal to an algebraic sum of the indefinite integrals of these functions*:

$$\int [f(x) + \varphi(x)]\,dx = \int f(x)\,dx + \int \varphi(x)\,dx.$$

4°. *A constant factor of the integrand expression may be taken outside the sign of the integral*:

$$\int af(x)\,dx = a\int f(x)\,dx.$$

5°. *If* $\int f(x)\,dx = F(x) + C$ *and* $u = \varphi(x)$ *is any known function having a continuous derivative, then*

$$\int f(u)\,du = F(u) + C.$$

Basic Integration Formulas (Tabular Integrals)

$$\int dx = x + C; \tag{11.1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1); \tag{11.2}$$

$$\int \frac{dx}{x} = \ln|x| + C; \tag{11.3}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C; \tag{11.4}$$

$$\int e^x\,dx = e^x + C; \tag{11.5}$$

$$\int \sin x\,dx = -\cos x + C; \tag{11.6}$$

$$\int \cos x\,dx = \sin x + C; \tag{11.7}$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C; \tag{11.8}$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C; \tag{11.9}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C; \tag{11.10}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C; \tag{11.11}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C; \tag{11.12}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C. \tag{11.13}$$

When applying formulas (11.3), (11.10), and (11.11), the sign of absolute value is written only when the expression standing under the sign of logarithm may have a negative value.

Each of the above formulas can be readily verified. The differentiation of the right-hand side yields the integrand expression.

2. Direct Integration. Direct integration is based on direct use of the table of integrals. Here, the following cases may occur:

(1) a given integral is found directly according to the appropriate tabular integral;

(2) taking advantage of Properties 3° and 4°, a given integral is reduced to one or several tabular integrals;

(3) after elementary identical transformations of the integrand function and application of Properties 3° and 4°, a given integral is reduced to one or several tabular integrals.

In Problems 1 through 8 we shall find the indicated integrals:

1. (1) $\int 5\,dx$; (2) $\int 6x^2\,dx$; (3) $\int 4(x^2 - x + 3)\,dx$;

(4) $\int 2(3x-1)^2\,dx$; (5) $\dfrac{x^3 + 3x^2 + 4x}{x}\,dx$.

○ (1) On the basis of Property 4°, we take the constant factor 5 outside the sign of integral, and, making use of Formula (11.1), we get

$$\int 5\,dx = 5\int dx = 5x + C.$$

(2) Using Property 4° and Formula (11.2), we obtain

$$\int 6x^2\,dx = 6\int x^2\,dx = 6\cdot\frac{x^{2+1}}{2+1} + C = 2x^3 + C.$$

Check: $d(2x^3 + C) = 6x^2\,dx$. We have obtained the integrand expression; consequently, the integral is found correctly.

(3) Using Properties 3° and 4° and Formulas (11.2) and (11.1), we have

$$\int 4(x^2 - x + 3)\,dx = 4\int x^2\,dx - 4\int x\,dx + 12\int dx$$

$$= 4\cdot\frac{x^3}{3} - 4\cdot\frac{x^2}{2} + 12x + C = \frac{4}{3}x^3 - 2x^2 + 12x + C.$$

The constant of integration C is equal to an algebraic sum of three constants of integration, since each integral has its own arbitrary constant ($C_1 - C_2 + C_3 = C$).

(4) $$\int 2(3x-1)^2\,dx = \int (18x^2 - 12x + 2)\,dx$$

$$= 18\int x^2\,dx - 12\int x\,dx + 2\int dx = 6x^3 - 6x^2 + 2x + C.$$

(5) $$\int \frac{x^3 + 3x^2 + 4x}{x}\,dx = \int (x^2 + 3x + 4)\,dx$$

$$= \int x^2\,dx + 3\int x\,dx + 4\int dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x + C. \bullet$$

2. (1) $\int x^{-4}\,dx$; (2) $\int \frac{dx}{\sqrt{x}}$.

○ Using Formula (11.2), we find

(1) $$\int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C;$$

(2) $$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-1/2+1} = 2x^{1/2} + C = 2\sqrt{x} + C. \bullet$$

3. (1) $\int \frac{3\,dx}{x}$; (2) $\int \frac{dx}{x+1}$; (3) $\int \frac{x\,dx}{x^2+1}$.

○ (1) By Formula (11.3), we find

$$\int \frac{3\,dx}{x} = 3\int \frac{dx}{x} = 3\ln|x| + C.$$

(2) Since $dx = d(1+x)$, we have

$$\int \frac{dx}{1+x} = \int \frac{d(1+x)}{1+x} = \ln|1+x| + C.$$

(3) Since $x\,dx=\frac{1}{2}\,d(x^2+1)$, we have

$$\int\frac{x\,dx}{x^2+1}=\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}=\frac{1}{2}\ln(x^2+1)+C.$$

We do not write the sign of the absolute value, since for any value of x the expression x^2+1 is positive. ●

4. (1) $\int 2^x\,dx$; (2) $\int 2^{x^2}x\,dx$; (3) $\int e^{-3x^2}x\,dx$.

○ (1) By Formula (11.4), for $a=2$ we get

$$\int 2^x\,dx=\frac{2^x}{\ln 2}+C.$$

(2) Since $x\,dx=(1/2)\,d(x^2)$, we have

$$\int 2^{x^2}x\,dx=\int 2^{x^2}\cdot\frac{1}{2}\,d(x^2)=\frac{1}{2}\int 2^{x^2}d(x^2)$$
$$=\frac{1}{2}\cdot\frac{2^{x^2}}{\ln 2}+C=\frac{2^{x^2}-1}{\ln 2}+C.$$

(3) Since $x\,dx=-(1/6)\,d(-3x^2)$, we have

$$\int e^{-3x^2}x\,dx=-\frac{1}{6}\int e^{-3x^2}d(-3x^2)=-\frac{1}{6}e^{-3x^2}+C.$$ ●

5. (1) $\int\sin(ax+b)\,dx$; (2) $\int\cos(5x-3)\,dx$; (3) $\int\tan x\,dx$.

○ (1) Since $d(ax+b)=a\,dx$, we have $dx=(1/a)\,d(ax+b)$. Consequently,

$$\int\sin(ax+b)\cdot\frac{1}{a}\,d(ax+b)=\frac{1}{a}\int\sin(ax+b)\,d(ax+b)$$
$$=-\frac{1}{a}\cos(ax+b)+C.$$

(2) Since $(1/5)\,d(5x-3)=dx$, we have

$$\int\cos(5x-3)\,dx=\frac{1}{5}\int\cos(5x-3)\,d(5x-3)=\frac{1}{5}\sin(5x-3)+C.$$

(3) Since $\sin x\,dx=-d(\cos x)$, we have

$$\int\tan x\,dx=\int\frac{\sin x\,dx}{\cos x}=-\int\frac{d(\cos x)}{\cos x}=-\ln|\cos x|+C.$$ ●

6. (1) $\int\frac{5\,dy}{\cos^2 y}$; (2) $\int\frac{x^2\,dx}{\cos^2 x^3}$; (3) $\int\frac{dz}{2\sin^2 z}$; (4) $\int\frac{x\,dx}{\sin^2(x^2+1)}$.

○ (1) By Formula (11.8), we find

$$\int\frac{5\,dy}{\cos^2 y}=5\int\frac{dy}{\cos^2 y}=5\tan y+C.$$

(2) Since $d(x^3)=3x^2\,dx$, we have $x^2\,dx=\frac{1}{3}\,d(x^3)$. Consequently,

$$\int\frac{x^2\,dx}{\cos^2 x^3}=\frac{1}{3}\int\frac{d(x^3)}{\cos^2 x^3}=\frac{1}{3}\tan x^3+C.$$

(3) By Formula (11.9), we find

$$\int\frac{dz}{2\sin^2 z}=\frac{1}{2}\int\frac{dz}{\sin^2 z}=-\frac{1}{2}\cot z+C.$$

(4) Since $x\,dx=(1/2)\,d(x^2+1)$, we have

$$\int\frac{x\,dx}{\sin^2(x^2+1)}=\frac{1}{2}\int\frac{d(x^2+1)}{\sin^2(x^2+1)}=-\frac{1}{2}\cot(x^2+1)+C. \bullet$$

7. (1) $\int\frac{dx}{x^2-4}$; (2) $\int\frac{dx}{\sqrt{x^2+4}}$.

○ (1) By Formula (11.10), we get

$$\int\frac{dx}{x^2-4}=\int\frac{dx}{x^2-2^2}=\frac{1}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right|+C=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C.$$

(2) By Formula (11.11), we find

$$\int\frac{dx}{\sqrt{x^2+4}}=\ln\left|x+\sqrt{x^2+4}\right|+C. \bullet$$

8. (1) $\int\frac{dx}{\sqrt{9-x^2}}$; (2) $\int\frac{dx}{\sqrt{9-16x^2}}$; (3) $\int\frac{dx}{16+x^2}$;

(4) $\int\frac{dx}{25+4x^2}$.

○ (1) By Formula (11.12), we find

$$\int\frac{dx}{\sqrt{9-x^2}}=\arcsin\frac{x}{3}+C.$$

(2) $$\int\frac{dx}{\sqrt{9-16x^2}}=\int\frac{dx}{4\sqrt{\frac{9}{16}-x^2}}$$

$$=\frac{1}{4}\int\frac{dx}{\sqrt{\left(\frac{3}{4}\right)^2-x^2}}=\frac{1}{4}\arcsin\frac{4x}{3}+C.$$

(3) By Formula (11.13), we find

$$\int\frac{dx}{16+x^2}=\frac{1}{4}\arctan\frac{x}{4}+C.$$

(4) $$\int\frac{dx}{25+4x^2}=\int\frac{dx}{4\left(\frac{25}{4}+x^2\right)}$$

$$=\frac{1}{4}\int\frac{dx}{\left(\frac{5}{2}\right)^2+x^2}=\frac{1}{4}\cdot\frac{2}{5}\arctan\frac{2x}{5}+C=\frac{1}{10}\arctan\frac{2x}{5}+C. \bullet$$

In Problems 9 through 26 find the indicated integrals:

9. (1) $\int a\,d\varphi$; (2) $\int (m-1)\,dy$; (3) $\int x^4\,dx$;

(4) $\int x^{m-1}\,dx$; (5) $\int x^{1-n}\,dx$.

10. (1) $\int 4t^3\,dt$; (2) $\int nx^{n-1}\,dx$; (3) $\int (4u^3-6u^2-4u+3)\,du$;

(4) $\int \left(\frac{4}{3}x^3-\frac{3}{4}x^2+5\right)dx$;

(5) $\int (4ax^3-6bx^2-4cx+d)\,dx$.

11. (1) $\int 3(2x^2-1)^2\,dx$; (2) $\int x^3(1+5x)\,dx$;

(3) $\int x^4(x-1)\,dx$; 4) $\int (2x-1)^3\,dx$.

12. (1) $\int \frac{u^2-u}{3u}\,du$; (2) $\int \frac{2\varphi-3\varphi^3}{5\varphi}\,d\varphi$.

13. (1) $\int (3x^{-4}+8x^{-5})\,dx$; (2) $\int (x^{-4}-x^{-3}-3x^{-2}+1)\,dx$.

14. (1) $\int \frac{dt}{t^2}$; (2) $\int \frac{x^n\,dx}{x}$; (3) $\int (5u^{3/2}-7u^{3/4})\,du$;

(4) $\int 5x\sqrt{x}\,dx$; (5) $\int x^{-2/3}\,dx$.

15. (1) $\int \frac{(x^{1/2}-x^{1/3}+x^{1/4})}{x^2}\,dx$; (2) $\int \frac{x^{3/4}+x^{2/3}+x^{1/2}}{x}\,dx$.

16. (1) $\int \frac{du}{\sqrt[3]{u^2}}$; (2) $\int \frac{d\varphi}{2\sqrt{\varphi}}$;

(3) $\int \frac{x\,dx}{\sqrt[3]{x^2}}$; (4) $\int \left(\frac{3}{t^2}-\frac{2}{\sqrt{t}}+\frac{4\sqrt[3]{t^2}}{t}\right)dt$.

17. (1) $\int \frac{ad\varphi}{\varphi}$; (2) $\int \frac{2\,dx}{x+3}$; (3) $\int \frac{3\,d\varphi}{2-\varphi}$.

(4) $\int \frac{x^2\,dx}{x^3+1}$; (5) $\int \frac{x^2\,dx}{a^3-x^3}$.

18 (1) $\int 5^x\,dx$; (2) $\int 4^{2x}\,dx$; (3) $\int 5^{x^3}x^2\,dx$.

19. (1) $\int (e^x+2x)\,dx$; (2) $\int (3^x-e^x-1)\,dx$.

(3) $\int e^{5x}\,dx$; (4) $\int e^{2x^2}x\,dx$; (5) $\int e^{-x^3}x^2\,dx$.

20. (1) $\int (\sin x-5)\,dx$; (2) $\int \frac{\sin 2x\,dx}{\cos x}$; (3) $\int \sin 6x\,dx$;

(4) $\int \sin(x/4)\,dx$; (5) $\int x\sin x^2\,dx$.

21. (1) $\int (4-3\cos x)\,dx$; (2) $\int \cos 4x\,dx$; (3) $\int \cos(x/6)\,dx$;
(4) $\int \cos(2-3x)\,dx$; (5) $\int x^2\cos x^3\,dx$.

22. (1) $\int \frac{\cos x\,dx}{3\sin x-1}$; (2) $\int \frac{\sin x\,dx}{2-\cos x}$; (3) $\int \frac{\cos x\,dx}{3+2\sin x}$.

23. (1) $\int \frac{dx}{2\cos^2 x}$; (2) $\frac{dx}{\cos^2 5x}$; (3) $\int \frac{dx}{\cos^2(ax+b)}$;
(4) $\int \frac{dx}{\sin^2(3x+2)}$; (5) $\int \frac{x\,dx}{\sin^2 x^2}$.

24. (1) $\int \frac{dx}{x^2-9}$; (2) $\int \frac{du}{u^2-25}$; (3) $\int \frac{dx}{\sqrt{x^2+9}}$;
(4) $\int \frac{dx}{\sqrt{x^2-16}}$.

25. (1) $\int \frac{2\,dx}{\sqrt{1-x^2}}$; (2) $\int \frac{du}{\sqrt{3-u^2}}$; (3) $\int \frac{dx}{\sqrt{5-4x^2}}$;
(4) $\int \frac{d\varphi}{\sqrt{9-2\varphi^2}}$; (5) $\int \frac{3\,dx}{\sqrt{16-9x^2}}$.

26. (1) $\int \frac{a\,dx}{1+x^2}$; (2) $\int \frac{dx}{25+x^2}$; (3) $\int \frac{3\,dz}{9+z^2}$; (4) $\int \frac{dx}{16+25x^2}$;
(5) $\int \frac{dx}{2+3x^2}$.

Sec. 11.2. Geometrical Applications of the Indefinite Integral

Finding a function from its derivative or from its differential is an indefinite problem, since $\int f(x)\,dx$ means an infinitude of primitive functions (or antiderivatives) of the form $y = F(x) + C$, any two of them only differing by the constant C which can take on any numerical values provided that no initial conditions is imposed on the function. In order to single out one definite function, certain initial conditions must be given. By the initial conditions we mean the particular values of x and y specified for the primitive function $y = F(x) + C$ by means of which we find the definite value of C satisfying these initial conditions.

27. Find the function whose derivative is $y' = 3x^2 - 6x + 2$.

○ We have $y' = 3x^2 - 6x + 2$ or $\frac{dy}{dx} = 3x^2 - 6x + 2$, i.e. $dy = (3x^2 - 6x + 2)\,dx$. We then find the integrals of the left-hand and right-hand members of the last equality:

$$\int dy = \int (3x^2-6x+2)\,dx;\quad y + C_1 = x^3 - 3x^2 + 2x + C_2.$$

Setting $C_2 - C_1 = C$, we get

$$y = x^3 - 3x^2 + 2x + C.$$

We have found the general expression for the functions whose derivative is $y' = 3x^2 - 6x + 2$. Henceforth, when integrating such expressions, we shall write the constant of integration only on the right-hand side. ●

28. Find the function whose derivative is $y' = 2x - 3$ if for $x = 2$ this function takes on the value equal to 6.

○ We have $y' = 2x - 3$, or $\frac{dy}{dx} = 2x - 3$, i.e. $dy = (2x - 3)\,dx$. Integrating both sides of the last equality, we find

$$\int dy = \int (2x-3)\,dx; \quad y = x^2 - 3x + C.$$

We now compute C for the given values $x = 2$ and $y = 6$. Substituting these values into the expression for the function, we get $6 = 2^2 - 3 \cdot 2 + C$, whence $C = 8$. Thus, the function satisfying the given initial conditions has the form $y = x^2 - 3x + 8$. ●

29. Find $\int (\cos x - \sin x)\,dx$ if for $x = \pi/2$ the value of the primitive function is equal to 6.

○ We have

$$\int (\cos x - \sin x)\,dx = \int \cos x\,dx - \int \sin x\,dx = \sin x + \cos x + C.$$

We compute C for the given initial conditions: $6 = \sin(\pi/2) + \cos(\pi/2) + C$, whence $C = 5$. Consequently, the primitive function $y = \sin x + \cos x + 5$. ●

30. Find the equation of the curve if the slope of the tangent line at each of its points (x, y) is equal to $2x$.

○ According to the hypothesis, $k = 2x$. It is known that $k = \tan\alpha = \frac{dy}{dx}$ consequently, $\frac{dy}{dx} = 2x$, i.e. $dy = 2x\,dx$. Integrating, we get

$$\int dy = \int 2x\,dx; \quad y = x^2 + C.$$

We have found the collection (family) of curves for which the slope of the tangent at any point is equal to $2x$. These curves differ from one another by an additive constant C. For $C = 0$ we obtain the parabola $y = x^2$ with the vertex at the origin, for $C = 1$ the parabola $y = x^2 + 1$ with the vertex at the point $(0, 1)$, for $C = -2$ the parabola $y = x^2 - 2$ with the vertex at the point $(0, -2)$, and so forth (Fig. 66). ●

31. Set up the equation of a line if the slope of the tangent line at any point of tangency is equal to y/x.

○ By the hypothesis, $k=\frac{y}{x}$; since $k=\frac{dy}{dx}$, we have $\frac{dy}{dx}=\frac{y}{x}$, whence, on separating the variables, we have $\frac{dy}{y}=\frac{dx}{x}$. Integrating, we find

$$\int \frac{dy}{y}=\int \frac{dx}{x}; \quad \ln y=\ln x+C.$$

For the conveniency of simplifications, we set the arbitrary constant to be equal to $\ln C$. Taking the antilogarithm, we get $y = Cx$ which is the equation of the families of straight lines passing through the origin. ●

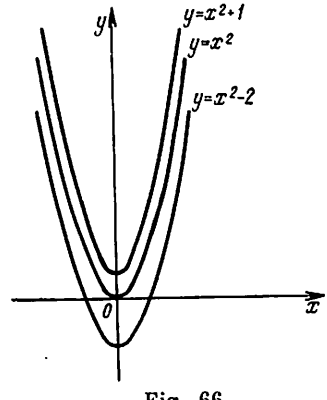


Fig. 66

32. Find the equation of the curve passing through the point A (0, 1) knowing that the tangent line at any point of the curve has a slope equal to the ordinate of the point of tangency.

○ By the hypothesis, we have $k=\frac{dy}{dx}=y$, i.e. $\frac{dy}{y}=dx$. Integrating, we get

$$\int \frac{dy}{y}=\int dx; \quad \ln y=x+C.$$

From the initial conditions we find $\ln 1 = 0 + C$, that is, $C = 0$; consequently, $y = e^x$.

33. Find the function: (1) whose derivative is $y' = 4x^3 - 2x + 3$; (2) whose differential $dy = (2x + 6)\,dx$.

34. Find the function: (1) whose derivative $y' = 2x - 5$ if for $x = -3$ this function takes on the value equal to 28; (2) vanishing for $x = 0$ if $y' = 3x^2 - 4x + 5$; (3) whose derivative $y' = 3e^x + 2x$ if for $x = 0$ this function takes on the value equal to 8.

35. Find: (1) $\int (\sin x+3\cos x)\,dx$ if for $x=\pi$ the primitive function is equal to 4; (2) $\int (\cos x-e^x+2x)\,dx$ if for $x=0$ the primitive function is equal to 3; (3) $\int \left(\frac{3}{1+x^2}+2\right)dx$ if for $x=0$ the primitive function is equal to zero; (4) $\int \left(\frac{1}{x^3}-x^2\right)dx$ if for $x=-1$ the primitive function is equal to 1/3.

36. Find the equation of the curve if the slope of the tangent at each of its points (x, y) is equal to: (1) $-3x$; (2) $x + 2$.

37. Set up the equation of the curve if the slope of the tangent at each of its points is equal to: (1) $-y/x$; (2) x/y; (3) $-x/y$.

38. Find the equation of the curve passing through the origin if the slope of the tangent at any point of the curve is equal to $x/3$.

39. Find the equation of the curve passing through the point M (1, 4) if the slope of the tangent to the curve at each of its points is equal to $3x^2 - 2x$.

40. Set up the equation of the curve passing through the point A (—1, 3) if the slope of the tangent to the curve at each of its points is equal to the triple square of the abscissa of the point of tangency.

41. Find the equation of the curve passing through the point A (0, e) if the slope of the tangent to the curve at any of its points is equal to y.

Sec. 11.3. Physical Applications of the Indefinite Integral

42. The velocity of a point moving in a straight line changes according to the equation $v = 3t^2 - 2t$. Find the law of its motion.

○ It is known that the velocity of straight-line motion of a point is equal to the derivative of the path s with respect to the time t, that is, $v = \frac{ds}{dt} = 3t^2 - 2t$, whence $ds = (3t^2 - 2t)\,dt$. Integrating, we find

$$\int ds = \int (3t^2 - 2t)\,dt; \quad s = t^3 - t^2 + C. \bullet$$

43. The velocity of straight-line motion of a point changes according to the equation $v = 3t^2 + 4$. Find the law of motion s if during the time $t = 2$ s the point covered 20 m.

○ Since $v = \frac{ds}{dt} = 3t^2 + 4$, we have $ds = (3t^2 + 4)\,dt$. Integrating, we get

$$\int ds = \int (3t^2 + 4)\,dt; \quad s = t^3 + 4t + C.$$

Making use of the initial conditions, we find $20 = 2^3 + 4 \cdot 2 + C$, that is, $C = 4$. Thus, the law of motion of the point has the form $s = t^3 + 4t + 4$. ●

44. Find the law of motion of a freely falling body with constant acceleration g if the body started its fall from rest.

○ It is known that acceleration a of a body moving in a straight line is the second derivative of the path s with respect to the time t, that is, $a = \frac{d^2s}{dt^2} = \frac{dv}{dt}$. Since $a = g$, we have $\frac{dv}{dt} = g$ whence $dv =$

$g\,dt$. Integrating, we obtain

$$\int dv = \int g\,dt; \quad v = gt + C_1.$$

Using the initial conditions $t = 0$, $v = 0$, we have $0 = g \cdot 0 + C_1$, that is, $C_1 = 0$. Thus, the velocity of the body changes according to the law $v = gt$.

Let us now find the equation of body's motion. Since $v = \frac{ds}{dt}$, we have $\frac{ds}{dt} = gt$, or $ds = gt\,dt$. Integrating, we get

$$\int ds = \int gt\,dt; \quad s = \frac{gt^2}{2} + C_2.$$

Using the initial conditions $t = 0$, $s = 0$, we have $0 = g \cdot 0^2/2 + C_2$, $C_2 = 0$. Thus, the law of motion of a freely falling body has the form $s = gt^2/2$. ●

45. A point moves in a straight line with acceleration $a = 6t - 12$. At the instant of time $t = 0$ (reference time) the initial velocity $v_0 = 9$ m/s; the distance (path) from the reference time $s_0 = 10$ m. Find: (1) the velocity and equation of motion of the point; (2) the values of acceleration, velocity, and path at the instant $t = 2s$; (3) the instant when the point has the lowest velocity.

○ (1) We find the velocity: $\frac{dv}{dt} = 6t - 12$, or $dv = (6t - 12)\,dt$. Integrating, we obtain

$$\int dv = \int (6t - 12)\,dt; \quad v = 3t^2 - 12t + C_1.$$

Making use of the initial conditions $t = 0$, $v_0 = 9$, we have $9 = 3 \cdot 0^2 - 12 \cdot 0 + C_1$, that is, $C_1 = 9$. Consequently, $v = 3t^2 - 12t + 9$.

We then find the equation of motion of the point: $\frac{ds}{dt} = 3t^2 - 12t + 9$ or $ds = (3t^2 - 12t + 9)\,dt$. Integrating, we find

$$\int ds = \int (3t^2 - 12t + 9)\,dt: \quad s = t^3 - 6t^2 + 9t + C_2.$$

Using the initial conditions $t = 0$, $s_0 = 10$, we have $10 = 0^3 - 6 \cdot 0^2 + 9 \cdot 0 + C_2$, that is, $C_2 = 10$. Thus, $s = t^3 - 6t^2 + 9t + 10$.

(2) We now find a, v, and s at $t = 2$: $a = 6 \cdot 2 - 12 = 0$; $v = 3 \cdot 2^2 - 12 \cdot 2 + 9 = -3$ (m/s); $s = 2^3 - 6 \cdot 2^2 + 9 \cdot 2 + 10 = 12$ (m).

(3) We finally investigate the function, determining the change in velocity, for maximum and minimum:

$$v = 3t^2 - 12t + 9, \; v' = 6t - 12, \; 6t - 12 = 0, \; t = 2,$$

$$v'' = 6 > 0.$$

Consequently, the point has the smallest velocity at $t = 2$ s. ●

46. Velocity of a point moving in a straight line changes according to the equation: (1) $v = t^2 - 8t + 2$; (2) $v = 4t - 3t^2$. Find the equation of motion.

47. Velocity of a point moving in a straight line is given by the formula $v = 2t - 3$. Find its equation of motion if by the reference instant it covered the path equal to 6 m.

48. Velocity of a point moving in a straight line is given by the formula $v = 3t^2 + 4t - 1$. Find its equation of motion if at the initial instant of time it was at the origin.

49. Velocity of a point moving in a straight line is given by the formula $v = 2 \cos t$. Find the equation of motion if at the instant $t = \pi/6$ the point was at a distance $s = 4$m from the origin.

50. A body is thrown vertically upward with an initial velocity v_0. Find the equation of motion of this body (air resistance may be neglected).

51. A point moves in a straight line with acceleration $a = 12t^2 + 6t$. Find the equation of motion if at the instant $t = 1$ s its velocity is $v = 8$ m/s and the path $s = 6$ m.

52. A point moves in a straight line with acceleration $a = -6t + 18$. At the instant of time $t = 0$ (the origin) the initial velocity $v_0 = 24$ m/s and the distance from the origin $s_0 = 15$ m. Find: (1) the velocity and the equation of motion; (2) the values of acceleration, velocity, and path at the instant $t = 2$ s; (3) the instant when the velocity has the greatest value.

Sec. 11.4. Integration by Change of Variable

The essence of integration by change of variable (by substitution) consists in transforming the integral $\int f(x)\,dx$ into the integral $\int F(u)\,du$, which is readily computed with the aid of a basic integration formula.

To find the integral $\int f(x)\,dx$, we replace the variable x by a new variable u with the aid of the substitution $x = \varphi(u)$. Differentiating this equality, we get $dx = \varphi'(u)\,du$. Replacing in the integrand x and dx by their values expressed in terms of u and du, we have

$$\int f(x)\,dx = \int f[\varphi(u)]\,\varphi'(u)\,du = \int F(u)\,du.$$

After the integral with respect to the new variable u is found it is reduced to the variable x with the aid of the substitution $u = \psi(x)$.

In Problems 53 through 57 we shall find the indicated integrals:

53. (1) $\int (3x+2)^5\,dx$; (2) $\int \frac{dx}{(4x+1)^4}$; (3) $\int (2x^3+1)^4 x^2\,dx$;

(4) $\int \frac{x\,dx}{(x^2+1)^3}$; (5) $\int \frac{x^2\,dx}{5x^3+1}$.

⊙ (1) We introduce the substitution $3x + 2 = u$. Differentiating, we have $3\,dx = du$, whence $dx = (1/3)\,du$. Substituting into the original integral instead of $3x + 2$ and dx their expressions, we get

$$\int (3x+2)^5\,dx = \frac{1}{3}\int u^5\,du = \frac{1}{3}\cdot\frac{u^6}{6} + C = \frac{1}{18}\,u^6 + C.$$

Replacing u by its expression in terms of x, we find

$$\int (3x+2)^5\,dx = \frac{1}{18}\,u^6 + C = \frac{1}{18}\,(3x+2)^6 + C.$$

Check: $d\left[\frac{1}{18}\,(3x+2)^6 + C\right] = \frac{6}{18}\,(3x+2)^5 \cdot 3dx = (3x+2)^5\,dx$.

(2) Setting $4x + 1 = u$ and differentiating, we get $4dx = du$, whence $dx = (1/4)\,du$. Consequently,

$$\int \frac{dx}{(4x+1)^4} = \int (4x+1)^{-4}\,dx = \frac{1}{4}\int u^{-4}\,du$$

$$= \frac{1}{4}\cdot\frac{u^{-3}}{-3} + C = -\frac{1}{-12u^3} + C = -\frac{1}{12\,(4x+1)^3} + C.$$

(3) We set $2x^3 + 1 = u$, whence $6x^2\,dx = du$, $x^2\,dx = (1/6)\,du$. Thus,

$$\int (2x^3+1)^4\,x^2\,dx = \frac{1}{6}\int u^4\,du = \frac{1}{6}\cdot\frac{u^5}{5} + C$$

$$= \frac{1}{30}\,u^5 + C = \frac{1}{30}\,(2x^3+1)^5 + C.$$

(4) Setting $x^2 + 1 = u$, we have $2x\,dx = du$, $x\,dx = (1/2)\,du$. Hence,

$$\int \frac{x\,dx}{(x^2+1)^3} = \int (x^2+1)^{-3}\,x\,dx = \frac{1}{2}\int u^{-3}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{-2}}{-2} + C = -\frac{1}{4u^2} + C = -\frac{1}{4\,(x^2+1)^2} + C.$$

(5) We put $5x^3 + 1 = u$, whence $15x^2\,dx = du$, $x^2\,dx = (1/15)\,du$. Therefore,

$$\int \frac{x^2\,dx}{5x^3+1} = \frac{1}{15}\int \frac{du}{u} = \frac{1}{15}\ln|u| + C = \frac{1}{15}\ln|5x^3+1| + C. \bullet$$

54. (1) $\int \tan kx\,dx$: (2) $\int \frac{du}{\sin u}$; (3) $\int \frac{du}{\cos u}$.

○ (1) We have $\int \tan kx\,dx = \int \frac{\sin kx\,dx}{\cos kx}$.

We set $\cos kx = u$, whence $-k \sin kx\,dx = du$, $\sin kx\,dx = -(1/k)\,du$. Consequently,

$$\int \tan kx\,dx = -\frac{1}{k}\int \frac{du}{u} = -\frac{1}{k}\ln|u| + C = -\frac{1}{k}\ln|\cos kx| + C.$$

(2) Since $\sin u = 2\sin(u/2)\cos(u/2)$, we have

$$\int \frac{du}{\sin u} = \int \frac{du}{2\sin(u/2)\cos(u/2)}.$$

Dividing and multiplying the denominator by $\cos(u/2)$, we get

$$\int \frac{du}{\sin u} = \frac{1}{2}\int \frac{du}{\tan(u/2)\cos^2(u/2)}.$$

We set $\tan(u/2) = z$; then $\frac{1}{\cos^2(u/2)} \cdot \frac{1}{2}\,du = dz$, i.e. $\frac{du}{\cos^2(u/2)} = 2dz$. Thus,

$$\int \frac{du}{\sin u} = \int \frac{dz}{z} = \ln|z| + C = \ln|\tan(u/2)| + C.$$

(3) We have $\int \frac{du}{\cos u} = \int \frac{du}{\sin\left(\frac{\pi}{2}+u\right)}$. Setting $\frac{\pi}{2}+u = z$, we then get $du = dz$.
Therefore

$$\int \frac{du}{\cos u} = \int \frac{dz}{\sin z} = \ln\left|\tan\frac{z}{2}\right| + C$$

$$= \ln\left|\tan\frac{\frac{\pi}{2}+u}{2}\right| + C = \ln\left|\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right| + C. \bullet$$

55. (1) $\int 3^{5x^2} x\,dx$. (2) $\int e^{-3x^2+1} x\,dx$.

○ (1) We put $5x^2 = u$, whence $10x\,dx = du$, $x\,dx = (1/10)\,du$. Hence,

$$\int 3^{5x^2} x\,dx = \frac{1}{10}\int 3^u\,du = \frac{1}{10}\frac{3^u}{\ln 3} + C = \frac{3^{5x^2}}{10\ln 3} + C.$$

(2) We set $-3x^2 + 1 = u$, whence $-6x\,dx = du$, $x\,dx = -(1/6)\,du$. Thus,

$$\int e^{-3x^2+1} x\,dx = -\frac{1}{6}\int e^u\,du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{-3x^2+1} + C. \bullet$$

56. (1) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$; (2) $\int \frac{3x\,dx}{\cos^2 2x^2}$.

○ (1) We set $\sqrt{x} = u$, whence $dx/(\sqrt{x}) = 2\,du$. Consequently,

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C.$$

(2) Wet set $2x^2 = u$, whence $4x\,dx = du$, $x\,dx = (1/4)\,du$. Thus,

$$\int \frac{3x\,dx}{\cos^2 2x^2} = \frac{3}{4}\int \frac{du}{\cos^2 u} = \frac{3}{4}\tan u + C = \frac{3}{4}\tan 2x^2 + C. \bullet$$

57. (1) $\int \frac{3^x\,dx}{\sqrt{25-9^x}}$; (2) $\int \frac{\cos x\,dx}{4+\sin^2 x}$.

○ (1) Setting $3^x=u$, we find $3^x \ln 3\,dx=du$, $\sqrt{25-9^x}=\sqrt{25-u^2}$. Consequently,

$$\int \frac{3^x\,dx}{\sqrt{25-9^x}}=\frac{1}{\ln 3}\int \frac{du}{\sqrt{5^2-u^2}}$$

$$=\frac{1}{\ln 3}\arcsin\frac{u}{5}+C=\frac{1}{\ln 3}\arcsin\frac{3^x}{5}+C.$$

(2) We set $\sin x=u$, whence $\cos x\,dx=du$. Thus,

$$\int \frac{\cos x\,dx}{4+\sin^2 x}=\int \frac{du}{4+u^2}=\frac{1}{2}\arctan\frac{u}{2}+C=\frac{1}{2}\arctan\frac{\sin x}{2}+C.$$ ●

In Problems 58 through 73 find the indicated integrals:

58. (1) $\int (7-2x)^3\,dx$; (2) $\int (5t-1)^4\,dt$; (3) $\int \frac{dx}{(4-3x)^2}$;

(4) $\int \frac{dz}{(5z+1)^3}$.

59. (1) $\int \sqrt[3]{(3x+1)^2}\,dx$; (2) $\int \sqrt{2x-1}\,dx$;

(3) $\int \sqrt[3]{(4-3t)^2}\,dt$;

(4) $\int \frac{dx}{\sqrt{(3x-1)^3}}$; (5) $\int \frac{dx}{\sqrt[3]{(3x-5)^2}}$.

60. (1) $\int (x^2+3)^5 x\,dx$; (2) $\int 4(x^4-1)^2 x^3\,dx$;

(3) $\int \frac{6z^2\,dz}{(1-2z^3)^4}$; (4) $\int \frac{x^3\,dx}{(5x^4+3)^5}$.

61. (1) $\int \sqrt{4x^3+1}\,x^2\,dx$; (2) $\int \sqrt{(x^4-1)^3}\,x^3\,dx$;

(3) $\int \sqrt{2\sin x-1}\cos x\,dx$ (substitution $2\sin x-1=u$);

(4) $\int \sqrt{e^x+1}\,e^x\,dx$ (substitution $e^x+1=u$).

62. (1) $\int \sqrt{(3z^4+2)^3}\,z^3\,dz$; (2) $\int \sqrt[3]{(1-3x^2)^4}\,x\,dx$.

63. (1) $\int \frac{x\,dx}{\sqrt{x^2+1}}$; (2) $\int \frac{x^3\,dx}{\sqrt[3]{(5x^4+2)^2}}$; (3) $\int \frac{x^2\,dx}{\sqrt{(x^3-1)^3}}$;

(4) $\int \frac{\cos x\,dx}{\sqrt{1-\sin x}}$; (5) $\int \frac{e^x\,dx}{(e^x+1)^3}$.

64. (1) $\int \frac{z^2\,dz}{1+z^3}$; (2) $\int \frac{e^{3x}\,dx}{e^{3x}+1}$; (3) $\int \frac{\cos x\,dx}{2\sin x+1}$.

65. (1) $\int \tan 3x\,dx$; (2) $\int \cot kx\,dx$; (3) $\int \cot (x/2)\,dx$.

66. (1) $\int \frac{dx}{\sin 2x}$; (2) $\int \frac{dx}{\sin (x/3)}$; (3) $\int \frac{dx}{\cos 3x}$;

(4) $\int \frac{dx}{\cos (x/2)}$.

67. (1) $\int \frac{dt}{t(1+\ln t)}$; (2) $\int \frac{(2-\ln t)\,dt}{t}$.

68. (1) $\int a^{x^4} x^3\,dx$; (2) $\int a^{2x} b^{2x}\,dx$.

69. (1) $\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$; (2) $\int xe^{-x^2}\,dx$;

(3) $\int e^{\sin x} \cos x\,dx$; (4) $\int \frac{e^{1/x}\,dx}{x^2}$.

70. (1) $\int t \sin (t^2-1)\,dt$; (2) $\int \sin (z/2)\,dz$;

(3) $\int x^3 \cos x^4\,dx$; (4) $\int \frac{\cos \sqrt{x}\,dx}{\sqrt{x}}$; (5) $\int x \cos (x^2+1)\,dx$.

71. (1) $\int \frac{dx}{\sqrt{x}\cos^2 \sqrt{x}}$; (2) $\int \frac{x^2\,dx}{\cos^2 x^3}$; (3) $\int \frac{d\varphi}{\sin^2 (\pi/3-\varphi)}$;

(4) $\int \frac{dx}{x^2 \sin^2 (1/x)}$; (5) $\int \frac{dx}{x \sin^2 \ln x}$.

72. (1) $\int \frac{e^{\varphi}\,d\varphi}{\sqrt{1-e^{2\varphi}}}$; (2) $\int \frac{dz}{z\sqrt{1-\ln^2 z}}$.

73. (1) $\int \frac{\sin x\,dx}{a+\cos^2 x}$; (2) $\int \frac{e^x\,dx}{1+e^{2x}}$;

(3) $\int \frac{x^2\,dx}{1+x^6}$; (4) $\int \frac{dx}{x(1+\ln^2 x)}$.

Sec. 11.5. Integration by Parts

Integrating both sides of the equality $d(uv) = u\,dv + v\,du$, we get

$$\int d(uv) = \int u\,dv + \int v\,du; \quad uv = \int u\,dv + \int v\,du,$$

whence

$$\int u\,dv = uv - \int v\,du. \tag{11.14}$$

This is the formula of integration by parts. With the aid of this formula, the computation of the integral $\int u\,dv$ is reduced to computing the integral $\int v\,du$ if the latter turns out to be simpler than the original one.

74. Find the following integrals:

(1) $\int x \sin x\,dx$; (2) $\int \frac{\ln x\,dx}{x^2}$; (3) $\int \sqrt{x^2+a^2}\,dx$.

○ (1) We set $u = x$, $dv = \sin x\,dx$; then $du = dx$, $\int dv = \int \sin x\,dx$, i.e. $v = -\cos x$. Using Formula (11.14), we get $\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$.

(2) We set $u = \ln x$, $dv = \frac{dx}{x^2}$; then $du = \frac{dx}{x}$, $\int dv = \int \frac{dx}{x^2}$ $= \int x^{-2}\,dx = -\frac{1}{x}$; $v = -\frac{1}{x}$. By Formula (11.14), we get Formula (11.14), we get

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x}\,\frac{dx}{x} = -\frac{\ln x}{x} + \int x^{-2}\,dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C.$$

(3) We set $u = \sqrt{x^2+a^2}$, $dv = dx$; then $du = \frac{x\,dx}{\sqrt{x^2+a^2}}$, $v = x$. By Formula (11.14), we get

$$\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int \frac{x^2\,dx}{\sqrt{x^2+a^2}}.$$

In the numerator of the integrand function of the last integral we add and subtract a^2 and represent this integral as a sum of two integrals:

$$\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\,dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}.$$

The last integral is found by the Formula (11.11):

$$\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx$$

$$+ a^2 \ln|x + \sqrt{x^2+a^2}| + C.$$

Transposing $\int \sqrt{x^2+a^2}\,dx$ from the right-hand side to the left-hand side, we get

$$2\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} + a^2 \ln|x + \sqrt{x^2+a^2}| + C,$$

or, finally

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x + \sqrt{x^2+a^2}| + C. \bullet$$

In Problems 75 through 79 find the indicated integrals:

75. (1) $\int x\cos x\,dx$; (2) $\int (1-x)\sin x\,dx$.

76. (1) $\int \frac{\ln x}{x^3}\,dx$; (2) $\int \ln^2 x\,dx$.

77. (1) $\int xe^x\,dx$; (2) $\int \frac{x\,dx}{\sin^2 x}$.

78. (1) $\int \arcsin x\,dx$; (2) $\int \arctan x\,dx$.

79. (1) $\int e^x \cos x\,dx$; (2) $\int e^x \sin x\,dx$.

80. Check the given equalities by integrating by parts:

(1) $$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C;$$

(2) $$\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{a^2-x^2} + C.$$

Sec. 11.6. Integrating Certain Trigonometric Functions

When computing integrals of the form $\int \sin^{2n} x\,dx$ or $\int \cos^{2n} x\,dx$ of an even power of sine or cosine, the following power lowering formulas are used:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

When computing integrals of the form $\int \sin^{2n+1} x\,dx$ or $\int \cos^{2n+1} x\,dx$ of an odd power of sine or cosine, it is necessary to separate one factor from the odd power and to introduce a new variable, setting $\cos x = t$ in the first integral and $\sin x = t$ in the second.

For computing integrals of the form $\int \sin \alpha x \cos \beta x\,dx$, $\int \sin \alpha x \sin \beta x\,dx$ and $\int \cos \alpha x \cos \beta x\,dx$ the following formulas are used:

$$\sin \alpha x \sin \beta x = \frac{1}{2}[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x],$$

$$\cos \alpha x \cos \beta x = \frac{1}{2}[\cos(\alpha-\beta)x + \cos(\alpha+\beta)x],$$

$$\sin \alpha x \cos \beta x = \frac{1}{2}[\sin(\alpha-\beta)x + \sin(\alpha+\beta)x].$$

In Problems 81 through 84 we shall find the indicated integrals:

81. (1) $\int \cos^2 x\,dx$; (2) $\int \cos^4 x\,dx$.

○ (1) Replacing $\cos^2 x$ by $(1+\cos 2x)/2$, we get

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

(2) $$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx.$$

In the last integral we replace $\cos^2 2x$ by $(1+\cos 4x)/2$; we then get

$$\int \cos^4 x\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \bullet$$

82. (1) $\int \tan^2 x\,dx$; (2) $\int \tan^4 x\,dx$,

○ (1) Taking advantage of the relationship $\tan^2 x = \frac{1}{\cos^2 x} - 1$, we get

$$\int \tan^2 x\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x + C.$$

(2) $$\int \tan^4 x\,dx = \int \tan^2 x \left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \int \frac{\tan^2 x\,dx}{\cos^2 x} - \int \tan^2 x\,dx.$$

We compute the first integral. Setting $\tan x = u$, we find $dx/(\cos^2 x) = du$ and, consequently,

$$\int \frac{\tan^2 x\,dx}{\cos^2 x} = \int u^2\,du = \frac{u^3}{3} = \frac{\tan^3 x}{3} + C.$$

Using the result of the preceding example, we, finally, get

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \tan x + x + C. \bullet$$

83. (1) $\int \sin^3 x\,dx$; (2) $\int \tan^3 x\,dx$.

○ (1) $$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

Let us set $\cos x = u$; then $-\sin x\,dx = du$, and, consequently,

$$\int \sin^3 x\,dx = -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

(2) $$\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx$$

$$= \int \left(\frac{1}{\cos^2 x} - 1\right) \tan x\,dx = \int \frac{\tan x\,dx}{\cos^2 x} - \int \tan x\,dx$$

$$= \int \tan x\,d(\tan x) - \int \frac{\sin x\,dx}{\cos x} = \frac{1}{2}\tan^2 x + \ln|\cos x| + C. \bullet$$

84. (1) $\int \sin 5x \sin 3x\,dx$; (2) $\int \cos 4x \cos x\,dx$;

(3) $\int \sin 7x \cos 3x\,dx$.

○ (1) $$\int \sin 5x \sin 3x\,dx = \frac{1}{2}\int (\cos 2x - \cos 8x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2x - \frac{1}{8}\sin 8x\right) + C = \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C.$$

(2) $$\int \cos 4x \cos x\,dx = \frac{1}{2}\int (\cos 3x + \cos 5x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right) + C = \frac{1}{6}\sin 3x + \frac{1}{10}\sin 5x + C.$$

(3) $$\int \sin 7x \cos 3x\,dx = \frac{1}{2}\int (\sin 4x + \sin 10x)\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x\right) + C = -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C. \bullet$$

In Problems 85 through 88 find the indicated integrals:

85. (1) $\int \sin^2 x\,dx$; (2) $\int \sin^4 x\,dx$

86. (1) $\int \cot^2 x\,dx$; (2) $\int \cot^4 x\,dx$

87. (1) $\int \cos^3 x\,dx$; (2) $\int \sin^5 x\,dx$;

(3) $\int \cos^5 x\,dx$; (4) $\int \cot^3 x\,dx$

88. (1) $\int \sin 3x \sin x\,dx$; (2) $\int \cos 5x \cos 3x\,dx$;

(3) $\int \sin 4x \cos 3x\,dx$.

Sec. 11.7. Miscellaneous Problems

89. Find the function whose derivative is $y' = \sin 2x - xe^{3x^2} + 1$.

90. Find $\int \frac{\sin x\,dx}{\cos x + e}$ if for $x = \pi/2$ the value of the primitive function is equal to 2

91. Set up the equation of the curve passing through the point A $(\pi/3, 2)$ if the slope of the tangent line to the curve at each of its points is equal to $\cos(x/2)$.

92. Velocity of a point moving in a straight line is given by the formula $v = \sin 2t$. Find the equation of motion if at the instant of time $t = \pi/6$ the point was at a distance 0.75 m from the origin.

93. A point moves in a straight line with acceleration $a = (-6t + 24)$ m/s^2. At the instant $t = 1$ s its velocity $v = 15$ m/s, and the path covered $s = 20$ m. Find: (1) the velocity equation; (2) the equation of motion of the point; (3) the acceleration, velocity, and path at the instant $t = 3$ s; (4) the instant of time at which the point will move with the highest velocity.

In Problems 94 through 99 find the indicated integrals:

94. $\int \frac{x^2\,dx}{\sqrt{x^3-1}}$. 95. $\int e^{\sin x} \cos x \sqrt{e^{\sin x}+1}\,dx$.

96. $\int \frac{\sin x\,dx}{1-\cos x}$. 97. $\int \frac{\cos x\,dx}{9+\sin^2 x}$.

98. $\sin^2 x \cos^2 x\,dx$. 99. $\int \sin 5x \cos x\,dx$.

QUIZ No. 1

Form I

Find the integrals:

(1) $\int \frac{x^2 + x\sqrt[3]{x} + \sqrt{x}}{x\sqrt{x}}\,dx$;

(2) $\int \left(\frac{2}{\sqrt{9-4x^2}} + \frac{1}{e^x}\right) dx$;

(3) $\int \frac{dx}{\sin x \cos x}$.

(4) Set up the equation of the curve passing through the point $(-2, 8)$ if the slope of the tangent at any point of tangency is equal to $2x - 4$.

(5) The velocity of a point moving in a straight line $v = 3t^2 + 6t - 4$. Find the equation of motion if during the time $t = 2$ s the point covered the path equal to 8 m.

Form II

Find the integrals:

(1) $\int \frac{\sqrt{x} - \sqrt[3]{x^2} - x^{-1/2}}{x\sqrt{x}}\,dx$;

(2) $\int \left(\frac{1}{\sqrt{3-x^2}} + \frac{1}{e^x}\right) dx$;

(3) $\int (4\sin^2 x \cos x - \cos x)\,dx$.

(4) Set up the equation of the curve passing through the point A $(\pi/3, 1/2)$ if the slope of the tangent line to the curve at each of its points is equal to $\sin x$.

(5) A point moves in a straight line with acceleration $a = 6t + 6$. Find the equation of motion if $s = 0$ at the instant of time $t = 0$, and at the instant $t = 3$ s the velocity $v = 40$ m/s.

Chapter 12

THE DEFINITE INTEGRAL

Sec. 12.1. The Definite Integral and Its Direct Computation

Let a function $f(x)$ be defined on the interval $[a, b]$. Let us divide the interval $[a, b]$ into n subintervals by the points $a < x_0 < x_1 < x_2 < \ldots < x_n = b$, choose an arbitrary point ξ_k on each of the elementary intervals (subinterval) $[x_{k-1}, x_k]$, and denote the length of each such subinterval by Δx_k. The sum of the form

$$\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k = f(\xi_1)\,\Delta x_1 + f(\xi_2)\,\Delta x_2 + \ldots + f(\xi_n)\,\Delta x_n$$

is called the *integral sum* for the function $f(x)$ on the interval $[a, b]$.

The *definite integral* of the function $f(x)$ on the interval $[a, b]$ is defined as the limit of the integral sum when the length of the greatest subinterval tends to zero:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k.$$

For any function $f(x)$, continuous on the interval $[a, b]$, there always exists the definite integral $\int_a^b f(x)\,dx$.

Consider the basic properties of the definite integral.

1°. *The definite integral of an algebraic sum of a finite number of functions is equal to the algebraic sum of the definite integrals of these functions:*

$$\int_a^b [f_1(x) \pm f_2(x)]\,dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx.$$

2°. *A constant factor can be taken outside the sign of the definite integral:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

3°. *Interchanging the limits of integration simply changes the sign of the definite integral:*

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

4°. *A definite integral with equal limits of integration is equal to zero:*

$$\int_a^a f(x)\,dx = 0.$$

5°. *The interval of integration may be split into parts:*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The definite integral of the function $f(x)$ in the case when it is possible to find the corresponding indefinite integral $F(x)$ is computed by the Newton-Leibniz formula:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

that is, *the value of the definite integral is equal to the difference of the values of any antiderivative of the integrand corresponding to the upper and lower limits of integration.*

In Problems 1 through 4 we shall compute the given definite integrals:

1. (1) $\int_0^1 x\,dx$; (2) $\int_2^3 x^2\,dx$; (3) $\int_{-1}^2 (x^2+2x+1)\,dx$.

○ By the Newton-Leibniz formula, we get

(1) $\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}(1^2-0^2) = \frac{1}{2}$;

(2) $\int_2^3 x^2\,dx = \frac{x^3}{3}\Big|_2^3 = \frac{1}{3}(3^3-2^3) = \frac{19}{3}$;

(3) $\int_{-1}^2 (x^2+2x+1)\,dx = \left[\frac{1}{3}x^3+x^2+x\right]_{-1}^2$

$= \left[\frac{1}{3}\cdot 2^3+2^2+2\right] - \left[\frac{1}{3}(-1)^3+(-1)^2+(-1)\right] = 9.$ ●

2. (1) $\int_{-1}^1 e^x\,dx$; (2) $\int_1^e \frac{dx}{x}$.

○ (1) $\int_{-1}^1 e^x\,dx = e^1\Big|_{-1}^1 = e^1 - e^{-1} = e - \frac{1}{e} = \frac{e^2-1}{e}$;

(2) $\int_1^e \frac{dx}{x} = \ln x\Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1.$ ●

3. (1) $\int_{\pi/6}^{\pi/2} \cos x\,dx$; (2) $\int_{\pi/4}^{\pi/3} \frac{dx}{\sin^2 x}$.

○ (1) $\int_{\pi/6}^{\pi/2} \cos x\,dx = \sin x\Big|_{\pi/6}^{\pi/2} = \sin\frac{\pi}{2} - \sin\frac{\pi}{6} = 1 - \frac{1}{2} = \frac{1}{2}$;

(2) $\int\limits_{\pi/4}^{\pi/3} \frac{dx}{\sin^2 x} = -\cot x \Big|_{\pi/4}^{\pi/3} = -\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right)$

$= -\left(\frac{\sqrt{3}}{3} - 1\right) = \frac{3-\sqrt{3}}{3}$. ●

4. (1) $\int\limits_{-1}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$; (2) $\int\limits_{0}^{1} \frac{dx}{1+x^2}$.

○ (1) $\int\limits_{-1}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-1}^{\sqrt{3}/2}$

$= \arcsin\frac{\sqrt{3}}{2} - \arcsin(-1) = \frac{\pi}{3} - \left(-\frac{\pi}{2}\right) = \frac{5\pi}{6}$;

(2) $\int\limits_{0}^{1} \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$. ●

In Problems 5 through 16 compute the indicated definite integrals:

5. (1) $\int\limits_{0}^{2} x^2\,dx$; (2) $\int\limits_{1}^{2} x^3\,dx$; (3) $\int\limits_{1}^{3} x^4\,dx$.

6. (1) $\int\limits_{-2}^{3} (4x^3 - 3x^2 + 2x + 1)\,dx$; (2) $\int\limits_{-1}^{0} (x^3 + 2x)\,dx$.

7. (1) $\int\limits_{1/2}^{1} \frac{dx}{x^3}$; (2) $\int\limits_{1/3}^{1/2} \frac{dx}{x^2}$; (3) $\int\limits_{0}^{4} \sqrt{x}\,dx$.

8. (1) $\int\limits_{1}^{8} \sqrt[3]{x^2}\,dx$; (2) $\int\limits_{8}^{27} \frac{dx}{\sqrt[3]{x}}$; (3) $\int\limits_{1}^{4} \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$;

(4) $\int\limits_{1}^{9} \frac{x-1}{\sqrt{x}}\,dx$.

9. (1) $\int\limits_{1}^{3} e^{2x}\,dx$; (2) $\int\limits_{0}^{1} e^{3x}\,dx$.

10. (1) $\int\limits_{3}^{6} \frac{dx}{x}$; (2) $\int\limits_{0}^{1} \frac{dx}{x+2}$; (3) $\int\limits_{2}^{3} \frac{dx}{x-1}$;

(4) $\int\limits_{1}^{2} \frac{dx}{x+3}$.

11. (1) $\int_0^{\pi/3} \sin x\,dx$; (2) $\int_{-\pi/4}^{\pi/4} \cos x\,dx$; (3) $\int_{-\pi/2}^{\pi/2} (\cos x - \sin x)\,dx$.

12. (1) $\int_0^{\pi/4} \cos 2x\,dx$; (2) $\int_0^{\pi/4} \sin 4x\,dx$; (3) $\int_0^{\pi} \cos (x/2)\,dx$.

13. (1) $\int_0^{\pi/4} \frac{4dx}{\cos^2 x}$; (2) $\int_{\pi/6}^{\pi/3} \frac{dx}{\sin^2 x}$.

14. (1) $\int_{\pi/6}^{\pi/3} \left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}\right) dx$; (2) $\int_{-\pi/4}^{\pi/4} \left(\frac{1}{\cos^2 x} - \sin x\right) dx$.

15. (1) $\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$; (2) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$; (3) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$.

16. (1) $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$; (2) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}$; (3) $\int_5^{5\sqrt{3}} \frac{dx}{25+x^2}$.

Sec. 12.2. Evaluating Definite Integrals by Change of Variable

When computing a definite integral by a change of variable (integration by substitution), the definite integral $\int_a^b f(x)\,dx$ is transformed with the aid of the substitution $u = \psi(x)$ or $x = \varphi(u)$ into a definite integral with respect to the new variable u. In this case, the limits of integration a and b are respectively replaced by the new limits of integration α and β which are found from the initial substitution.

From the first substitution the new limits of integration are computed directly: $\alpha = \psi(a)$, $\beta = \psi(b)$.

From the second substitution the new limits of integration are found by solving the equations $a = \varphi(\alpha)$ and $b = \varphi(\beta)$ with respect to α and β.

Thus, we have

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(u)]\,\varphi'(u)\,du = \int_\alpha^\beta F(u)\,du.$$

17. Evaluate the following definite integrals:

(1) $\int_2^3 (2x-1)^3\,dx$; (2) $\int_1^2 \frac{dx}{\sqrt{5x-1}}$; (3) $\int_0^1 (2x^3+1)^4 x^2\,dx$;

$$(4)\quad \int_{\sqrt{2}/3}^{\sqrt{3}/3} \frac{dx}{\sqrt{4-9x^2}}$$

○ (1) Let us introduce a new variable of integration with the aid of the substitution $2x - 1 = u$. Differentiating, we have $2dx = du$, whence $dx = (1/2)\,du$. We then find the new limits of integration. Substituting the values $x = 2$ and $x = 3$ into the relationship $2x - 1 = u$, we get the respective values of the lower and upper limits: $u_l = 2\cdot 2 - 1 = 3$, $u_u = 2\cdot 3 - 1 = 5$. Consequently,

$$\int_2^3 (2x-1)^3\,dx = \frac{1}{2}\int_3^5 u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4}\Big|_3^5 = \frac{1}{8}(5^4-3^4) = 68.$$

(2) Let us set $5x - 1 = u$; then $5dx = du$, $dx = (1/5)\,du$. Computing the new limits of integration, we have: $u_e = 5\cdot 1 - 1 = 4$, $u_u = 5\cdot 2 - 1 = 9$. Therefore

$$\int_1^2 \frac{dx}{\sqrt{5x-1}} = \int_1^2 (5x-1)^{-1/2}\,dx = \frac{1}{5}\int_4^9 u^{-1/2}\,du$$

$$= \frac{2}{5}u^{1/2}\Big|_4^9 = \frac{2}{5}(9^{1/2}-4^{1/2}) = \frac{2}{5}.$$

(3) Let us set $2x^3 + 1 = u$; then $6x^2\,dx = du$, $x^2\,dx = (1/6)\,du$. Computing the new limits of integration, we have: $u_l = 2\cdot 0^3 + 1 = 1$, $u_u = 2\cdot 1^3 + 1 = 3$. Thus,

$$\int_0^1 (2x^3+1)^4 x^2\,dx = \frac{1}{6}\int_1^3 u^4\,du = \frac{1}{6}\cdot\frac{u^5}{5}\Big|_1^3 = \frac{1}{30}(3^5-1^5) = 8\frac{1}{15}.$$

(4) We transform the radicand: $4 - 9x^2 = 4\,[1 - (3x/2)^2]$. Set $3x/2 = u$, whence $dx = (2/3)\,du$. Find the new limits of integration $u_l = (3/2)\cdot(\sqrt{2}/3) = \sqrt{2}/2$, $u_u = (3/2)\cdot(\sqrt{3}/3) = \sqrt{3}/2$. Consequently,

$$\int_{\sqrt{2}/3}^{\sqrt{3}/3} \frac{dx}{\sqrt{4-9x^2}} = \frac{2}{3}\cdot\frac{1}{2}\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\arcsin u\,\Big|_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \frac{1}{3}\left(\arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{\sqrt{2}}{2}\right) = \frac{1}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{36}.\quad ●$$

In Problems 18 through 29 evaluate with the aid of substitutions the indicated definite integrals:

18. (1) $\int_4^5 (4-x)^3\,dx$; (2) $\int_0^1 \frac{dx}{(3x+1)^4}$.

19. (1) $\int_0^3 \sqrt[3]{3x-1}\,dx$; (2) $\int_1^5 \sqrt{(2x-1)^3}\,dx$;

(3) $\int_{-2}^5 \frac{dx}{\sqrt[3]{(x+3)^2}}$.

20. (1) $\int_{-1}^2 (x^2-1)^3\,x\,dx$; (2) $\int_0^2 \frac{4x\,dx}{(x^2-1)^3}$; (3) $\int_0^2 9\sqrt{x^3+1}\,x^2\,dx$;

(4) $\int_{\sqrt{3}}^2 \frac{2}{3}\sqrt[3]{(x^4-8)^2}\,x^3\,dx$;

(5) $\int_{\sqrt{5}}^{2\sqrt{2}} \frac{x\,dx}{\sqrt{3x^2+1}}$; (6) $\int_1^{\sqrt{3}} \frac{32x\,dx}{(x^2+1)^5}$.

21. (1) $\int_0^{\pi/2} \sqrt{3\sin x+1}\cos x\,dx$; (2) $\int_{3\pi/2}^{2\pi} \sqrt{1-\cos x}\sin x\,dx$.

22. (1) $\int_0^{\pi/3} \frac{\sin x\,dx}{3-\cos x}$; (2) $\int_0^{\pi/2} \frac{\cos x\,dx}{2+\sin x}$; (3) $\int_{\pi/2}^{\pi/3} \frac{\sin t\,dt}{1+\cos t}$.

23. (1) $\int_0^{\pi/6} e^{\sin x}\cos x\,dx$; (2) $\int_0^1 e^{x^2}\,x\,dx$;

(3) $\int_0^{\sqrt[3]{2}} 3e^{x^3}\,x^2\,dx$.

24. (1) $\int_{\pi/12}^{\pi/8} \sin 2x\,dx$; (2) $\int_0^{\pi} \sin\frac{x}{3}\,dx$; (3) $\int_{2\pi/9}^{\pi/3} \sin\left(3x-\frac{\pi}{2}\right)dx$.

25. (1) $\int_{\pi/6}^{\pi/4} \cos\left(2x-\frac{\pi}{6}\right)dx$; (2) $\int_{\pi/18}^{\pi/12} \cos 3x\,dx$;

(3) $\int_{2\pi/3}^{4\pi/3} \cos\frac{x}{4}\,dx$.

26. (1) $\int_{\pi/8}^{\pi/6} \frac{dx}{\cos^2 2x}$; (2) $\int_0^{\pi/12} \frac{dx}{\cos^2 3x}$; (3) $\int_{\pi/2}^{\pi} \frac{dx}{\cos^2(x/3)}$.

27. (1) $\int_{\pi/18}^{\pi/9} \frac{dx}{\sin^2 3x}$; (2) $\int_0^{\pi/12} \frac{dx}{\sin^2(\pi/6+x)}$; (3) $\int_{\pi/2}^{\pi} \frac{dx}{\sin^2(x/3)}$.

28. (1) $\int\limits_{3/2}^{3\sqrt{2}/2} \frac{\sqrt{2}\,dx}{\sqrt{9-2x^2}}$; (2) $\int\limits_{\sqrt{3}/3}^{2\sqrt{3}/3} \frac{dx}{\sqrt{4-3x^2}}$;

(3) $\int\limits_{\sqrt{3}/2}^{3\sqrt{2}/4} \frac{dx}{\sqrt{3-2x^2}}$.

29. (1) $\int\limits_{3/4}^{3\sqrt{3}/4} \frac{4\,dx}{9+16x^2}$; (2) $\int\limits_{0}^{\sqrt{6}/6} \frac{dx}{1+2x^2}$; (3) $\int\limits_{1/2}^{3/2} \frac{dx}{3+4x^2}$.

Sec. 12.3. Integration by Parts in the Definite Integral

If the functions $u(x)$ and $v(x)$ and their derivatives $u'(x)$ and $v'(x)$ are continuous in the interval $[a, b]$, then the formula of integration by parts for the definite integral has the form

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

30. Compute $\int\limits_e^4 x \ln x\,dx$.

○ Let us set $u = \ln x$, $dv = x\,dx$; then $du = \frac{dx}{x}$, $v = \frac{x^2}{2}$. Consequently,

$$\int_e^4 x \ln x\,dx = \frac{x^2}{2}\ln x\Big|_e^4 - \frac{1}{2}\int_e^4 x^2\,\frac{dx}{x}$$

$$= 8\ln 4 - \frac{e^2}{2} - \frac{1}{4}x^2\Big|_e^4 = 8\ln 4 - \frac{e^2}{2} - 4 + \frac{e^2}{4} = 8\ln 4 - 4 - \frac{e^2}{4}.$$ ●

In Problems 31 and 32 evaluate the indicated definite integrals, applying the formula of integration by parts:

31. (1) $\int\limits_0^1 \arcsin x\,dx$; (2) $\int\limits_0^{\pi/2} x\cos x\,dx$; (3) $\int\limits_0^1 x\arctan x\,dx$.

32. (1) $\int\limits_1^e \ln^2 x\,dx$; (2) $\int\limits_0^1 xe^{-x}\,dx$.

Sec. 12.4. Approximate Evaluation of Definite Integrals

(1) Rectangle Formulas:

$$\int_a^b y\,dx \approx \frac{b-a}{n}(y_0+y_1+y_2+\ldots+y_{n-1}) \text{ (with deficit);}$$

$$\int_a^b y\,dx \approx \frac{b-a}{n}(y_1+y_2+y_3+\ldots+y_n) \text{ (with excess).}$$

(2) Trapezoid Formula:

$$\int_a^b y\,dx \approx \frac{b-a}{n}\left(\frac{y_0+y_n}{2}+y_1+y_2+\ldots+y_{n-1}\right).$$

(3) Formula of Parabolic Trapezoids (Simpson's Formula):

$$\int_a^b y\,dx \approx \frac{b-a}{6n}[y_0+y_{2n}+4(y_1+y_3+\ldots+y_{2n-1}) + 2(y_2+y_4+\ldots+y_{2n-2})].$$

33. Using Simpson's formula, compute $\int_0^1 \frac{dx}{1+x^2}$, taking $n=2$.

○ We have

$$\int_0^1 \frac{dx}{1+x^2}=\frac{1-0}{6\cdot 2}[y_0+y_4+4(y_1+y_3)+2y_2].$$

Since $y=f(x)=1/(1+x^2)$, $y_0=f(0)=1$, $y_1=f(1/4)=16/17$, $y_2=f(1/2)=4/5$, $y_3=f(3/4)=16/25$, $y_4=f(1)=1/2$, we have

$$\int_0^1 \frac{dx}{1+x^2}\approx\frac{1}{12}\left[1+\frac{1}{2}+4\left(\frac{16}{17}+\frac{16}{25}\right)+2\cdot\frac{4}{5}\right]=0.78539.$$

The exact value of the integral is $\pi/4=0.78540$; the relative error $\varepsilon=0.00127\%$. ●

34. Evaluate approximately the following definite integrals:

(1) $\int_1^2 \frac{dx}{x}$ by the rectangle formula ($n=10$);

(2) $\int_1^2 \frac{dx}{x}$ by the trapezoid formula ($n=10$);

(3) $\int_0^{2\pi} x \sin x \, dx$ by the rectangle formula $(n = 12)$;

(4) $\int_0^{\pi/4} \frac{\sin x}{x} \, dx$ by the trapezoid formula $(n = 5)$;

(5) $\int_0^{\pi/2} \frac{\sin x}{x} \, dx$ by Simpson's formula $(2n = 10)$.

Chapter 13
APPLICATIONS OF THE DEFINITE INTEGRAL

Sec. 13.1. Applying the Definite Integral to Computing Various Quantities. The Area of a Plane Figure

1. Applying the Definite Integral to Computing Various Quantities. The definite integral is widely used for computing various geometrical and physical quantities. A certain quantity u corresponding to the interval $[a, b]$ of change of the independent variable x is computed according to the following scheme:

1. Let the quantity u receive the increment $\Delta u \approx f(x)\,\Delta x$ corresponding to the change of x by a small quantity Δx; $f(x)$ is regarded as a function of x, given or determined from the conditions of the problem (Fig. 67).

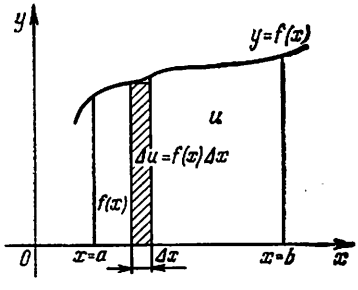


Fig. 67

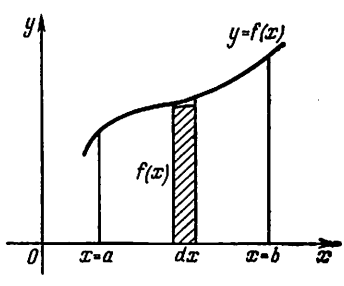


Fig. 68

2. Replacing the increment Δu by the differential du (the principal part of the increment Δu) and Δx by the differential dx ($\Delta x = dx$), we get

$$du = f(x)\,dx.$$

3. Integrating this equality in the limits from $x = a$ to $x = b$ we find

$$u = \int_a^b f(x)\,dx.$$

2. Computing the Area of a Plane Figure. Let us find the area S of a curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis, and two straight lines $x = a$ and $x = b$, where $a \leqslant x \leqslant b$, $f(x) \geqslant 0$ (Fig. 68).

Since the differential of a variable area is the area of the rectangle with base dx and altitude $f(x)$, that is $dS = f(x)\,dx$, integrating this equality in the limits from a to b, we get

$$S = \int_a^b f(x)\,dx. \tag{13.1}$$

If a curvilinear trapezoid adjoins to the y-axis so that $c \leqslant y \leqslant d$, $x = \varphi(y) \geqslant 0$ (Fig. 69), then the differential of the variable area S is equal to $dS = f(y)\,dy$, whence

$$S = \int_c^d \varphi(y)\,dy. \tag{13.2}$$

If the curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis, and the straight lines $x = a$ and $x = b$ lies below the x-axis (Fig. 70), then its area is found by the formula

$$S = \int_a^b |f(x)|\,dx. \tag{13.3}$$

If the figure bounded by the curve $y = f(x)$, the x-axis, and the straight lines $x = a$ and $x = b$ is situated on both sides of the x-axis (Fig. 71), then

$$S = \int_a^c f(x)\,dx + \int_c^b |f(x)|\,dx. \tag{13.4}$$

Let, finally, the figure S be bounded by two intersecting curves $y = f_1(x)$ and $y = f_2(x)$ and the straight lines $x = a$ and $x = b$, where $a \leqslant x \leqslant b$ and

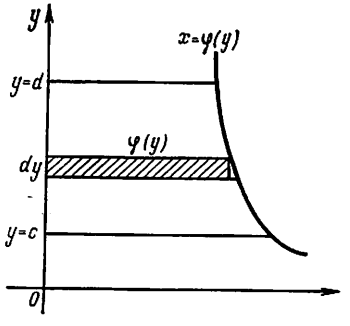


Fig. 69

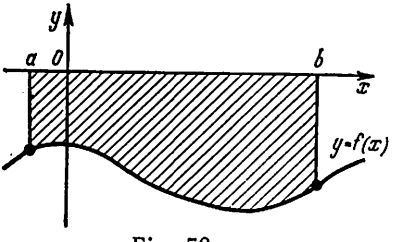


Fig. 70

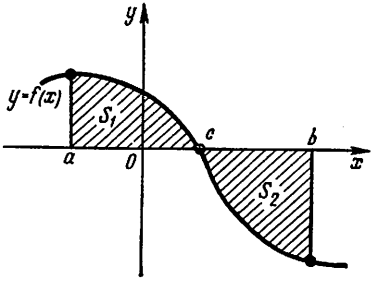


Fig. 71

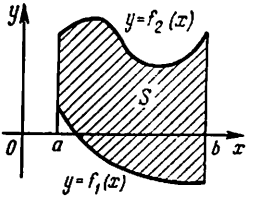


Fig. 72

$f_1(x) \leqslant f_2(x)$ (Fig. 72). Then its area is found by the formula

$$S = \int_a^b [f_2(x) - f_1(x)]\,dx. \tag{13.5}$$

In Problems 1 through 11 we shall compute the areas of the figures bounded by the indicated lines:

1. $x + 2y - 4 = 0$, $y = 0$, $x = -3$, and $x = 2$.

○ Let us first construct the figure. We construct the straight line $x + 2y - 4 = 0$ using the two points: $A\ (4, 0)$ and $B\ (0, 2)$ (Fig. 73). Expressing y in terms of x, we get $y = -0.5x + 2$. By Formula (13.1), where $f(x) = -0.5x + 2$, $a = -3$, $b = 2$, we find

$$S = \int_{-3}^{2} (-0.5x + 2)\,dx = [-0.25x^2 + 2x]_{-3}^{2} = 11.25 \text{ (square units)}.$$

To check the obtained result, let us compute the area of the trapezoid M_1MNN_1 in the usual way. We find: $|M_1M| = f(-3) =$

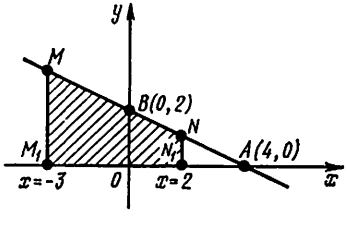


Fig. 73

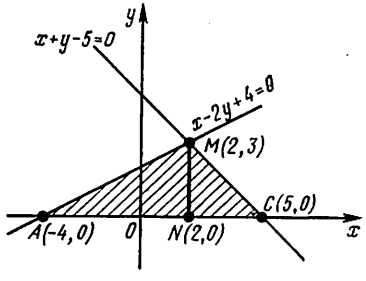


Fig. 74

$-0.5\,(-3) + 2 = 3.5$, $|N_1N| = f(2) = -0.5 \cdot 2 + 2 = 1$, $|M_1N_1| = 5$. Consequently, $S = 0.5\,(3.5 + 1) \cdot 5 = 11.25$ (square units). ●

2. $x - 2y + 4 = 0$, $x + y - 5 = 0$, and $y = 0$.

○ Let us construct the figure (Fig. 74). Construct the straight line $x - 2y + 4 = 0$: $y = 0$, $x = -4$, $A\ (-4,\ 0)$; $x = 0$, $y = 2$, $B\ (0,\ 2)$. Construct the straight line $x + y - 5 = 0$: $y = 0$, $x = 5$, $C\ (5,\ 0)$; $x = 0$, $y = 5$, $D\ (0,\ 5)$.

Solving the system of equations, we find the point of intersection of the straight lines:

$$\begin{cases} x - 2y + 4 = 0, \\ x + y - 5 = 0, \end{cases} \quad x = 2,\ y = 3,\ M\ (2,\ 3).$$

To compute the desired area, let us partition the triangle AMC into two triangles AMN and NMC, since as x changes from A to N the area is bounded by the straight line $x - 2y + 4 = 0$ and when x varies from N to C, by the straight line $x + y - 5 = 0$.

For the triangle AMN we have: $x - 2y + 4 = 0$; $y = 0.5x + 2$, that is, $f(x) = 0.5x + 2$, $a = -4$ and $b = 2$. For the triangle NMC we have: $x + y - 5 = 0$; $y = -x + 5$, that is, $f(x) = -x + 5$, $a = 2$ and $b = 5$.

Computing the area of each of the triangles and adding together the obtained results, we find:

$$S_{\triangle AMN} = \int_{-4}^{2} (0.5x+2)\,dx = [0.25x^2+2x]_{-1}^{2} = 9 \text{ (square units)};$$

$$S_{\triangle NMC} = \int_{2}^{5} (-x+5)\,dx = [-0.5x^2+5x]_{2}^{5} = 4.5 \text{ (square units)};$$

$$S = S_{\triangle AMN} + S_{\triangle NMC} = 9+4.5 = 13.5 \text{ (square units)}.$$

Check: $S_{\triangle AMC} = 0.5\,|AC| \cdot |NM| = 0.5 \cdot 9 \cdot 3 = 13.5$ (square units). ●

3. $y = x^2$, $y = 0$, $x = 2$, and $x = 3$.

○ Here, it is required to compute the area of the curvilinear trapezoid bounded by the parabola $y = x^2$, straight lines $x = 2$ and $x = 3$, and the x-axis (Fig. 75). By Formula (13.1), we find

$$S = \int_{2}^{3} x^2\,dx = \frac{x^3}{3}\Big|_{2}^{3} = 6\frac{1}{3} \text{ (square units)}.$$

4. $y = -x^2 + 4$ and $y = 0$.

○ We first construct the figure (Fig. 76). The desired area is enclosed between the parabola $y = -x^2 + 4$ and the x-axis.

Now, we find the points at which the parabola cuts the x-axis. Setting $y = 0$, we find $x = \pm 2$. Since the given figure is symmetric with respect to the y-axis, we shall compute the area of the figure situated on the right of the y-axis and then double the result thus obtained:

$$S_1 = \int_{0}^{2} (-x^2+4)\,dx = \left[-\frac{x^3}{3} + 4x\right]_{0}^{2} = 5\frac{1}{3} \text{ (square units)};$$

$$S = 2S_1 = 2 \cdot 5\frac{1}{3} = 10\frac{2}{3} \text{ (square units)}. ●$$

5. $y^2 = x$, $y \geqslant 0$, $x = 1$, and $x = 4$.

○ In this problem, it is required to compute the area of the curvilinear trapezoid bounded by the upper branch of the parabola $y^2 = x$, the x-axis, and the straight lines $x = 1$ and $x = 4$ (Fig. 77). By Formula (13.1), where $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, we find

$$S = \int_{1}^{4} \sqrt{x}\,dx = \int_{1}^{4} x^{1/2}\,dx = \frac{2x^{3/2}}{3}\Big|_{1}^{4} = \frac{2}{3}(4^{3/2} - 1^{3/2})$$

$$= \frac{2}{3}(8-1) = 4\frac{2}{3} \text{ (square units)}. ●$$

6. $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

○ The desired area is bounded by half-wave of the sine curve and the x-axis (Fig. 78). We have

$$S=\int_0^{\pi}\sin x\,dx=-\cos x\Big|_0^{\pi}=-\cos\pi+\cos 0$$

$$=1+1=2 \text{ (square units).} \bullet$$

7. $y=-6x$, $y=0$, and $x=4$.

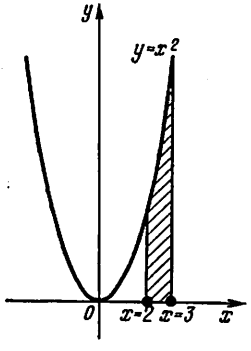


Fig. 75

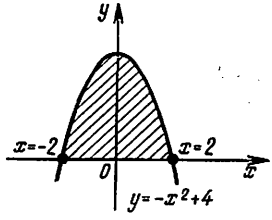


Fig. 76

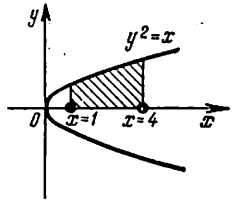


Fig. 77

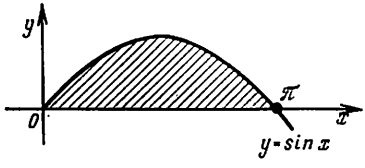


Fig. 78

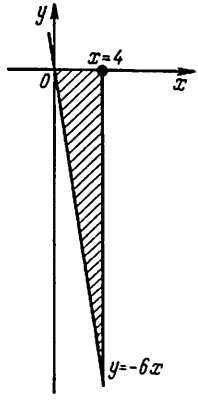


Fig. 79

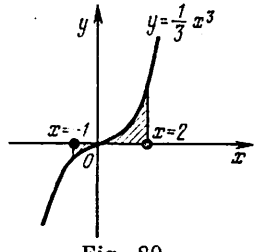


Fig. 80

○ The given figure is situated below the x-axis (Fig. 79). Consquently, its area is found by Formula (13.3):

$$S=\left|-\int_0^4 6x\,dx\right|=|[-3x^2]_0^4|=|-48|=48 \text{ (square units).} \bullet$$

8. $y=(1/3)x^3$, $y=0$, $x=-1$, and $x=2$.

○ We plot the curve $y=(1/3)x^3$ from the given points (Fig. 80). The figure bounded by the given lines is situated on both

sides of the x-axis. Thus, the area of the figure is found by formula (13.4):

$$S=\left|\int_{-1}^{0}\frac{1}{3}x^3\,dx\right|+\int_{0}^{2}\frac{1}{3}x^3\,dx=\left|\left[\frac{x^4}{12}\right]_{-1}^{0}\right|+\left[\frac{x^4}{12}\right]_{0}^{2}$$

$$=\left|-\frac{1}{12}\right|+\frac{16}{12}=1\frac{5}{12}\ \text{(square units)}.\ \bullet$$

9. $x^2+y^2=r^2$.

○ Here, it is required to compute the area bounded by the circumference $x^2+y^2=r^2$, that is, the area of a circle of radius r

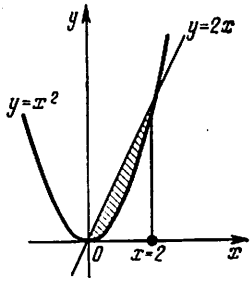


Fig. 81

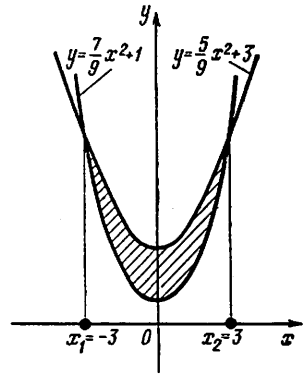


Fig. 82

with centre at the origin. We shall first find one-fourth of this area, taking the limits of integration from 0 to r:

$$S_1=\int_0^r\sqrt{r^2-x^2}\,dx.$$

An integral of this kind is given in Problem 80 (2) of Chapter 11:

$$S_1=\left[\frac{r^2}{2}\arcsin\frac{x}{r}+\frac{x}{2}\sqrt{r^2-x^2}\right]_0^r=\frac{r^2}{2}\arcsin 1=\frac{r^2}{2}\cdot\frac{\pi}{2}=\frac{\pi r^2}{4}.$$

Consequently, $S=4S_1=\pi r^2$. ●

10. $y=x^2$ and $y=2x$.

○ The given figure is bounded by the parabola $y=x^2$ and straight line $y=2x$ (Fig. 81). To determine the points of intersection of the given lines, we have to solve the system of equations

$$\begin{cases} y=x^2, \\ y=2x, \end{cases}$$

whence we find $x^2 - 2x = 0 \Leftrightarrow x(x-2) = 0 \Leftrightarrow \left[\begin{array}{l} x = 0 \\ x = 2 \end{array}\right.$. Using Formula (13.5) for finding the desired area, we obtain

$$S = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ (square units).} \bullet$$

11. $7x^2 - 9y + 9 = 0$ and $5x^2 - 9y + 27 = 0$.

○ Let us rewrite the equations of the parabolas in the form $y = (7/9)x^2 + 1$ and $y = (5/9)x^2 + 3$, and construct these parabolas (Fig. 82). To find the points of their intersection, we solve the system

$$\left\{\begin{array}{l} y = (7/9)x^2 + 1, \\ y = (5/9)x^2 + 3, \end{array}\right.$$

whence $x_1 = -3$, $x_2 = 3$. Since the figure is symmetric with respect to the y-axis, we shall find one-half of its area, taking the limits of integration from 0 to 3, and then double the obtained result:

$$S_1 = \int_0^3 \left[\left(\frac{5}{9}x^2 + 3\right) - \left(\frac{7}{9}x^2 + 1\right)\right] dx$$

$$= \int_0^3 \left(2 - \frac{2}{9}x^2\right) dx = 2\left[x - \frac{x^3}{27}\right]_0^3 = 2(3-1) = 4 \text{ (square units);}$$

$$S = 2S_1 = 8 \text{ (square units).} \bullet$$

In Problems 12 through 30 compute the areas of the figures bounded by the indicated lines:

12. (1) $x - y + 2 = 0$, $y = 0$, $x = -1$, and $x = 2$; (2) $2x - 3y + 6 = 0$, $y = 0$, and $x = 3$.

13. (1) $x - y + 3 = 0$, $x + y - 1 = 0$, and $y = 0$; (2) $x - 2y + 4 = 0$, $x + 2y - 8 = 0$, $y = 0$, $x = -1$, and $x = 6$.

14. (1) $y = x^2$, $y = 0$, $x = 0$, and $x = 3$; (2) $y = 3x^2$, $y = 0$, $x = -3$, and $x = 2$.

15. (1) $y = x^2 + 1$, $y = 0$, $x = -1$, and $x = 2$; (2) $y = 0.5x^2 + 2$, $y = 0$, $x = 1$, and $x = 3$; (3) $y = -(1/3)x^2 + 3$, $y = 0$, $x = 0$, and $x = 3$.

16. $y^2 = x$, $y \geqslant 0$, $x = 0$, and $x = 3$.

17. (1) $y = -x^2 - 2x + 8$, $y = 0$; (2) $y = -(2/9)x^2 + (4/3)x$, $y = 0$; (3) $y = -x^2 + 6x - 5$, $y = 0$, $x = 2$, and $x = 3$.

18. (1) $y = 1/x$, $y = 0$, $x = 1$, and $x = 3$; (2) $y = 2/x$, $y = 0$, $x = 2$, and $x = 4$.

19. (1) $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$; (2) $y = \tan x$, $y = 0$, $x = 0$, and $x = \pi/3$; (3) $y = \tan x$, $y = \tan x$, $y = 0$, $x = \pi/6$, and $x = \pi/3$.

20. (1) $y = -3x$, $y = 0$, and $x = 2$; (2) $y = 2x$, $y = 0$, and $x = -3$; (3) $x - 2y - 6 = 0$, $y = 0$; (4) $x - 2y - 5 = 0$, $y = -2x$, and $y = 0$.

21. (1) $y = -3x^2$, $y = 0$, $x = 1$, and $x = 2$; (2) $y = -x^2 - 1$, $y = 0$, $x = -2$, and $x = 1$; (3) $y = x^2 - 4$, and $y = 0$.

22. (1) $y = x^3$, $y = 0$, $x = -2$ and $x = 2$; (2) $y = 4x^3$, $y = 0$, $x = -1$, and $x = 2$; (3) $y = x^3 - x$, $y = 0$, $x = -1$, and $x = 1$; (4) $y = x^3 - 4x$, $y = 0$, $x = -2$, and $x = 2$.

23. (1) $y^2 = 4x$, $x = 1$, and $x = 9$; (2) $y^2 = 9x$, and $x = 4$.

24. (1) $x^2 + y^2 = 9$; (2) $x^2/a^2 + y^2/b^2 = 1$; (3) $x^2/16 + y^2/9 = 1$.

25. (1) $y = \sin x$, $y = 0$, $x = -\pi/2$, and $x = \pi$; (2) $y = \sin x$, $y = 0$, $x = 0$, and $x = 2\pi$.

26. (1) $y = x^2$ and $y = -3x$; (2) $y = x^2$ and $y = 2x + 8$; (3) $y = x^2$ and $y = x + 2$; (4) $y = x^2 + 2$ and $y = 6$.

27. (1) $y = 0.5x^2 - 4x + 10$ and $y = x + 2$; (2) $y = x^2 - 2x + 3$ and $y = 3x - 1$; (3) $y = (1/3)x^2 - 2x + 4$ and $y = -x + 10$; (4) $y = 0.5x^2 + 2x + 4$ and $y = x + 8$.

28. (1) $y = 2x^2 + 1$ and $y = x^2 + 10$; (2) $y = -1.5x^2 + 9x - 7.5$ and $y = -x^2 + 6x - 5$.

29. (1) $y = x^2$ and $y = 2 - x^2$; (2) $y = x^2$ and $x = y^2$.

30. (1) $x^2 = 3y$ and $y = x$; (2) $y = x^2 - 6x + 9$ and $3x - y - 9 = 0$.

QUIZ No. 1

Form I

Evaluate the following integrals

(1) $\int_0^2 \frac{4x\,dx}{\sqrt{1+2x^2}}$;

(2) $\int_{\pi/3}^{\pi/2} \frac{3\,dx}{2\cos^2(x/2)}$;

(3) $\int_{\sqrt{3}}^{3} \frac{dx}{3+x^2}$.

Find the areas of the figures bounded by the indicated lines:

(4) $y = -x^2 + x + 6$ and $y = 0$;

(5) $y = x^2 - 8x + 18$, $y = -2x + 18$.

Form II

Evaluate the following integrals

(1) $\int_0^8 (\sqrt{2x} + \sqrt[3]{x})\,dx$;

(2) $\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{2\sin x + 1}}$;

(3) $\int_{1/3}^{\sqrt{3}/3} \frac{dx}{\sqrt{4-9x^2}}$.

Find the areas of the figures bounded by the indicated lines:

(4) $y = -x^2 + 2x + 3$ and $y = 0$;

(5) $y = -x^2 + 10x - 16$, $y = x + 2$.

Sec. 13.2. Computing the Path Covered by a Point

The path covered by a point moving nonuniformly in a straight line with variable velocity $v = f(t) \geqslant 0$ during the time interval from t_1 to t_2 is computed by the formula

$$s = \int_{t_1}^{t_2} f(t)\, dt. \tag{13.6}$$

31. The velocity of a moving point changes according to the equation $v = (3t^2 + 2t + 1)$ m/s. Find the path covered by the point during 10 seconds from the start.

○ According to the hypothesis, $f(t) = 3t^2 + 2t + 1$, $t_1 = 0$, $t_2 = 10$. By Formula (13.6), we find

$$s = \int_0^{10} (3t^2 + 2t + 1)\, dt = [t^3 + t^2 + t]_0^{10} = 10^3 + 10^2 + 10 = 1110 \text{ (m)}. \bullet$$

32. A point moves with velocity $v = (9t^2 - 8t)$ m/s. Find the path covered by the point during the fourth second.

○ According to the hypothesis, $f(t) = 9t^2 - 8t$, $t_1 = 3$, $t_2 = 4$. Consequently,

$$s = \int_3^4 (9t^2 - 8t)\, dt = [3t^3 - 4t^2]_3^4 = 83 \text{ (m)}. \bullet$$

33. The velocity of a moving point is $v = (12t - 3t^2)$ m/s. Find the path covered by the point from the beginning of motion to the instant of stop.

○ The velocity of the point is equal to zero both at the instant of the beginning of motion and at the instant it ceases moving. Let us determine the instant at which the point will stop. To this end, we solve the equation $12t - 3t^2 = 0$, whence $t(4 - t) = 0$, $t_1 = 0$, $t_2 = 4$. Now, by Formula (13.6), we find

$$s = \int_0^4 (12t - 3t^2)\, dt = [6t^2 - t^3]_0^4 = 32 \text{ (m)}. \bullet$$

34. Two bodies started moving at the same time from the same point in the same direction along a straight line. The first body moves with velocity $v = (6t^2 + 2t)$ m/s, the second with velocity $v = (4t + 5)$ m/s. What will be the distance between the bodies in 5 seconds?

○ It is obvious that the desired quantity is the difference of the paths covered by the first and second bodies during 5 seconds:

$$s_1 = \int_0^5 (6t^2 + 2t)\, dt = [2t^3 + t^2]_0^5 = 275 \text{ (m)},$$

$$s_2 = \int_0^5 (4t+5)\,dt = [2t^2+5t]_0^5 = 75 \text{ (m)},$$

$$s_1 - s_2 = 275 - 75 = 200 \text{ (m)}. \bullet$$

35. Two bodies move from the same point along a straight line. The first body moves with velocity $v = (3t^2 - 6t)$ m/s, the second with velocity $v = (10t + 20)$ m/s. At what instant and at what distance from the initial point will they meet?

○ According to the hypothesis, the bodies start moving from the same point, therefore they will cover equal distances before they meet. So, we find the motion equation for each of the bodies:

$$s_1 = \int (3t^2-6t)\,dt = t^3-3t^2; \quad s_2 = \int (10t+20)\,dt = 5t^2+20t.$$

The constants of integration for the initial conditions $t = 0$, $s = 0$ are equal to zero. The bodies will meet provided that $s_1 = s_2$, whence $t^3 - 3t^2 = 5t^2 + 20t$ or $t^3 - 8t^2 - 20t = 0$. Let us solve this equation:

$$t(t^2 - 8t - 20) = 0,$$

that is, $t_1 = 0$, $t_2 = -2$, $t_3 = 10$. Thus, the bodies will meet at the instant $t = 10$ s.

Substituting the value $t = 10$ into the equality determining the motion equation of any of the two bodies (say, the first one), we find the path covered by each body before they meet: $s_1 = s_2 = 10^3 - 3 \cdot 10^2 = 700$ (m). ●

36. A body is thrown from Earth's surface vertically upward with velocity $v = (39.2 - 9.8t)$ m/s. Find the greatest height reached by the body.

○ The body will reach the greatest height at the instant of time t, that is, $39.2 - 9.8t = 0$, whence $t = 4$ s. By formula (13.6) we find

$$s = \int_0^4 (39.2-9.8t)\,dt = [39.2t-4.9t^2]_0^4 = 78.4 \text{ (m)}. \bullet$$

37. A point moves with velocity $v = (6t^2 + 4)$ m/s. Find the path covered by the point during five seconds from the beginning of motion.

38. The velocity of a moving point $v = (2t + 8t^{-2})$ m/s. Find the path covered by the point during the 2nd second.

39. The velocity of a moving point $v = (18t - 3t^2)$ m/s. Find the path covered by the point during the time interval from the start to the instant when the point ceased moving.

40. The velocity of a moving point $v = (24t - 6t^2)$ m/s. Find: (1) the path covered by the point during three seconds from the beginning of motion; (2) the path covered by the point during the third second; (3) the path covered by the point during the time interval from the start to the instant the body ceased moving.

41. Two bodies started moving simultaneously from the same point in the same direction along a straight line. The first body moves with velocity $v = 3t^2$ m/s, the second with $v = (6t^2 + 10)$ m/s. What will be the distance between them in ten seconds?

42. Two bodies started moving from the same point along a straight line. The first body moves with velocity $v = (3t^2 + 4t)$ m/s, the second with velocity $v = (6t + 12)$ m/s. At what instant and at what distance from the initial point will they meet?

43. A body is thrown from Earth's surface vertically upward with velocity $v = (29.4 - 9.8t)$ m/s. Find the greatest height reached by the body.

Sec. 13.3. Computing the Work Done by a Force

The work done by a variable force $f(x)$ in displacing a material point along the x-axis from $x = a$ to $x = b$ is found by the formula

$$A = \int_a^b f(x)\,dx \tag{13.7}$$

When solving problems on computing the work done by a force, we frequently use Hooke's law:

$$F = kx \tag{13.8}$$

where F is the applied force (N), x the absolute elongation of a spring (m) caused by the force F, and k is a proportionality factor (N/m).

44. The compression x of a helical spring is proportional to the applied force F. Compute the work done by the force F in compressing the spring by 0.04 m if to compress it by 0.01 m a force of 10 N is required.

○ Since $x = 0.01$ m for $F = 10$ N, substituting these values into the equality (13.8), we get $10 = k \cdot 0.01$, whence $k = 1000$ N/m. Substituting now the value of k into the same equality, we find $F = 1000x$, that is, $f(x) = 1000x$. The desired work is found by formula (13.7) (we set here $a = 0$, $b = 0.04$):

$$A = \int_0^{0.04} 1000x\,dx = 500x^2 \Big|_0^{0.04} = 0.8 \text{ (J)}. \quad \bullet$$

45. A spring at rest has a length of 0.2 m. A force of 50 N stretches the spring by 0.01 m. What work must be done to stretch the spring from 0.22 m to 0.32 m?

○ Using the equality (13.8), we have $50 = 0.01\ k$, that is, $k = 5000$ N/m. We now find the limits of integration: $a = 0.22 - 0.2 = 0.02$ (m), $b = 0.32 - 0.2 = 0.12$ (m). Now, by Formula (13.7) we get

$$A = \int_{0.02}^{0.12} 5000x\,dx = 5000\,\frac{x^2}{2}\Big|_{0.02}^{0.12}$$

$$= 2500\,(0.0144 - 0.0004) = 2500 \cdot 0.014 = 35\ (\text{J}). \bullet$$

46. The work done in compressing a spring by 0.05 m is equal to 25 J. What work must be done to compress the spring by 0.1 m?

○ Knowing the magnitude of compression of the spring (0.05 m) and the work done (25 J), we take advantage of formula (13.7):

$$25 = \int_0^{0.05} kx\,dx = k\,\frac{x^2}{2}\Big|_0^{0.05} = 0.00125\ k,$$

whence $k = 25/0.00125 = 20{,}000$ (N/m). By the same formula, we now find:

$$A = \int_0^{0.1} 20{,}000x\,dx = 20{,}000\,\frac{x^2}{2}\Big|_0^{0.1} = 20{,}000 \cdot \frac{0.1}{2} = 100\ (\text{J}). \bullet$$

47. To stretch a spring by 0.04 m, it is necessary to perform the work equal to 20 J. By what length is it possible to stretch the spring performing the work equal to 80 J?

○ Since we know the elongation of the spring (0.04 m) and the work done (20 J), using Formula (13.7), we have

$$20 = \int_0^{0.04} kx\,dx = k\,\frac{x^2}{2}\Big|_0^{0.04} = 0.0008k,$$

whence $k = 20/0.0008 = 25{,}000$ (N/m).

Let x_1 be the elongation of the spring corresponding to the performed work of 80 J. Then

$$80 = \int_0^{x_1} 25{,}000x\,dx = 25{,}000\,\frac{x^2}{2}\Big|_0^{x_1} = 12{,}500x_1^2,$$

whence $x_1^2 = 80/12{,}500 = 16/2500$; $x_1 = 4/50 = 0.08$ m. ●

48. A cylinder fitted with a movable piston, the area of whose cross-section is S square units, is filled with gas. Assuming that the gas contained in the cylinder, when gaining its volume, obeys Boyle's (Marriotte's) law $pV = k$ = constant, compute the work done by the force of gas pressure when its volume increases from V_0 to V_1 (gas temperature is maintained at a constant level).

○ Let x (m) be the distance covered by the piston (Fig. 83). Suppose that when x is changed by a small quantity dx, the pressure exerted by the piston remains unchanged; in this case, the volume V changes by the quantity ΔV. Then the work ΔA done by the force of gas pressure when passing the small distance dx is expressed by the approximate equality $\Delta A \approx pS\,dx$. Since $p = k/V$ and $S\,dx = \Delta V$, we have

$$\Delta A \approx \frac{k}{V}\Delta V = k\frac{\Delta V}{V}.$$

Replacing the increments ΔV and ΔA by the differentials dV and dA, we get

$$dA = k\frac{dV}{V}.$$

Integrating this equality between the limits from V_0 to V_1, we find

$$A = k \ln (V_1/V_0). \bullet$$

49. A spring is stretched by 0.02 m under the action of the force equal to 60 N. What work will be done by this force in stretching the spring by 0.12 m?

50. Under the action of the force equal to 80 N a spring is elongated by 0.02 m. The initial length of the spring is equal to 0.15 m. What work must be done to stretch the spring by 0.2 m?

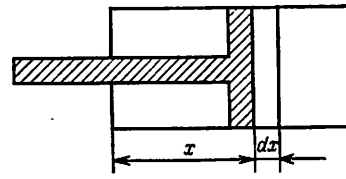


Fig. 83

51. The length of a spring at rest is 0.1 m. A force equal to 20 N stretches it by 0.01 m. What work must be done to stretch the spring from 0.12 to 0.14 m?

52. In compressing a spring by 0.05 m, the work equal to 30 J is done. What work is it necessary to do to compress the spring by 0.08 m?

53. To compress a spring by 0.02 m, it is necessary to perform the work equal to 16 J. By what length is it possible to compress the spring performing the work equal to 100 J?

54. A cylindrical vessel of volume $V_0 = 0.2$ m^3 is filled with free air at a normal pressure $P_0 = 101.325$ N/m^2. Then the air is compressed by a piston to a volume equal to 0.05 m^3. What work is done in the process of compression if the air temperature remains unchanged?

Sec. 13.4. Computing the Work Done in Hoisting a Load

55. A cylindrical tank with radius of base equal to 0.5 m and height equal to 2 m is filled with water. Compute the work which is necessary to be done in order to pump the water out of the tank.

○ At depth x, we single out a horizontal layer whose height is dx (Fig. 84). The work A to be done in order to raise the water layer of weight P to the height x is equal to Px.

A change in the depth x by a small quantity dx will cause a change in the volume V by a quantity $dV = \pi r^2\,dx$ and a change in the weight P by the quantity* $dP = 9807\,\pi r^2\,dx$; in this case the work A will change by the quantity $dA = 9807\,\pi r^2 x\,dx$. Integrating this equality when x varies from 0 to H, we get

$$A = \int_0^H 9807\pi r^2 x\,dx = 4903\pi r^2 H^2 = 4903\pi \cdot 0.25 \cdot 2^2 = 4903\pi \text{ (J)}. \quad \bullet$$

56. Compute the work to be done in order to pump the water out of the conic reservoir shown in Fig. 85. The reservoir is filled with

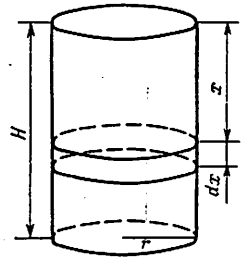


Fig. 84

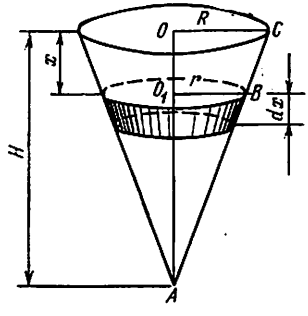


Fig. 85

water to the brim. The radius of the base $R = 1$ m, the altitude of the cone is equal to 2 m.

○ At depth x, we single out a horizontal layer whose height is dx (see Fig. 85). The work A done in raising the water layer of weight P depends on the height x to which it is raised. A change in the depth x by a small quantity dx will cause a change in the volume V by the quantity

$$\Delta V = \pi r^2\,dx \qquad (*)$$

(the elementary layer is taken for a cylinder in view of the fact that dx is small enough; r is the radius of the layer base). Let us express r in terms of the variable x and the constants R and H. From the similarity of the triangles AOC and AO_1B we have $r : R = (H - x) : H$, whence $r = \frac{R}{H}(H - x) = R - \frac{R}{H}x$. Substituting the value of r

* Since the density of water is equal to 1000 kg/m³, the weight of the water contained in the volume of 1 m³ amounts to $9.807 \cdot 1000 = 9807$ N. Therefore the weight dP of the water layer in the volume dV is equal to $9807\pi r^2\,dx$.

from the last equality into the expression (*), we get

$$\Delta V = \pi \left(R - \frac{R}{H} x\right)^2 dx.$$

The weight ΔP of the water layer in the volume ΔV (density of water is 1000 kg/m^3) is equal to

$$\Delta P = 9807\pi \left(R - \frac{R}{H} x\right)^2 dx.$$

With a change in P by the quantity ΔP the work A will change by the quantity

$$dA = 9807\pi \left(R - \frac{R}{H} x\right)^2 x\, dx. \qquad (**)$$

Integrating the equality (**) as x changes from 0 to H, we obtain

$$A = \int_0^H 9807\pi \left(R - \frac{R}{H} x\right)^2 x\, dx = \int_0^H 9807\pi R^2 \left(x - \frac{2x^2}{H} + \frac{x^3}{H^2}\right) dx$$

$$= 9807\pi R^2 \left[\frac{x^2}{2} - \frac{2x^3}{3H} + \frac{x^4}{4H^2}\right]_0^H = \frac{9807}{12} \pi R^2 H^2.$$

Substituting the numerical values of R and H, we find

$$A = 9807\pi \cdot 1^2 \cdot \frac{2^2}{12} = 3269\pi \text{ (J)}. \bullet$$

57. A rectangular reservoir whose base is a square 3 m on a side, and height is equal to 2 m is filled with water. Compute the work which must be done in pumping the water out of the reservoir.

58. A cylindrical vessel with radius of base equal to 2 m and height equal to 3 m is filled with water. Compute the work which is necessary to be done in order to pump the water out of the vessel.

59. Compute the work which is necessary to be done in order to pump the water out of the pit having the shape of a cone (with the vertex at the bottom) whose height $H = 1$ m, and radius of base $R = 2$ m.

60. Compute the work which must be done in pumping the water out of a boiler having the shape of a hemisphere of radius $R = 1$ m.

61. Compute the work done in pumping the water out of a brim-full boiler having the shape of a paraboloid of revolution (with its vertex downward). The depth of the boiler $H = 1$ m, the radius of the base $R = 2$ m.

Sec. 13.5. Computing the Pressure Force of a Fluid

The magnitude of the pressure force of a fluid exerted on a horizontal surface element depends on the depth of submergence x of this element, that is, on the distance of the element from the top surface of the fluid.

The pressure force (N) acting on a horizontal surface element is computed by the formula

$$P = 9807\delta S x,$$

where δ is fluid density, kg/m^3; S, the area of the surface element, m^2; x, the depth of submergence of the surface element, m.

If the surface element, acted upon by the fluid pressure, is not horizontal, then the pressure on it varies with depth, consequently, the pressure force exerted by the fluid on the surface element is a function of the depth of its submergence $P(x)$.

62. Compute the pressure force of water on a vertical rectangular lock whose base is 20 m and height is 5 m (the water level coincides with the upper edge of the lock).

○ At a depth x let us take a horizontal strip of width dx (Fig. 86). The pressure force P on the lock wall is a function of x. A change in

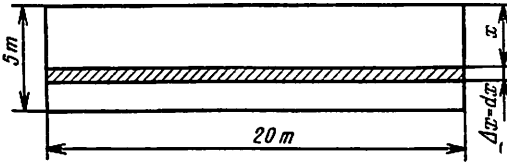


Fig. 86

depth by a small quantity dx will cause a change in the pressure force P by a small quantity ΔP. Differentiating the variable P, we get the approximate value (the principal part) dP of the increment ΔP.

We then find the approximate value of the pressure force on this strip: $\Delta P = 9.807\ \delta x\ \Delta S = 9807\ x \cdot 20\ \Delta x$. But $dP \approx \Delta P$. Integrating dP, as x changes from 0 to 5, we get

$$P = 9807 \cdot 20 \int_0^5 x\,dx = 9807 \cdot 10x^2 \Big|_0^5 = 2.45 \text{ (MN)}. \bullet$$

63. Compute the pressure force of water on a vertical dam having the shape of an isosceles trapezoid with bases a and b ($a > b$) and altitude h.

○ Let the hatched strip be at a depth x and have the dimensions y and dx (Fig. 87). The approximate value of the pressure force of water on this strip is $\Delta P \approx xy\,dx = dP$. Let us express the variable y in terms of x and the trapezoid dimensions a, b, and h. From the similarity of the triangles ADE and ANM we have $|DE| : |NM| = |AE| : |AM|$. Since $|DE| = (a - b)/2$, $|NM| = (y - b)/2$, $|AE| = h$ and $|AM| = h - x$, substituting these values into the proportion, we obtain

$$\frac{a-b}{y-b} = \frac{h}{h-x}, \text{ whence } y = a - \frac{a-b}{h}x.$$

Consequently,

$$dP = x\left(a - \frac{a-b}{h}\,x\right)dx.$$

Integrating dP as x changes from 0 to h, we find

$$P = \int_0^h \left(ax - \frac{a-b}{h}\,x^2\right)dx = \left[\frac{ax^2}{2} - \frac{a-b}{3h}\,x^3\right]_0^h = \frac{h^2(a+2b)}{6}. \bullet$$

64. A triangular plate with base equal to 0.2 m and height 0.4 m is submerged vertically into water so that its vertex lies on the water surface, the base being parallel to the latter. Compute the water pressure on the plate.

○ Let us separate a horizontal strip of width dx at a depth x (Fig. 88). A change in the depth x by a small quantity dx will cause

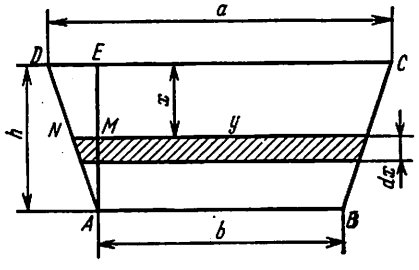


Fig. 87

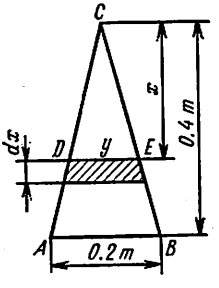


Fig. 88

a change in the pressure force P by a small quantity dP. The area of strip $\Delta S = y\,dx$. From the similarity of the triangles ABC and DEC we have $y : 0.2 = x : 0.4$, whence $y = 0.5x$. Consequently, $\Delta S = 0.5x\,dx$. The elementary pressure force (N) will amount to

$$dP = 9.807\,\delta x\,\Delta S = 9807\,x \cdot 0.5x\,dx = 4903.5x^2\,dx.$$

Integrating dP, as x changes from 0 to 0.4, we get

$$P = 4903.5\int_0^{0.4} x^2\,dx = 4903.5\,\frac{x^3}{3}\Big|_0^{0.4} = 1634.5 \cdot 0.4^3 \approx 104.6\ (\text{N}). \bullet$$

65. Compute the pressure force of water acting on a vertical rectangular wall whose base is equal to 2 m and height to 4 m. The water level coincides with the upper edge of the wall.

66. Compute the water pressure on a vertical wall having the shape of an isosceles trapezoid. The upper base of the trapezoid coinciding with the water level is equal to 4.5 m, the lower base to 3 m; the wall being 2 m high.

67. A triangular plate with base equal to 0.4 m and height to 0.6 m is dipped vertically into water so that its base is found on the water surface. Compute the water pressure on the plate.

68. A cylindrical glass is filled with oil. Compute the oil pressure on the lateral surface of the glass if its height $h = 0.08$ m and the radius of the base $r = 0.04$ m. The density of oil is 900 kg/m^3.

69. A cylindrical glass is filled with mercury. Compute the mercury pressure on the lateral surface of the glass if its height is 0.1 m and the radius of base is 0.04 m. The density of mercury is 13,600 kg/m^3.

70. Compute the water pressure on the bottom and walls of an aquarium the sides of whose base are 0.8 m and 0.5 m, its height being equal to 0.3 m. The aquarium is brim-full with water.

Sec. 13.6. Arc Length of a Plane Curve

Let the plane curve AB (Fig. 89) be given by the equation $y = f(x)$ $(a \leqslant x \leqslant b)$, $f(x)$ and $f'(x)$ being continuous functions in the interval $[a, b]$. Then the differential dl of the arc length AB is expressed by the formula

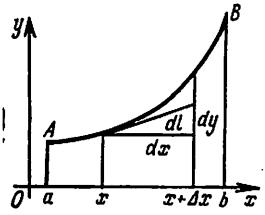


Fig. 89

$$dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \text{or} \quad dl = \sqrt{1 + [f'(x)]^2}\,dx,$$

and the length of the arc AB is computed by the formula

$$L = \int_a^b dl = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx, \qquad (13.9)$$

where a and b are the values of the independent variable x at the points A and B.

If a curve is given by the equation $x = \varphi(y)$ $(c \leqslant y \leqslant d)$, then the length of the arc AB is computed by the formula

$$L = \int_c^d \sqrt{1 + [\varphi'(y)]^2}\,dy, \qquad (13.10)$$

where c and d are the values of the independent variable y at the points A and B.

71. Find the length of a circle $x^2 + y^2 = r^2$.

○ Differentiating the equation of the circle, we have

$$2x + 2y \cdot \frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -\frac{x}{y}.$$

By Formula (13.9), we compute the arc length of one-fourth of the circle, taking the limits of integration from 0 to r:

$$L/4 = \int_0^r \sqrt{1 + \left(-\frac{x}{y}\right)^2}\,dx = \int_0^r \sqrt{\frac{x^2+y^2}{y^2}}\,dx$$

$$= \int_0^r \sqrt{\frac{r^2}{y^2}}\,dx = r \int_0^r \frac{dx}{\sqrt{r^2-x^2}} = r \arcsin \frac{x}{r}\Big|_0^r = \frac{\pi r}{2}.$$

The length of a circle $C = 4L = 4(\pi r/2) = 2\pi r$. ●

72. Find the arc length of the parabola $y = x^2/2$ between the points O (0; 0) and A ($\sqrt{3}$; 3/2).

○ Differentiating the equation of the parabola, we get $\frac{dy}{dx} = x$. We then compute the desired arc length:

$$L = \int_0^{\sqrt{3}} \sqrt{1+x^2}\,dx = \left[\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)\right]_0^{\sqrt{3}} \approx 2.4 \text{ (units of length).} \bullet$$

73. Find the arc length of the parabola $y^2 = 4x$ between the points O (0, 0) and A (5/4, $\sqrt{5}$).

○ To compute the desired arc length, we use Formula (13.10), that is, we take the variable y for the argument. We find $x = \frac{1}{4}y^2$; $\frac{dx}{dy} = \frac{1}{2}y$; consequently,

$$L = \int_0^{\sqrt{5}} \sqrt{1+\left(\frac{1}{2}y\right)^2}\,dy = \frac{1}{2}\int_0^{\sqrt{5}} \sqrt{4+y^2}\,dy$$

$$= \frac{1}{2}\left[\frac{1}{2}y\sqrt{y^2+4} + 2\ln\left(y+\sqrt{y^2+4}\right)\right]_0^{\sqrt{5}}$$

$$\approx 2.64 \text{ (units of length).} \bullet$$

74. Find the arc length of the catenary $y = a(e^{x/a} + e^{-x/a})/2$ between the points A (0, a) and B (a, $a(e^2+1)/(2e)$).

○ From the equation of the catenary we find $\frac{dy}{dx} = \frac{1}{2}(e^{x/a} - e^{-x/a})$. Consequently,

$$L = \int_0^a \sqrt{1+\left[\frac{1}{2}(e^{x/a}-e^{-x/a})\right]^2}\,dx = \int_0^a \sqrt{\left(\frac{1}{2}e^{x/a}+\frac{1}{2}e^{-x/a}\right)^2}\,dx$$

$$= \frac{1}{2}\int_0^a (e^{x/a}+e^{-x/a})\,dx = \frac{1}{2}a(e^{x/a}-e^{-x/a})\Big|_0^a$$

$$= \frac{a}{2}\left(e-\frac{1}{e}\right) \text{ (units of length).} \bullet$$

In Problems 75 through 80 find the arc lengths of the given curves between the indicated points:

75. The parabola $y = x^2$ between the points O (0, 0) and A ($\sqrt{3}/2$, 3/4).

76. The parabola $y = 4 - x^2$ between the points of its intersection with the x-axis.

77. The parabola $y^2 = x$ between the points O (0, 0) and A (3/4, $\sqrt{3}/2$).

78. The semicubical parabola $y^2 = x^3$ between the points O (0; 0) and A (4/3, $8\sqrt{3}/9$).

79. The semicubical parabola $9y^2 = 4x^3$ between the points O (0; 0) and A (3, $2\sqrt{3}$).

80. The catenary $y = (e^x + e^{-x})/2$ between the points A (0, 1) and B (1/2, $(e^{1/2} + e^{-1/2})/2$).

QUIZ No. 2

Form I

(1) The velocity of a moving point is $v = (3t^2 - 2t - 3)$ m/s. Find the path covered by the point during the 2nd second.

(2) Compute the work done in compressing a spring by 0.06 m if for its compression by 0.01 m a force of 10 N is required.

(3) Compute the work done in compressing a spring by 0.04 m if for its compression by 0.02 m the work equal to 40 J was performed.

(4) Compute the work done in pumping the water out of a cylindrical tank ($R = 2$ m, $H = 1$ m) filled with water to the brim (the weight of 1 m^3 of water is approximately equal to 9807 N).

(5) Compute the water pressure on a vertical surface having the shape of a triangle whose base is equal to 5 m and altitude to 3 m. The water level coincides with triangle's vertex.

Form II

(1) The velocity of a moving point is $v = (36t - 12t^2)$ m/s. Find the path covered by the point during its motion.

(2) Compute the work done in stretching a spring by 0.05 m if for its stretching by 0.02 m a force of 40 N is required.

(3) To stretch a spring by 0.03 m the work of 12 J must be done. By what length can the spring be stretched by doing the work equal to 48 J?

(4) Compute the work done in pumping the water out of the reservoir having the shape of a right parallelepiped with bases 3 and 4 m long and altitude equal to 2 m filled with water to the brim (the weight of 1 m^3 of water is approximately equal to 9807 N).

(5) Compute the water pressure on a vertical surface having the shape of a triangle whose base is equal to 6 m and altitude to 2 m. The water level coincides with triangle's base.

Chapter 14
COMPLEX NUMBERS

Sec. 14.1. Complex Numbers and Their Geometrical Interpretation

Complex numbers are numbers of the form $a + bi$, where a and b are real numbers, and the number i defined by the equality $i^2 = -1$ is the so-called *unit imaginary number* if for these numbers the notions of equality and operations of addition and multiplication are defined in the following way:

(1) two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are said to be *equal* if $a_1 = a_2$ and $b_1 = b_2$;

(2) the *sum* of two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is defined as the complex number $(a_1 + a_2) + (b_1 + b_2)\, i$;

(3) the *product* of two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is defined as the complex number $(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\, i$.

The notation of a complex number in the form $z = a + bi$ is called an *algebraic form* of notation of a complex number.

A complex number is defined as a number consisting of two parts, one *real* and the other *imaginary*. By the above expression $a + bi$, we mean that a represents the real part, and bi represents the imaginary part. The real part is separated from the imaginary part by a plus or minus sign. The coefficient b itself is real. The real part should always be written first, the imaginary part second.

Any real number a is contained in the set of complex numbers, it can be written as: $a = a + 0 \cdot i$. The numbers 0, 1, and i are written in the respective form: $0 = 0 + 0 \cdot i$, $1 = 1 + 0 \cdot i$, and $i = 0 + 1 \cdot i$.

For $a = 0$ the complex number $a + bi$ becomes a pure imaginary number bi.

If $z = a + bi$, then the complex number $a - bi$ is termed the *conjugate complex number* of z and is denoted by $\bar{z}$, that is, $\bar{z} = \overline{a + bi} = a - bi$. Or in other words, two complex numbers $z = a + bi$ and $\bar{z} = a - bi$ having equal real parts and equal and opposite imaginary parts are called *conjugate complex numbers* or *complex conjugates*.

Complex numbers of the form $a + bi$ and $-a - bi$ are said to be *opposite* to each other.

The number $\sqrt{a^2 + b^2}$ is called the *modulus* (or the absolute value) of the complex numbe $z = a + bi$:

$$| z | = | a + bi | = \sqrt{a^2 + b^2}. \tag{14.1}$$

The modulus of a complex number is always a real nonnegative number: $| z | \geqslant 0$ and $| z | = 0$ if and only if $z = 0$.

From the definition of the modulus of a complex number it follows that for any complex numbers z, z_1, z_2 the following relationships are valid:

(1) $| z_1 \cdot z_2 | = | z_1 | \cdot | z_2 |$; (2) $\left| \frac{z_1}{z_2} \right| = \left| \frac{z_1}{z_2} \right|$ if $z_2 \neq 0$;

(3) $| z^n | = | z |^n$ for any integer n (for $n < 0$ it is assumed that $z \neq 0$).

The complex number $z = a + bi$ can be represented by a point in the plane with coordinates (a, b) (Fig. 90), whose abscissa is the real part of z and ordinate the imaginary part of z. In this representation the axis of abscissas is called the *real axis* (or the axis of reals) and the axis of ordinates the *imaginary axis* (the axis of imaginaries), the plane Oxy itself being referred to as the *complex plane*.

To each point in the plane with coordinates (a, b) there corresponds one and only one vector applied to the origin (the point O (0, 0)) with terminus at

the point $M\ (a,\ b)$. Therefore the complex number $a + bi$ can be represented as the vector $\overrightarrow{OM} = \vec{z}$ with the initial point at the point $z = 0$ and terminus at the point $z = a + bi$.

The geometrical interpretation of the complex number yields the following properties:

1°. *The length of the vector* $\vec{z}$ *is equal to* $|\ z\ |$.

2°. *The points* $z = a + bi$ *and* $z = a - bi$ *are symmetric with respect to the real axis.*

3°. *The points* z *and* $-z$ *are symmetric with respect to the point* $z = 0$.

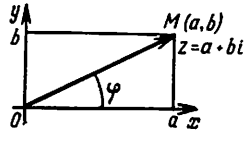


Fig. 90

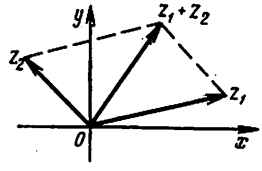


Fig. 91

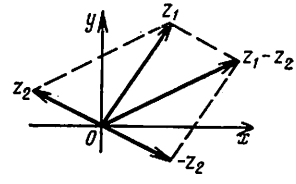


Fig. 92

4°. *The number* $z_1 + z_2$ *is represented geometrically as a vector constructed according to the rule of addition of the vectors corresponding to the points* z_1 *and* z_2 (Fig. 91).

5°. *The distance between the points* z_1 *and* z_2 *is equal to* $|\ z_1 - z_2\ |$ (Fig. 92).

The angle φ between the real axis Ox and the vector $\overrightarrow{OM}$ measured from the positive direction of the real axis is called the *argument* of the complex number $z = 0$ (see Fig. 90). If the angle is measured anticlockwise, then its size is regarded to be positive, otherwise it is negative.

The argument φ of the complex number $z = a + bi$ is denoted as

$$\varphi = \arg z \quad \text{or} \quad \varphi = \arg (a + bi). \tag{14.2}$$

For the number $z = 0$ the argument is not defined.

The argument of a complex number is defined ambiguously; any complex number $z \neq 0$ has infinitely many arguments differing from one another by a number multiple of 2π. The smallest (in modulus) value of the argument from the interval $(-\pi,\ \pi)$ is called the *principal value* of the argument.

From the definition of trigonometric functions it follows that if $\varphi = \arg (a + bi)$, then the following equalities take place:

$$\cos \varphi = a/\sqrt{a^2 + b^2} = a/r, \quad \sin \varphi = b/\sqrt{a^2 + b^2} = b/r. \tag{14.3}$$

The converse is also true, that is, if both Equalities (14.3) are fulfilled, then $\varphi = \arg (a + bi)$. Thus, all the values of the argument φ can be found by solving simultaneously Equations (14.3).

The values of the argument of a complex number $z = a + bi \neq 0$ can also be found in the following way:

(1) determine the quadrant in which the point $z = a + bi$ is situated (use the geometric interpretation of the number $z = a + bi$);

(2) find the angle φ in this quadrant, solving one of Equations (14.3) or the equation

$$\tan \varphi = b/a; \tag{14.4}$$

(3) find all the values of the argument of the number z, using the formula

$$\arg z = \varphi + 2\pi k,\ k \in \mathbf{Z}.$$

1. Construct the radius vectors corresponding to the following complex numbers: (1) $z = 2$; (2) $z = -3$; (3) $z = 3i$; (4) $z = -2i$; (5) $z = 2 + 3i$.

○ (1) M_1 (2, 0); (2) M_2 (—3, 0); (3) M_3 (0, 3); (4) M_4 (0, —2); (5) M_5(2, 3) (Fig. 93). ●

2. Find the real numbers x and y from the condition of equality of two complex numbers: $-2 + 5ix - 3iy = 9i + 2x - 4y$.

○ In both sides of the equality, we separate the real and imaginary parts of the given complex numbers:

$$-2 + (5x - 3y)\, i = 2x - 4y + 9i.$$

Now, using the equality of the complex numbers, we form the system

$$\begin{cases} 2x - 4y = -2, \\ 5x - 3y = 9, \end{cases}$$

whose solution yields: $x = 3$, $y = 2$. ●

3. Find the modulus and the principal value of the argument of each of the indicated numbers: (1) $z = i$; (2) $z = -5i$; (3) $z = 1 + i$; (4) $z = 2 - 2i$.

○ (1) Here $a = 0$, $b = 1$. By Formula (14.1), we get $r = \sqrt{0^2 + 1^2} = 1$; $\varphi = \pi/2$, since the vector representing the given number lies on the positive semi-axis Oy.

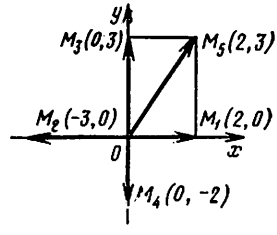


Fig. 93

(2) Here, $a = 0$, $b = -5$; we find $r = \sqrt{0^2 + (-5)^2} = 5$; $\varphi = -\pi/2$, since the vector representing the given number lies on the negative semi-axis Oy.

(3) Here, $a = 1$, $b = 1$ (the point representing the given number lies in Quadrant I); $r = \sqrt{1^2 + 1^2} = \sqrt{2}$; $\tan\varphi = b/a = 1$; $\varphi = \pi/4$.

(4) Here, $a = 2$, $b = -2$ (Quadrant IV); $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\tan\varphi = b/a = -1$; $\varphi = -\pi/4$. ●

4. Find all the values of the argument of the complex numbers: (1) $z = -4$; (2) $z = 1 - i$.

○ (1) Here, $a = -4$, $b = 0$; we find $\arg(-4) = \pi + 2\pi k$, $k \in \mathbf{Z}$.

(2) Here, $a = 1$, $b = -1$; we find $\arg(1 - i) = -\pi/4 + 2\pi k$, $k \in \mathbf{Z}$. ●

5. Construct the radius vectors corresponding to the following complex numbers: (1) $z = 2 - 3i$; (2) $z = -2 + 3i$; (3) $z = -2 - 3i$; (4) $z = \sqrt{2} + \sqrt{3}i$; (5) $z = 2 - \sqrt{3}i$.

6. Given the numbers: (a) $z = 3 + i$; (2) $z = 3 - i$; (3) $z = -3 + i$; (4) $z = -3 - i$; (5) 3; (6) —3; (7) $-i$; (8) i. Mention the numbers conjugate and opposite to the given numbers.

7. Given in the coordinate plane is a circle with centre at the origin and radius equal to 1 (Fig. 94). What numbers correspond to the points A_1, A_2, A_3, A_4, A_5, A_6 lying at the vertices of the regular hexagon inscribed in this circle.

8. Given the point representing the number $-3 + 2i$. What numbers represent the points symmetric to the given point with respect to: (1) the real axis; (2) the imaginary axis; (3) the origin?

9. Find the real numbers x and y from the condition of equality of two complex numbers: (1) $9 + 2ix + 4iy = 10i + 5x - 6y$; (2) $2ix + 3iy + 17 = 3x + 2y + 18i$; (3) $5x - 2y + (x + y)\,i = 4 + 5i$.

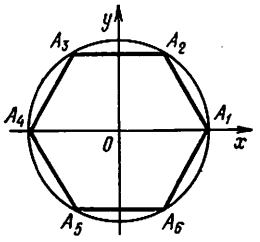


Fig. 94

10. Using the condition of equality of two complex numbers, find x and y from the relationships: (1) $x^2 - 5(x - 1) + 4i = yi - 1$; (2) $\frac{1}{x} - 4yi = 4$.

11. Find the real values of x for which the equality $(x^2 + 1)\,i + 3 = x(x - 2i) - 2x$ is true.

12. Find the modulus and the principal value of the argument of the following complex numbers: (a) $z = 3$; (2) $z = -3$; (3) $z = 3i$; (4) $z = -3i$; (5) $z = -2 - 2i$; (b) $z = 1 + i\sqrt{3}$; (7) $z = 1 - i\sqrt{3}$; (8) $z = -\sqrt{3} + i$.

13. What is the argument of: (1) a pure imaginary number; (2) any negative number; (3) any positive number; (4) zero?

14. The argument of the complex number $a+bi$ is equal to φ. What is the argument of the number $a - bi$?

15. Find the set of points of the coordinate plane: (1) whose modulus is equal to 2; (2) whose argument is equal to $3\pi/4$.

16. Find all the values of the argument of the complex numbers: (1) $z = -1 + i$; (2) $z = \sqrt{3} - i$.

Sec. 14.2. Operations with Complex Numbers Given in Algebraic Form

With complex numbers we perform the same operations as with real numbers. The operations of addition and multiplication are given in the definition of the complex number (see Sec. 14.1).

Considering the subtraction and division of complex numbers as operations respectively inverse to addition and multiplication we obtain the rules for subtracting and dividing complex numbers:

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)\,i;$$

$$\frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\,i.$$

17. Perform the indicated operations: (1) $(4 + 2i) + (1 + 5i)$; (2) $(3 + 5i) - (6 + 3i)$.

○ (1) By the rule for adding complex numbers, we get

$$(4 + 2i) + (1 + 5i) = (4 + 1) + (2 + 5)\, i = 5 + 7i.$$

(2) By the rule for subtracting complex numbers, we get

$$(3 + 5i) - (6 + 3i) = (3 - 6) + (5 - 3)\, i = -3 + 2i.$$

The addition (subtraction) of complex numbers is reduced to adding (subtracting) the vectors representing these numbers. The operations on the given vectors are illustrated geometrically in Fig. 95. ●

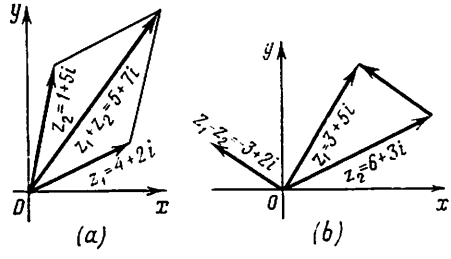


Fig. 95

18. Show that the following equalities hold true: $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$, where $k \in \mathbf{Z}$.

○ Since $i^2 = -1$, we have $i^3 = i^2 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, ... Consequently, we get the four alternating values: $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$, $k \in \mathbf{Z}$. ●

19. Compute: (1) i^{16}; (2) i^{25}; (3) i^{15}; (4) $(-i)^8$; (5) $(-i)^7$.

○ (1) $i^{16} = i^{4 \cdot 4} = 1$; (2) $i^{25} = i^{4 \cdot 6 + 1} = i$; (3) $i^{15} = i^{4 \cdot 3 + 3} = i^3 = -i$; (4) $(-i)^8 = i^8 = i^{4 \cdot 2} = 1$, (5) $(-i)^7 = -i^7 = -i^{4+3} = -i^3 = -(-i) = i$. ●

20. Perform the indicated operations: (1) $2i \cdot 3i$; (2) $(2 - 3i) \times (2 + 3i)$; (3) $(5 - 4i)(3 + 2i)$.

○ (1) $2i \cdot 3i = 6i^2 = -6$; (2) $(2 - 3i)(2 + 3i) = 4 - 9i^2 = 4 + 9 = 13$;

(3) By the rule for multiplying complex numbers, we get

$$(5 - 4i)(3 + 2i) = [5 \cdot 3 - (-4) \cdot 2] + i\,[5 \cdot 2 + 3\,(-4)] = 23 - 2i.$$

The multiplication can be performed using the rule for multiplying polynomials:

$$(5 - 4i)(3 + 2i) = 15 + 10i - 12i + 8 = 23 - 2i. \quad ●$$

21. Perform the indicated operations

(1) $\frac{2}{3i}$; (2) $\frac{1}{1+i}$; (3) $\frac{1+i}{1-i}$; (4) $\frac{2-3i}{4+5i}$.

○ (1) Multiplying the dividend and divisor by i, we get

$$\frac{2}{3i}=\frac{2i}{3i\cdot i}=\frac{2i}{-3}=-\frac{2}{3}i.$$

(2) We multiply the dividend and divisor by the factor conjugate to the divisor:

$$\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{1-i^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i.$$

(3) $\frac{1+i}{1-i}=\frac{(1+i)(1+i)}{(1-i)(1+i)}=\frac{1+2i+i^2}{1-i^2}=\frac{2i}{2}=i.$

(4) $\frac{2-3i}{4+5i}=\frac{(2-3i)(4-5i)}{(4+5i)(4-5i)}=\frac{8-10i-12i+15i^2}{16+25}$

$$=\frac{-7-22i}{41}=-\frac{7}{41}-\frac{22}{41}i.$$ ●

22. Compute $(1+i)^8$.

○ Using the relationship $(1+i)^2=2i$, we get $(1+i)^8=[(1+i)^2]^4=(2i)^4=16i^4=16$. ●

23. Perform the indicated operations: (1) $(3+i)+(-3-8i)$; (2) $(5-4i)+(7+4i)$; (3) $(-6+2i)+(-6-2i)$; (4) $(0.2+0.1i)+(0.8-1.1i)$; (5) $(2-3i)+(5+6i)+(-3-4i)$; (6) $(1-i)-(7-3i)-(2+i)+(6-2i)$.

24. Compute: (1) $i^6+i^{20}+i^{30}+i^{36}+i^{54}$; (2) $i+i^2+i^3+i^4+i^5$; (3) $i+i^{11}+i^{21}+i^{31}+i^{41}$; (4) $i\cdot i^2\cdot i^3\cdot i^4$; (5) $\frac{1}{i^3}+\frac{1}{i^5}$; (6) $\frac{1}{i^{13}}+\frac{1}{i^{23}}+\frac{1}{i^{33}}$.

25. Perform the indicated operations: (1) $-i\sqrt{5}\cdot 4i\sqrt{5}$; (2) $(5-3i)\cdot 2i$; (3) $(3+4i)(3-4i)$; (4) $(5+3i)(2-5i)$; (5) $(-2-i)\times(1+i)$; (6) $4+2i+(-1+6i)(6-i)$; (7) $(3-2i)(5+4i)-7i+1$; (8) $\left(\frac{2}{3}-\frac{1}{3}i\right)\left(\frac{1}{3}+\frac{4}{3}i\right)$; (9) $(0.2-0.3i)(0.5+0.4i)$.

26. Perform the indicated operations: (1) $\frac{1}{i}$; (2) $\frac{1}{1-i}$; (3) $\frac{1-i}{1+i}$; (4) $\frac{3-2i}{1+3i}$; (5) $\frac{(1-2i)(2+i)}{3-2i}$; (6) $\frac{2+3i}{(4+i)(2-2i)}$; (7) $\frac{(3+2i)(2-i)}{(2+3i)(1+i)}$; (8) $\frac{a+bi}{a-bi}$; (9) $\frac{(a+bi)(b+ai)}{b-ai}$; (10) $\frac{\sqrt{5}+i}{\sqrt{5}-2i}$; (11) $\frac{1-3i}{i-2}+\frac{4i+1}{3i-1}$; (12) $\frac{a+bi}{a-bi}-\frac{a-bi}{a+bi}$.

27. Factorize into complex factors: (1) m^2+n^2; (2) $4m^2+9n^2$; (3) $\frac{a^2}{9}+\frac{b^2}{16}$; (4) $m+n$; (5) $2+\sqrt{3}$; (6) $1+\sin^2\alpha$; (7) 3.

28. Compute: (1) $(1-i)^{12}$; (2) $(1+i)^{17}$; (3) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$; (4) $\left(\frac{\sqrt{2}}{2}+\frac{1}{2}i\right)^2$; (5) $(1+i)^{-2}$; (6) $(1-i)^{-3}$; (7) $\left(\frac{\sqrt{2}+i\sqrt{2}}{2}\right)^{-8}$.

Sec. 14.3. Operations with Complex Numbers Given in Trigonometric Form

1. Trigonometric Form of the Complex Number. Let $r = |a+bi| = \sqrt{a^2+b^2}$ be the modulus, and φ be one of the values of the argument of the complex number $a+bi$. Since it follows from the relationships (14.3) that $a = r\cos\varphi$, $b = r\sin\varphi$, we have

$$a+bi = r(\cos\varphi + i\sin\varphi). \tag{14.5}$$

Thus, any complex number $a+bi \neq 0$ can be written according to Formula (14.5), where r is the modulus, and φ is one of the values of the argument of this number.

The converse is also true: if the complex number $a+bi$ is represented in the form (14.5), where $r > 0$, then $r = |a+bi|$, $\varphi = \arg(a+bi)$.

The representation of a complex number in the form

$$z = r(\cos\varphi + i\sin\varphi),$$

where $r > 0$ is called the *trigonometric form* of the notation of a complex number.

To represent the complex number $z = a+bi$ in trigonometric form, it is necessary to find: (1) the modulus of this number; (2) one of the values of the argument of this number. Since $\arg z$ is many-valued, the trigonometric form of a complex number is also ambiguous.

2. Operations with Complex Numbers Represented in Trigonometric Form. The product of two complex numbers $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ is found by the formula

$$r_1(\cos\varphi_1 + i\sin\varphi_1)\cdot r_2(\cos\varphi_2 + i\sin\varphi_2) = r_1r_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)], \tag{14.6}$$

that is,

$$|z_1z_2| = |r_1\cdot r_2| = |z_1|\cdot|z_2|, \quad \arg(z_1z_2) = \varphi_1+\varphi_2.$$

Thus, *as two complex numbers represented in trigonometric form are multiplied, their moduli are multiplied, while the arguments are added together.*

The quotient of two complex numbers $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ is found by the formula

$$\frac{r_1(\cos\varphi_1+i\sin\varphi_1)}{r_2(\cos\varphi_2+i\sin\varphi_2)} = \frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)], \tag{14.7}$$

that is,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \qquad \arg\frac{z_1}{z_2} = \varphi_1-\varphi_2.$$

Thus, *as complex numbers represented in trigonometric form are divided, their moduli are divided, while the arguments are subtracted.*

To raise the complex number $r(\cos\varphi + i\sin\varphi)$ to the nth power, use is made of the formula

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi), \quad n \in \mathbf{Z}, \tag{14.8}$$

which is known as *De Moivre's formula.*

To extract the nth root of the complex number $r(\cos\varphi + i\sin\varphi)$ the following formula is used:

$$z_k = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right), \qquad (14.9)$$

where $\sqrt[n]{r}$ is an arithmetic root, $k = 0, 1, 2, \ldots, n-1$.

29. Represent the following numbers in trigonometric form: (1) 2; (2) $6i$; (3) $-2 + 2\sqrt{3}i$; (4) $2 - 2i$; (5) $-\sqrt{3} - i$.

○ (1) Here, $a = 2$, $b = 0$, $r = 2$. Since the vector representing the number 2 lies on the positive semi-axis Ox, the principal value of the argument $\varphi = 0$; consequently,

$$2 = 2(\cos 0 + i\sin 0)$$

or

$$2 = 2(\cos 2\pi k + i\sin 2\pi k),\ k \in \mathbf{Z}.$$

(2) Here, $a = 0$, $b = 6$, $r = 6$. Since the vector representing the number $6i$ lies on the positive semi-axis Oy, the principal value of the argument $\varphi = \pi/2$; therefore

$$6i = 6[\cos(\pi/2) + i\sin(\pi/2)]$$

or

$$6i = 6[\cos(\pi/2 + 2\pi k) + i\sin(\pi/2 + 2\pi k)],\ k \in \mathbf{Z}.$$

(3) Here $a = -2$, $b = 2\sqrt{3}$, $r = 4$. The point representing the number z lies in Quadrant II; $\tan\varphi = 2\sqrt{3}/(-2) = -\sqrt{3}$, $\varphi = 2\pi/3$. Hence,

$$-2 + 2\sqrt{3}i = 4[\cos(2\pi/3) + i\sin(2\pi/3)]$$

or

$$-2 + 2\sqrt{3}i = 4[\cos(2\pi/3 + 2\pi k) + i\sin(2\pi/3 + 2\pi k),\quad k \in \mathbf{Z}.$$

(4) Here $a = 2$, $b = -2$, $r = 2\sqrt{2}$. The point representing the number z lies in Quadrant IV; $\tan\varphi = -1$, $\varphi = -\pi/4$. Therefore

$$2 - 2i = 2\sqrt{2}[\cos(-\pi/4) + i\sin(-\pi/4)]$$

or

$$2 - 2i = 2\sqrt{2}[\cos(-\pi/4 + 2\pi k) + i\sin(-\pi/4 + 2\pi k),\quad k \in \mathbf{Z}.$$

(5) Here $a = -\sqrt{3}$, $b = -1$, $r = 2$. The point representing the number z lies in Quadrant III; $\tan\varphi = 1/\sqrt{3}$, $\varphi = -5\pi/6$, then

$$-\sqrt{3} - i = 2[\cos(-5\pi/6) + i\sin(-5\pi/6)]$$

or

$$-\sqrt{3} - i = 2[\cos(-5\pi/6 + 2\pi k) + i\sin(-5\pi/6 + 2\pi k),\quad k \in \mathbf{Z}.$$ ●

30. Represent the following numbers in algebraic form:

(1) $z = 2(\cos 2\pi + i\sin 2\pi)$; (2) $z = \sqrt{2}[\cos(3\pi/4) + i\sin(3\pi/4)]$.

○ (1) Substituting the values $\cos 2\pi = 1$, $\sin 2\pi = 0$ into the given equality, we get $z = 2(1 + i \cdot 0) = 2$.

(2) We have

$$z = \sqrt{2}[\cos(3\pi/4) + i \sin(3\pi/4)]$$
$$= \sqrt{2}[-\sqrt{2}/2 + i(\sqrt{2}/2)] = -1 + i. \bullet$$

31. Find the product

$$2[\cos(\pi/6) + i \sin(\pi/6)] \cdot 3[\cos(\pi/12) + i \sin(\pi/12)].$$

○ By formula (14.6), we obtain

$$2[\cos(\pi/6) + i \sin(\pi/6)] \cdot 3[\cos(\pi/12) + i \sin(\pi/12)] =$$
$$2 \cdot 3[\cos(\pi/6 + \pi/12) + i \sin(\pi/6 + \pi/12) = 6[\cos(\pi/4) +$$
$$i \sin(\pi/4)] = 6[(\sqrt{2}/2) + i(\sqrt{2}/2)] = 3\sqrt{2} + 3i\sqrt{2}. \bullet$$

32. Perform the operation of division:

$$10[\cos(3\pi/4) + i \sin(3\pi/4)] \div 2[\cos(\pi/4) + i \sin(\pi/4)].$$

○ By Formula (14.7), we find

$$10[\cos(3\pi/4) + i \sin(3\pi/4)] \div 2[\cos(\pi/4) + i \sin(\pi/4)] =$$
$$(10/2)[\cos(3\pi/4 - \pi/4) + i \sin(3\pi/4 - \pi/4)] =$$
$$5[\cos(\pi/2) + i \sin(\pi/2)] = 5(0 + i) = 5i. \bullet$$

33. Raise to the indicated power: (1) $[\cos(\pi/6) + i \sin(\pi/6)]^6$; (2) $[3/2 - (\sqrt{3}/2)i]^{10}$.

○ (1) By De Moivre's formula, we get

$$[\cos(\pi/6) + i \sin(\pi/6)]^6 = \cos[6 \cdot (\pi/6)] + i \sin[6 \cdot (\pi/6)] =$$
$$\cos \pi + i \sin \pi = -1 + i \cdot 0 = -1.$$

(2) Let us represent the given number in trigonometric form. Here, $a = 3/2$, $b = -\sqrt{3}/2$, i.e. $r = \sqrt{(3/2)^2 + (-\sqrt{3}/2)^2} = \sqrt{3}$. The point representing the given number lies in Quadrant IV, therefore $\tan \varphi = (-\sqrt{3}/2) \div (3/2) = -\sqrt{3}/3$, that is, $\varphi = -\pi/6$. Thus

$$3/2 - (\sqrt{3}/2)i = \sqrt{3}[\cos(-\pi/6) + i \sin(-\pi/6)].$$

Consequently,

$$\{\sqrt{3}[\cos(-\pi/6) + i \sin(-\pi/6)\}^{10} = 3^5[\cos(-10 \cdot \pi/6)\pi +$$
$$i \sin(-10 \cdot \pi/6) = 3^5[\cos(5\pi/3) - i \sin(5\pi/3)] =$$
$$3^5[\cos(5\pi/3 - 2\pi) - i \sin(5\pi/3 - 2\pi)] =$$
$$243[\cos(\pi/3) + i \sin(\pi/3)] = 243[1/2 + i(\sqrt{3}/2)] =$$
$$121.5(1 + i\sqrt{3}). \bullet$$

34. Applying De Moivre's formula, prove the validity of the following identities:

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi; \quad \sin 2\varphi = 2\sin\varphi\cos\varphi;$$
$$\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi; \quad \sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi.$$

○ Setting in the relationship (14.8) $r = 1$ and $n = 2$, we get

$$(\cos\varphi + i\sin\varphi)^2 = \cos 2\varphi + i\sin 2\varphi,$$

or

$$\cos^2\varphi + 2i\cos\varphi\sin\varphi - \sin^2\varphi = \cos 2\varphi + i\sin 2\varphi.$$

From the condition of equality of two complex numbers it follows that

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi; \quad \sin 2\varphi = 2\sin\varphi\cos\varphi.$$

Analogously setting in (14.8) $r = 1$ and $n = 3$, we have

$$(\cos\varphi + i\sin\varphi)^3 = \cos 3\varphi + i\sin 3\varphi,$$

that is,

$$\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\sin^2\varphi\cos\varphi - i\sin^3\varphi = \cos 3\varphi + i\sin 3\varphi.$$

From the condition of equality of two complex numbers, it follows that

$$\cos 3\varphi = \cos^3\varphi - 3\sin^2\varphi\cos\varphi =$$
$$\cos^3\varphi - 3(1 - \cos^2\varphi)\cos\varphi = 4\cos^3\varphi - 3\cos\varphi;$$
$$\sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi =$$
$$3(1 - \sin^2\varphi)\sin\varphi - \sin^3\varphi = 3\sin\varphi - 4\sin^3\varphi. \bullet$$

35. Extract the roots of the complex numbers: (1) $\sqrt{i}$; (2) $\sqrt[3]{1}$.

○ (1) Let us represent the number i in trigonometric form: $i = 0 + 1\cdot i = \cos(\pi/2) + i\sin(\pi/2)$. By Formula (14.9), we get

$$z_k = \sqrt{i} = \sqrt{\cos(\pi/2) + i\sin(\pi/2)} = \cos\frac{\pi/2 + 2\pi k}{2} + i\sin\frac{\pi/2 + 2\pi k}{2}$$
$$= \cos(\pi/4 + \pi k) + i\sin(\pi/4 + \pi k), \quad k = 0, 1;$$

if $k = 0$, then $z_0 = \cos(\pi/4) + i\sin(\pi/4) = \sqrt{2}/2 + (\sqrt{2}/2)i$;

if $k = 1$, then $z_1 = \cos(\pi/4 + \pi) + i\sin(\pi/4 + \pi) = -\cos(\pi/4)$
$-i\sin(\pi/4) = -\sqrt{2}/2 - (\sqrt{2}/2)i$.

(2) Let us represent the number 1 in trigonometric form: $1 = \cos 0 + i\sin 0$. By Formula (14.9), we find

$$z_k = \sqrt[3]{1} = \sqrt[3]{\cos 0 + i\sin 0} = \cos\frac{0 + 2\pi k}{3} + i\sin\frac{0 + 2\pi k}{3}$$
$$= \cos(2\pi k/3) + i\sin(2\pi k/3), \quad k = 0, 1, 2;$$

if $k = 0$, then $z_0 = \cos 0 + i \sin 0 = 1$;
if $k = 1$, then $z_1 = \cos (2\pi/3) + i \sin (2\pi/3) = -1/2 + (\sqrt{3}/2)\, i$;
if $k = 2$, then $z_2 = \cos (4\pi/3) + i \sin (4\pi/3) = -1/2 - (\sqrt{3}/2)\, i$.

36. Represent the following complex numbers in trigonometric form: (1) $3i$; (2) $-1 + i$; (3) $1 - i\sqrt{3}$; (4) $\sqrt{3} - i$; (5) $\sqrt{3}/2 - (1/2)\, i$; (6) $-3 + 4i$.

37. Represent the following numbers in algebraic form:
(1) $5 [\cos (\pi/2) + i \sin (\pi/2)]$; (2) $4 [\cos (-\pi/3) + i \sin (-\pi/3)]$;
(3) $\cos \pi + i \sin \pi$; (4) $2 [\cos (\pi/4) + i \sin (\pi/4)]$;
(5) $3 (\cos 0 + i \sin 0)$.

38. Find the following products:
(1) $3 [\cos (\pi/8) + i \sin (\pi/8)] \cdot [\cos (5\pi/24) + i \sin (5\pi/24)]$;
(2) $2 [\cos (\pi/3) + i \sin (\pi/3)] \cdot [5 \cos (-\pi/4) + i \sin (-\pi/4)]$;
(3) $(\cos 5 + i \sin 5) \cdot (\cos 2 + i \sin 2)$;
(4) $[\cos (2\pi/3) + i \sin (2\pi/3)] \cdot [\cos (-\pi/2) + i \sin (-\pi/2)]$;
(5) $4 (\cos 10^\circ + i \sin 10^\circ) \cdot 2 (\cos 35^\circ + i \sin 35^\circ)$;
(6) $[\cos (5\pi/6) + i \sin (5\pi/6)] \cdot [\cos (2\pi/3) + i \sin (2\pi/3)]$.

39. Perform the operation of multiplication, using the trigonometric form of the complex number:

(1) $\left(\frac{1}{4} + \frac{1}{4}\, i\right) \left(-\frac{\sqrt{2}}{6} + \frac{i\sqrt{6}}{6}\right)$; (2) $(1 + i\sqrt{3})(-2 - 2i\sqrt{3})$;
(3) $(1 + i)(3 + 3i\sqrt{3})$; (4) $(6 + 2i\sqrt{3})(-3 - 3i)$;
(5) $(5 + 5i) (\cos 15^\circ + i \sin 15^\circ)$;
(6) $3 [\cos (-\pi/8) + i \sin (-\pi/8)] \cdot (3 + \sqrt{3}i)$.

40. Perform the division in trigonometric form:
(1) $3 [\cos (3\pi/4) + i \sin (3\pi/4)] : [\cos (\pi/2) + i \sin (\pi/2)]$;
(2) $(\cos 210^\circ + i \sin 210^\circ) : (\cos 150^\circ + i \sin 150^\circ)$;
(3) $[\cos (-\pi/3) + i \sin (-\pi/3)] : [\cos (-\pi/6) + i \sin (-\pi/6)]$;
(4) $(\cos 150^\circ + i \sin 150^\circ) : [\cos (-120^\circ) + i \sin (-120^\circ)]$.

41. Raise to the indicated power:

(1) $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\, i\right)^6$; (2) $\left[2 \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}\right)\right]^8$;
(3) $(\cos 35^\circ + i \sin 35^\circ)^{-12}$.

42. Compute: (1) $(1 - i)^{12} + (1 + i)^{12}$; (2) $\frac{(1+i)^8 - (1-i)^8}{(1+i)^8 \cdot (1-i)^8}$.

43. Take the root: (1) $\sqrt[3]{-1}$; (2) $\sqrt[4]{-1}$; (3) $\sqrt[3]{i}$; (4) $\sqrt[4]{4}$; (5) $\sqrt[4]{-2 + 2i\sqrt{3}}$; (6) $\sqrt[6]{1}$.

Sec. 14.4. Exponential Function with a Complex Exponent. Euler's Formulas

The power e^z with the complex exponent $z = x + iy$ is defined by the equality

$$e^z = \lim_{n\to\infty} \left(1+\frac{z}{n}\right)^n.$$

It is possible to prove that

$$e^z = e^x (\cos y + i \sin y),$$

that is,

$$e^{x+iy} = e^x (\cos y + i \sin y). \tag{14.10}$$

In particular, for $x = 0$ we get the relationship

$$e^{iy} = \cos y + i \sin y, \tag{14.11}$$

which is called *Euler's formula.*

The basic rules of operation with exponents are applicable to complex exponents as well; for instance, as numbers are multiplied their exponents are added together, in the operation of division the exponents are subtracted, when raising to a power the exponents are multiplied.

The exponential function has a period equal to $2\pi i$, that is, $e^{z+2\pi i} = e^z$. In particular, for $z = 0$ we obtain the relationship $e^{2\pi i} = 1$.

The trigonometric form of the complex number $z = r(\cos\varphi + i \sin\varphi)$ can be replaced by *exponential form*:

$$z = re^{\varphi i}. \tag{14.12}$$

Multiplication, division, raising to an integral positive power, and extracting the corresponding root for complex numbers represented in exponential form are performed by the following formulas:

$$r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 \cdot r_2 e^{i(\varphi_1+\varphi_2)}; \tag{14.13}$$

$$\frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}; \tag{14.14}$$

$$(re^{i\varphi})^n = r^n e^{in\varphi}; \tag{14.15}$$

$$\sqrt[n]{re^{i\varphi}} = \sqrt[n]{r} \cdot e^{\frac{\varphi+2\pi k}{n} i} \quad (k=0, 1, 2, \ldots, n-1) \tag{14.16}$$

Euler's formula (14.11) establishes relation between the two trigonometric functions and the exponential function. Replacing in this formula y by φ and by $-\varphi$, we get

$$e^{\varphi i} = \cos\varphi + i \sin\varphi, \quad e^{-\varphi i} = \cos\varphi - i \sin\varphi.$$

Adding and subtracting these equalities, we get

$$\cos\varphi = (e^{\varphi i} + e^{-\varphi i})/2, \tag{14.17}$$

$$\sin\varphi = (e^{\varphi i} - e^{-\varphi i})/(2i) \tag{14.18}$$

These two simple formulas, also called Euler's formulas, expressing the two trigonometric functions in terms of exponential functions make it possible to obtain the basic trigonometric formulas in algebraic way.

44. Find: (1) $e^{i\pi/4}$; (2) $e^{\pi e^{-i\pi/2}}$; (3) $e^{2+i\pi}$.

By Formula (14.11), we get

(1) $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \sqrt{2}/2 + i(\sqrt{2}/2)$;

(2) $e^{\pi e^{-i\pi/2}} = e^{\pi[\cos(-\pi/2)+i\sin(-\pi/2)]}$

$= e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1$;

(3) By Formula (14.10), we get $e^{2+i\pi} = e^2(\cos\pi + i\sin\pi) = -e^2$. ●

45. Find: (1) $\cos i$; (2) $\cos(1-i)$.

○ By Formula (14.17), we get

(1) $\cos i = \dfrac{e^{i^2}+e^{-i^2}}{2} = \dfrac{e^{-1}+e}{2} = \dfrac{e^2+1}{2e}$;

(2) $\cos(1-i) = \dfrac{e^{i(1-i)}+e^{-i(1-i)}}{2} = \dfrac{e^{i+1}+e^{-i-1}}{2} =$

$\dfrac{1}{2}[e(\cos 1 + i\sin 1) + e^{-1}(\cos(-1) + i\sin(-1)] =$

$\dfrac{1}{2}[e\cos 1 + ei\sin 1 + e^{-1}\cos 1 - e^{-1}i\sin 1] =$

$\dfrac{1}{2}[(e+e^{-1})\cos 1 + i(e-e^{-1})\sin 1] = \dfrac{e^2+1}{2e}\cos 1 + i\dfrac{e^2-1}{2e}\sin 1$. ●

46. Show that for the complex variable z the following formulas are valid: (1) $\sin^2 z + \cos^2 z = 1$; (2) $\sin 2z = 2\sin z\cos z$;

(3) $\cos 2z = \cos^2 z - \sin^2 z$.

○ (1) Squaring both members of Equalities (14.17) and (14.18) and then adding them termwise, we get

$$\cos^2 z + \sin^2 z = \frac{(e^{zi}+e^{-zi})^2}{4} + \frac{(e^{zi}-e^{-zi})^2}{-4} = \frac{e^{2zi}+2+e^{-2zi}-e^{2zi}+2-e^{-2zi}}{4} = 1.$$

(2) Multiplying the left-hand and right-hand sides of (14.17) and (14.18), we obtain

$$2\sin z\cos z = 2\,\frac{e^{zi}+e^{-zi}}{2}\cdot\frac{e^{zi}-e^{-zi}}{2i} = \frac{e^{2zi}-e^{-2zi}}{2i} = \sin 2z.$$

(3) Squaring both sides of (14.17) and (14.18) and subtracting them termwise, we get

$$\cos^2 z - \sin^2 z = \frac{(e^{zi}+e^{-zi})^2}{4} - \frac{(e^{zi}-e^{-zi})^2}{-4} = \frac{e^{2zi}+2+e^{-2zi}+e^{2zi}-2+e^{-2zi}}{4} = \frac{e^{2zi}+e^{-2zi}}{2} = \cos 2z.$$ ●

47. Represent the numbers: (1) $z = 2i$; (2) $z = -1 + i$ in exponential form.

○ (1) Here $a = 0$, $b = 2$, $r = 2$, $\varphi = \pi/2$. By Formula (14.12), we get $z = 2e^{i\pi/2}$.

(2) Here $a = -1$, $b = 1$, $r = \sqrt{2}$, $\tan\varphi = -1$, $\varphi = 3\pi/4$. By Formula (14.12), we have $z = \sqrt{2}e^{3\pi i/4}$. ●

48. Representing the numbers $z_1 = 1 + i$ and $z_2 = 1 - i\sqrt{3}$ in exponential form, compute: (1) $z_1 z_2$; (2) z_1/z_2; (3) z_1^6; (4) $\sqrt[4]{z_1}$.

○ For the number $z_1 = 1 + i$ we have: $a = 1$, $b = 1$, $r = \sqrt{2}$, $\varphi = \pi/4$, i.e. $z_1 = \sqrt{2}e^{i\pi/4}$. For the number $z_2 = 1 - i\sqrt{3}$ we have: $a = 1$, $b = -\sqrt{3}$, $r = 2$, $\varphi = -\pi/3$, i.e. $z_2 = 2e^{-i\pi/3}$.

(1) By Formula (14.13), we find $z_1 z_2 = \sqrt{2}e^{i\pi/4} \cdot 2e^{-i\pi/3} = 2\sqrt{2}e^{-i\pi/12}$.

(2) By Formula (14.14), we obtain

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\,e^{i\pi/4}}{2e^{-i\pi/3}} = \frac{\sqrt{2}}{2}\,e^{i\pi/4-(-i\pi/3)} = \frac{\sqrt{2}}{2}\,e^{7\pi/12}.$$

(3) By Formula (14.15) we have $z_1^6 = (\sqrt{2}\,e^{i\pi/4})^6 = 8e^{i3\pi/2}$.

(4) By Formula (14.16), we find

$$z_k = \sqrt[4]{z_1} = \sqrt[4]{\sqrt{2}\,e^{i\pi/4}} = \sqrt[8]{2}\,e^{(\pi/4+2\pi k)i/4},\quad k = 0,\ 1,\ 2,\ 3;$$

if $k = 0$, then $z_0 = \sqrt[8]{2}\,e^{i\pi/16}$;

if $k = 1$, then $z_1 = \sqrt[8]{2}\,e^{(\pi/4+2\pi)i/4} = \sqrt[8]{2}\,e^{9\pi i/16}$;

if $k = 2$, then $z_2 = \sqrt[8]{2}\,e^{(\pi/4+4\pi)i/4} = \sqrt[8]{2}\,e^{17\pi i/16} = \sqrt[8]{2}\,e^{-15\pi i/16}$;

if $k = 3$, then $z_3 = \sqrt[8]{2}\,e^{(\pi/4+6\pi)i/4} = \sqrt[8]{2}\,e^{25\pi i/16} = \sqrt[8]{2}\,e^{-7\pi i/16}$. ●

49. Find: (1) e^i; (2) $e^{i\pi}$; (3) e^{1+i}; (4) $e^{i\pi/2}$; (5) $e^{i\pi/3}$; (6) e^{4+3i}; (7) e^{2-i}; (8) e^{3i-2}.

50. Find: (1) $\sin i$; (2) $\cos(1 + i)$; (3) $\sin(1 - i)$.

51. Show that for the complex variable z the following equalities hold:

$$\cos(-z) = \cos z;\quad \sin(-z) = -\sin z;$$

$$\sin 3z = 3\sin z - 4\sin^3 z;\quad \sin^2 z = \frac{1-\cos 2z}{2};\quad \cos^2 z = \frac{1+\cos 2z}{2}.$$

52. Represent the following numbers in exponential form: (1) 1; (2) $\sqrt{3} + i$; (3) $3 + i\sqrt{3}$; (4) $-\sqrt{2} + i\sqrt{6}$.

53. Representing the numbers $z_1 = \sqrt{3} + i$ and $z_2 = \sqrt{2} + i\sqrt{2}$ in exponential form, compute: (1) $z_1 z_2$; (2) z_2/z_1; (3) z_2^4; (4) $\sqrt[3]{z_1}$; (5) $\sqrt[4]{z_2}$.

Sec. 14.5. Miscellaneous Problems

54. Perform the multiplication: (1) $(a\sqrt{b} + bi\sqrt{a})(-a\sqrt{b} - bi\sqrt{a})$; (2) $(\sqrt{a/b} - i\sqrt{b/a}) \cdot (\sqrt{b/a} + i\sqrt{a/b})$.

55. Factor: (1) $4x^2+25$; (2) a^2+12; (3) $\sqrt{5}+9$; (4) $\sqrt{2}+\sqrt{3}$.

56. Reduce the given fractions: (1) $\frac{9m^2+4n^2}{3m-2in}$; (2) $\frac{x+1}{\sqrt{x}+i}$; (3) $\frac{a+b}{\sqrt{a}+i\sqrt{b}}$.

57. Perform the indicated division: (1) $\frac{\sqrt{27}+i\sqrt{8}}{\sqrt{3}+i\sqrt{2}}$; (2) $\frac{m\sqrt{n}-ni\sqrt{m}}{n\sqrt{m}+mi\sqrt{n}}$.

58. Find the modulus and argument of the complex number $\frac{3i-1}{2i+1}$.

59. Check the equality $\frac{\sqrt{m}+i\sqrt{n}}{\sqrt{m}-i\sqrt{n}}-\frac{\sqrt{n}+i\sqrt{m}}{\sqrt{n}-i\sqrt{m}}=\frac{2(m-n)}{m+n}$.

60. Find the numerical value of the given polynomial: (1) $x^{15}+x^{14}+3x^{12}-x^{10}+x^7$ for $x=i$; (2) x^3+x^2+x+1 for $x=1+i$.

61. Perform the indicated operations: (1) $\left(\frac{4}{\sqrt{3}+i}\right)^2$; (2) $\left(i\frac{\sqrt{2}}{2}+\frac{1}{2}\right)^2$.

62. Prove the validity of the indicated formulas by squaring them:

(1) $\sqrt{a+bi}+\sqrt{a-bi}=\sqrt{2(\sqrt{a^2+b^2}+a)}$;

(2) $\sqrt{a+bi}-\sqrt{a-bi}=i\sqrt{2(\sqrt{a^2+b^2}-a)}$.

63. Show that if $a=-0.5(1+i\sqrt{3})$, $b=0.5(-1+i\sqrt{3})$, then (a) $a^3=1$; (2) $b^3=1$; (3) $a^2=b$; (4) $b^2=a$.

64. Prove that $x^3+y^3-3xy=-1$ if $x=-0.5+1.5i$ and $y=-0.5-1.5i$.

65. Perform the indicated operations:

(1) $\frac{1}{(1+i)^2}+\frac{1}{(1-i)^2}$; (2) $\frac{1}{(1+i)^4}+\frac{1}{(1-i)^3}$; (3) $\frac{1+i}{1-i}+\frac{1-i}{1+i}$.

66. Perform the indicated operations in trigonometric and exponential forms:

(1) $5[\cos(\pi/6)-i\sin(\pi/6)]\cdot[\cos(\pi/4)+i\sin(\pi/4)]$;

(2) $8[\cos(\pi/3)+i\sin(\pi/3)]\div 4[\cos(\pi/12)+i\sin(\pi/12)]$.

67. Compute with the aid of De Moivre's formula:

(1) $[\cos(\pi/24)+i\sin(\pi/24)]^6$; (2) $[\cos(\pi/10)+i\sin(\pi/10)]^{10}$.

68. Prove that $(\cos\varphi-i\sin\varphi)^{-1}=\cos\varphi-i\sin\varphi$.

69. Compute: (1) $\sqrt[4]{i}$; (2) $\sqrt{1+i}$.

70. Solve the following binomial equations: (1) $x^3 - 8 = 0$; (2) $8x^3 - 27 = 0$; (3) $x^3 + 125 = 0$; (4) $27x^3 + 1 = 0$; (5) $x^4 + 81 = 0$; (6) $x^6 - 64 = 0$.

71. Set up a quadratic equation with real coefficients whose roots are the following numbers: (1) i and $-i$; (2) $3 + i$ and $3 - i$; (3) $1 - i\sqrt{5}$ and $1 + i\sqrt{5}$.

72. Solve the biquadratic equation $x^4 + x^2 + 1 = 0$, performing the operation of evolution (finding the root) in trigonometric form.

73. Solve the following equations: (1) $x^6 - 28x^3 + 27 = 0$; (2) $(2x + 3)^6 - 9(2x + 3)^3 + 8 = 0$.

QUIZ No. 1

Form I

(1) Find the modulus and argument of the number $\frac{8+2i}{5-3i}$.

(2) Perform the indicated operations: $\frac{5+2i}{2-5i} - \frac{3-4i}{4+3i}$.

(3) Raise to the indicated power, applying De Moivre's formula: $(-1+i\sqrt{3})^9$.

(4) Find the root: $\sqrt{2+2i\sqrt{3}}$.

(5) Solve the equation: $x^4 - 4x^2 + 16 = 0$.

Form II

(1) Find the modulus and argument of the number $\frac{5+i}{2+3i}$.

(2) Perform the indicated operations: $\frac{4+3i}{3-4i} - \frac{5-4i}{4+5i}$.

(3) Raise to the indicated power, applying De Moivre's formula: $\left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right)^6$.

(4) Find the root: $\sqrt[3]{8}$.

(5) Solve the equation: $x^4 - 2x^2 + 4 = 0$.

Chapter 15
DIFFERENTIAL EQUATIONS

Sec. 15.1. First-order Differential Equations with Variables Separable

A *differential equation* is an equation connecting the independent variable x, the unknown function y, and its derivatives or differentials.

A differential equation is symbolized as follows:

$F(x, y, y') = 0$, $F(x, y, y'') = 0$, $F(x, y, y', y'', \ldots, y^{(n)}) = 0$.

A differential equation is said to be *ordinary* if the unknown function depends solely on one independent variable.

The *order* of a differential equation is the order of the highest derivative (or differential) which appears in the given equation.

Any function which reduces the differential equation to an identity when substituted for the dependent variable is called a *solution* (or *integral*) of a differential equation.

The *general solution* (or the *general integral*) of a differential equation is a solution in which the number of essential arbitrary constants is equal to the order of the differential equation. Thus, the general solution of a first-order differential equation contains one arbitrary constant.

A *particular solution* of a differential equation is a solution obtained from the general solution by giving particular values to the arbitrary constants. The values of arbitrary constants are found for definite initial values of the argument and function.

The graph of a particular solution of a differential equation is called an *integral curve* of the equation. To the general solution of differential equation there corresponds a family of integral curves.

A *differential equation* of the first order is an equation containing derivatives (or differentials) of not higher than the first order.

A *differential equation with variables separable* is defined as an ordinary differential equation of the form

$$\frac{dy}{dx} = f(x)\,\varphi(y).$$

To solve this equation, it is first necessary to separate the variables:

$$\frac{dy}{\varphi(y)} = f(x)\,dx,$$

and then to integrate both sides of the obtained equality:

$$\int \frac{dy}{\varphi(y)} = \int f(x)\,dx.$$

1. Find the general solution of the equation $x(1 + y^2)\,dx = y\,dy$.

○ Separating the variables, we have

$$x\,dx = \frac{y\,dy}{1+y^2}.$$

We then integrate both sides of the obtained equation:

$$\int x\,dx = \int \frac{y\,dy}{1+y^2}; \quad \frac{x^2}{2} = \frac{1}{2}\ln(1+y^2) + \frac{1}{2}\ln C$$

Since the arbitrary constant C can take on any numerical values, for the convenience of further transformations, we have written $(1/2)\ln C$

instead of C. Taking the antilogarithms of the last equality, we get

$$x^2 = \ln [C(1+y^2)].$$

This is just the general solution of the given equation. ●

2. Find the particular solution of the equation $s \tan t\, dt + ds = 0$ satisfying the initial conditions $s = 4$ for $t = \pi/3$.

○ Separating the variables, we have

$$\tan t\, dt + \frac{ds}{s} = 0.$$

Integrating both sides of the obtained equation, we get:

$$\int \tan t\, dt + \int \frac{ds}{s} = \ln C; \quad -\ln \cos t + \ln s = \ln C,$$

or

$$\ln s = \ln C + \ln \cos t, \quad s = C \cos t.$$

This is the general solution of the given equation. To find the value of the arbitrary constant C, we substitute the values $t = \pi/3$ and $s = 4$ into the expression for the general solution: $4 = C \cos(\pi/3)$ or $4 = C/2$, whence $C = 8$.

Consequently, the desired particular solution satisfying the indicated initial conditions has the form $s = 8 \cos t$. ●

In Problems 3 through 6 find the general solution for each of the given equations:

3. (1) $x^2\, dx = 3y^2\, dy$; (2) $\sqrt{x}\, dy = \sqrt{y}\, dx$; (3) $\frac{dy}{\sqrt{x}} = \frac{3\, dx}{\sqrt{y}}$; 4) $(1+y)\, dx = (x-1)\, dy$.

4. (1) $xy\, dx = (1+x^2)\, dy$; (2) $y^2\, dx + (x-2)\, dy = 0$.

5. (1) $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$; (2) $x^2 dy - (2xy + 3y)\, dx = 0$.

6. (1) $(1+y^2)\, dx - \sqrt{x}\, dy = 0$; (2) $\sqrt{1-x^2}\, dy - x\sqrt{1-y^2}\, dx = 0$.

In Problems 7 through 14 find the particular solutions of the given equations satisfying the indicated initial conditions:

7. (1) $y\, dy = x\, dx$; $y = 4$ for $x = -2$; (2) $x\, dy = y\, dx$; $y = 6$ for $x = 2$.

8. $ds = (3t^2 - 2t)\, dt$; $s = 4$ for $t = 2$.

9. $\frac{dy}{x^2} = \frac{dx}{y^2}$; $y = 2$ for $x = 0$.

10. $\frac{dy}{x-1} = \frac{dx}{y-2}$; $y = 4$ for $x = 0$.

11. $(1+y)\, dx = (1-x)\, dy$; $y = 3$ for $x = -2$.

12. $(1+x)\, y\, dx + (1-y)\, x\, dy = 0$; $y = 1$ for $x = 1$.

13. $y^2\, dx = e^x\, dy$; $y = 1$ for $x = 0$.

14. $\frac{dx}{\cos^2 x \cos y} = \cot x \sin y\, dy$; $y = \pi$ for $x = \pi/3$.

Sec. 15.2. Problems on Setting up Differential Equations

15. Find the equation of motion of a body displacing along the x-axis if it started from the point M (4; 0) with velocity $v = 2t + 3t^2$.

○ For motion in a straight line, velocity is the derivative of path with respect to time. Denoting the path by x, we have $v = \frac{dx}{dt}$; then

$$\frac{dx}{dt} = 2t + 3t^2, \text{ or } dx = (2t + 3t^2)\,dt.$$

Integrating, we get $x = t^2 + t^3 + C$. Using the initial conditions, we find C. Since $x = 4$ for $t = 0$, substituting these values into the general solution, we find $C = 4$. Thus, the equation of motion of the body has the form $x = t^2 + t^3 + 4$. ●

16. Set up the equation of the curve passing through the point M (2; —3) and having the tangent with the slope $4x - 3$.

○ By the hypothesis, we have

$$\frac{dy}{dx} = 4x - 3 \text{ or } dy = (4x - 3)\,dx.$$

Integrating, we get $y = 2x^2 - 3x + C$. Using the initial conditions $x = 2$ and $y = -3$, we find $C = -5$. Consequently, the desired equation has the form $y = 2x^2 - 3x - 5$. ●

17. An open tank is filled with water. At the beginning, the water temperature was equal to 70°, and in ten minutes it turns out to be equal to 65°, the ambient temperature being equal to 15°. Determine: the temperature of the water in the tank in 30 minutes from the initial instant; the instant of time when the water temperature will be equal to 20°.

○ We denote by T the water temperature at the instant t. The rate of cooling of the water contained in the tank is the rate of change of the function relating t and T, that is, the derivative $\frac{dT}{dt}$.

The quantity $\frac{dT}{dt}$ is proportional to the difference between the water temperature and the ambient temperature, that is, $k\,(T - 15°)$, where k is a proportionality factor. Consequently, $\frac{dT}{dt} = k\,(T - 15°)$. Separating the variables, we have $\frac{dT}{T - 15°} = k\,dt$. We now integrate the obtained equation:

$$\int \frac{dT}{T - 15°} = \int k\,dt; \quad \ln (T - 15°) = kt + C,$$

or

$$T - 15° = e^{kt+C} = e^{kt}e^{C} = e^{kt}C_1,$$

whence

$$T = C_1 e^{kt} + 15°. \qquad (*)$$

This relationship just expresses the rate of cooling of the water.

We find C_1 for the initial conditions: $T = 70°$ at $t = 0$. We have

$$70 = C_1 e^{k \cdot 0} + 15 \text{ or } 55° = C_1 e^0 = C_1, \text{ i.e. } C_1 = 55°.$$

Substituting the found value of C_1 into the equality (*), we get

$$T = 55° e^{kt} + 15°. \qquad (**)$$

Let us now find k. By the hypothesis, $T = 65°$ at $t = 10$ min. Substituting these values into the relationship (**), we get

$$65° = 55° e^{k \cdot 10} + 15° \text{ or } 50° = 55° e^{10k} \text{ or } 10/11 = e^{10k}.$$

Taking the logarithms of the last equality, we have

$$\log 10 - \log 11 = 10\, k \log e,$$

whence

$$k = \frac{1 - \log 11}{10 \log e} = \frac{1 - 1.0414}{10 \cdot 0.4343} = -\frac{0.0414}{4.343} = -0.009532.$$

Substituting the value of k into the relationship (**), we obtain the law of cooling relating the variables t and T:

$$T = 55° e^{-0.009532t} + 15°. \qquad (***)$$

We then find the water temperature in 30 minutes from the initial instant. For this purpose, we substitute the value $t = 30$ into the equation (***):

$$T = 55° e^{-0.009532 \cdot 30} + 15° \text{ or } T = 55° e^{-0.286} + 15°.$$

Performing calculations, we get:

$x = 55 \cdot e^{-0.286}$, $\log x = \log 55 - 0.286 \log e = 1.7404$

$- 0.286 \cdot 0.4343 = 1.7404 - 0.1242 = 1.6162$. $x = 41.32 \approx 41$;

then

$$T = 41° + 15° = 56°.$$

We finally find how much time it takes for the temperature of the water contained in the tank to reach 20°. Substituting the value $T = 20°$ into the relationship (***), we get

$$20° = 55° e^{-0.009532t} + 15° \text{ or } 5° = 55° e^{-0.009532t},$$

whence

$e^{-0.009532t} = 1/11 \approx 0.0909$ or $-0.009532t \log e$

$= \log 0.0909 = \bar{2}.9586,$

that is,

$$t = -\frac{\bar{2}.9586}{0.009532 \cdot 0.4343} = \frac{1.041}{0.009532 \cdot 0.4343} \approx 251 \text{ min} = 4 \text{ hrs } 11 \text{ min}. \bullet$$

18. Experience suggests that the rate of decay of radium at every instant of time is proportional to the initial mass of radium. At the

initial instant of time ($t = 0$) there was R_0 grams of radium. Derive the formula for computing the mass of radium at any instant of time t.

○ Let the proportionality factor k be known ($k > 0$). Let us denote by R the mass of the remaining radium at the instant t. The rate of decay of radium is the rate of change of the function relating t and R, that is, the derivative $\frac{dR}{dt}$. By the hypothesis $\frac{dR}{dt} = -kR$. The minus sign indicates that R is a decreasing function, consequently, $\frac{dR}{dt} < 0$ and $kR > 0$, since $k > 0$ and $R > 0$.

Separating the variables and integrating, we get

$$\frac{dR}{R} = -k\,dt; \quad \int \frac{dR}{R} = -\int k\,dt;$$

$$\ln R = -kt + \ln C, \quad \ln R - \ln C = -kt, \quad \ln (R/C) = -kt.$$

Taking the antilogarithms of the last equality we find $R/C = e^{-kt}$ or

$$R = Ce^{-kt}. \qquad (*)$$

This relationship expresses the equation of radium decay.

Let us find the constant quantity C under the initial conditions $R = R_0$ for $t = 0$. Substituting these values into the equality (*), we get $R_0 = Ce^{-k \cdot 0}$, that is, $C = R_0$. Consequently, the desired function has the form $R = R_0 e^{-kt}$. ●

19. A disk rotates in a liquid. Its angular speed decreases due to friction, the friction force being proportional to the angular speed. Find: (1) the rotational speed of the disk at the instant $t = 120$ s, if at $t = 0$ it rotated with a speed of 12 rad/s and at $t = 10$ s its speed became equal to 8 rad/s; (2) the instant of time when the rotational speed of disk will turn out to be equal to 1 rad/s.

○ Let ω be the angular speed of the disk at the instant t. Then the deceleration of disk rotation under the action of the friction force is equal to $\frac{d\omega}{dt}$. By the hypothesis, $\frac{d\omega}{dt} = k\omega$, where k is a proportionality factor. Separating the variables and integrating, we get

$$\frac{d\omega}{\omega} = k\,dt, \quad \int \frac{d\omega}{\omega} = k \int dt, \quad \ln \omega = kt + C,$$

whence $\omega = e^{kt+C} = e^{kt}e^{C}$ or

$$\omega = C_1 e^{kt}. \qquad (*)$$

We then find the constant C_1 under the initial conditions $\omega = 12$ rad/s for $t = 0$. Substituting these values into the equality (*), whence $12 = C_1 e^{k \cdot 0}$, that is, $12 = C_1$. Thus,

$$\omega = 12e^{kt}. \qquad (**)$$

Find the numerical value of k from the following data: $t = 10$ s and $\omega = 8$ rad/s. Substituting these values into the equality (**), we get $8 = 12e^{k \cdot 10}$, whence

$$e^{10k} = 2/3, \quad 10k \log e = \log 2 - \log 3,$$

$$k = \frac{\log 2 - \log 3}{10 \log e} = -\frac{\log 3 - \log 2}{10 \log e} = -\frac{0.4771 - 0.3010}{10 \cdot 0.4343} = -0.0405.$$

Substituting the value of k into the equality (**), we get

$$\omega = 12e^{-0.0405t}. \qquad (***)$$

We then find the rotational speed of the disk at the instant $t = 120$ s. Substituting the value $t = 120$ into the equality (***) we obtain

$$\omega = 12e^{-0.0405 \cdot 120} = 12e^{-4.9} = 0.09 \text{ (rad/s)}.$$

We finally determine at what instant of time the disk will rotate with a speed of 1 rad/s. Substituting the value $\omega = 1$ into the relationship (***), we have

$$1 = 12e^{-0.0405t}; \quad e^{-0.0405t} = \frac{1}{12}; \quad -0.0405t \log e = \log 1 - \log 12,$$

$$t = \frac{\log 12}{0.0405 \log e} = 61 \text{ (s)}. \bullet$$

20. Find the equation of motion of a body displacing along the y-axis if it began moving from the point M (0, 6) with the velocity $v = 4t - 6t^2$.

21. Set up the equation of the curve passing through the point M (2, —1) and having a tangent with the slope $k = 1/(2y)$.

22. The ambient temperature is equal to 20°. During 40 minutes a body is cooled from 80° to 30°. What temperature will the body have in 30 minutes after the initial measurement?

23. Radium decays at a rate directly proportional to its initial amount. In how many years will one-half of its initial amount decay? Put $k = 0.00044$ (time is measured in years).

24. Decelerating action of friction on a disk rotating in a liquid is proportional to its angular speed. At what instant of time will the rotational speed of the disk turn out to be equal to 2 rad/s if at $t = 0$ it rotates with a speed of 20 rad/s, and at $t = 8$ s with a speed of 16 rad/s?

Sec. 15.3. Linear Differential Equations of the First Order

An equation of the form

$$\frac{dy}{dx} + f(x)y + \varphi(x) = 0,$$

where $f(x)$ and $\varphi(x)$ are functions of x is called a *linear differential equation of the first order*. In a particular case $f(x)$ and $\varphi(x)$ may be constant quantities.

This equation is reduced to an equation with variables separable with the aid of the substitution $y = uz$, where u and z are new functions of x.

25. Find the general solution of the equation $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^3$.

○ This is a linear equation: here $f(x) = -2/(x+1)$, $\varphi(x) = -(x+1)^3$. We set $y = uz$ and differentiate this equality with respect to x:

$$\frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx}$$

Substituting now the expressions for y and $\frac{dy}{dx}$ in the given equation, we get

$$u\frac{dz}{dx} + z\frac{du}{dx} - \frac{2uz}{x+1} = (x+1)^3,$$

or

$$u\frac{dz}{dx} + z\left(\frac{du}{dx} - \frac{2u}{x+1}\right) = (x+1)^3. \qquad (*)$$

Since one of the auxiliary functions u or z may be chosen arbitrarily, we take, as u, one of the particular solutions of the equation $\frac{du}{dx} - \frac{2u}{x+1} = 0$. Separating the variables in this equation and integrating, we have

$$\frac{du}{u} - \frac{2\,dx}{x+1} = 0, \quad \int\frac{du}{u} = 2\int\frac{dx}{x+1}; \quad \ln u = 2\ln(x+1),$$

$$u = (x+1)^2$$

(we regard the constant C to be equal to zero, since we find one of the particular solutions).

We now substitute the expression for u into the equation (*); then we get the equation

$$(x+1)^2\frac{dz}{dx} = (x+1)^3 \quad \text{or} \quad \frac{dz}{dx} = x+1.$$

Hence we find

$$\int dz = \int (x+1)\,dx; \quad z = \frac{(x+1)^2}{2} + C.$$

Knowing u and z, we now get the general solution of the given equation:

$$y = uz = (x+1)^2\left[\frac{(x+1)^2}{2} + C\right] = \frac{(x+1)^4}{2} + C(x+1)^2. \bullet$$

26. Find the particular solution of the equation: $\cos x\,dy + y\sin x\,dx = dx$, if $y = 1$ for $x = 0$.

○ Dividing all the terms of the given equation by $\cos x\,dx$, we get the equation

$$\frac{dy}{dx}+y\tan x=\frac{1}{\cos x}, \qquad (*)$$

which is a linear one. Setting $y=uz$; we have $\frac{dy}{dx}=u\frac{dz}{dx}+z\frac{du}{dx}$. Substituting the expressions for y and $\frac{dy}{dx}$ into the equation (*), we have

$$u\frac{dz}{dx}+z\frac{du}{dx}+uz\tan x=\frac{1}{\cos x},$$

or

$$u\frac{dz}{dx}+z\left(\frac{du}{dx}+u\tan x\right)=\frac{1}{\cos x}. \qquad (**)$$

For finding u, we get the equation

$$\frac{du}{dx}+u\tan x=0, \quad \text{i.e.} \quad \frac{du}{u}+\tan x\,dx=0,$$

whence

$$\int\frac{du}{u}=-\int\tan x\,dx;\ \ln u=\ln\cos x;\ u=\cos x.$$

Substituting the expression for u into the equation (*), we have

$$\cos x\frac{dz}{dx}=\frac{1}{\cos x} \text{ or } \frac{dz}{dx}=\frac{1}{\cos^2 x},$$

$$\text{i.e.} \quad z=\tan x+C.$$

Consequently, the general solution of the given equation is written as follows:

$$y=uz=\cos x\,(\tan x+C)=\sin x+C\cos x.$$

Making use of the initial conditions $y=1$, $x=0$, we have $1=\sin 0+C\cos 0$, whence $C=1$. Thus, the desired particular solution has the form $y=\sin x+\cos x$. ●

In Problems 27 and 28 find the general solution of each of the indicated equations:

27. (1) $\frac{dy}{dx}-2y-3=0$; (2) $\frac{dy}{dx}=y+1$; (3) $x\frac{dy}{dx}-x^2+2y=0$.

28. (1) $\frac{dy}{dx}+xy=x$; (2) $\frac{dy}{dx}-\frac{2y}{x+1}=(x+1)^2$; (3) $\frac{dy}{dx}-y\cot x=\sin x$.

In Problems 29 through 31 find the particular solutions of the given equations satisfying the indicated initial conditions:

29. $\frac{dy}{dx}-\frac{3y}{x}=e^x x^3$; $y=e$ for $x=1$.

30. $\frac{dy}{dx}+\frac{2y}{x}=\frac{1}{x^2}$; $y=1$ for $x=2$.

31. $\frac{dy}{dx}\cos^2 x=\tan x-y$; $y=0$ for $x=0$.

Sec. 15.4. Incomplete Second-order Differential Equations

An equation involving derivatives (or differentials) of not higher than second order is said to be of second order and is called a *second-order differential equation*. Its general form is written as follows:

$$\frac{d^2y}{dx^2}=f\left(x,\ y,\ \frac{dy}{dx}\right).$$

The general solution of a second-order differential equation contains two arbitrary constants.

Let us consider incomplete second-order differential equations by way of examples.

32. Find the general solution of the equation $\frac{d^2y}{dx^2}=\sin x$.

○ This is an incomplete second-order differential equation of the form $\frac{d^2y}{dx^2}=f(x)$. We set $\frac{dy}{dx}=z$, then the given equation can be written in the form

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\sin x, \quad \text{i.e.} \quad \frac{dz}{dx}=\sin x,$$

whence $dz=\sin x\,dx$. Integrating the last equality, we obtain

$$\int dz=\int \sin x\,dx, \quad \text{i.e.} \quad z=-\cos x+C_1.$$

Consequently,

$$\frac{dy}{dx}=-\cos x+C_1, \quad \text{i.e.} \quad dy=(-\cos x+C_1)\,dx.$$

Integrating once again, we find

$$\int dy=\int(-\cos x+C_1)\,dx \quad \text{or} \quad y=-\sin x+C_1x+C_2.$$

This is just the general solution of the given equation. ●

33. Find the particular solution of the equation $\frac{d^2s}{dt^2}=12t$ if $s=2$ and $\frac{ds}{dt}=20$ for $t=0$.

○ Setting $\frac{ds}{dt} = z$, we rewrite the given equation in the form $\frac{dz}{dt} = 12t$, whence we get

$$dz = 12t\,dt, \quad \int dz = 12 \int t\,dt, \quad z = 6t^2 + C_1.$$

Thus,

$$\frac{ds}{dt} = 6t^2 + C_1, \qquad (*)$$

that is $ds = (6t^2 + C_1)\,dt$. Integrating, we find the general solution of the given equation:

$$s = 2t^3 + C_1 t + C_2. \qquad (**)$$

Let us now find the particular solution of the equation. Substituting the initial data into the equations (*) and (**), we get the system of equations:

$$\begin{cases} 20 = 6 \cdot 0 + C_1, \\ 2 = 2 \cdot 0 + C_1 \cdot 0 + C_2, \end{cases}$$

that is $C_1 = 20$, $C_2 = 2$. Thus, the desired particular solution has the form $s = 2t^3 + 20t + 2$. ●

34. Find the particular solution of the equation $\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$ if $y = \frac{3}{2}$ and $\frac{dy}{dx} = 1$ for $x = 0$.

○ This is an incomplete second-order differential equation of the form $\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}\right)$. We set $\frac{dy}{dx} = z$; then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ and hence $\frac{dz}{dx} = 2z$. Separating the variables in this equation and integrating, we get

$$\frac{dz}{z} = 2\,dx; \quad \int \frac{dz}{z} = 2 \int dx, \quad \ln z = 2x + C_1; \quad z = e^{2x+C_1}.$$

Consequently,

$$\frac{dy}{dx} = e^{2x+C_1}, \qquad (*)$$

that is, $dy = e^{2x+C_1}\,dx$. Integrating, we find the general solution of the given equation:

$$y = (1/2)\,e^{2x+C_1} + C_2. \qquad (**)$$

To find the sought for particular solution, we substitute the initial data into the relationships (*) and (**):

$$\begin{cases} 1 = e^{2 \cdot 0 + C_1}, \\ 3/2 = (1/2)\,e^{2 \cdot 0 + C_1} + C_2 \end{cases} \quad \text{or} \quad \begin{cases} 1 = e^{C_1}, \\ 3/2 = (1/2)\,e^{C_1} + C_2, \end{cases}$$

whence $C_1 = 0$, $C_2 = 1$. Thus, the desired particular solution has the form $y = (1/2)\,e^{2x} + 1$. ●

35. Find the particular solution of the equations $\frac{d^2y}{dx^2} = \frac{1}{x+2}\frac{dy}{dx}$ if $y = 2$ and $\frac{dy}{dx} = 8$ for $x = 2$.

○ This is an incomplete second-order differential equation of the form $\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$. We set $\frac{dy}{dx} = z$, then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$. Substituting the expressions for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the given equation, we get $\frac{dz}{dx} = \frac{1}{x+2}\,z$. Separating the variables and integrating, we have

$$\frac{dz}{z} = \frac{dx}{x+2}; \quad \int \frac{dz}{z} = \int \frac{dx}{x+2}, \quad \ln z = \ln(x+2) + \ln C_1,$$

whence $z = C_1(x + 2)$. Consequently,

$$\frac{dy}{dx} = C_1(x+2). \qquad (*)$$

Now we can find the general solution of the given equation:

$$y = (1/2)\,C_1x^2 + 2C_1x + C_2. \qquad (**)$$

The desired particular solution is found by substituting the initial data into the equations (*) and (**)

$$\begin{cases} 8 = C_1(2+2), \\ 2 = (1/2)\,C_1 \cdot 2^2 + 2C_1 \cdot 2 + C_2, \end{cases}$$

whence $C_1 = 2$ and $C_2 = -10$. Thus, the desired particular solution has the form $y = x^2 + 4x - 10$. ●

In Problems 36 through 42 find the particular solutions of the given equations satisfying the indicated initial conditions:

36. $\frac{d^2y}{dx^2} = 0$; $y = 2$ for $x = 0$, $y = 3$ for $x = 1$.

37. $\frac{d^2y}{dx^2} = 4$; $y = 0$, for $x = 0$, $y = 1$ for $x = 1$.

38. $\frac{d^2s}{dt^2} = 6t$; $s = 0$ and $\frac{ds}{dt} = 10$ for $t = 0$.

39. $\frac{d^2s}{dt^2} = 18t + 2$; $s = 4$ and $\frac{ds}{dt} = 5$ for $t = 0$.

40. $\frac{d^2\theta}{d\omega^2} = \omega^2$; $\theta = 0$ and $\frac{d\theta}{d\omega} = 12$ for $\omega = 0$.

41. $\frac{d^2y}{dx^2} = \frac{dy}{dx}$; $y = 2$ and $\frac{dy}{dx} = 1$ for $x = 0$.

42. $\frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx}$; $y = 6$ and $\frac{dy}{dx} = 1$ for $x = 2$.

43. The acceleration of a freely falling body satisfies the equation $\frac{d^2s}{dt^2} = g$ ($g \approx 9.8$ m/s^2). Find the equation stating the law by which the body falls if $s = s_0$ and $\frac{ds}{dt} = v_0$ at the instant of time $t = 0$.

Sec. 15.5. Homogeneous Linear Differential Equations of the Second Order with Constant Coefficients

A homogeneous linear differential equation of the second order with constant coefficients is defined as an equation of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{15.1}$$

where p and q are constants.

In order to find the general solution of Equation (15.1), we set up the *characteristic equation*

$$r^2 + pr + q = 0, \tag{15.2}$$

which is obtained from Equation (15.1) by replacing $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y by the corresponding powers of r, the function y itself being replaced by unity.

Then the general solution of differential Equation (15.1) is constructed depending on the roots r_1 and r_2 of the characteristic equation (15.2). Here, three cases may occur.

I. The roots r_1 and r_2 are real and distinct. In this case the general solution of Equation (15.1) has the form

$$y = C_1e^{r_1x} + C_2e^{r_2x}. \tag{15.3}$$

II. The roots r_1 and r_2 are real and equal: $r_1 = r_2 = r$. Then the general solution of Equation (15.1) is written as follows

$$y = (C_1 + C_2x)\,e^{rx}. \tag{15.4}$$

III. The roots r_1 and r_2 are complex conjugates: $r_1 = \alpha + \beta i$; $r_2 = \alpha - \beta i$. In this case the general solution of Equation (15.1) is written in the following way:

$$y = e^{\alpha x}\,(C_1 \cos \beta x + C_2 \sin \beta x). \tag{15.5}$$

44. Solve the equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$.

○ We set up the characteristic equation and find its roots: $r^2 - 7r + 10 = 0$; $r_1 = 2$, $r_2 = 5$. Since the roots of the characteristic equation are real and distinct, the general solution of the given differential equation is written according to Formula (15.3): $y = C_1e^{2x} + C_2e^{5x}$. ●

45. Find the particular solution of the equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = 0$ if $y = 1$ and $\frac{dy}{dx} = -1$ for $x = 0$.

○ Set up the characteristic equation $r^2 - 5r = 0$, whence $r_1 = 0$, $r_2 = 5$. Since the roots of the characteristic equation are real and distinct, the general solution of the given differential equation has

the form

$$y = C_1 e^{0 \cdot x} + C_2 e^{5x}, \quad \text{i.e.} \quad y = C_1 + C_2 e^{5x}.$$

To find the desired particular solution, we have to determine the values of the constants C_1 and C_2. Substituting the values $x = 0$, $y = 1$ into the general solution, we get $1 = C_1 + C_2$.

Differentiating the general solution and substituting into the obtained expression the values $x = 0$, $\frac{dy}{dx} = -1$, we have $\frac{dy}{dx} = 5C_2 e^{5x}$; $-1 = 5C_2$. Hence we find: $C_2 = -1/5$, $C_1 = 1 - C_2 = 6/5$. Thus, the desired particular solution has the form: $y = 6/5 - (1/5)\, e^{5x}$. ●

46. Solve the equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$.

○ Set up the characteristic equation and find its roots: $r^2 - 8r + 16 = 0$; $r_1 = r_2 = 4$. The characteristic equation has equal real roots; therefore, according to Formula (15.4), the general solution of the given differential equation is written in the form $y = (C_1 + C_2 x)\, e^{4x}$. ●

47. Find the particular solution of the equation $y'' + 8y' + 16y = 0$ if $y = 1$ and $y' = 1$ for $x = 0$.

○ Since the characteristic equation $r^2 + 8r + 16 = 0$ has equal real roots $r_1 = r_2 = -4$, the general solution of the given differential equation is written in the form

$$y = (C_1 + C_2 x)\, e^{-4x} = C_1 e^{-4x} + C_2 x e^{-4x}.$$

Differentiating the general solution, we have

$$y' = -4C_1 e^{-4x} + C_2 e^{-4x} - 4C_2 x e^{-4x}.$$

Substituting the initial data into the expressions for y and y', we get the system of equations

$$\begin{cases} 1 = C_1 e^0 + C_2 \cdot 0 \cdot e^0, \\ 1 = -4C_1 e^0 + C_2 e^0 - 4C_2 \cdot 0 \cdot e^0 \end{cases} \quad \text{or} \quad \begin{cases} 1 = C_1, \\ 1 = -4C_1 + C_2, \end{cases}$$

whence $C_1 = 1$ and $C_2 = 5$. Consequently, the desired particular solution has the form $y = e^{-4x} + 5xe^{-4x}$. ●

48. Solve the equation $y'' - 6y' + 25y = 0$.

○ Set up the characteristic equation and find its roots: $r^2 - 6r + 25 = 0$; $r_1 = 3 + 4i$, $r_2 = 3 - 4i$; here $\alpha = 3$, $\beta = 4$. Since the characteristic equation has two complex conjugate roots, the general solution of the differential equation is written in accord with Formula (15.5): $y = e^{3x}(C_1 \cos 4x + C_2 \sin 4x)$. ●

49. Find the particular solution of the equation $y'' - 6y' + 13 = 0$ if $y = 1$ and $y' = 5$ for $x = 0$.

○ Since the characteristic equation $r^2 - 6r + 13 = 0$ has complex conjugate roots $r_1 = 3 + 2i$ and $r_2 = 3 - 2i$, the general

solution of the given differential equation is written as follows:

$$y = e^{3x}(C_1 \cos 2x + C_2 \sin 2x).$$

Differentiating the general solution, we have

$$y' = 3e^{3x}(C_1 \cos 2x + C_2 \sin 2x) + e^{3x}(-2C_1 \sin 2x + 2C_2 \cos 2x)$$
$$= e^{3x}(3C_1 \cos 2x + 3C_2 \sin 2x - 2C_1 \sin 2x + 2C_2 \cos 2x)$$
$$= e^{3x}[(3C_1 + 2C_2)\cos 2x + (3C_2 - 2C_1)\sin 2x].$$

We now substitute the initial data into the expressions for y and y':

$$\begin{cases} 1 = e^0(C_1 \cos 0 + C_2 \sin 0), \\ 5 = e^0[(3C_1 + 2C_2)\cos 0 + (3C_2 - 2C_1)\sin 0] \end{cases} \text{ or } \begin{cases} 1 = C_1, \\ 5 = 3C_1 + 2C_2, \end{cases}$$

whence $C_1 = 1$ and $C_2 = 1$. Thus, the desired particular solution has the form: $y = e^{3x}(\cos 2x + \sin 2x)$. ●

In Problems 50 through 52 solve the indicated equations:

50. (1) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$; (2) $y'' - 8y' + 15y = 0$; (3) $y'' + 5y' + 6 = 0$.

51. (1) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 0$; (2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$; (3) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$.

52. (1) $\frac{d^2y}{dx^2} - 9y = 0$; (2) $\frac{d^2y}{dx^2} - 16y = 0$; (3) $\frac{d^2y}{dx^2} - y = 0$.

53. Find the indicated particular solutions of the given equations:

(1) $\frac{d^2y}{dx^2} - 1 = 0$; $y = 2$ and $\frac{dy}{dx} = 0$ for $x = 0$;

(2) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$; $y = 8$ and $\frac{dy}{dx} = 0$ for $x = 0$;

(3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 20 = 0$; $y = \frac{9}{5}$ and $\frac{dy}{dx} = 0$ for $x = 0$.

54. Solve the indicated equations:

(1) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$; (2) $y'' + 2y' + y = 0$;

(3) $y'' + 10y' + 25y = 0$.

55. Find the indicated particular solutions of the following equations:

(1) $y'' - 10y' + 25y = 0$; $y = 2$ and $y' = 8$ for $x = 0$;

(2) $y'' + 6y' + 9y = 0$; $y = 1$ and $y' = 2$ for $x = 0$.

56. Solve the following equations:

(1) $y'' + 9y = 0$; (2) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$; (3) $y'' + 4y' + 7y = 0$.

57. Find the indicated particular solutions of the given equations:

(1) $y'' + 9y = 0$; $y = 1$ and $y' = -6$ for $x = \pi/3$;

(2) $y'' - 4y' + 5y = 0$; $y = 1$ and $y' = -1$ for $x = 0$.

Sec. 15.6. Miscellaneous Problems

In Problems 58 through 65 find the indicated particular solution of each of the following equations:

58. $ds - s \cot t\, dt = 0$; $s = 2$ for $t = \pi/2$.

59. $(1 - y)\, dx + (1 + x)\, dy = 0$; $y = 3$ for $x = 1$.

60. $(1 - x^2)\, dy = xy\, dx$; $y = 1$ for $x = 0$.

61. $x^2\, dy + (x - 1)\, y\, dx = 0$; $y = 1$ for $x = 1$.

62. $\frac{dy}{dx} - 2y - 4 = 0$; $y = -1$ for $x = 0$.

63. $\frac{dy}{dx} + y = \frac{1}{e^x}$; $y = 5$ for $x = 0$.

64. $\frac{d^2s}{dt^2} = -\frac{1}{t}\frac{ds}{dt}$; $s = 2$ and $\frac{ds}{dt} = 1$ for $t = 1$.

65. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$; $y = 4$ and $\frac{dy}{dx} = -4$ for $x = 0$.

QUIZ No. 1

Form I

Find the particular solution of each of the listed differential equations:

(1) $4xy\, dx = (x^2 + 1)\, dy$; $y = 4$ for $x = 1$.

(2) $y' + 4y - 2 = 0$; $y = 1.5$ for $x = 0$.

(3) $\frac{d^2s}{dt^2} = 6t - 4$; $s = 5$ and $\frac{ds}{dt} = 6$ for $t = 2$.

(4) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$; $y = 5$ and $\frac{dy}{dx} = 0$ for $x = 0$.

(5) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13 = 0$; $y = 2$ and $\frac{dy}{dx} = 1$ for $x = 0$.

Form II

(1) $(x^2 + 1)\, dy = xy\, dx$; $y = 2$ for $x = \sqrt{3}$.

(2) $y' = 4y - 2$; $y = 1.5$ for $x = 0$.

(3) $\frac{d^2s}{dt^2} = 6t + 8$; $s = 12$ and $\frac{ds}{dt} = -5$ for $t = -2$.

(4) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$; $y = 3$ and $\frac{dy}{dx} = 0$ for $x = 0$.

(5) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13 = 0$; $y = 3$ and $\frac{dy}{dx} = 11$ for $x = 0$.

Chapter 16
ELEMENTS OF COMBINATORICS AND THEORY OF PROBABILITY

Sec. 16.1. Elements of Combinatorics

Problems dealing with the computation of all possible distinct combinations made up of a finite number of elements according to a certain rule are called *combinatorial*. The branch of mathematics in which such problems are solved is called *combinatorics* (or *combinatorial mathematics*).

1. Permutations. Each different arrangement of all or a part of a finite set of elements (things) is called a *permutation*.

The total number of permutations of n elements is denoted by P_n. This number is equal to the product of all successive natural numbers (integers) from 1 to n inclusive:

$$P_n = 1 \cdot 2 \cdot 3 \ldots (n-1)\, n. \tag{16.1}$$

The product $1 \cdot 2 \cdot 3 \ldots (n-1)n$ is customarily denoted by the symbol $n!$ (read "n factorial"); here we assume $0! = 1$, $1! = 1$. Therefore the equality (16.1) can be rewritten in the form

$$P_n = n! \tag{16.2}$$

2. Ordered Sets and Arrangements. A set in which its elements are arranged in the given order is said to be *ordered*. Let there be given a finite set consisting of n elements. Any of its ordered m-element subset ($m \leqslant n$) is called an *arrangement* of n elements taken m at a time.

The number of arrangements of n elements taken m at a time is denoted by A_n^m. It is found by the formula

$$A_n^m = n\,(n-1)\,(n-2) \ldots (n-m+1), \tag{16.3}$$

which can be written in the form

$$A_n^m = \frac{n!}{(n-m)!} \tag{16.4}$$

If $m = n$, then we obtain the formula

$$A_n^n = P_n = n! \tag{16.5}$$

When solving problems, use is frequently made of the equality

$$A_n^{m+1} = (n-m)\, A_n^m. \tag{16.6}$$

3. Combinations. Let there be given a finite set consisting of n elements. Any of its m-element subset ($m \leqslant n$) is called a *combination* of n elements taken m at a time.

The total number of distinct combinations of n elements taken m at a time is symbolized by C_n^m. It is found by the formula

$$C_n^m = \frac{A_n^m}{P_m}, \tag{16.7}$$

which may also be written in the form

$$C_n^m = \frac{n!}{m!\,(n-m)!}, \tag{16.8}$$

or

$$C_n^m = \frac{n\,(n-1) \ldots (n-m+1)}{m!}. \tag{16.9}$$

In addition, when solving combinatorial problems, use is made of the following formulas expressing the basic properties of combinations:

$$C_n^m = C_n^{n-m} \quad (0 \leqslant m \leqslant n); \tag{16.10}$$

$$C_n^m + C_n^{m+1} = C_{n+1}^{m+1}. \tag{16.11}$$

1. Write all possible permutations of the elements of the indicated sets if: (1) $A = \{1\}$; (2) $A = \{5, 6\}$; (3) $\{a, b, c\}$.

○ (1) (1); $P_1 = 1$; (2) (5, 6); (6, 5); $P_2 = 1 \cdot 2 = 2$; (3) (a, b, c); (a, c, b): (b, a, c); (b, c, a); (c, a, b); (c, b, a); $P_3 = 1 \cdot 2 \cdot 3 = 6$. ●

2. Evaluate the following expressions: (1) $5! + 6!$; (2) $52!/50!$.

○ (1) $5! + 6! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 + 720 = 840$;

(2) $\dfrac{52!}{50!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \ldots 1}{50 \cdot 49 \ldots 1} = 52 \cdot 51 = 2652$. ●

3. Find the number of arrangements: (1) of 10 elements taken 4 at a time; (2) of $n + 4$ elements taken $n - 2$ at a time.

○ By Formula (16.3), we get:

(1) $A_{10}^4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$;

(2) $A_{n+4}^{n+2} = (n + 4)(n + 3) \ldots [n + 4 - (n - 2) + 1] = (n + 4)(n + 3) \ldots 8 \cdot 7$. ●

4. Solve the equation $A_n^5 = 30A_{n-2}^4$.

○ Using Formula (16.3) for the number of arrangements, rewrite the given equation as follows:

$$n(n-1)(n-2)(n-3)(n-4) = 30(n-2)(n-3)(n-4)(n-5).$$

Taking into account that $n \geqslant 6$, divide both parts by $(n - 2) \times (n - 3)(n - 4)$; further we have

$$(n(n-1) = 30(n-5)) \Leftrightarrow (n^2 - 31n + 150 = 0) \Leftrightarrow (n_1 = 6,\ n_2 = 25).$$ ●

5. Compute: (1) C_{15}^{13}, (2) $C_6^4 + C_5^0$.

○ By Formula (16.8), we get:

(1) $C_{15}^{13} = \dfrac{15!}{13!\,(15-13)!} = \dfrac{15 \cdot 14 \cdot 13 \cdot 12 \ldots 1}{13 \cdot 12 \ldots 1 \cdot 2 \cdot 1} = 15 \cdot 7 = 105$;

(2) $C_6^4 + C_5^0 = \dfrac{6!}{4!\,(6-4)!} + 1 = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} + 1 = 15 + 1 = 16$. ●

6. Solve the system of equations $\begin{cases} C_x^y = C_x^{x+2}, \\ C_x^2 = 66. \end{cases}$

○ Let us solve the second equation:

$$\left(C_x^2 = 66\right) \Leftrightarrow \left(\frac{x(x-1)}{1 \cdot 2} = 66\right) \Leftrightarrow (x^2 - x - 132 = 0) \Leftrightarrow (x_1 = -11,$$

$x_2 = 12$). Since $x > 2$, $x_1 = -11$ does not satisfy the conditions of the problem.

Substituting $x = 12$ into the first equation of the system, we get $C_{12}^{y} = C_{12}^{y+2}$. Using the property $C_m^n = C_m^{m-n}$, we have $C_{12}^{y} = C_{12}^{12-y}$. Consequently, $12 - y = y + 2$, whence $y = 5$. Thus, we obtain the *answer*: $x = 12$, $y = 5$. ●

7. Write all possible permutations of the elements of the set $A = \{a, b, c, d\}$.

8. Evaluate the following expressions:

$$(1)\ \frac{10!-8!}{89};\quad (2)\ \frac{5!+6!}{4!};\quad (3)\ 6!\,(7!-3!).$$

9. Prove the following identities:

$$(1)\ \frac{(m+4)!}{m!} = (m+1)(m+2)(m+3)(m+4);\quad (2)\ \frac{(n-2)!}{(n-4)!} = (n-2)(n-3).$$

10. Reduce the fractions: $(1)\ \frac{n!}{(n-2)!}$; $(2)\ \frac{(n-3)!}{n!}$; $(3)\ \frac{2m(2m-1)}{(2m)!}$.

11. Perform the indicated operations:

$$(1)\ \frac{1}{n!}+\frac{1}{(n+1)!};\quad (2)\ \frac{1}{(n+1)!}-\frac{1}{n!};\quad (3)\ \frac{n(n-1)(n-2)(n-3)(n-4)}{(n-3)!}.$$

12. How many elements must be contained in a set so that the number of all possible permutations of the elements of this set does not exceed 100? is not less than 200?

13. In how many ways can a list of 10 men be made?

14. In how many ways can 12 rooms be distributed to form 12 study rooms?

15. Find the number of arrangements: $(1)\ A_{15}^{3}$; $(2)\ A_{m-1}^{m-5}$.

16. Compute: $(1)\ A_7^3 + A_6^3 + A_5^3$; $(2)\ (A_6^5 + A_6^4)/A_6^3$; $(3)\ A_5^2 \cdot A_4^2 \cdot A_3^2$.

17. Thirty students interchanged photographs with one another. What is the total number of the photographs interchanged?

18. Solve the following equations:

$$(1)\ A_{n-2}^3 = 4A_{n-3}^2;\quad (2)\ 20A_{n-2}^3 = A_n^5;\quad (3)\ A_n^4 = 15A_{n-2}^3.$$

19. Solve the following equations:

$$(1)\ A_7^3 = 42x;\quad (2)\ x/A_x^3 = 1/12;\quad (3)\ A_{m+1}^3 = 5m(m+1);$$
$$(4)\ (A_x^4 + A_x^3)/A_x^2 = 13;\quad (5)\ A_{2x}^3 = 14A_x^3.$$

20. Compute: $(1)\ C_{12}^{10}$; $(2)\ C_{100}^{98}$; $(3)\ C_8^5$; $(4)\ C_{100}^{100} + C_{100}^{1}$.

21. Check the following equalities:

$$(1)\ C_n^6 = A_n^{n-6}/P_{n-6};\quad (2)\ C_{15}^6 = C_{15}^9;\quad (3)\ C_{10}^5 + C_{10}^6 = C_{11}^6;\quad (4)\ C_{14}^9 + C_{14}^{10} = C_{15}^{10}.$$

22. Check the indicated equalities: $(1)\ C_{20}^{12} = A_{20}^8/P_8$; $(2)\ C_{15}^4 - C_{15}^3 = C_{16}^4/2$.

23. The number of combinations of n elements taken 3 at a time is one-fifth of the number of combinations of $n + 2$ elements taken 4 at a time. Find n.

24. Solve the following systems of equations:

$$(1)\ \begin{cases} C_n^m = C_n^{m+2}, \\ C_n^2 = 153; \end{cases} \quad (2)\ \begin{cases} C_m^n = C_m^{n+1}, \\ A_m^2 = 20. \end{cases}$$

Sec. 16.2. The Classical and Statistical Definitions of Probability

The study of every phenomenon on the basis of observations and/or experiment involves realization of a set (complex) of conditions (trials). Any result or outcome of a trial is called an *event*.

If, upon realization of a set of conditions, an event may or may not occur, then it is called *random*. An event that unavoidably occurs is called *certain* (or *sure*), and if it definitely cannot occur upon realization of a set of conditions, then it is called *impossible*.

Two events are called *mutually exclusive* (or *incompatible*) if their joint occurrence is impossible. Events are termed *compatible* if upon every realization of a set of conditions the occurrence of one of them does not exclude the occurrence of the other in the same trial.

Two events are termed *contrary* (or *opposite*) if, under the conditions of a trial, being its only outcomes, they are mutually exclusive.

The probability of an event is regarded as the measure of objective possibility of occurrence of a random event.

The number M of occurrences of a certain event A in N trials is called the *absolute frequency*, and the ratio M/N, the *relative frequency*, that is

$$W(A)=M/N. \tag{16.12}$$

Statistical Definition of Probability. The *probability* of occurrence of an event is defined as a constant quantity near which the observed values of frequency are grouped.

Classical Definition of Probability. The *probability* of event A is defined as the ratio of the number of possible trial outcomes m favourable to the occurrence of the given event A to the number n of all possible outcomes (mutually exclusive, the only possible, and equally possible), that is,

$$P(A) = m/n. \tag{16.13}$$

The probability of any event cannot be less than zero and greater than unity, that is, $0 \leqslant P(A) \leqslant 1$. To an impossible event there corresponds the probability $P(A) = 0$, while to a certain event the probability $P(A) = 1$.

25. Having fired 100 shots, a rifleman hitted the target 86 times. Find the relative frequency of hitting the target by this rifleman.

○ Here $M = 100$, $N = 86$. Using Formula (16.12), we get $W(A) = 86/100 = 0.86$. ●

26. From an urn containing 5 white balls and 3 black ones one ball is taken out. Find the probability that the ball will turn out to be black.

○ Let us denote by A the event consisting in the occurrence of a black ball. The total number of outcomes possible $n = 5 + 3 = 8$. The number of outcomes favourable to the occurrence of event A is equal to 3. By formula (16.13), we get $P(A) = m/n = 3/8$. ●

27. From an urn containing 12 white and 8 black balls two balls are drawn at random. What is the probability that both balls will turn out to be black?

○ We denote the event consisting in the occurrence of two black balls by A. The total number of possible outcomes n is equal to the number of combinations of 20 balls taken 2 at a time:

$$n = C_{20}^2 = \frac{20 \cdot 19}{1 \cdot 2} = 190.$$

The number of outcomes m favourable to the occurrence of event A is equal to

$$m = C_8^2 = \frac{8 \cdot 7}{1 \cdot 2} = 28.$$

By Formula (16.13), we find the probability of the occurrence of two black balls: $P(A) = m/n = 28/190 = 14/95$. ●

28. (A Problem on Sampling). There are M defective items in a lot consisting of N items. From this lot, n items are selected at random. What is the probability that there will be m defective items among them?

○ An elementary outcome consists in sampling any n items from their total number N. The total number of such outcomes is equal to the number of combinations of N items taken n at a time, that is, to C_N^n.

Event A is extracting n items m of which are defective. An outcome favourable to the occurrence of event A is the occurrence of a group of n items consisting of $n - m$ high-grade items and m defective items. The number of such groups is equal to $C_{N-M}^{n-m} C_M^m$, since a group of m defective items can be formed in C_M^m ways, and a group of $n - m$ high-grade items in C_{N-M}^{n-m} ways. In this case, any group of quality items can be combined with any group of defective items.

The sought-for probability of event A is equal to the ratio of the number of outcomes favourable to this event to the number of all elementary outcomes:

$$P(A) = \frac{C_{N-M}^{n-m} \cdot C_M^m}{C_N^n}. \quad ●$$

29. Among the first 4000 numbers of natural series, there are 551 prime numbers. Find the relative frequency of the occurrence of a prime number.

30. Among 1000 new-born children, there were 512 boys. Find the relative frequency of birth of boys.

31. An urn contains 7 white and 5 black balls. Find the probability that: (1) a ball taken out at random will turn out to be black; (2) two balls taken out at random will turn out to be black.

Sec. 16.3. Theorems on Addition of Probabilities

The Theorem on Addition of Probabilities of Mutually Exclusive Events. *The probability of the occurrence of one or several pairwise incompatible events, no matter which, is equal to the sum of the probabilities of these events:*

$$P(A + B) = P(A) + P(B); \tag{16.14}$$

$$P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots + P(A_n). \tag{16.15}$$

The Theorem on Addition of Probabilities of Compatible Events. *The probability of the occurrence of at least one of two compatible events is equal to the sum of the probabilities of these events less the probability of their simultaneous occurrence:*

$$P(A + B) = P(A) + P(B) - P(AB). \tag{16.16}$$

For three compatible events the following formula holds true:

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC). \tag{16.17}$$

The event opposite to event A (that is, the nonoccurrence of event A) is denoted by $\overline{A}$. The sum of the probabilities of two opposite events is equal to unity:

$$P(A) + P(\overline{A}) = 1. \tag{16.18}$$

The probability of the occurrence of the event A computed on the hypothesis that the event B has already happened is called the *conditional probability* of the event A under the condition B and is denoted by $P_B(A)$ or $P(A/B)$.

If the events A and B are independent, then

$$P(B) = P_A(B) = P_{\overline{A}}(B). \tag{16.19}$$

Events $A, B, C, \ldots$ are defined as *statistically independent* if the probability of each of them remains unchanged in connection with the occurrence or nonoccurrence of other separate events or any of their combinations.

32. A box contains 20 parts arranged in a random way, five of them being standard. A worker takes out three parts at random. Find the probability that at least one of the taken parts will turn out to be standard (event A).

○ First Method. It is obvious that at least one of the taken parts will turn out to be standard if there will occur any of the three mutually exclusive events: B—one part is standard, two are not; C—two parts are standard, one is nonstandard; and D—three parts are standard.

Thus, the event A can be represented as the sum of the three events: $A = B + C + D$. By the addition theorem, we have: $P(A) = P(B) + P(C) + P(D)$. Hence (see Problem 28),

$$P(B) = \frac{C_5^1 \cdot C_{15}^2}{C_{20}^3} = \frac{5}{1} \cdot \frac{15 \cdot 14}{1 \cdot 2} \cdot \frac{1 \cdot 2 \cdot 3}{20 \cdot 19 \cdot 18} = \frac{35}{76};$$

$$P(C) = \frac{C_5^2 \cdot C_{15}^1}{C_{20}^3} = \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{15}{1} \cdot \frac{1 \cdot 2 \cdot 3}{20 \cdot 19 \cdot 18} = \frac{5}{38};$$

$$P(D) = \frac{C_5^3}{C_{20}^3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \cdot \frac{1 \cdot 2 \cdot 3}{20 \cdot 19 \cdot 18} = \frac{1}{114}.$$

Adding together the found results, we get $P(A) = 35/76 + 5/38 + 1/114 = 137/228$.

Second Method. The events A (at least one of the three taken parts turned out to be standard) and $\bar{A}$ (none of the taken parts turned out to be standard) are opposite; therefore $P(A) + P(\bar{A}) = 1$, or $P(A) = 1 - P(\bar{A})$.

The probability of the occurrence of the event $\bar{A}$

$$P(\bar{A}) = \frac{C_{15}^3}{C_{20}^3} = \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3} \cdot \frac{1 \cdot 2 \cdot 3}{20 \cdot 19 \cdot 18} = \frac{91}{228}.$$

Consequently, the sought-for probability $P(A) = 1 - P(\bar{A}) = 1 - 91/228 = 137/228$. ●

33. Find the probability that a two-digit number taken at random will turn out to be multiple either of 3 or 5, or of both of these numbers simultaneously.

○ Let A be the event consisting in that the randomly taken number is multiple of 3, and let B be the event consisting in that this number is multiple of 5. Find $P(A + B)$. Since A and B are compatible events, we take advantage of Formula (16.16):

$$P(A + B) = P(A) + P(B) - P(AB).$$

The total number of two-digit numbers is 90: 10, 11, . . ., 98, 99. Of them, 30 are multiple of 3 (favourable to the occurrence of the event A); 18 are multiple of 5 (favourable to the occurrence of the event B), and 6 are multiple of both 3 and 5 (favourable to the occurrence of the event AB). Thus,

$P(A) = 30/90 = 1/3$, $P(B) = 18/90 = 1/5$,

$$P(AB) = 6/90 = 1/15,$$

that is

$$P(A + B) = 1/3 + 1/5 - 1/15 = 7/45. \quad ●$$

34. Placed in a box in a random order are 10 parts, 4 of which are standard. A quality inspector took 3 parts at random. Compute the probability that at least one of the taken parts turned out to be standard.

35. An urn contains 10 white, 15 black, 20 blue, and 25 red balls. Compute the probability that the extracted ball will turn out to be white, black, or red.

Sec. 16.4. Theorems on Multiplication of Probabilities

The Theorem on Multiplication of Probabilities of Independent Events. *The probability of simultaneous occurrence of two independent events is equal to the product of the probabilities of these events*

$$P(AB) = P(A) \cdot P(B). \qquad (16.20)$$

The probability of occurrence of several statistically independent events is computed by the formula

$$P(A_1A_2 \ldots A_n) = P(A_1) \cdot P(A_2) \ldots P(A_n). \quad (16.21)$$

The Theorem on Multiplication of Probabilities of Dependent Events *The probability of simultaneous occurrence of two dependent events is equal to the product of one of them by the conditional probability of the other*:

$$P(AB) = P(A) \cdot P_A(B) = P(B) \cdot P_B(A). \quad (16.22)$$

36. One urn contains four white and eight black balls, and the other urn three white and nine black balls. One ball is then taken out from each of the urns. Compute the probability that both balls will turn out to be white.

○ Let A be the occurrence of a white ball from the first urn, and let B be the occurrence of a white ball from the second urn. It is obvious that A and B are independent events. We find $P(A) = 4/12 = 1/3$, $P(B) = 3/12 = 1/4$. By Formula (16.20), we get

$$P(AB) = P(A) \cdot P(B) = (1/3) \cdot (1/4) = 1/12. \quad \bullet$$

37. A box contains 12 parts, 8 of which are standard. A worker takes at random two parts. Compute the probability that both parts will turn out to be standard.

○ Let A denote the first taken standard part, and let B denote the second taken standard part. The probability that the first part is standard is $P(A) = 8/12 = 2/3$. The probability that the second taken part will turn out to be standard, provided that the first part was standard, that is, the conditional probability of the event B, is $P_A(B) = 7/11$.

The probability that both taken parts will turn out to be standard is found by the theorem of multiplication of the probabilities of dependent events:

$$P(AB) = P(A) \cdot P_A(B) = (2/3) \cdot (7/11) = 14/33. \quad \bullet$$

38. An operator attends to two independently working automatic machines. The probability that the first machine will not require the attention of the operator during an hour is equal to 0.8, and for the second machine this probability is equal to 0.7. Find the probability that neither of the machines will require the attention of the operator during an hour.

39. An urn contains six balls three of which are white. Two balls are taken out at random. Compute the probability that both balls will turn out to be white.

Sec. 16.5. The Formula of Total Probability. Bayes's Formula

Let events (hypotheses) $B_1, B_2, \ldots, B_n$ form a complete group of events, and on the occurrence of each of them, say B_i, event A may occur with a certain conditional probability $P_{B_i}(A)$. Then the probability of the occurrence of the event A is equal to the sum of the products of the probabilities of each of the hypotheses by the corresponding conditional probability of the event A:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + \ldots + P(B_n) \cdot P_{B_n}(A), \quad (16.23)$$

where $P(B_1) + P(B_2) + \ldots + P(B_n) = 1$.

Formula (16.23) is called the *formula of total probability.*

Suppose the event A may occur only under the condition of the occurrence of one of the incompatible events (hypotheses) $B_1, B_2, \ldots, B_n$, which form a complete group of events. If the event A has already happened, then the probabilities of the hypotheses can be revalued by *Bayes's formula* (or the *formula for the probability of hypotheses*)

$$P_A(B_i) = \frac{P(B_i) \cdot P_{B_i}(A)}{P(A)}, \quad (16.24)$$

where $P_A(B_i)$ is the probability of each of the hypotheses after the trial which results in the occurrence of the event A; $P_{B_i}(A)$ is the conditional probability of the event A on the occurrence of the event B_i, and $P(A)$ is found by the formula of total probability (16.23).

40. A stock-room received workpieces machined on three lathes. 40% of the total amount of the received workpieces were manufactured on the first lathe, 35% on the second lathe, and 25% on the third; 90% of the workpieces manufactured on the first lathe were of the first grade, on the second lathe 80% were of the first grade, and on the third lathe 70% were of the first grade. What is the probability that a workpiece taken at random will turn out to be of the first grade?

○ Let us introduce the following notation: B_1 is a workpiece manufactured on the first lathe, B_2 on the second lathe, and B_3 on the third lathe; event A consists in that the workpiece turned out to be of the first grade. It follows from the hypothesis that $P(B_1) = 0.4$, $P(B_2) = 0.35$, $P(B_3) = 0.25$, $P_{B_1}(A) = 0.9$, $P_{B_2}(A) = 0.8$ and $P_{B_3}(A) = 0.7$. Consequently,

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A) =$$
$$0.4 \cdot 0.9 + 0.35 \cdot 0.8 + 0.25 \cdot 0.7 = 0.815. \quad \bullet$$

41. The first box contains eight white and six black balls, and the second box, ten white and four black balls. A box and a ball are chosen at random. It is known that the ball taken out is black. Find the probability that the first box was chosen.

○ Let us introduce the following notation: B_1 denotes the event consisting in that there was chosen the first box; B_2—there was chosen the second box, A means that when carrying out two successive

trials of choosing a box and a ball there was taken out a black ball. Then $P(B_1) = 1/2$, $P(B_2) = 1/2$. The probability of extracting a black ball after the first box was chosen is $P_{B_1}(A) = 6/14 = 3/7$. The probability of extracting a black ball after the second box was chosen is $P_{B_2}(A) = 4/14 = 2/7$.

Using the formula of total probability, we find the probability that the extracted ball has turned out to be black:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A)$$
$$= (1/2) \cdot (3/7) + (1/2) \cdot (2/7) = 5/14.$$

The sought-for probability that a black ball was taken out of the first box is computed by Bayes's formula:

$$P_A(B_1) = \frac{P(B_1) \cdot P_{B_1}(A)}{P(A)} = \frac{(1/2) \cdot (3/7)}{5/14} = \frac{3}{5}. \bullet$$

42. In an urn containing three balls a white ball was placed, and then one ball was taken out of this urn at random. Find the probability that the extracted ball will turn out to be white if all possible hypotheses concerning the initial set of balls (by colour) are equally possible.

43. A supply of manufactured parts is placed in a box: 16 parts from the first bay, 24 from the second bay, and 20 from the third. The probability that a part manufactured at the second bay is of excellent quality is equal to 0.6, and for the parts manufactured at the first and third bays the respective probabilities are equal to 0.8 each. Find the probability that a randomly chosen part will turn out to be of excellent quality.

44. Two automatic machines produce equal parts which are fed to a common conveyer. The output of the first machine is three times the output of the second. The first machine produces, on the average, 80% of parts of the first grade, and the second—90%. A part randomly taken from the conveyer turned out to be of the first grade. Find the probability that this part had been produced by the first machine.

45. There are three batches of parts, each containing 30 pieces. The number of standard parts in the first, second, and third batches is equal to 30, 25, and 20, respectively. A part extracted at random from an arbitrarily chosen batch turned out to be standard. The part is then returned to its batch, and another part is extracted at random which also proved to be standard. Find the probability that the two parts were extracted from the third batch.

Sec. 16.6. Trial Recurrence. Bernoulli's Formula

If trials are carried out in which the probability of occurrence of event A in each trial is independent of the outcomes of other trials, then such trials are said to be *independent with respect to the event A*.

The probability that in n independent trials in each of which the probability of the occurrence of event A is equal to p (where $0 < p < 1$) the event A will occur exactly k times (no matter in what succession) is found by the *Bernoulli formula*

$$P_n(k) = \frac{n!}{k!\,(n-k)!}\,p^k q^{n-k} = C_n^k p^k q^{n-k}, \text{ where } q = 1-p \qquad (16.25)$$

46. The probability of hitting a target by one shot is $p = 0.8$. Find the probability that of ten shots seven will hit the target.

○ Here $n = 10$, $k = 7$, $p = 0.8$, $q = 0.2$. Using the Bernoulli formula, we find

$$P_{10}(7) = \frac{10!}{7!\,(10-7)!} \cdot (0.8)^7 \cdot (0.2)^{10-7} = \frac{10!}{7!3!} \cdot (0.8)^7 \cdot (0.2)^3 = 0.2. \quad \bullet$$

47. The probability of hitting a target by one shot is equal to 0.8. Find the probability that of four shots three will hit the target.

48. The germination of seeds is estimated by a probability of 0.8. What is the probability that of five sown seeds at least four will spring?

49. An average of 4% of machined parts are defective. What is the probability that of every 30 parts tested for defects two will turn out to be faulty?

Sec. 16.7. Miscellaneous Problems

50. Solve the following equations:

(1) $\frac{(n+2)!}{n!} = 108$; (2) $\frac{(k+1)!}{(k-1)!} = 42$; (3) $8C_{2n+1}^{n+1} = 5C_{2n+2}^{n+2}$;

(4) $13C_{2n}^{n+1} = 8C_{2n+1}^{n-1}$; (5) $C_n^3 = \frac{4}{15}C_{n+2}^4$.

51. Solve the inequalities: (1) $\frac{(n-1)!}{(n-3)!} < 20$; (2) $\frac{(n-1)!}{(n-3)!} > 30$.

52. The number of combinations of n elements taken 4 at a time is to the number of combinations of $n + 2$ elements taken 5 at a time as 5 to 18 (5 : 18). Find n.

53. A box contains six white and ten black balls. Two balls are taken out of the box at random. Find the possibility that both balls will turn out to be black.

54. There are two urns. The first urn contains ten white and two black balls, and the second urn has four white and eight black balls. One ball was taken out from each urn. What is the probability that both balls will turn out to be black?

55. The letters "a", "l", "e", "b", and "t" are written on separate cards. After they are intermingled, one card is taken at a time and is placed side by side in succession. Compute the probability that the word "table" will be formed by these letters.

56. Three boys shoot at a target. The probabilities of hitting the target for the first, second, and third boys are respectively equal to 3/4, 4/5, and 9/10. Find the probability that the target will be hitted by at least one boy.

57. It is known that 3/4 of the students of a group are second-class sportsmen, and 1/4 of them are first-class sportsmen. Find the probability that among three randomly chosen students two are second-class and one is first-class sportsmen.

58. The probability to win in a lottery is equal to 0.15. What is the probability that at least one of four lottery tickets will win?

59. In a chess tournament, there are ten participants having equal chances of any outcome in each game (one for each two participants). What is the probability that one of the participants will win all the games?

60. A box contains 60 standard and 40 nonstandard parts. Find the probability that of two parts taken out at random one will turn out to be standard, the other nonstandard.

QUIZ No. 1

Form I

(1) Prove the identity:

$$C_n^9 + C_n^8 = C_{n+1}^9.$$

(2) Solve the equation

$$\frac{n!}{(n-5)!} = \frac{20n!}{(n-3)!}.$$

(3) Solve the equation

$$5C_{2n}^{n-1} = 8C_{2n-1}^{n}.$$

(4) Rolled-up labels are numbered with all two-digit numbers. One label is then taken at random. What is the probability that the number of the taken label consists of equal digits?

(5) A box contains parts, of which 12 were manufactured on the first machine tool, 20 on the second, and 16 on the third. The probability that the parts made on the first, second, and third machines are of high quality is equal to 0.9, 0.8, and 0.6, respectively. Find the probability that a part extracted at random will turn out to be of high quality.

Form II

(1) Prove the identity:

$$C_{n+3}^5 + C_{n+3}^4 = C_{n+1}^5,$$

(2) Solve the equation

$$\frac{(2n)!}{(2n-3)!} = \frac{40n!}{(n-1)!}.$$

(3) Solve the equation

$$7C_{2n-2}^{n-2} = 3C_{2n-1}^{n-1}.$$

(4) An urn contains 12 balls, 3 of which are white and 9 are black. What is the probability that a ball taken at random will turn out to be black?

(5) Two continuous production lines produce the same workpieces which are then transported to the quality control department. The output of the first production line is twice the output of the second. The first line produces, on the average, 70% of the workpieces of the first grade, while the second line—90%. A workpiece taken at random for inspection at the quality control department turned out to be of the first grade. Find the probability that this workpiece was produced on the first line.

Part III
GEOMETRY

Chapter 17
VECTORS IN PLANE

Sec. 17.1. Basic Notions and Definitions

1. Translation. A transformation of figure F in the plane in which its arbitrary point with the coordinates (x, y) goes into a point with the coordinates $(x + a, y + b)$, where a and b are constants, is called *translation*. Translation is given by the formulas

$$x' = x + a, \quad y' = y + b, \tag{17.1}$$

where (x', y'), are the coordinates of the point into which the point (x, y) is translated.

2. Directed Line-segment. A line segment is said to be *directed* if one of its end points is considered to be the initial point and the other the terminal point (or terminus).

3. The Notion of Vector. In geometry, a *vector* is any directed line-segment.

A vector given by a pair of noncoincident points (A, B) is denoted as $\overrightarrow{AB}$. In this symbol, A is the initial point and B is the terminal point (or terminus) of the given vector.

The distance $|\overrightarrow{AB}|$ is called the *length* (or *modulus*) of the vector $\overrightarrow{AB}$.

Vectors are also denoted by small boldface letters $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{x}, \mathbf{y}$.

4. The Null Vector. The vector $\overrightarrow{AA}$ whose end points coincide is called the *null vector*. The length of a null vector is equal to zero. The notion of direction is not introduced for the null vector.

Every vector, different from a null vector is completely characterized by its direction and length.

5. Collinear Vectors. Two vectors are said to be *collinear* if they lie on one and the same straight line or on parallel lines. The null vector is considered collinear with any vector.

If two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then they can have the same direction or they can have opposite directions. In the first case, the vectors $\mathbf{a}$ and $\mathbf{b}$ are in the *same direction* which is symbolized by $(\mathbf{a}\uparrow\uparrow\mathbf{b})$, in the second case, the vectors $\mathbf{a}$ and $\mathbf{b}$ are in *opposite directions*: $(\mathbf{a}\uparrow\downarrow\mathbf{b})$.

6. Equality of Vectors. Two vectors are said to be *equal* if they are made coincident by a translation, that is, if there exists a translation which carries the initial and terminal points of one vector into the initial and terminal points of the other, respectively.

In other words, equal vectors are in the same direction and have the same absolute value (modulus), that is, if $\mathbf{a} = \mathbf{b}$, then $\mathbf{a}\uparrow\uparrow\mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}|$ and conversely, if two vectors are in the same direction and have the same modulus, then they are equal, that is, if $\mathbf{a}\uparrow\uparrow\mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}|$, then $\mathbf{a} = \mathbf{b}$.

Any vector is equal to itself: $\mathbf{a} = \mathbf{a}$.

If $\mathbf{a} = \mathbf{b}$ and $\mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{c}$.

7. Constructing a Vector from a Given Point. From any point in the plane we can construct only one vector equal to a given vector. The construction of

the vector $\overrightarrow{MN}$, equal to the vector a is called the *laying off* of the vector a from the point M (Fig. 96).

In order to construct the vector $\overrightarrow{MN} = \mathbf{a}$, draw from the point M a ray in the same direction with the vector a and, on it, lay off a line segment MN such that $|MN| = |\mathbf{a}|$. Then $\overrightarrow{MN} = \mathbf{a}$.

1. A translation carries the point (2, 3) into the point (—3, 2). Into what point does it carry the point (5, —2)?

○ Using Formulas (17.1), we find the values of a and b corresponding to the translation of the point (2, 3) into the point (—3, 2):

$$-3 = 2 + a, \quad 2 = 3 + b, \quad \text{that is,} \quad a = -5, \quad b = -1.$$

Then, by Formulas (17.1), we get:

$$x' = 5 - 5 = 0, \quad y' = -2 - 1 = -3,$$

that is, the point (5, —2) goes into the point (0, —3) (Fig. 97). ●

2. A translation carries the point (—4, 1) into the point (2, —3). Into what point does it carry the point (5, 5)?

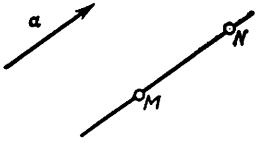


Fig. 96

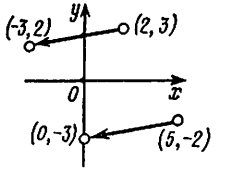


Fig. 97

3. A translation carries the origin into the point (—3, —5). Into what figure will it carry the triangle ABC with vertices A (—2, 6), B (4, 8), C (5, 3)?

4. Given a parallelogram. Carry out the translation which maps the point of intersection of its diagonals into one of its vertices.

5. How many vectors are specified by all possible ordered pairs of points formed from the vertices of (1) a triangle; (2) a parallelogram?

6. Given the following pairs of points: (1) (—2, —3), (5, 4); (2) (6, —2), (13, 5); (3) (—8, —5), (—1, 1). Indicate which pairs define equal vectors.

7. Given a parallelogram $ABCD$; its diagonals intersect at the point O. Which pairs formed from the points A, B, C, D, and O define one and the same vector.

Sec. 17.2. Addition and Subtraction of Vectors. Multiplication of a Vector by a Number

1. Addition of Vectors. In order to construct the sum of two given vectors a and b, it is necessary to choose an arbitrary point A and to construct from this point the vector $\overrightarrow{AB} = \mathbf{a}$, and then lay off the vector $\overrightarrow{BC} = \mathbf{b}$ from the point B.

The vector $\overrightarrow{AC}$ is the desired sum: $\mathbf{a} + \mathbf{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \mathbf{c}$ (Fig. 98).

The vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is called the *resultant vector*, $\mathbf{a}$ and $\mathbf{b}$ being the *component vectors*. This method of construction is termed the *triangle rule*. This rule may be formulated in a different way: if A, B, and C are arbitrary points in the plane, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

This equality is called the *three-point rule*.

The sum of two given vectors $\mathbf{a}$ and $\mathbf{b}$ can also be constructed in the following way. Laying off from an arbitrary point O (Fig. 99) the vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, we construct the parallelogram $OACB$. Then the vector $\overrightarrow{OC}$ (where

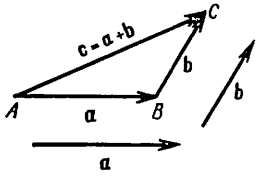


Fig. 98

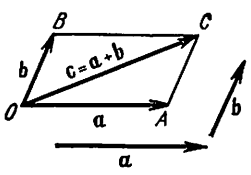


Fig. 99

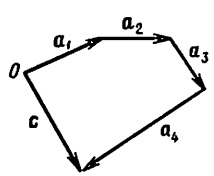


Fig. 100

$|\ OC\ |$ is the diagonal of the parallelogram) is the sought-for sum: $\mathbf{a} + \mathbf{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC} = \mathbf{c}$. This method of construction is called the *parallelogram rule*.

In order to construct the sum of n given vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, proceed as follows: starting from an arbitrary origin O (see Fig. 100) lay off the vector $\mathbf{a}_1$, then from the terminus of the vector $\mathbf{a}_1$ lay off the vector $\mathbf{a}_2$, . . ., finally, from the terminus of the vector $\mathbf{a}_{n-1}$ lay off the vector $\mathbf{a}_n$. Then the vector whose initial point coincides with the initial point of the vector $\mathbf{a}_1$ and the terminus with the terminus of the vector $\mathbf{a}_n$ is the desired sum: $\mathbf{c} = \mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$.

Note the properties of the sum of vectors.

1°. $\mathbf{a} + 0 = \mathbf{a}$ (*null vector absorbing law*).

2°. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (*commutative law*).

3°. $\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (*associative law*).

2. Subtraction of Vectors. Two vectors are said to be *opposite* if their sum is equal to the null vector. The vector opposite to the vector $\mathbf{a}$ is denoted by $-\mathbf{a}$. Thus, $\mathbf{a} + (-\mathbf{a}) = 0$.

Nonvanishing opposite vectors have equal lengths and opposite directions (Fig. 101).

The vector $\mathbf{c}$ is called the *difference* of the vectors $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{c} + \mathbf{b} = \mathbf{a}$.

To subtract the vector $\mathbf{b}$ from the vector $\mathbf{a}$, it suffices to add to the vector $\mathbf{a}$ the vector opposite to the vector $\mathbf{b}$, that is, $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ (Fig. 102).

Another method of constructing the difference between the vectors $\mathbf{a}$ and $\mathbf{b}$ consists in the following. Laying off from an arbitrary point O the vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = -\mathbf{b}$ (vector opposite to the vector $\mathbf{b}$), we get $\overrightarrow{OB} = \mathbf{a} - \mathbf{b}$ (Fig. 103).

3. Multiplication of a Vector by a Number. The *product of a* nonvanishing *vector* $\mathbf{a}$ *by a number* m is defined as a vector having the direction of the vector $\mathbf{a}$ if $m > 0$ and an opposite direction if $m < 0$. The length of this vector is equal to the product of the length of the vector $\mathbf{a}$ and the absolute value of the number m.

The product of the vector $\mathbf{a}$ by the number m is denoted by $m\mathbf{a}$. For any m and $\mathbf{a}$ the vectors $m\mathbf{a}$ and $\mathbf{a}$ are collinear and $|\ m\mathbf{a}\ | = |\ m\ | \cdot |\ \mathbf{a}\ |$.

Note the properties of multiplication of a vector by a number:
1°. $m(n\mathbf{a}) = (mn)\,\mathbf{a}$ (*associative law*).
2°. $m\mathbf{a} + n\mathbf{a} = (m + n)\,\mathbf{a}$ (*first distributive law*).
3°. $m\mathbf{a} + m\mathbf{b} = m(\mathbf{a} + \mathbf{b})$ (*second distributive law*).
4°. $0 \cdot \mathbf{a} = m \cdot 0 = 0$ (*law of absorption of zero and the null vector*).

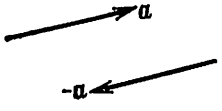


Fig. 101

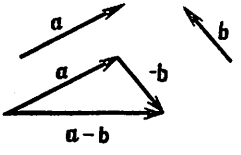


Fig. 102

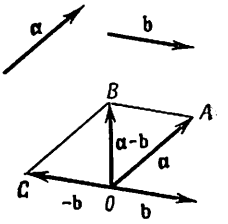


Fig. 103

4. Angle Between Two Vectors The *angle between two* nonvanishing *vectors* $\mathbf{a}$ and $\mathbf{b}$ is defined as the angle between the directions of these vectors: $\widehat{(\mathbf{a}, \mathbf{b})} = \varphi$, where $\varphi \in [0, 180°]$.

Particular cases: (1) if $\mathbf{a} \uparrow\uparrow \mathbf{b}$, then $\widehat{(\mathbf{a}, \mathbf{b})} = 0$; (2) if $\mathbf{a} \uparrow\downarrow \mathbf{b}$, then $\widehat{(\mathbf{a}, \mathbf{b})} = 180°$.

8. Given: $\mathbf{a}$, $\mathbf{b}$, $\widehat{(\mathbf{a}, \mathbf{b})}$. Find the modulus of the vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

○ By the theorem of cosines, we have $c^2 = a^2 + b^2 - 2ab \cos C$. Since $\cos C = [180° - \widehat{(\mathbf{a}, \mathbf{b})}] = -\cos \widehat{(\mathbf{a}, \mathbf{b})}$, we have

$$c = \sqrt{a^2 + b^2 + 2ab \cos \widehat{(\mathbf{a}, \mathbf{b})}}. \quad \bullet$$

9. Find the modulus of the resultant $\mathbf{R}$ of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ and the angles formed by the resultant with the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ if $F_1 = 4$ N and $F_2 = 6$ N, and $\varphi = 60°$ (Fig. 104).

○ Using the formula from the preceding problem, we find

$$R = \sqrt{4^2 + 6^2 + 2 \cdot 4 \cdot 6 \cos 60°} = \sqrt{16 + 36 + 24} = \sqrt{76} = 8.72\ (N).$$

By the theorem of sines, we have

$$F_2/\sin \alpha = F_1/\sin \beta = R/\sin (180° - \varphi) = R/\sin \varphi.$$

Consequently,

$$\sin \alpha = (F_2 \sin \varphi)/R = (6 \sin 60°)/8.72 = 0.596; \quad \alpha = 36.6°;$$

$$\sin \beta = (F_1 \sin \varphi)/R = (4 \sin 60°)/8.72 = 0.397; \quad \beta = 23.4°.$$

Computing check: $\alpha + \beta = 36.6° + 23.4° = 60°$. ●

10. Given a nonvanishing vector $\overrightarrow{OA}$. From the point O lay off the following vectors: (1) $3\overrightarrow{OA}$; (2) $-2\overrightarrow{OA}$; (3) $0.5\overrightarrow{OA}$; (4) $-0.75\overrightarrow{OA}$; (5) $\sqrt{2}\overrightarrow{OA}$.

○ Figure 105 represents the vectors $3\overrightarrow{OA}$ and $-2\overrightarrow{OA}$. The construction of the remaining vectors is left to the reader. ●

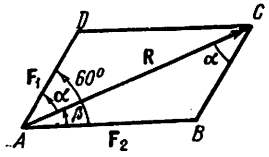


Fig. 104

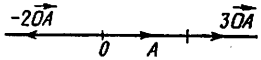


Fig. 105

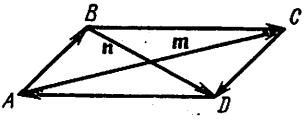


Fig. 106

11. The vectors $\overrightarrow{AC} = \mathbf{m}$ and $\overrightarrow{BD} = \mathbf{n}$ serve as the diagonals of the parallelogram $ABCD$. Express the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{DA}$ in terms of $\mathbf{m}$ and $\mathbf{n}$ (Fig. 106).

○ By the definition of the sum and difference of vectors, we have: $\overrightarrow{BC} + \overrightarrow{CD} = \mathbf{n}$ and $\overrightarrow{BC} - \overrightarrow{CD} = \mathbf{m}$. Adding together these equalities, we get $BC = (\mathbf{m} + \mathbf{n})/2$. Then, we find

$$\overrightarrow{CD} = \mathbf{n} - \overrightarrow{BC} = \mathbf{n} - (\mathbf{m} + \mathbf{n})/2 = (\mathbf{n} - \mathbf{m})/2,$$

$$AB = -(\mathbf{n} - \mathbf{m})/2 = (\mathbf{m} - \mathbf{n})/2,\ \overrightarrow{DA} = -\overrightarrow{BC} = -(\mathbf{m} + \mathbf{n})/2.$$ ●

12. Given the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, construct the vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

13. What condition must be satisfied by three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ so that a triangle can be formed from them?

14. Find the modulus of the resultant of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ and the angles formed by the resultant with the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ if the moduli of the forces are equal to $F_1 = 8$ N and $F_2 = 5$ N, and the angle between them equals 30°.

15. For what values of m does the length of the vector $m\mathbf{a}$ (where $\mathbf{a} \neq 0$) satisfy the following conditions: (1) $|m\mathbf{a}| = |\mathbf{a}|$; (2) $|m\mathbf{a}| > |\mathbf{a}|$; (3) $|m\mathbf{a}| < |\mathbf{a}|$?

16. In the triangle ABC, the medians AA_1 and CC_1 intersect at the point M. Find the factor m if: (1) $\overrightarrow{A_1C} = m \cdot \overrightarrow{BC}$; (2) $\overrightarrow{C_1B} = m \cdot \overrightarrow{C_1A}$; (3) $\overrightarrow{AM} = m \cdot \overrightarrow{MA_1}$.

17. M is the midpoint of the side AB of the triangle ABC. Express $\overrightarrow{CM}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

18. Prove that the median of a trapezoid is equal to one-half the sum of its bases.

19. Given a regular hexagon $ABCDEF$, O is its centre, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. Find $\overrightarrow{OC}$, $\overrightarrow{OD}$, $\overrightarrow{OE}$, and $\overrightarrow{OF}$.

Sec. 17.3. The Rectangular Coordinate System

1. The Axis. Angle Between a Vector and an Axis. A straight line with the positive direction chosen and a unit of length measurement established is defined as an *axis*.

A vector **e** having the length $|\mathbf{e}| = 1$ and direction coinciding with that of an axis is called the *unit vector* of this axis.

If $\mathbf{a} \neq 0$ and **e** is a vector of unit length whose direction coincides with the direction of the vector **a**, then $\mathbf{a} = |\mathbf{a}| \cdot \mathbf{e}$.

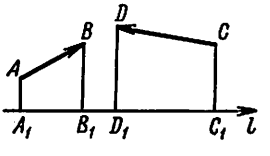


Fig. 107

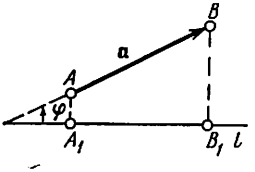


Fig. 108

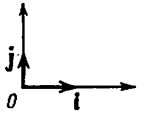


Fig. 109

The *angle between* a nonvanishing *vector* $\mathbf{a} \neq 0$ and an *axis* l is defined as the angle between the directions of the axis and the vector: $\widehat{(\mathbf{a}, l)} = \varphi$, where $\varphi \in [0, 180°]$.

2. Projection of a Vector on an Axis. The *projection of a vector on an axis* is a directed line-segment on the axis whose initial point is the projection of the initial point of the given vector and the terminus is the projection of its terminus. The length of this directed line-segment is taken with the plus sign if the vector is in the same direction as the axis or with the minus sign if in the opposite direction (Fig. 107).

The projection of the vector $\mathbf{a} \neq 0$ on the axis l is equal to the product of the length of the vector by the cosine of the angle φ between the axis and the vector (Fig. 108):

$$\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cdot \cos \varphi. \tag{17.2}$$

Note the properties of projections:

$$\mathrm{pr}_l (\mathbf{a} + \mathbf{b}) = \mathrm{pr}_l \mathbf{a} + \mathrm{pr}_l \mathbf{b}; \quad \mathrm{pr}_l (m\mathbf{a}) = m \cdot \mathrm{pr}_l \mathbf{a}.$$

3. The Rectangular Coordinate System. Let there be given in a plane a pair of mutually perpendicular unit vectors **i** and **j** laid off from some origin (point O in Fig. 109). Such pair of vectors is said to be a *rectangular basis* in the plane. The origin O together with the rectangular basis (**i**, **j**) form a *rectangular coordinate system* in the plane. In this case, the point O is called the *origin of coordinates*, the vectors **i** and **j** being termed the *coordinate vectors*.

4. Coordinates of a Vector. The vector directed from the origin of coordinates to an arbitrary point M lying in the (x, y)-plane is called the *radius vector* of the point M and is denoted by $\mathbf{r} : \overrightarrow{OM} = \mathbf{r}$ (Fig. 110).

The projections of the vector **r** on the coordinate axes, that is, $\mathrm{pr}_x \mathbf{r} = x$ and $\mathrm{pr}_y \mathbf{r} = y$ are the *coordinates of the vector* (see Fig. 110).

The coordinates of the vector **r** are briefly written as follows: $\mathbf{r} = (x, y)$.

The coordinates of the radius vector $\mathbf{r} = \overrightarrow{OM}$ are at the same time the coordinates of the point M, that is, the terminus of the radius vector **r**.

If the initial point of the vector $\mathbf{a} = \overrightarrow{AB}$ does not coincide with the origin, then the coordinates of the vector **a** and those of its terminus are different (Fig. 111). In this case, the projections of the vector $\mathbf{a} = \overrightarrow{AB}$ on the coordinate

axes are respectively equal to $x = x_B - x_A$ and $y = y_B - y_A$, that is,

$$\mathbf{a} = \overrightarrow{AB} = (x, y) = (x_B - x_A, y_B - y_A). \tag{17.3}$$

5. Decomposition of a Vector Along the Coordinate Axes. The decomposition of the vector a in the basis (i, j) has the form

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j}, \tag{17.4}$$

where i is the unit vector of the x-axis and j is the unit vector of the y-axis (Fig. 112). The numbers x and y are the coordinates of the vector a in the basis (i, j).

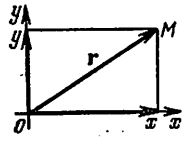


Fig. 110

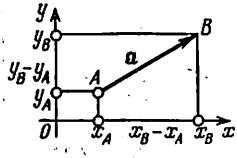


Fig. 111

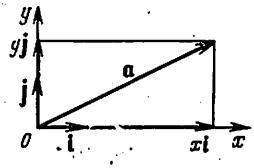


Fig. 112

The vectors $x\mathbf{i}$ and $y\mathbf{j}$ are said to be *components* of the vector a along the respective coordinate axes.

If $A(x_A, y_A)$ is the initial point of the vector a, and $B(x_B, y_B)$ is its terminal point, then the decomposition of the vector a is written in the form

$$\mathbf{a} = \overrightarrow{AB} = (x_B - x_A)\,\mathbf{i} + (y_B - y_A)\,\mathbf{j}. \tag{17.5}$$

6. The Rules of Operations Involving Vectors Specified by Their Coordinates. If two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ are given in the basis (i, j), then:

(1) When two (or more) vectors are added, their coordinates are also added, that is, $\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2)$;

(2) A similar rule holds for the subtraction of vectors, that is, $\mathbf{a} - \mathbf{b} = (x_1 - x_2, y_1 - y_2)$;

(3) When multiplying a vector by a number, multiply all the coordinates by that number, that is, $m\mathbf{a} = (mx_1, my_1)$.

7. Collinearity Condition of Two Vectors. The collinearity condition of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ has the form

$$x_1 = mx_2, \quad y_1 = my_2, \tag{17.6}$$

that is, *if the respective coordinates of two vectors are proportional, then the vectors are collinear.*

If $m > 0$, then the vectors a and b are in the same direction; if $m < 0$, then the vectors are in opposite directions.

20. Given: $\mathrm{pr}_l \mathbf{a} = -2$; $\mathrm{pr}_l \mathbf{b} = 1$. Compute: (1) $\mathrm{pr}_l (2\mathbf{a} + \mathbf{b})$; (2) $\mathrm{pr}_l (3\mathbf{a} - 2\mathbf{b})$.

○ Using the properties of projections, we get

(1) $\mathrm{pr}_l (2\mathbf{a} + \mathbf{b}) = 2\mathrm{pr}_l \mathbf{a} + \mathrm{pr}_l \mathbf{b} = 2 \cdot (-2) + 1 = -3$;

(2) $\mathrm{pr}_l (3\mathbf{a} - 2\mathbf{b}) = 3\mathrm{pr}_l \mathbf{a} - 2\mathrm{pr}_l \mathbf{b} = 3 \cdot (-2) - 2 \cdot 1 = -8$. ●

21. Find the projection of the vector a onto the axis l forming with the vector an angle of 60° if $|\mathbf{a}| = 6$.

○ By Formula (17.2), we obtain $\mathrm{pr}_l \mathbf{a} = 6 \cdot \cos 60° = 3$. ●

22. Construct: (1) the vector $\mathbf{a} = (3, -2)$; (2) the vector $\mathbf{a} = \overrightarrow{AB}$ if $A\ (-1, -2)$, $B\ (4, 3)$.

○ (1) We construct the radius vector with terminus at the point $M\ (3, -2)$; $\overrightarrow{OM}$ is the desired radius vector (Fig. 113).

(2) We construct the points $A\ (-1, -2)$—the initial point of the vector, and $B\ (4, 3)$—the terminal point of the vector (Fig. 114); $\overrightarrow{AB}$ is the desired vector. ●

23. Find the coordinates of the vector $\mathbf{a} = \overrightarrow{AB}$ if $A\ (-1, -2)$, $B\ (4, 5)$.

○ By Formula (17.3), we get $\mathbf{a} = \overrightarrow{AB} = (4 - (-1), 5 - (-2)) = (5, 7)$. ●

24. Express in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ the following vectors: (1) $\mathbf{a} = (-2, 4)$; (2) $\mathbf{a} = \overrightarrow{AB}$, $A\ (-2, -1)$, $B\ (4, -3)$.

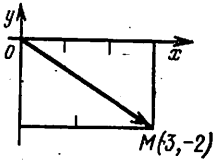


Fig. 113

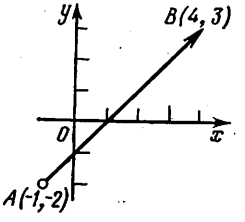


Fig. 114

○ (1) Here, $x = -2$, $y = 4$. By Formula (17.4), we get $\mathbf{a} = -2\mathbf{i} + 4\mathbf{j}$.

(2) By Formula (17.5), we find

$$\mathbf{a} = \overrightarrow{AB} = (4 - (-2))\,\mathbf{i} + (-3 - (-1))\,\mathbf{j} = 6\mathbf{i} - 2\mathbf{j}. \quad ●$$

25. Check to see whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear; if they are, then answer whether they are in the same direction. The vectors are specified by the respective points: (1) $A\ (1, 1)$, $B\ (7, 3)$, $C\ (-4, -5)$, and $D\ (5, -2)$; (2) $A\ (2, 1)$, $B\ (-4, 4)$, $C\ (-1, -1)$, and $D\ (7, -5)$; (3) $A\ (2, 1)$, $B\ (6, 5)$, $C\ (3, -1)$, and $D\ (7, -2)$.

○ (1) Using Formula (17.3), we find the coordinates of the given vectors: $\overrightarrow{AB} = (6, 2)$, $\overrightarrow{CD} = (9, 3)$. Applying the relationships (17.6), we establish that the respective coordinates of these vectors are proportional: $6/9 = 2/3 = m > 0$; consequently, the given vectors are collinear and in the same direction.

(2) Analogously, we obtain: $\overrightarrow{AB} = (-6, 3)$, $\overrightarrow{CD} = (8, -4)$, $(-6)/8 = 3/(-4) = m < 0$; consequently, the vectors are collinear and are in opposite directions.

(3) We have $\overrightarrow{AB} = (4, 4)$, $\overrightarrow{CD} = (4, -1)$; since $4/4 \neq (-1)/4$, the respective coordinates are not proportional and, consequently, are not collinear. ●

26. Given: $\mathrm{pr}_l \mathbf{a} = 3$, $\mathrm{pr}_l \mathbf{b} = -1$. Compute: (1) $\mathrm{pr}_l (\mathbf{a} - \mathbf{b})$; (2) $\mathrm{pr}_l \left(\frac{1}{3}\mathbf{a} + \mathbf{b}\right)$.

27. Prove that if for two nonparallel axes l and n the relationships $\mathrm{pr}_l \mathbf{a} = 0$ and $\mathrm{pr}_n \mathbf{a} = 0$ are fulfilled, then $\mathbf{a} = 0$.

28. In what case is the projection of a vector on an axis: (1) equal to zero; (2) equal (in absolute value) to the length of the given vector?

29. Vectors $\mathbf{a}$ and $\mathbf{b}$ are symmetric with respect to the straight line l. By what relationship are the projections of these vectors on the axis n connected if this axis is: (1) parallel to the line l; (2) perpendicular to the line l?

30. Given two vectors: $\mathbf{a}$ and $\mathbf{b}$. For what position of the axis l does the equality $\mathrm{pr}_l \mathbf{a} = \mathrm{pr}_l \mathbf{b}$ hold true?

31. Find the projection of the vector $\mathbf{a}$ on the axis l forming with the vector an angle of: (1) 45°; (2) 120°; (3) 150°.

32. Construct the following vectors: (1) $\mathbf{a} = (-2, 4)$; (2) $\mathbf{b} = (3, 2)$; (3) $\mathbf{a} = \overrightarrow{AB}$ if A $(-1, -1)$, B $(4, 1)$; (4) $\mathbf{c} = \overrightarrow{CD}$ if C $(0, 2)$, D $(4, 0)$.

33. Find the coordinates of the indicated vectors: (1) $\mathbf{a} = \overrightarrow{AB}$, A $(-2, -2)$, B $(4, -1)$; (2) $\mathbf{b} = \overrightarrow{BC}$, B $(1, -3)$, C $(4, -5)$.

34. Given the vectors: $\mathbf{a} = (-2, -3)$, $\mathbf{b} = (5, 0)$, $c = (3, -5)$. Find the coordinates of the following vectors: (1) $\mathbf{a} + \mathbf{b}$; (2) $\mathbf{a} - \mathbf{c}$; (3) $\mathbf{a} + \mathbf{b} - \mathbf{c}$; (4) $2\mathbf{a}$; (5) $3\mathbf{a} - \mathbf{c}$; (6) $\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$.

35. Express in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ the following vectors: (1) $\mathbf{a} = (-2, -4)$; (2) $\mathbf{a} = \overrightarrow{AB}$, A $(-1, 2)$, B $(-2, -6)$.

36. Given three points: A $(-2, -3)$, B $(2, 4)$, and C $(5, 1)$. Decompose the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ into the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

37. Check to see whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear; if they are, then check whether they are in the same direction. The vectors are specified by the respective points: (1) A $(-3, 6)$, B $(1, 2)$, C $(4, -6)$ and D $(-2, 0)$; (2) A $(-3, 1)$, B $(3, 3)$, C $(-2, -3)$, and D $(6, -1)$; (3) A $(-3, -6)$, B $(-1, 2)$, C $(3, -5)$, and D $(5, 3)$.

Sec. 17.4. The Length of a Vector. The Distance Between Two Points in the Plane. The Angle Between a Coordinate Axis and a Vector

The length of the radius vector $\mathbf{a} = (x, y)$ is found by the formula

$$|\mathbf{a}| = \sqrt{x^2 + y^2}. \tag{17.7}$$

The length of the vector $\mathbf{a} = \overrightarrow{AB} = (x_B - x_A;\ y_B - y_A)$ is found by the formula

$$|\overrightarrow{AB}| = |AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}. \tag{17.8}$$

The distance between two points in the plane is also computed with the aid of this formula.

The angles formed by the vector $\mathbf{a} = \overrightarrow{AB}$ and the x- and y-axes are found by the formulas

$$\cos\alpha = \frac{x_B - x_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}};$$

$$\cos\beta = \frac{y_B - y_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}}. \tag{17.9}$$

38. Find the length of the vector $\overrightarrow{AB}$ if A (1, 1) and B (4, —3).

○ By Formula (17.8), we find $|\overrightarrow{AB}| = \sqrt{(4-1)^2 + (-3-1)^2} = 5$. ●

39. Find the unit vector in the same direction as the vector: (1) $\mathbf{a} = (3, 4)$; (2) $\mathbf{b} = (-6, -8)$.

○ (1) We find the length of the given vector: $|\mathbf{a}| = \sqrt{3^3 + 4^2} = 5$. The unit vector $\mathbf{e}$, having the same direction as the vector $\mathbf{a}$, is equal to $\mathbf{e} = \mathbf{a}/|\mathbf{a}| = (1/5)\,\mathbf{a}$. Each projection of the vector $\mathbf{e}$ is also one-fifth of the corresponding projection of the vector $\mathbf{a}$, therefore $\mathbf{e} = (3/5)\,\mathbf{i} + (4/5)\,\mathbf{j}$ or $\mathbf{e} = (3/5, 4/5)$.

(2) Analogously, we get $\mathbf{e} = (-0.6, -0.8)$. ●

40. Find the cosines of the angles formed by the given vectors with the coordinate axes: (1) $\mathbf{a} = \overrightarrow{AB}$, A (2, —3), B (1, 4); (2) $\mathbf{b} = \overrightarrow{BC}$, B (—1, 1), C (2, 5).

○ (1) By Formulas (17.9), we find:

$$\cos\alpha = \frac{1-2}{\sqrt{(1-2)^2 + (4+3)^2}} = -0.1\sqrt{2};$$

$$\cos\beta = \frac{4+3}{\sqrt{(1-2)^2 + (4+3)^2}} = 0.7\sqrt{2}.$$

(2) In this case, we get $\cos\alpha = 0.6$, $\cos\beta = 0.8$. ●

41. Given three points: A (4, 0), B (7, 4), and C (—4, 6). Find the lengths of the following vectors: (1) $\overrightarrow{AB}$; (2) $\overrightarrow{BC}$; (3) $\overrightarrow{CA}$.

42. Compute the perimeter of the triangle whose vertices lie at the points: (1) A (4, 0), B (7, 4), and C (—4, 6); (2) A (6, 7); B (3, 3), and C (1, —5).

43. Find the cosines of the angles formed by the given vectors with the coordinate axes: (1) $\mathbf{a} = \overrightarrow{AB}$, A (—2, —3), B (3, 9); (2) $\mathbf{b} = \overrightarrow{BC}$, B (4, —1), C (0, 2).

44. Find the point equidistant from the points: (1) A (7, —1), B (—2, 2), and C (—1, —5); (2) A (10, 7), B (—4, —7), and C (12, —7).

45. Find the centre of the circle passing through the points A (—1, 9), B (—8, 2), C (9, 9) and the length of its radius.

46. The distance from the point M, lying on the x-axis, to the point N (10, 5) is equal to 13. Find the point M.

47. The distance from the point B, lying on the y-axis, to the point A (3, —1) is equal to 5. Find the point B.

48. Compute the coordinates of the point of the y-axis equidistant from the points: (1) A (—4, 0) and B (—3, —7); (2) A (—3, —1) and B (6, 2).

49. Find the point on the x-axis equidistant from the points: (1) A (5, 13) and B (—12, —4); (2) A (0, 6) and B (2, —4).

50. Compute the coordinates of the point M equidistant from the coordinate axes and from the point: (1) A (—8, —1); (2) A (4, 2).

51. Find the point M whose distance from the axis of ordinates and from the point A (8, 6) is equal to 5.

52. Compute the coordinates of the point M whose distance from the axis of abscissas and from the point A (1, 2) is equal to 10.

Sec. 17.5. Dividing a Line Segment in a Given Ratio

If the line segment AB is divided by the point C in the ratio $|AC| : |CB| = \lambda$, then the coordinates of the point C are found by the formulas

$$x_C = \frac{x_A + \lambda x_B}{1+\lambda}, \quad y_C = \frac{y_A + \lambda y_B}{1+\lambda}. \tag{17.10}$$

If $\lambda = 1$, then we obtain the formulas for finding the coordinates of the midpoint of a line segment:

$$x_C = \frac{x_A + x_B}{2}, \quad y_C = \frac{y_A + y_B}{2}. \tag{17.11}$$

53. The line segment whose end points are A (—11, 1) and B (9, 11) is divided in the ratio 2 : 3 : 5 (as measured from A to B). Find the division points.

○ Let us denote the division points (from A to B) by C and D. By the hypothesis, $x_A = -11$, $x_B = 9$, $y_A = 1$, $y_B = 11$, and $|AC| : |CD| : |DB| = 2 : 3 : 5$. The point C divides the line-

segment $[AB]$ in the ratio $\lambda = \left|\frac{AC}{CB}\right| = \frac{2}{3+5} = \frac{2}{8} = \frac{1}{4}$; hence,

$$x_C = \frac{-11+(1/4)\cdot 9}{1+1/4} = -7; \quad y_C = \frac{1+(1/4)\cdot 11}{1+1/4} = 3; \quad C(-7, 3).$$

The point D serves as the midpoint of $[AB]$, therefore

$$x_D = \frac{-11+9}{2} = -1; \quad y_D = \frac{1+11}{2} = 6; \quad D(-1, 6). \bullet$$

54. Given the vertices of a triangle: $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$. Find the point of the intersection of the three medians of this triangle (or the median point).

○ As is known, the medians of a triangle intersect at a point which is the centroid of the triangle (also called the median point). This point divides each of the medians in the ratio 2 : 1.

Let us find the point D which is the midpoint of the side BC : $x_D = (x_B + x_C)/2$, $y_D = (y_B + y_C)/2$.

We then find the point M at which the three medians intersect; to this end, we divide the median AD in the ratio $\lambda = 2 : 1 = 2$ (from A to D):

$$x_M = \frac{x_A+\lambda x_D}{1+\lambda} = \frac{x_A+2\cdot(x_B+x_C)/2}{1+2} = \frac{x_A+x_B+x_C}{3};$$

$$y_M = \frac{y_A+\lambda y_D}{1+\lambda} = \frac{y_A+2\cdot(y_B+y_C)/2}{1+2} = \frac{y_A+y_B+y_C}{3}.$$

Thus, the coordinates of the median point of a triangle are equal to the arithmetic mean of the like coordinates of its vertices. ●

55. The line segment AB is given by the points $A(-9, -3)$ and $B(1, 2)$. To what point C should the line segment AB be extended so that $|AB| : |BC| = 5 : 3$?

○ By the hypothesis, $x_A = -9$, $x_B = 1$, $y_A = -3$, $y_B = 2$, $\lambda = |AB| : |BC| = 5 : 3$. It is required to find $C(x_C, y_C)$.

For the point $B(1, 2)$ that divides the line segment AC in the given ratio we get

$$1 = \frac{-9+(5/3)x_C}{1+5/3}, \quad 2 = \frac{-3+(5/3)y_C}{1+5/3},$$

whence $x_C = 7$, $y_C = 5$, that is, $C(7, 5)$. ●

56. Compute the coordinates of the point C which is the midpoint of the line segment AB if: (1) $A(5, -4)$ and $B(-1, 2)$; (2) $A(6, -3)$ and $B(-2, -7)$.

57. A point C divides the line segment AB in the ratio 3 : 5 (from A to B). The points $A(2, 3)$ and $B(10, 11)$ serve as end points of the line segment. Find the point C.

58. The line segment whose end points lie at the points $A(3, -2)$ and $B(10, -9)$ is divided by a point C in the ratio 2 : 5. Find the point C.

59. The line segment whose end points lie at the points A (—5, —2) and B (4, 2.5) is divided in the ratio 3 : 4 : 2 from A to B. Find the division points.

60. One of the end points of a line segment is the point A (—3, —5), and its midpoint is C (3, —2). Find the other end point—the point B.

61. Find the median point of a triangle if its vertices lie at the points: (1) A (7, —4), B (—1, 8), and C (—12, —1); (2) A (—4, 2), B (2, 6), and C (0, —2).

62. The points A (—8, —5) and B (10, 4) serve as the end points of a line segment. Find the points C and D dividing this segment into three equal parts.

63. The point C (3, 5) divides the line segment AB in the ratio $|AC| : |CB| = 3 : 4$. Find the other end point of the segment (point A) if one of its end points is represented by the point B (—1, 1).

64. The point C (—2, 1) divides the line segment AB in the ratio $|AC| : |CB| = 2 : 1$. Find the other end point of the segment (point B) if one of its end points is represented by the point A (—10, 5).

65. A line segment is specified by the points A (—4, 7) and B (—3, 5). Find on the extension of the segment AB a point C such that $|AB| : |BC| = 1 : 7$.

66. A line segment is given by the points A (—5, —2) and B (—1, 0). To what point C is it necessary to extend it so that $|AB| : |BC| = 2 : 5$?

Sec. 17.6. Scalar Product of Two Vectors

The *scalar product* of two nonvanishing vectors is defined as the number equal to the product of the lengths of these vectors by the cosine of the angle between them.

The scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$ is symbolized as $\mathbf{a} \cdot \mathbf{b}$. Thus, by definition

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos \widehat{(\mathbf{a}, \mathbf{b})}. \tag{17.12}$$

The scalar product of two vectors is equal to the absolute value of one of them multiplied by the projection of the other vector on the direction of the first:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \operatorname{pr}_{\vec{a}} \mathbf{b} = |\mathbf{b}| \cdot \operatorname{pr}_{\vec{b}} \mathbf{a}. \tag{17.13}$$

Note the following properties of a scalar product:

1°. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (*commutative property*);

2°. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (*distributive property*);

3°. $m(\mathbf{a} \cdot \mathbf{b}) = (m \cdot \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m \cdot \mathbf{b})$ (*associative property*).

The *scalar square* of the vector $\mathbf{a}$ is defined as the scalar product $\mathbf{a} \cdot \mathbf{a}$. The scalar square of a vector is equal to the square of its length:

$$\mathbf{a} \cdot \mathbf{a} = a^2. \tag{17.14}$$

For two nonvanishing vectors $\mathbf{a}$ and $\mathbf{b}$ to be mutually perpendicular, it is necessary and sufficient that their scalar product be equal to zero:

$$(\mathbf{a} \neq 0,\ \mathbf{b} \neq 0,\ \mathbf{a} \cdot \mathbf{b} = 0) \Leftrightarrow \mathbf{a} \perp \mathbf{b}. \tag{17.15}$$

For the two vectors **a** and **b** to be collinear, it is necessary and sufficient that the following relationship be fulfilled:

$$\mathbf{ab} = \pm |\mathbf{a}|\,|\mathbf{b}| \tag{17.16}$$

(the plus sign if **a** and **b** are in the same direction; and the minus sign if they are in opposite directions).

The scalar product of the vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is expressed in terms of their coordinates by the formula

$$\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2. \tag{17.17}$$

The angle between two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b}$ (x_2, y_2) is found by the formula

$$\cos\widehat{(\mathbf{a}, \mathbf{b})} = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}. \tag{17.18}$$

It follows from this formula that if the vectors **a** and **b** are mutually perpendicular, then

$$x_1x_2 + y_1y_2 = 0. \tag{17.19}$$

67. The vectors **a** and **b** form an angle $\widehat{(\mathbf{a}, \mathbf{b})} = 60°$. Knowing that $|\mathbf{a}| = 6$, $|\mathbf{b}| = 3$, compute $(2\mathbf{a} + \mathbf{b})\cdot(2\mathbf{a} - 3\mathbf{b})$.

○ Using Formulas (17.12) and (17.14), we get

$$(2\mathbf{a} + \mathbf{b})\cdot(2\mathbf{a} - 3\mathbf{b}) = 4\mathbf{a}^2 - 6|\mathbf{a}|\,|\mathbf{b}|\cdot\cos 60° + 2|\mathbf{a}|\,|\mathbf{b}|\cos 60° -$$
$$- 3\mathbf{b}^2 = 4\cdot 6^2 - 6\cdot 6\cdot 3\cdot 0.5 + 2\cdot 6\cdot 3\cdot 0.5 - 3\cdot 3^2 = 81. \bullet$$

68. Find the scalar product of the vectors $\mathbf{a} = (-3, 2)$ and $\mathbf{b} = (4, 3)$.

○ By Formula (17.17), we get $\mathbf{a}\cdot\mathbf{b} = (-3)\cdot 4 + 2\cdot 3 = -6$. ●

69. Compute the angle between the vectors $\mathbf{a} = (-4, 3)$ and $\mathbf{b} = (3, -4)$.

○ By Formula (17.18), we find

$$\cos\widehat{(\mathbf{a}, \mathbf{b})} = \frac{-4\cdot 3 + 3\cdot(-4)}{\sqrt{(-4)^2+3^2}\cdot\sqrt{3^2+(-4)^2}} = -0.96;\ \widehat{(\mathbf{a}, \mathbf{b})} = 163.7°. \bullet$$

70. Check whether the indicated vectors are mutually perpendicular: (1) $\mathbf{a} = (-3, 2)$ and $\mathbf{b} = (4, 6)$; (2) $\mathbf{c} = (3/4, -1/5)$ and $\mathbf{d} = (4/3, 5)$; (3) $\mathbf{p} = (-2, 5)$ and $\mathbf{q} = (3, 1)$.

○ By Formula (17.19), we find:

(1) $\mathbf{a}\cdot\mathbf{b} = (-3)\cdot 4 + 2\cdot 6 = 0$, i.e. $\mathbf{a} \perp \mathbf{b}$;

(2) $\mathbf{c}\cdot\mathbf{d} = (3/4)\cdot(4/3) + (-1/5)\cdot 5 = 0$, i.e. $\mathbf{c} \perp \mathbf{d}$;

(3) $\mathbf{p}\cdot\mathbf{q} = (-2)\cdot 3 + 5\cdot 1 = -1 \neq 0$, i.e. $\mathbf{p} \not\perp \mathbf{q}$. ●

71. Find the work done by the force **F** over the path **s** if $|\mathbf{F}| = 3$, $|\mathbf{s}| = 8$, and $\widehat{(\mathbf{F}, \mathbf{s})} = 60°$.

The work A is computed by the formula: $A = \mathbf{F}\cdot\mathbf{s} = |\mathbf{F}|\cdot|\mathbf{s}|\cdot\cos\widehat{(\mathbf{F}, \mathbf{s})}$, where **F** is the vector of the acting force, **s** is the vector of the path. We have $A = 3\cdot 8\cdot\cos 60° = 12$ (units of work). ●

72. The vectors **a** and **b** form an angle of 120°. Knowing that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$, compute $(2\mathbf{a} - 3\mathbf{b})^2$.

73. The vectors **a** and **b** form an angle of 150°. Knowing that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, compute $(\mathbf{a} + \mathbf{b}) \cdot 4\mathbf{a}$.

74. Find the scalar product of the indicated vectors: (1) $\mathbf{a} = (2, 4)$ and $\mathbf{b} = (4, 1)$; (2) $\mathbf{c} = (1/2, 2/5)$ and $\mathbf{d} = (2/3, 5/6)$.

75. Given the points A (—2, 4), B (3, —6), C (4, —2), and D (1, 5). Compute the scalar product $\overrightarrow{AB} \cdot \overrightarrow{CD}$.

76. Find the scalar product of the vectors: (1) $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$; (2) $\overrightarrow{AB}$ and $\overrightarrow{BC}$ if A (1, 3), B (—2, —4), and C (4, —3).

77. Compute the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if A (3, 1), B (7, 4), C (3, 2), and D (6, 6).

78. Find the cosine of the angle between the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ if $\mathbf{a} = (2, 3)$ and $\mathbf{b} = (1, 1)$.

79. Check whether the indicated vectors are mutually perpendicular: (1) $\mathbf{a} = (-3, 5)$ and $\mathbf{b} = (5, 3)$; (2) $\mathbf{c} = (2, -4)$ and $\mathbf{d} = (-4, -2)$; (3) $\mathbf{p} = (-3, -4)$ and $\mathbf{q} = (5, 1)$.

80. What condition must be satisfied by the vectors **a** and **b** for the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ to be mutually perpendicular?

81. Compute the work done by the force $\mathbf{F} = (2, -1)$ if the point of its application, when moving in a straight line, is displaced from the position M (1, —2) to the position N (5, —5).

Sec. 17.7. Transformations of Rectangular Coordinates

1. Translation of the Coordinate Axes. Let there be two systems of rectangular coordinates with different origins whose axes are parallel and in the same direction. Then the coordinates of one and the same point in these systems are related by the following formulas:

$$x = x_1 + x_0, \quad y = y_1 + y_0, \tag{17.20}$$

where x and y are the coordinates of the point in the original (old) system; x_1 and y_1, its coordinates in the new system; x_0 and y_0, the coordinates of the new origin O_1 with respect to the original system (Fig. 115).

2. Rotation of the Coordinate Axes (without changing the position of the origin). Let the axes Ox and Oy be rotated through an angle α to the positions Ox_1 and Oy_1 (Fig. 116). Then x and y are the initial coordinates of a point referred to the old coordinate system, α is the angle of rotation, and x_1 and y_1 are the coordinates of the same point referred to the new coordinate system. In this case, the formulas of rotation are

$$x = x_1 \cos\alpha - y_1 \sin\alpha, \quad y = x_1 \sin\alpha + y_1 \cos\alpha \tag{17.21}$$

and

$$x_1 = x \cos\alpha + y \sin\alpha, \quad y_1 = -x \sin\alpha + y \cos\alpha. \tag{17.22}$$

82. The coordinates of a point in the new coordinate system are $x_1 = 3$ and $y_1 = -1$, and the coordinates of the new origin with the

direction of the axes retained are $x_0 = 2$ and $y_0 = -3$. Find the coordinates of the point in the original system.

○ By Formulas (17.20), we get

$$x = x_1 + x_0 = 3 + 2 = 5, \quad y = y_1 + y_0 = -1 + (-3) = -4. \bullet$$

83. The coordinates of a point in the old system are $x = -4$ and $y = 2$, and the coordinates of the new origin with the direction of the old axes retained are $x_0 = 3$ and $y_0 = -1$. Find the coordinates of the point in the new system.

○ By Formulas (17.20), we find

$$x_1 = x - x_0 = -4 - 3 = -7, \quad y_1 = y - y_0 = 2 - (-1) = 3. \bullet$$

84. A certain point is referred to two coordinate systems xOy and $x_1O_1y_1$ having the same directions of the axes; the respective

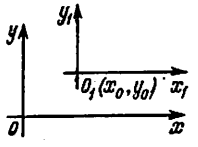


Fig. 115

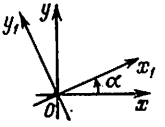


Fig. 116

coordinates being $(-2, 3)$ and $(-7, 6)$. Find the coordinates of the origin of either of these systems with respect to each other.

○ Setting $x = -2$, $y = 3$, and $x_1 = -7$, $y_1 = 6$, by Formulas (17.20) we obtain $-2 = -7 + x_O$, $3 = 6 + y_O$, that is, $x_O = 5$, $y_O = -3$. The coordinates of the new origin in the system xOy are: O $(5, -3)$.

Interchanging x and x_1, y and y_1 in Formulas (17.20), we get $-7 = -2 + x_{O_1}$, $6 = 3 + y_{O_1}$ that is, $x_{O_1} = -5$, $y_{O_1} = 3$. The coordinates of the new origin in the system $x_1O_1y_1$ are: O_1 $(-5, 3)$. ●

85. Given the point M $(-2\sqrt{3}, 4)$. Find the coordinates of this point in the new coordinate system obtained by rotating the axes through an angle of 60° with the origin remaining unchanged.

○ By Formulas (17.22), we get

$$x_1 = -2\sqrt{3}\cos 60° + 4\sin 60°, \quad y_1 = 2\sqrt{3}\sin 60° + 4\cos 60°,$$

that is, $x_1 = \sqrt{3}$, $y_1 = 5$. ●

86. In the system rotated through an angle of 45° with respect to the original system the point $(-2, 4)$ is given. Find the coordinates of this point referred to the original system.

○ By Formulas (17.21), we get

$$x = -2\cos 45° - 4\sin 45°, \quad y = -2\sin 45° + 4\cos 45°,$$

that is, $x = -3\sqrt{2}$, $y = \sqrt{2}$. ●

87. The coordinates of a certain point in the new system are: $x_1 = -2$ and $y_1 = 4$. Find the coordinates of this point in the old system if, without changing the directions of the axes, the origin is translated to the point: (1) (—3, 5); (2) (4, —2); (3) (—1, —3), (4) (2, 1).

88. The coordinates of a certain point in the original system are: $x = -2$ and $y = -3$. Find the coordinates of this point in the new system if, without changing the directions of the axes, the origin is translated to the point: (1) (3, 2); (2) (—3, 2); (3) (3, —2); (4) (—3, —2).

89. A certain point is referred to two coordinate systems: xOy and $x_1O_1y_1$ having the same directions of the axes; the respective coordinates being (—4, 7) and (—8, 3). Find the coordinates of the origin of either of these systems with respect to each other.

90. Two coordinate systems have the same directions of the axes. The first system, when referred to the second, has the origin at the point (—3, 5). Find the coordinates of the origin of the second system with respect to the first.

91. Given a point M (x, y). How will the coordinates of this point change if the axis of ordinates is taken for the axis of abscissas, and the axis of abscissas for the axis of ordinates?

92. Given the point A (4, —2). Find the coordinates of this point in the new coordinate system if the original axes are turned through an angle of: (1) 45°; (2) 30°.

93. Find the coordinates of a point referred to the original system if this point has the coordinates $(2\sqrt{3}, 2)$ in the system rotated with respect to the original system through an angle of: (1) 30°; (2) 60°.

Sec. 17.8. Polar Coordinates

The elements of a polar coordinate system are: (1) the fixed point, O in Fig. 117, is the pole; (2) the ray emanating from the point O is the polar axis (Op in the figure); (3) the scale unit for measuring the length l.

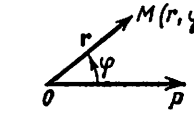


Fig. 117

The position of the point M in the plane is given by the distance of this point from the pole, that is, by the length of the radius vector expressed in the adopted units of measurement and the angle between the radius vector and polar axis. The numbers φ and r are called the *polar coordinates* of the point M; r is the *polar radius* and φ the *polar angle*. The polar coordinates of a point are written in the following way: M (r, φ).

If the point M coincides with the pole, then $r = 0$, and the value of φ is not defined. For any other point in the plane $r > 0$ and the value of φ is defined with an accuracy to a term multiple of 2π.

If the pole of a polar coordinate system is situated at the origin of a rectangular coordinate system, and the positive semiaxis Ox coincides with the polar axis, then the rectangular coordinates of a point are expressed in terms of its polar coordinates by the formulas

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{17.23}$$

For expressing the polar coordinates of a point in terms of its rectangular coordinates, the following formulas are used:

$$r=\sqrt{x^2+y^2},\quad \sin\varphi=y/\sqrt{x^2+y^2},\quad \cos\varphi=x/\sqrt{x^2+y^2}, \tag{17.24}$$

$$\tan\varphi=y/x. \tag{17.25}$$

94. Construct the point M $(2, \pi/4)$ in the polar coordinate system.

○ Through the pole O we draw the ray Om at an angle $\pi/4$ to the polar axis Op. On the ray Om, lay off the line segment OM whose length is equal to two scale units. The point M is the desired point (Fig. 118). ●

95. Find the rectangular coordinates of the point A $(4, \pi/3)$.

○ By Formulas (17.23), we get:

$$x=4\cos(\pi/3)=2,\quad y=4\sin(\pi/3)=2\sqrt{3}.$$ ●

96. Find the polar coordinates of the point whose rectangular coordinates are $(-3, 3\sqrt{3})$.

○ By Formulas (17.24), we obtain:

$$r=\sqrt{(-3)^2+(3\sqrt{3})^2}=6,$$

$$\sin\varphi=3\sqrt{3}/6=\sqrt{3}/2,\quad \cos\varphi=-3/6,\quad =-1/2.$$

Taking into account the signs of the sine and cosine, we find the polar angle $\varphi=2\pi/3$. ●

97. Compute the distance between two points A (r_1, φ_1) and B (r_2, φ_2) given in polar coordinates.

○ By the theorem of cosines, we have

$$|AB|=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\varphi_2-\varphi_1)}.$$ ●

98. Construct the following points in the polar coordinate system: (1) $(3, \pi/3)$; (2) $(4, 3\pi/4)$; (3) $(2, -\pi/6)$; (4) $(5, -3\pi/4)$.

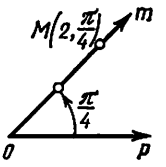


Fig. 118

99. Find the polar coordinates of the points symmetric to the points $(2, \pi/4)$, $(4, 2\pi/3)$, and $(1, -\pi/6)$ with respect to: (1) the pole; (2) the polar axis.

100. Find the rectangular coordinates of the indicated points: (1) $(2, \pi/2)$; (2) $(2\sqrt{3}, -\pi/3)$; (3) $(\sqrt{2}, -\pi/4)$; (4) $(2, 3\pi/4)$.

101. Find the polar coordinates of the following points: (1) $(0, 5)$; (2) $(0, -\sqrt{3})$; (3) $(-\sqrt{3}, -1)$; (4) $(1, -1)$.

102. Is it possible that the rectangular and polar coordinates of a point are represented by one and the same pair of numbers?

103. Compute the distance between the two points: A $(6, \pi/3)$ and B $(5, 2\pi/3)$.

Sec. 17.9. Miscellaneous Problems

In Problems 104 through 113 carry out the proofs using vectors:

104. Prove that the three medians of any triangle can serve as the sides of the triangle.

105. Prove that the line segment joining the midpoints of two sides of a triangle is equal to one-half the length of the third side.

106. Given: a parallelogram $ABCD$ and an arbitrary point O in the plane. Prove that $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$.

107. Prove that the midpoints of the sides of an arbitrary quadrilateral are the vertices of a parallelogram.

108. Prove that the three altitudes of a triangle intersect at one point.

109. Prove that the sum of the squared diagonals of a parallelogram is equal to the sum of the squares of all of its sides.

110. Prove that the diagonals of a rhombus are mutually perpendicular.

111. Prove that in an isosceles trapezoid the diagonals are equal to each other.

112. Prove that in a right trapezoid the difference between the squared diagonals is equal to the difference of the squared bases.

113. Prove that the inscribed angle subtended by the diameter of a circle is a right one.

114. Find the points symmetric to the point A (—2, —3) with respect to: (1) the origin; (2) the x-axis; (3) the y-axis; (4) the angle bisectors of Quadrants I and III.

115. Given a line segment with the end points A (—3, 1) and B (—1, 7). Find the end points of the line segment symmetric to the given with respect to: (1) the origin; (2) the x-axis; (3) the y-axis; (4) the angle bisectors of Quadrants II and IV.

116. Given a triangle with the vertices A (3, 2), B (7, 4), and C (1, 6). Find the vertices of the triangle symmetric to the given triangle with respect to: (1) the origin; (2) the x-axis; (3) the y-axis; (4) the angle bisectors of Quadrants I and III.

117. The line segment with the end points A (—2, 4) and B (6, 12) is bisected at the point C. Find the length of the line segment CD, where D (7, —4).

118. Given the coordinates of the midpoints of the side of a triangle: (2, 1), (0, —4), and (—4, —1). Find the vertices of the triangle.

119. A line segment is given by the points A (4, 6) and B (1, 3). To what point C is it necessary to extend this line segment in the direction from A to B in order to obtain a line segment AC whose length will be three times the length of the line segment AB?

120. The line segment AB is divided into five equal parts. One

end point of the segment is the point A (8, 6), the second division point (from A to B) is D (2, 4). Find the point B.

121. Divide the line segment whose end points are A (—6, 8) and B (9, 12) in the same ratio (from A to B) in which are the lengths of the radius vectors of these points.

122. A triangle ABC with the medians AM_1, BM_2, and CM_3 is constructed on the vectors $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. Express the vectors $\overrightarrow{AM}_1$, $\overrightarrow{BM}_2$, and $\overrightarrow{CM}_3$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Find $|AM_1|$ if the coordinates of the vertices of the triangle are: A (2, 5), B (4, —4), and C (12, —2).

123. The line segment AB is given by its end points A (—7, —1) and B (9, 7) and is divided by the point C in the ratio 5 : 3 (from A to B). Find the length of the line segment joining the point C to the point which is symmetric to it with respect to the origin.

124. The line segment AB is divided into seven equal parts. The fourth division point (from A to B) is F (4, 1). Find the point A if B (13, 4).

125. On the angle bisectors of Quadrants I and III, find the point equidistant from the points A (7, 2) and B (2, —13).

126. The points A (3, 2), B (—2, 1) and C (1, —4) serve as the vertices of a parallelogram, A and C being opposite vertices. Find the fourth vertex D.

127. The points A (—3, 1) and B (1, 3) are adjacent vertices of a parallelogram. Its diagonals intersect at the point M (1, —2). Find two other vertices.

QUIZ No. 1

Form I

(1) Find the coordinates of the vector $\overrightarrow{AB}$ if A (—2, —3) and B (1, 4).

(2) The point C (2, 3) divides the line segment $[AB]$ in the ratio 1 : 4 (from A to B). Find the point A if B (—6, —1).

(3) Find the point M equidistant from the coordinate axes and from the given point A (4, —2).

(4) Compute the angle between the vectors $\mathbf{a} = (-3, 4)$ and $\mathbf{b} = (4, 3)$.

(5) Prove that if O is the median point of the triangle ABC, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$.

Form II

(1) Given the points A (—3, —4) and B (2, 5). Decompose the vector $\overrightarrow{AB}$ into the unit vectors $\mathbf{i}$ and $\mathbf{j}$ of the coordinate axes.

(2) The line segment AB is given by the points A (7, —4) and B (—8, 1), and is divided by the point C in the ratio 1 : 4 (from A to B). Find the point C.

(3) A line segment is given by the points A (—10, 4) and B (5, —1). To what point C is it necessary to extend it so that $|AB| : |BC| = 5 : 1$?

(4) Compute the cosine of the angle between the vectors $\mathbf{a} = (3, 4)$ and $\mathbf{b} = (5, 12)$.

(5) In the triangle ABC, the median AM is drawn. Prove that $2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{AC}$.

Chapter 18
A STRAIGHT LINE IN THE PLANE AND ITS EQUATIONS

Sec. 18.1. General Equation of a Straight Line. The Vector Equation and Canonical Equation of a Straight Line

1. The General Equation of a Straight Line. An equation of the first degree in two variables x and y, that is, an equation of the form

$$Ax + By + C = 0 \tag{18.1}$$

provided that the coefficients A and B are not both zero, is called the *general equation* of a straight line.

Note some particular cases of the general equation of a straight line:

Value of coefficient	Form of equation	Position of straight line
$C=0$	$Ax+By=0$ $(y=kx)$	Passes through the origin
$A=0$	$By+C=0$ $(y=b)$	Parallel to the x-axis
$B=0$	$Ax+C=0$ $(x=a)$	Parallel to the y-axis
$A=C=0$	$y=0$	Coincides with the x-axis
$B=C=0$	$x=0$	Coincides with the y-axis

2. The Vector Equation of a Straight Line. Let l be a straight line in the plane xOy (Fig. 119), $M_0(x_0, y_0)$ a point on this line, and $\mathbf{n} = (A, B)$ a non-vanishing vector perpendicular to the line l (it is called the *normal vector* of the straight line). If $M(x, y)$ is an arbitrary point on the line l, distinct from M_0, then the vector $\overrightarrow{M_0M} = \mathbf{r} - \mathbf{r}_0 = (x - x_0, y - y_0)$ is perpendicular to the vector $\mathbf{n} = (A, B)$, that is, the scalar product of these vectors is equal to zero:

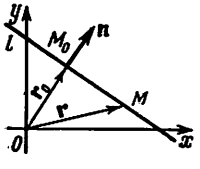


Fig. 119

$$\mathbf{n}(\mathbf{r} - \mathbf{r}_0) = 0. \tag{18.2}$$

The above equation is called the *vector equation* of a straight line. Rewriting it in the coordinate form, we get the equation

$$A(x - x_0) + B(y - y_0) = 0. \tag{18.3}$$

3. The Canonical Equation of a Straight Line. Let $M_0(x_0, y_0)$ be a point given on a straight line and $\mathbf{q} = (m, n)$ a vector collinear to the line (it is called the *direction vector* of the line). If $M(x, y)$ is an arbitrary point on the line, then the vectors $\overrightarrow{M_0M} = (x - x_0, y - y_0)$ and $\mathbf{q} = (m, n)$ are collinear, that is, the coordinates of these vectors are proportional:

$$(x - x_0)/m = (y - y_0)/n. \tag{18.4}$$

Equation (18.4) is called the *canonical equation* of a straight line.

1. Check whether the points A (3, 14), B (4, 13), C (—3, 0), and D (0, 7) belong to the straight line $7x - 3y + 21 = 0$.

○ If the coordinates of a point satisfy the equation, that is, turn it into an identity, then this point belongs to the given line; and if the coordinates of the point do not satisfy the equation, then the point does not belong to the line.

Substituting the coordinates of the point A for the variables x and y into the given equation $7x - 3y + 21 = 0$, we get the identity $7 \cdot 3 - 3 \cdot 14 + 21 = 0$; consequently, the point A (3, 14) belongs to the given straight line.

Analogously, we make sure that the points C and D belong to the given line, and the point B does not belong. ●

2. Construct the straight line $3x + 4y - 12 = 0$.

○ To construct the straight line, we find the coordinates of the points of intersection of the line with x- and y-axes. Setting $y = 0$, we get

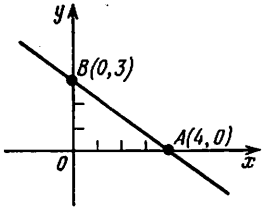


Fig. 120

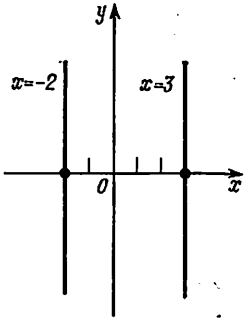


Fig. 121

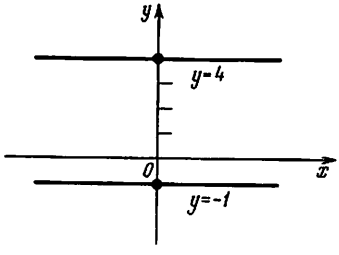


Fig. 122

$3x - 12 = 0$, $x = 4$, A (4, 0). For $x = 0$ we obtain $4y - 12 = 0$, $y = 3$, B (0, 3). Through the points A and B we draw the desired line (Fig. 120). ●

3. Construct the following straight lines: (1) $x = 3$; $x = -2$; $x = 0$; (2) $y = 4$; $y = -1$; $y = 0$.

○ (1) On the x-axis we take the points $x = 3$ and $x = -2$. Through these points we then draw straight lines parallel to the y-axis (Fig. 121). The line $x = 0$ is the y-axis.

(2) On the y-axis we take the points $y = 4$ and $y = -1$. Through these points we then draw straight lines parallel to the x-axis (Fig. 122). The line $y = 0$ is the x-axis. ●

4. A straight line, parallel to the x-axis, passes through the point (—2, 2). Write the equation of this line.

○ The equation of a straight line parallel to the x-axis has the form $y = b$. The ordinate of the point through which the desired line passes is equal to 3; consequently, the equation of the line under consideration is $y = 3$ or $y - 3 = 0$. ●

5. Set up the equation of the straight line passing through the point M_0 (3, —5) and perpendicular to the vector $\mathbf{n} = (4, 2)$.

○ Let M (x, y) be an arbitrary point of the desired line. The vector

$\overrightarrow{M_0M} = (x - 3,\ y + 5)$ is perpendicular to the vector $\mathbf{n} = (4, 2)$. Since the vectors are mutually perpendicular, their scalar product is equal to zero: $\mathbf{n} \cdot \overrightarrow{M_0M} = 0$. Writing the product of these vectors in coordinate form, we get

$$(4; 2)\ (x - 3; y + 5) = 0 \Leftrightarrow 4\ (x - 3) + 2\ (y + 5) = 0$$
$$\Leftrightarrow 4x + 2y - 2 = 0 \Leftrightarrow 2x + y - 1 = 0.$$

The equation of the desired line has the form: $2x + y - 1 = 0$. ●

6. Write the equation of the straight line passing through the origin and the point M (2, 3).

○ The vector $\overrightarrow{OM} = (2, 3)$ is collinear to the required straight line. To set up the equation of the line under consideration, let us use the canonical equation of the straight line (18.4). Substituting the values $m = 2$, $n = 3$, $x_0 = 0$, and $y_0 = 0$ into Equation (18.4), we get

$$(x/2 = y/3) \Leftrightarrow (3x - 2y = 0). \quad ●$$

7. Compute the length of the line segment $3x + 4y - 24 = 0$ enclosed between the coordinate axes.

○ We find the coordinates of the points of intersection of the given line with the coordinate axes: $y = 0$, $x = 8$, A (8, 0); $x = 0$, $y = 6$, B (0, 6). Consequently, the length of the line segment AB is equal to $|AB| = \sqrt{64 + 36} = 10$. ●

8. On the straight line $2x + y - 6 = 0$ find the point M equidistant from the points A (3, 5) and B (2, 6).

○ Denoting the coordinates of the point M by (x_M, y_M), we get:

$$|MA| = \sqrt{(x_M - 3)^2 + (y_M - 5)^2},\ |MB| = \sqrt{(x_M - 2)^2 + (y_M - 6)^2}.$$

Since $|MA| = |MB|$, we have

$$\sqrt{(x_M - 3)^2 + (y_M - 5)^2} = \sqrt{(x_M - 2)^2 + (y_M - 6)^2}.$$

After squaring and simplifying, we get

$$x_M - y_M + 3 = 0.$$

The point M (x_M, y_M) belongs to the straight line $2x + y - 6 = 0$, consequently, its coordinates satisfy this equation.

Solving the system of equations

$$\begin{cases} x_M - x_M + 3 = 0, \\ 2x_M + y_M - 6 = 0, \end{cases}$$

we find: $x_M = 1$, $y_M = 4$; M (1, 4). ●

9. Construct the straight lines: (1) $2x - 5y + 10 = 0$; (2) $4x + 6y - 3 = 0$.

10. Construct the following straight lines: (1) $x = 4$; (2) $x = -3$, (3) $y = 2$; (4) $y = -4$.

11. Construct the figure bounded by the lines $x = -2$, $x = 0$, $y = -3$, and $y = 0$. Compute the area of this figure.

12. (1) A straight line, parallel to the x-axis, passes through the point (3, —4). Write the equation of this line.

(2) A straight line, parallel to the y-axis, passes through the point (—6, 0). Write the equation of this line.

13. (1) The lengths of the sides of a rectangle are equal to 3 and 4. Write the equations for all of its sides if it is situated in Quadrant III so that two of its sides lie on the coordinate axes, the shorter side lying on the y-axis.

(2) Write the equations of the sides of a square if it is situated in Quadrant I, and two of its vertices have the coordinates A (2, 0) and B (5, 0).

14. Set up the equation of the straight line passing through the given point M_0 and perpendicular to the given vector $\mathbf{n}$: (1) M_0 (—2, —3), $\mathbf{n} = (4, -5)$; (2) M_0 (1, —1), $\mathbf{n} = (-3, 4)$.

15. Set up the equation of the straight line passing through the given point M_0 and perpendicular to the given vector $\overrightarrow{AB}$: (1) M_0 (—2, —3), A (—5, 2), B (—1, 4); (2) M_0 (2, 2), A (1, —3), B (6, —5); (3) M_0 (—2, —3), A (2, 1), B (1, 5).

16. Set up the equation of the straight line passing through the origin and the given point: (1) (—4, —1); (2) (5, —4).

17. Compute the length of the segment of the line $4x + 3y - 36 = 0$ enclosed between the coordinate axes.

18. (1) On the straight line $x - 2y - 4 = 0$ find the point equidistant from the points A (5, —1) and B (2, —4).

(2) On the straight line $3x + 4y + 20 = 0$ find the point equidistant from the points A (—8, —3) and B (—5, —6).

Sec. 18.2. Intercept Form of the Equation of a Straight Line

If a straight line intercepts, on the coordinate axes, line segments a, b (not equal to zero), then it may be given by the equation

$$\frac{x}{a}+\frac{y}{b}=1, \tag{18.5}$$

where a and b are, respectively, the abscissa and ordinate of the points of intersection of the straight line with the x- and y-axes.

Equation (18.5) is the *intercept form of the equation of a straight line.*

19. Construct the straight line $\frac{x}{2}-\frac{y}{3}=1$.

○ Let us rewrite the given equation as follows:

$$\frac{x}{2}+\frac{y}{-3}=1,$$

that is, $a = 2$ and $b = -3$. Thus, we obtain the points A (2, 0) and B (0, —3). The line drawn through the points A and B is the desired one (Fig. 123). ●

20. Transform the general equation of the straight line $3x - 4y + 2 = 0$ into the intercept form.

○ We carry out the following transformations:

$$3x - 4y = -2; \quad \frac{3x}{-2} - \frac{4y}{-2} = 1; \quad \frac{x}{-2/3} + \frac{y}{1/2} = 1. \bullet$$

21. Write the equation of the straight line cutting the x-axis at the point (3, 0) and the y-axis at the point (0, 5).

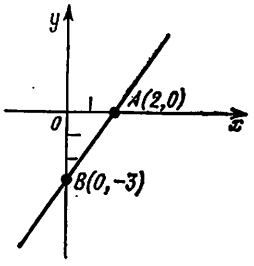


Fig. 123

○ By the hypothesis, $a = 3$ and $b = 5$. Consequently, the desired equation has the form $x/3 + y/5 = 1$. ●

22. Construct the following straight lines: (1) $x/2 + y/6 = 1$; (2) $x/5 - y/4 = 1$; (3) $-x/3 + y/2 = 1$; (4) $-x/6 - y/3 = 1$.

23. Transform the equations of the following straight lines into the intercept form: (1) $x + y - 3 = 0$; (2) $2x + 3y + 1 = 0$; (3) $2x + 3y - 6 = 0$; (4) $3x - 4y + 12 = 0$.

24. Write the intercept form of the straight line if it intersects the coordinate axes at the points: (1) A (—2, 0) and B (0, 3); (2) A (3, 0) and B (0, —4).

25. Find the lengths of the segments enclosed between the points of intersection with the coordinate axes for the following straight lines: (1) $x/6 + y/8 = 1$; (2) $x/12 - y/16 = 1$; (3) $x/9 - y/12 = 1$.

Sec. 18.3. Slope-intercept Form of the Equation of a Straight Line

The *slope-intercept form of the equation of a straight line* is:

$$y = kx + b, \tag{18.6}$$

where $k = \tan \alpha$ is the slope equal to the tangent of the angle at which the straight line is inclined to the x-axis and b is the ordinate of the point at which the line cuts the y-axis.

If $\alpha = 0$, then $k = 0$, that is, the straight line is parallel to the x-axis.

For $\alpha = 90°$ there is no k, that is, the straight line perpendicular to the x-axis has no slope.

If we take a point A (x_A, y_A) on a straight line passing through the origin, then

$$k = \tan \alpha = y_A/x_A. \tag{18.7}$$

26. Construct the following straight lines: $y = 3x$; $y = x$; $y = (1/2)x$; $y = -3x$.

○ The position of a straight line in the plane is determined by two points; but for a straight line passing through the origin one point

is already known, therefore it suffices to find one more point from its equation. Joining this point to the origin, we obtain the desired line.

We first construct the straight line $y = 3x$. Setting $x = 1$, we find $y = 3 \cdot 1 = 3$. Joining the point A (1, 3) to the origin, we get the desired line.

Analogously, we construct the remaining lines: $y = x$, B (1, 1); $y = (1/2)\,x$, C (1, 1/2); $y = -3x$, D (1, —3) (Fig. 124). ●

27. Construct the straight line $y = 2x + 8$.

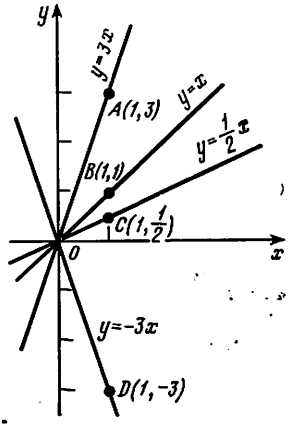


Fig. 124

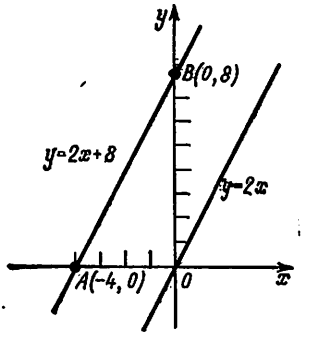


Fig. 125

○ First Method. We construct the straight line $y = 2x$. The line $y = 2x + 8$ passes parallel to the line $y = 2x$ by 8 scale units above the origin (Fig. 125).

Second Method. We find the points at which the line intersects the coordinate axes. Setting $y = 0$, we get $x = -4$, that is, A (—4, 0). Setting now $x = 0$, we find $y = 8$, that is, B (0, 8). Through the points A and B we draw the desired line (Fig. 125). ●

28. Compute the angles of slope (inclination) to the x-axis for the following straight lines: (1) $y = x$; (2) $y = -x$; (3) $y = 3x$; (4) $y = -2x$; (5) $y = mx$.

○ (1) $y = x$, $k = \tan \alpha = 1$, $\alpha = 45°$;

(2) $y = -x$, $k = \tan \alpha = -1$, $\alpha = 135°$;

(3) $y = 3x$, $k = \tan \alpha = 3$, $\alpha = 71.6°$;

(4) $y = -2x$, $k = \tan \alpha = -2$, $\alpha = 180° - 63.4° = 116.6°$;

(5) $y = mx$, $k = \tan \alpha = m$, $\alpha = \arctan m$ if $m \geqslant 0$; $\alpha = \pi - \arctan m$ if $m < 0$. ●

29. Compute the angle of inclination of the straight line $3x + 2y + 6 = 0$ to the x-axis.

○ Solving the equation $3x + 2y + 6 = 0$ with respect to y,

we get $y = (-3/2)\,x - 3$, whence $k = \tan\alpha = -3/2$; $\alpha = 180° - 56.3° = 123.7°$. ●

30. Write the equation of the straight line passing through the origin if its slope: (1) $k = 5$; (2) $k = -3$.

○ To set up the equation of the desired line, it suffices to substitute the numerical value of k into the equation $y = kx$. We have: (1) $y = 5x$ or $5x - y = 0$; (2) $y = -3x$ or $3x + y = 0$. ●

31. Set up the equation of the straight line passing through the origin and making with the x-axis an angle of: (1) 0; (2) $\pi/4$; (3) 120°; (4) arctan (—3).

○ (1) $k = \tan 0° = 0$; $y = 0$ is the equation of the x-axis;

(2) $k = \tan(\pi/4) = 1$; $y = x$ or $x - y = 0$;

(3) $k = \tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$; $y = -\sqrt{3}x$ or $\sqrt{3}x + y = 0$;

(4) $k = \tan(\arctan(-3)) = -3$; $y = -3x$ or $3x + y = 0$. ●

32. Set up the equation of the straight line passing through the origin and the point A (—2, 3).

○ To set up the equation of the straight line passing through the origin, it is necessary to know k. The latter is found from Relationship (18.7): $k = y_A/x_A = -3/2$; substituting the value of k into the equation $y = kx$, we get $y = (-3/2)\,x$ or $3x + 2y = 0$.

The value of k can also be computed by substituting the coordinates of the point A (—2, 3) into the equation $y = kx$ for the variables x and y: $3 = k(-2)$, whence $k = -3/2$. ●

33. Find the coordinates of the point A if the slope of the straight line passing through the origin and the point A is equal to 3/4 and the distance between the origin and the point A equals 10 units of length.

○ From (18.7) we have $y_A/x_A = 3/4$. On the other hand, the length of the line segment OA is equal to $\sqrt{x_A^2 + y_A^2} = 10$. Solving the system

$$\begin{cases} y_A/x_A = 3/4, \\ \sqrt{x_A^2 + y_A^2} = 10. \end{cases}$$

we get ($x_A = 8$; $y_A = 6$) or ($x_A = -8$; $y_A = -6$), that is, the conditions of the problem are satisfied by two points: (8, 6) and (—8, —6). ●

34. Set up the equation of the straight line passing through the point (3, 4) and intercepting a line segment $b = 2$ on the y-axis.

○ To set up the required equation of the straight line, it is necessary to find k. Substituting, instead of the variables x and y, the coordinates of the given point and the value of b, we get $4 = k \cdot 3 + 2$, whence $k = 2/3$. Thus, the required equation has the form $y = (2/3)\,x + 2$. ●

35. Set up the equation of the straight line passing through the point (2, 6) and making an angle of arctan 5 with the x-axis.

○ To set up the desired equation, it is necessary to compute k and b. We first find the slope $k = \tan(\arctan 5) = 5$. To compute b, we substitute the coordinates of the given point and the found value of k into the equation (18.6); we get $6 = 5 \cdot 2 + b$, whence $b = -4$. The desired equation has the form $y = 5x - 4$. ●

36. Construct the following straight lines: (1) $y = 5x$; (2) $y = -(1/3)x$; (3) $y = 4x + 3$; (4) $y = -x + 2$.

37. Find the angles of inclination to the x-axis for the following straight lines: (1) $y = (\sqrt{3}/3)x$; (2) $y = -3x$; (3) $y = 7x - 8$; (4) $y = -2.9x + 3$; (5) $3x + 5y + 20 = 0$; (6) $29x - 10y + 10 = 0$.

38. Set up the equation of the straight line passing through the origin if its slope is: (1) $k = -1$; (2) $k = 4$.

39. Set up the equation of the straight line if its slope is $k = 2/3$ and $b = -1/2$.

40. Set up the equation of the straight line passing through the origin and making with the x-axis an angle of: (1) 60°; (2) $\pi/6$; (3) 135°; (4) arctan 3; (5) arctan (—5).

41. Set up the equation of the straight line for which $b = 3$ and the angle of inclination to the x-axis is: (1) $\alpha = 45°$; (2) $\alpha = 120°$; (3) $\alpha = \arctan 5$.

42. Write the equation of the straight line for which $b = -2$ and the angle of inclination to the x-axis is: (1) $\alpha = 30°$; (2) $\alpha = 135°$; (3) $\alpha = \arctan 2$.

43. Set up the equation of the straight line passing through the origin and the point: (1) A (3, —6); (2) A (—1, —5).

44. (1) The distance of the point P from the origin is equal to 5 units of length. The slope of the straight line passing through the origin and the point P is equal to 3/4. Find the point P.

(2) The diagonal of a rectangle whose sides lie on the positive directions of the coordinate axes is equal to 20 units of length. The slope of the diagonal is 4/3. Find the vertices of the rectangle.

45. Set up the equation of the straight line passing through the point (—5, —2) and intercepting the line segment $b = -12$ on the y-axis.

46. Set up the equation of the straight line: (1) passing through the point (5, —7) and making an angle arctan (—2) with the x-axis; (2) passing through the point (—1, —4) and making an angle of 135° with the x-axis.

Sec. 18.4. The Equation of a Straight Line (Point-slope Form)

The *equation of a straight line passing through a given point* $A(x_A, y_A)$ *in a given direction* has the form

$$y - y_A = k(x - x_A), \tag{18.8}$$

where $k = \tan\alpha$ is the slope of the line.

Equation (18.8) may be considered as the equation of a *pencil of lines*, that is, the set of lines passing through one point $A\ (x_A, y_A)$. Note that only one line of all those passing through the point A, namely, the one perpendicular to the x-axis is not expressed by the equation of the form (18.8). Its equation has the form $x = x_A$.

47. Set up the equation of the straight line passing through the point A (5, —1) and having the slope $k = 3$.

○ According to the hypothesis, $x_A = 5$, $y_A = -1$, $k = 3$. Substituting these values into Equation (18.8), we get

$$y + 1 = 3(x - 5) \quad \text{or} \quad 3x - y - 16 = 0. \quad \bullet$$

48. Set up the equation of the straight line passing through the point (—3, —2) and making an angle of arctan 2 with the x-axis.

○ By hypothesis, $x_A = -3$, $y_A = -2$. We find $k = \tan(\arctan 2) = 2$. Substituting these values into Equation (18.8), we get

$$y + 2 = 2(x + 3) \quad \text{or} \quad 2x - y + 4 = 0. \quad \bullet$$

49. (1) Set up the equation of the straight line: (1) passing through the point (—1, —1) and having the slope $k = 1$; (2) passing through the point (2, 0) and having the slope $k = -2$.

50. (1) Set up the equation of the straight line: (1) passing through the point (4, —5) and making an angle of arctan (—3) with the x-axis; (2) passing through the point (2, 3) and making an angle of 45° with the x-axis; (3) passing through the point (0, 5) and making an angle of 135° with the x-axis.

Sec. 18.5. The Equation of a Straight Line Through Two Points (Two-point Form)

The *equation of a straight line passing through two given points* $A\ (x_A, y_A)$ and $B\ (x_B, y_B)$ has the form

$$y - y_A = \frac{y_B - y_A}{x_B - x_A}(x - x_A). \tag{18.9}$$

The slope of the straight line passing through the points A and B is found from the relationship

$$k_{AB} = \frac{y_B - y_A}{x_B - x_A}. \tag{18.10}$$

51. Set up the equation of the straight line passing through the points A (2, —3) and B (—1, 4).

○ By hypothesis, $x_A = 2$, $x_B = -1$, $y_A = -3$, and $y_B = 4$. Substituting these values into Equation (18.9), we obtain

$$y + 3 = \frac{4+3}{-1-2}(x - 2) \quad \text{or} \quad 7x + 3y - 5 = 0. \quad \bullet$$

52. Find the angle of inclination to the x-axis made by the straight line passing through the points A (2, 3) and B (—3, 1).

○ By hypothesis, $x_A = 2$, $x_B = -3$, $y_A = 3$, and $y_B = 1$. By Formula (18.10), we find

$$k = \frac{y_B - y_A}{x_B - x_A} = \frac{1-3}{-3-2} = \frac{2}{5} = 0.4,$$

whence $\alpha = \arctan 0.4 = 21.8°$. ●

53. Find the segments intercepted on the coordinate axes by the straight line passing through the points A (6, 2) and B (—3, 8).

○ Substituting the coordinates of the points A (6, 2) and B (—3, 8) into Equation (18.9), we get

$$y - 2 = \frac{8-2}{-3-6}(x-6).$$

We reduce this equation to the intercept form:

$$y - 2 = (-2/3)(x - 6), \quad y - 2 = (-2/3)x + 4,$$

$$(2/3)x + y = 6;$$

$$\frac{(2/3)x}{6} + \frac{y}{6} = 1, \quad \frac{x}{9} + \frac{y}{6} = 1.$$

Consequently, $a = 9$ and $b = 6$. ●

54. A straight line passing through the point (—5, 1) intercepts on the y-axis the segment $b = 6$. Write the equation of this line.

○ The desired line intersects the y-axis at the point (0, 6). Substituting the coordinates of the points A (—5, 1) and B (0, 6) into the equation (18.9), we get the required equation

$$y - 1 = \frac{6-1}{0+5}(x+5) \quad \text{or} \quad x - y + 6 = 0. \; ●$$

55. Set up the equation of the straight line passing through the indicated two points: (1) A (—1, —1) and B (—2, —2); (2) A (3, 0) and B (0, 4).

56. Write the equations of the sides of the triangle whose vertices lie at the following points: (1) A (—3, —2), B (1, 5), and C (8, —4); (2) (—1, —3), (3, 5), and (4, 0).

57. (1) A triangle is given by its vertices: A (—3, 4), B (—4, —3), and C (8, 1). Write the equation of the median AD.

(2) A triangle is given by its vertices: A (2, 5), B (—6, —4), and C (6, —3). Write the equation of the median BD.

58. Find the angle of inclination to the x-axis made by the straight line passing through the points A (—3, —3) and B (2, 1).

59. (1) A straight line passes through the points A (—1, —6) and B (7, 2). Find the segments intercepted by this line on the x- and y-axes.

(2) Moving in a straight line, a point passed through the positions A (12, —1) and B (3, 2). At what point will it cut the y-axis?

60. (1) A straight line passing through the point (—4, —1) inter-

sects the y-axis at the point (0, 3). Write the equation of this line.

(2) A straight line passing through the point (—2, 4) intercepts on the x-axis a segment $a = 2$. Write the equation of this line.

Sec. 18.6. The Intersection of Two Straight Lines

If we are given two intersecting lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, then, to compute the coordinates of the point of intersection of the given lines, we have to solve the system of equations of these lines.

61. Find the point of intersection of the straight lines $3x - 4y + 11 = 0$ and $4x - y - 7 = 0$.

○ Solving the system of equations

$$\begin{cases} 3x - 4y + 11 = 0, \\ 4x - y - 7 = 0, \end{cases}$$

we get $x = 3$ and $y = 5$. Consequently, (3, 5) is the point of intersection of these lines. ●

62. Given the equations of the sides of a triangle: $x + 3y - 3 = 0$, $3x - 11y - 29 = 0$, and $3x - y + 11 = 0$. Find the vertices of this triangle.

○ To compute the coordinates of the vertices of the given triangle, it is necessary to solve the following three systems of equations:

$$\begin{cases} x + 3y - 3 = 0; \\ 3x - 11y - 29 = 0, \end{cases} \quad \begin{cases} 3x - 11y - 29 = 0, \\ 3x - y + 11 = 0; \end{cases} \quad \begin{cases} 3x - y + 11 = 0, \\ x + 3y - 3 = 0. \end{cases}$$

The respective solutions are: (1) $x = 6$, $y = -1$; (2) $x = -5$, $y = -4$ and (3) $x = -3$, $y = 2$. Consequently, the points (6, —1), (—5, —4), and (—3, 2) are the three vertices of the given triangle. ●

63. Write the equation of the straight line perpendicular to the vector $\mathbf{n} = (4, -3)$ and passing through the point of intersection of the straight lines: $x + 11y - 27 = 0$ and $6x - 7y - 16 = 0$.

○ We first find the point M_1 of intersection of the given lines:

$$\begin{cases} x + 11y - 27 = 0, \\ 6x - 7y - 16 = 0 \end{cases} \Leftrightarrow (x = 5,\ y = 2); \quad M_1(5,\ 2).$$

Writing then the scalar product of the vectors $\mathbf{n} = (4, -3)$ and $\overrightarrow{M_1M} = (x - 5,\ y - 2)$ in coordinate form and simplifying, we get the desired equation:

$$\mathbf{n} \cdot \overrightarrow{M_1M} = (4, -3)(x - 5,\ y - 2) = 0 \Leftrightarrow$$
$$4(x - 5) - 3(y - 2) = 0 \Leftrightarrow 4x - 3y - 14 = 0. \quad ●$$

64. Find the points of intersection of the indicated pairs of straight lines: (1) $y = 3x$ and $x + y + 4 = 0$; (2) $x - 2y - 8 = 0$ and $x + y - 2 = 0$.

65. Find the vertices of the triangle if its sides are given by the following equations: (1) $4x + 3y + 20 = 0$, $6x - 7y - 16 = 0$, and $x - 5y + 5 = 0$; (2) $7x + 3y - 25 = 0$, $2x - 7y - 15 = 0$, and $9x - 4y + 15 = 0$.

66. Set up the equation of the straight line perpendicular to the given vector and passing through the point of intersection of the given lines: (1) $\mathbf{n} = (-3, 2)$, $2x + 3y - 17 = 0$, $x + y - 6 = 0$; (2) $\mathbf{n} = (-4, -5)$, $3x + y - 10 = 0$, $2x + y - 6 = 0$.

Sec. 18.7. The Angle Between Two Straight Lines

The angle φ between two straight lines given by the general equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ is computed by the formula

$$\cos\varphi = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}. \tag{18.11}$$

The angle φ between two straight lines given by the equations with slopes $y = k_1x + b_1$ and $y = k_2x + b_2$ is computed by the formula

$$\tan\varphi = \frac{k_2 - k_1}{1 + k_2k_1}. \tag{18.12}$$

The angle φ between two straight lines given by the canonical equations $(x - x_1)/m_1 = (y - y_1)/n_1$ and $(x - x_2)/m_2 = (y - y_2)/n_2$ is computed by the formula

$$\cos\varphi = \frac{m_1m_2 + n_1n_2}{\sqrt{m_1^2 + n_1^2} \cdot \sqrt{m_2^2 + n_2^2}}. \tag{18.13}$$

Formulas (18.11)-(18.13) determine the value of the trigonometric function of one of the two angles (acute or obtuse) between the given lines. To find the acute angle between the lines, the expressions in the right-hand side should be taken in the absolute value.

67. Find the acute angle between the straight lines $y = 5x$ and $y = 2x$ (Fig. 126).

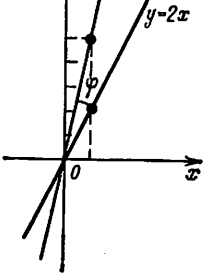


Fig. 126

○ The slopes of the given lines are equal to 5 and 2, respectively. We take advantage of Formula (18.12), taking its right-hand side in the absolute value:

$$\tan\varphi = \left|\frac{2-5}{1+2\cdot5}\right| = \frac{3}{11} = 0.273, \quad \varphi \approx 15.3°. \bullet$$

68. Find the acute angle between the straight lines $5x - 12y - 16 = 0$ and $3x + 4y - 12 = 0$.

○ By Formula (18.11), we find

$$\cos\varphi = \left|\frac{5\cdot3 - 12\cdot4}{\sqrt{5^2 + (-12)^2} \cdot \sqrt{3^2 + 4^2}}\right| = \frac{33}{65}; \quad \varphi = \arccos\frac{33}{65} \approx 59.5°. \bullet$$

69. Find the acute angle between the straight lines $(x - 5)/(-24) = (y - 2)/7$ and $(x + 4)/8 = (y - 3)/15$.

○ By Formula (18.13), we find

$$\cos\varphi=\left|\frac{-24\cdot 8+7\cdot 15}{\sqrt{(-24)^2+7^2}\sqrt{8^2+15^2}}\right|=\frac{87}{425};\quad \varphi=\arccos\frac{87}{425}\approx 78.2^\circ.\ \bullet$$

70. Given a triangle with the vertices A $(-6, -1)$, B $(4, 6)$, and C $(2, 1)$. Find the interior angles of this triangle.

○ We first find the slopes of the sides of the triangle:

$$k_{AB}=\frac{y_B-y_A}{x_B-x_A}=\frac{6-(-1)}{4-(-6)}=\frac{7}{10};$$

$$k_{BC}=\frac{y_C-y_B}{x_C-x_B}=\frac{1-6}{2-4}=\frac{5}{2};$$

$$k_{CA}=\frac{y_A-y_C}{x_A-x_C}=\frac{-1-1}{-6-2}=\frac{1}{4};$$

and then its angles:

$$\tan\widehat{A}=\frac{k_{AB}-k_{CA}}{1+k_{AB}k_{CA}}=\frac{7/10-1/4}{1+(7/10)\cdot(1/4)}=0.383,\quad \widehat{A}=21^\circ;$$

$$\tan\widehat{B}=\frac{k_{BC}-k_{AB}}{1+k_{BC}k_{AB}}=\frac{5/2-7/10}{1+(5/2)\cdot(7/10)}=0.655;\quad \widehat{B}=33.2^\circ;$$

$$\tan\widehat{C}=\frac{k_{BC}-k_{CA}}{1+k_{BC}k_{CA}}=\frac{5/2-1/4}{1+(5/2)\cdot(1/4)}=1.39;\quad \widehat{C}=54.2^\circ.$$

Adding together the found values of the angles, we get: $21^\circ + 33.2^\circ + 54.2^\circ = 108.4^\circ$. The sum of the interior angles of the given triangle turns out to be less than 180° because our computation yielded two interior angles and one exterior angle adjacent to the desired interior one. By construction, we make sure that the angle C is obtuse (Fig. 127) and is equal to $180^\circ - 54.2^\circ = 125.8^\circ$. Then the sum of the interior angles of the triangle will amount to $21^\circ + 33.2^\circ + 125.8^\circ = 180^\circ$.

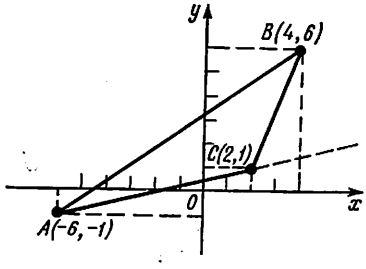


Fig. 127

Without constructing the triangle, it is easy to show that one of the interior angles of the given triangle is obtuse. Now, we are going to compute the lengths of the sides of the triangle:

$$|AB|=\sqrt{(-6-4)^2+(-1-6)^2}=\sqrt{149};$$

$$|BC|=\sqrt{(4-2)^2+(6-1)^2}=\sqrt{29};$$

$$|AC|=\sqrt{(-6-2)^2+(-1-1)^2}=\sqrt{68}.$$

It is known that if in a triangle the square of the longest side is greater than the sum of the squares of two other sides, then this

triangle is obtuse. Since $|AB|^2 > |BC|^2 + |AC|^2$, the side AB lies opposite the obtuse angle. ●

71. Write the equation of the straight line passing through the point A (2, 3) and making an angle of arctan (4/3) with the straight line $2x - y - 1 = 0$.

○ It is obvious that the problem has two solutions, since the slope of the given line in Formula (18.12) can be equal to both k_1 and k_2.

(1) $k_1 = 2$; $\tan\left(\arctan\frac{4}{3}\right) = \frac{k_2 - 2}{1 + 2k_2}$ or $\frac{4}{3} = \frac{k_2 - 2}{1 + 2k_2}$. Solving this equation, we get $k_2 = -2$. The desired equation has the form

$$y - 3 = -2(x - 2) \quad \text{or} \quad 2x + y - 7 = 0.$$

(2) $k_2 = 2$; $\tan\left(\arctan\frac{4}{3}\right) = \frac{2 - k_1}{1 + 2k_1}$, whence $k_1 = \frac{2}{11}$; thus we have

$$y - 3 = \frac{2}{11}(x - 2) \quad \text{or} \quad 2x - 11y + 29 = 0. \; ●$$

72. A triangle is given by its vertices: A (2, −1), B (−7, 3), and C (−1, −5). Write the equation of the bisector of the angle C.

○ Let us first find the point M of intersection of the bisector of the angle C with the side AB. As is known, an angle bisector of a triangle divides the opposite side into parts proportional to the lengths of the adjacent sides of the triangle. Consequently, $\lambda = |BM| : |MA| = |CB| : |CA|$.

Since $|CB| = \sqrt{(-1 + 7)^2 + (-5 - 3)^2} = 10$, $|CA| = \sqrt{(-1 - 2)^2 + (-5 + 1)^2} = 5$, we have $\lambda = 10/5 = 2$. We then compute the coordinates of the point M:

$$x_M = \frac{-7 + 2 \cdot 2}{1 + 2} = -1, \quad y_M = \frac{3 + 2(-1)}{1 + 2} = \frac{1}{3}, \quad M\left(-1, \frac{1}{3}\right).$$

The abscissas of the points C and M are equal to each other, consequently, the bisector of the angle C is parallel to the y-axis: $x = -1$ or $x + 1 = 0$. ●

73. Find the acute angle between the following pairs of straight lines: (1) $y = 3x$ and $y = -x$; (2) $2x - 3y + 6 = 0$ and $3x - y - 3 = 0$; (3) $x/5 + y/2 = 1$ and $x/3 + y/4 = 1$; (4) $3x + 4y - 12 = 0$ and $15x - 8y - 45 = 0$.

74. Find the acute angle between the straight lines $(x - 1)/5 = (y - 4)/12$ and $(x + 3)/3 = (y + 2)/4$.

75. Find the interior angles of a triangle if its sides are given by the following triples of equations: (1) $7x + 4y + 9 = 0$, $x - 8y + 27 = 0$, and $2x - y - 6 = 0$; (2) $6x - y + 13 = 0$, $3x + 7y - 1 = 0$, and $3x - 8y - 31 = 0$; (3) $3x - 2y - 1 = 0$, $5x + 4y - 31 = 0$, and $x - 8y - 15 = 0$.

76. Find the acute angle between two straight lines if: (1) one of them passes through the points A_1 (4, 2) and B_1 (1, —7), and the other through the points A_2 (—1, 3) and B_2 (8, 6); (2) one of them passes through the points A_1 (—6, 7) and B_1 (2, —5), and the other through the points A_2 (—5, 2) and B_2 (1, 1).

77. Find the acute angle between two straight lines having a common point M (—2, —1) if one of them passes through the point A (3, 3) and the other through the point B (3, —2).

78. Find the interior angles of a triangle if its vertices are represented by the points: (1) A (—6, —3), B (6, 7), and C (2, —1); (2) A (0, 4), B (4, —2) and C (—4, —2).

79. Given a triangle with the vertices A (6, 8), B (2, —4), and C (—6, 4). Find the angle between the side AB and the median drawn from the vertex A.

80. Find the acute angle between: (1) the straight line $3x + 2y + 4 = 0$ and the straight line passing through the points A (4, —3) and B (2, —2); (2) the straight line $x + 2y - 4 = 0$ and the straight line passing through the points A (1, 5) and B (—4, 3).

81. Find the acute angle between two straight lines passing through the origin and the points by which the line segment $x + 3y - 9 = 0$, enclosed between the coordinate axes, is divided in the ratio 1 : 3 : 2 in the direction from the point of its intersection with the x-axis to the point of intersection with the y-axis.

82. Find the acute angle between two straight lines passing through the point C (8, 7) and through the points by which the line segment $3x + 2y - 18 = 0$, enclosed between the coordinate axes, is divided into three equal parts.

83. Find the acute angle between two straight lines passing through the point M (—6, —8) and the points by which the line segment $2x + y + 10 = 0$, enclosed between the coordinate axes, is divided in the ratio 1 : 2 : 2 in the direction from the point of its intersection with the x-axis to the point of intersection with the y-axis.

84. (1) Write the equation of the straight line: (1) passing through the point (—2, 5) and making an angle of arctan (1/7) with the straight line $3x - y + 4 = 0$; (2) passing through the origin and making an angle of 45° with the straight line $x - y + 1 = 0$. Find the point of intersection of this line and the given one.

85. Two straight lines passing through the origin form an angle of arctan (1/3) with each other. The ratio of the slopes of these lines is equal to 2/7. Write the equations of these lines.

86. Two straight lines passing through the origin form an angle of arctan (7/9) with each other. The ratio of the slopes of the lines is 9/2. Write the equations of these lines.

87. A triangle is given by its vertices: A (—6, —2), B (4, 8), and C (2, —10). Set up the equation of the bisector of the angle A.

Sec. 18.8. The Parallelism Condition of Two Straight Lines

The parallelism condition of two straight lines given by the general equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ has the form

$$A_1/A_2 = B_1/B_2.$$

The parallelism condition of two straight lines given by the slope-intercept equations $y = k_1x + b_1$ and $y_2 = k_2x + b_2$ has the form

$$k_2 = k_1.$$

The parallelism condition of two straight lines given by their canonical equations $(x - x_1)/m_1 = (y - y_1)/n_1$ and $(x - x_2)/m_2 = (y - y_2)/n_2$ has the form

$$m_1/m_2 = n_1/n_2.$$

88. Write the equation of the straight line passing through the point M $(-2, 4)$ parallel to the straight line $2x - 3y + 6 = 0$ (Fig. 128).

○ Rewriting the equation of the given line in the form $y = (2/3)x + 2$, we find its slope $k_1 = 2/3$. Since the given and desired lines are parallel, their slopes are equal, that is, $k_2 = k_1 = 2/3$. The sought-for line passes through the point M $(-2, 4)$ and has the slope $k_2 = 2/3$. Therefore its equation is written in the form

$$y - 4 = \frac{2}{3}(x + 2) \quad \text{or}$$

$$2x - 3y + 16 = 0. \ ●$$

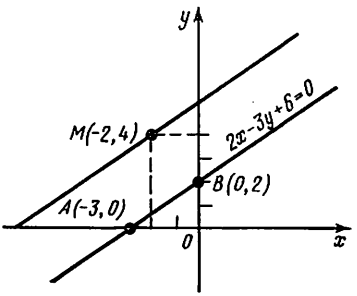


Fig. 128

89. Check whether the following pairs of straight lines are parallel: (1) $2x - 3y + 4 = 0$ and $10x - 15y - 7 = 0$; (2) $25x + 20y - 8 = 0$ and $5x + 4y + 4 = 0$; (3) $y = -2x + 8$ and $y = -2x + 1$; (4) $y = 3x + 4$ and $y = -3x + 2$.

90. For what value of the parameter a will the straight lines $(x - 1)/2 = (y + 4)/5$ and $(x + 6)/4 = (y - 2)/a$ be parallel?

91. Write the equation of the straight line: (1) passing through the point A $(-3, 2)$ parallel to the line $5x - 3y + 21 = 0$; (2) passing through the point A $(-1, -4)$ parallel to the line $x/4 + y/3 = 1$.

92. Write the equation of the straight line: (1) passing through the point M $(-3, -1)$ parallel to the line (AB), where A $(-2, 6)$ and B $(3, -1)$; (2) passing through the point $(1, -4)$ parallel to the line (AB), where A $(-3, 1)$ and B $(3, 2)$.

93. Write the equation of the straight line: (1) passing through the point of intersection of the straight lines $x + y - 4 = 0$ and

$x - y = 0$ parallel to the line $x - 4y + 4 = 0$; (2) passing through the point of intersection of the straight lines $x/6 + y/3 = 1$ and $x/3 + y/6 = 1$ parallel to the line $x - 2y - 6 = 0$.

Sec. 18.9. The Perpendicularity Condition of Two Straight Lines

The perpendicularity condition of two straight lines given by the general equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ has the form

$$A_1A_2 + B_1B_2 = 0.$$

The perpendicularity condition of two lines given by the slope-intercept form of the equations $y = k_1x + b_1$ and $y = k_2x + b_2$ has the form

$$k_2 = -1/k_1 \quad \text{or} \quad k_2k_1 = -1.$$

The perpendicularity condition of two straight lines given by the canonical equations $(x - x_1)/m_1 = (y - y_1)/n_1$ and $(x - x_2)/m_2 = (y - y_2)/n_2$ has the form

$$m_1m_2 + n_1n_2 = 0.$$

The distance d from the point M_1 (x_1, y_1) to the straight line $Ax + By + C = 0$ is computed by the formula

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}. \tag{18.14}$$

94. Write the equation of the straight line passing through the point M (2, 3) perpendicular to the straight line $5x - 4y - 20 = 0$ (Fig. 129).

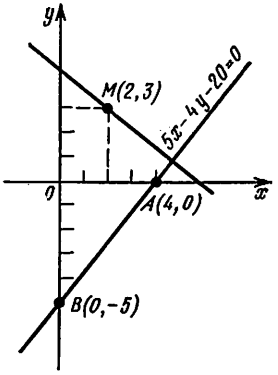


Fig. 129

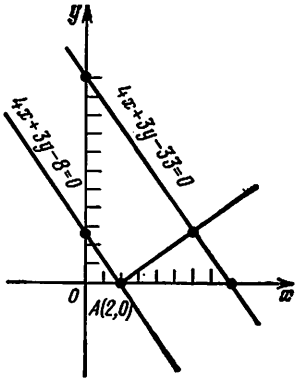


Fig. 130

○ We find the slope of the given line $k_1 = 5/4$. Then the slope of the required line $k_2 = -4/5$ and, consequently, its equation has the form

$$y - 3 = -\frac{4}{5}(x - 2) \quad \text{or} \quad 4x + 5y - 23 = 0. \ ●$$

95. Find the distance from the point M (6, 8) to the straight line $4x + 3y + 2 = 0$.

○ By Formula (18.14), we get

$$d = \frac{|4 \cdot 6 + 3 \cdot 8 + 2|}{\sqrt{4^2 + 3^2}} = 10. \bullet$$

96. Find the distance between two parallel straight lines $4x + 3y - 8 = 0$ (AB) and $4x + 3y - 33 = 0$ (CD).

○ On the line AB we take an arbitrary point, say, the point A (2, 0) of intersection of this line with the x-axis (Fig. 130). By Formula (18.14), we find the distance from the point A (2, 0) to the line $4x + 3y - 33 = 0$:

$$d = \frac{|4 \cdot 2 + 3 \cdot 0 - 33|}{\sqrt{4^2 + 3^2}} = 5. \bullet$$

97. Check whether the following pairs of straight lines are mutually perpendicular:
(1) $3x - 4y + 12 = 0$ and $4x + 3y - 6 = 0$; (2) $4x + 5y - 8 = 0$ and $3x - 2y + 4 = 0$; (3) $(x - x_1)/2 = (y - y_1)/3$ and $(x - x_2)/3 = (y - y_2)/(-2)$; (4) $(x - x_1)/5 = (y - y_1)/(-4)$ and $(x - x_2)/4 = (y - y_2)/5$.

98. For what value of the parameter k are the straight lines $y = 5x - 4$ and $y = kx - 2$ mutually perpendicular?

99. Write the equation of the straight line: (1) passing through the point M (4, —3) perpendicular to the straight line $5x - 2y + 10 = 0$; (2) passing through the point M (—4, 1) perpendicular to the straight line $x/5 - y/6 = 1$; (3) passing through the origin perpendicular to the line $2x + 3y - 12 = 0$.

100. Write the equation of the straight line passing through the point (2, 4) perpendicular to the line MN, where M (—2, 6) and N (3, —3).

101. A straight line passes through the points (—4, 1) and (2, —5). Through the point of its intersection with the y-axis and perpendicular to the given line there passes another straight line. Write the equations of these lines.

102. Write the equation of the straight line passing through the origin perpendicular to the straight line intersecting the x-axis at the point (2, 0) and the y-axis at the point (0, —6).

103. Write the equation of the straight line passing through the point of intersection of the straight lines $x + 2y + 4 = 0$ and $3x - y - 9 = 0$ perpendicular to the line $x + y - 7 = 0$.

104. A straight line passes through the point of intersection of the straight lines $x + y - 5 = 0$ and $x - y + 3 = 0$ perpendicular to the line intersecting the x-axis at the point (—2, 0) and the y-axis at the point (0, —3). Write the equation of this line.

105. A straight line passes through the midpoint of the line segment AB perpendicular to the latter. Write the equation of this line if: (1) A $(-2, 1)$, B $(4, 4)$; (2) A $(-1, 4)$, B $(3, -2)$.

106. A straight line passes through the midpoint of the line-segment $3x - 7y + 21 = 0$ enclosed between the coordinate axes perpendicular to this segment. Write the equation of the line.

107. Set up the equations of the three altitudes of the triangle whose vertices are represented by the following triples of points: (1) $(-4, 2)$, $(6, 5)$, and $(1, -4)$; (2) $(2, -3)$, $(7, 2)$, and $(-8, -2)$; (3) $(4, 2)$, $(6, -5)$, and $(-5, 4)$.

108. Write the equations of the altitudes of a triangle given the equations of its sides: (1) $11x + 2y - 21 = 0$, $8x - 3y + 7 = 0$ and $3x + 5y + 21 = 0$; (2) $2x - y + 5 = 0$, $x + y - 5 = 0$ and $x - 2y - 5 = 0$; (3) $3x - 10y + 28 = 0$, $5x + 4y + 26 = 0$ and $4x - 3y - 4 = 0$.

109. Find the distance: (1) from the point M $(-2, 4)$ to the straight line $4x - 3y - 5 = 0$; (2) from the point $(4, 6)$ to the straight line $3x + 4y + 14 = 0$.

110. Find the distance between two parallel lines: (1) $4x + 3y + 33 = 0$ and $4x + 3y - 17 = 0$; (2) $12x + 5y - 101 = 0$ and $12x + 5y + 68 = 0$.

Sec. 18.10. Miscellaneous Problems

111. For what value of the slope k does the straight line $y = kx + 9$ pass through the point of intersection of the lines $x - y + 5 = 0$ and $x + 2y + 2 = 0$?

112. A straight line passes through the point M $(2, 5)$ and makes an angle of arctan 3 with the x-axis. Find on this line the point with abscissa -2.

113. The segment of the straight line $x + 2y - 4 = 0$ enclosed between the coordinate axes is divided by two straight lines passing through the origin in the ratio $1 : 2 : 1$. Write the equations of these lines.

114. The segment of the straight line $x/9 + y/3 = 1$ enclosed between the coordinate axes is divided into three equal parts by two straight lines passing through the origin. Write the equations of these lines.

115. Given the equations of the sides of a triangle: $6x - 5y + 8 = 0$, $4x + y - 38 = 0$, and $x - 3y - 3 = 0$. Find the equations of its medians.

116. Given the equations of the sides of a triangle: $4x - 5y + 22 = 0$, $5x - 2y + 2 = 0$, and $x + 3y + 14 = 0$. Find the equation of the straight line passing through the point of intersection of its medians and point $(1, -3)$.

117. On the straight line $2x + 3y - 18 = 0$ find the point which is situated three times farther from the y-axis than from the x-axis.

118. Write the equation of the straight line passing through the origin and making with the x-axis an angle twice the size of the angle made by the line $y = 0.2x$ with the x-axis.

119. Write the equation of the straight line passing through the point (8, 5) and making with the x-axis an angle twice the size of the angle formed by the line $x - 4y + 4 = 0$ with the x-axis.

120. Find the equations of the straight lines passing through the point (—7, 8) at an angle of 45° to the straight line $3x - 5y + 15 = 0$.

121. Find the equations of the perpendiculars to the straight line $5x - 4y - 20 = 0$ erected at the points of its intersection with the coordinate axes.

122. A triangle is given by its vertices: A (—5, —2), B (7, 6), and C (5, —4). Find: (1) the equation of the side AB; (2) the equation of the median drawn from the vertex A; (3) the equation of the altitude drawn from the vertex C; (4) the angles B and C; (5) the centroid of this triangle.

123. Given two parallel lines: $x - y - 7 = 0$ and $x - y + 3 = 0$. Set up the equation of the line, parallel to the given lines, which divides the distance between them in the ratio 3 : 2 (in the direction from the line with a lesser initial ordinate to the line with a greater initial ordinate).

124. To the straight line passing through the points A (—4, 2) and B (8, 4) a perpendicular is drawn through the point which divides the distance AB (from A to B) in the ratio 3 : 4. Find the equation of the perpendicular.

125. Given the equations of two sides of a rhombus: $3x - 10y + 37 = 0$ and $9x + 2y - 17 = 0$ and the equation of one of its diagonals $3x - 2y - 19 = 0$. Find the equations of two other sides of the rhombus and its second diagonal.

126. Given the equations of two sides of a parallelogram: $3x - 2y + 12 = 0$ and $x - 3y + 11 = 0$ and the point of intersection of its diagonals (2, 2). Write the equations of two other sides of the parallelogram and its diagonals.

127. Two sides emanating from one vertex of a parallelogram are given by the equations $5x - 3y + 28 = 0$ and $x - 3y - 4 = 0$; the coordinates of the opposite vertex of the parallelogram are (10, 6). Write the equations of two other sides of the parallelogram and its diagonals.

128. Two opposite vertices of a square are situated at the points A (—1, 1) and C (5, 3). Find the equations of the sides and diagonals of this square.

129. Find the equations of the legs of a right isosceles triangle if the equation of its hypotenuse is $x - 2y - 3 = 0$ and the vertex of the right angle lies at the point C (1, 6).

130. A light beam, emanating from the point A (3, 10), reflects from the straight line $2x + y - 6 = 0$ and then passes through the point B (7, 2). Find the equations of the incident and reflected beams.

QUIZ No. 1

Form I

A triangle is given by its vertices: A (—7, 3), B (2, —1), and C (—1, —5). Find: (1) the equation of the line AM parallel to the side BC; (2) the equation of the median AD; (3) the equation of the altitude BF; (4) the angle B; (5) the equation of the angle bisector CN.

Form II

A triangle is given by its vertices: A (—8, —2), B (2, 10), and C (4, 4). Find: (1) the equation of the line BN parallel to the side AC; (2) the equation of the median CD; (3) the equation of the altitude AE; (4) the angle B; (5) the centroid of this triangle.

Chapter 19

SECOND-ORDER CURVES

Sec. 19.1. Sets of Points in the Plane

To the equation with the variables x and y there corresponds a certain line in the plane as the set of points whose coordinates satisfy this equation. Conversely: to a line in the plane representing a set of points there corresponds a certain equation with the variables x and y.

To set up the equation of a set of points in the plane using the conditions of a problem, it is necessary to establish the relationship between the variables x and y (the coordinates of an arbitrary point belonging to this set of points) and constant quantities (parameters) and to describe this relationship by an equation.

1. Find the equation of the set of points in the plane equidistant from the points A (2, 4) and B (4, 6).

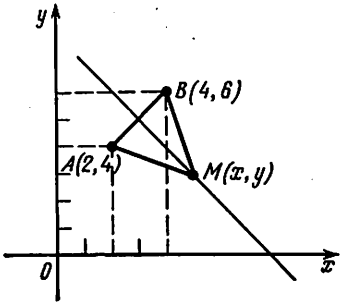


Fig. 131

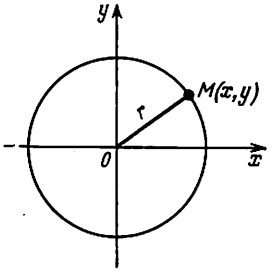


Fig. 132

○ Let the point M belong to the sought-for set of points (Fig. 131). Then $|MA| = |MB|$. Since $|MA| = \sqrt{(x-2)^2 + (y-4)^2}$, $|MB| = \sqrt{(x-4)^2 + (y-6)^2}$, we have

$$\sqrt{(x-2)^2+(y-4)^2} = \sqrt{(x-4)^2 + (y-6)^2}.$$

On squaring both sides and carrying out obvious simplifications, we get

$$(x-2)^2 + (y-4)^2 = (x-4)^2 + (y-6)^2 \quad \text{or}$$

$$x + y - 8 = 0.$$

The straight line $x + y - 8 = 0$ is just the set of points possessing the property indicated in the conditions of the problem. As is known, this line is the perpendicular to the midpoint of the line-segment AB. ●

2. Find the set of points in the plane situated at a distance r from the origin.

○ It follows from the hypothesis that for any point $M(x, y)$ belonging to the desired set the equality $|OM| = r$ is valid. Since

$|OM| = \sqrt{x^2 + y^2}$, hence it follows

$$\sqrt{x^2 + y^2} = r \quad \text{or} \quad x^2 + y^2 = r^2.$$

The desired set of points is the circle of radius r centred at the origin (Fig. 132). ●

3. Write the equation of the set of points in the plane the ratio of the distances of which from the point A (1, 0) and the straight line $x = 9$ is $\lambda = 1/3$.

○ It follows from the conditions of the problem that for any point $M(x, y)$ of the required set the relationship $|MA| : |MB| = 1/3$ (Fig. 133). Since $|MA| = \sqrt{(x-1)^2 + y^2}$, $|MB| = \sqrt{(x-9)^2} = |x - 9|$, we have

$$\frac{\sqrt{(x-1)^2 + y^2}}{|x-9|} = \frac{1}{3} \quad \text{or} \quad 3\sqrt{(x-1)^2 + y^2} = |x-9|.$$

Squaring both sides and simplifying, we get

$9(x-1)^2 + 9y^2 = |x-9|^2$, i.e. $8x^2 + 9y^2 = 72$

or $x^2/9 + y^2/8 = 1$. ●

4. Write the equation of the set of points in the plane equidistant from the points A (—4, 2) and B (6, —8).

5. Write the equation of the set of points in the plane whose distance from the point A (6, 0) is three times the distance from the point B (2, 0).

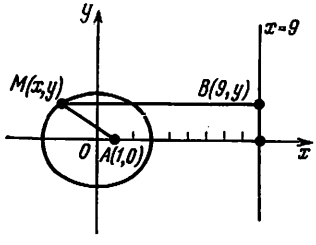


Fig. 133

6. Find the equation of the set of points in the plane the sum of the squared distances of which from the two given points A (—6, 0) and B (6, 0) is a constant quantity equal to 104.

7. Write the equation of the set of points in the plane the sum of whose squared distances from the two given points A (0, —2) and B (0, 2) is a constant quantity equal to 33.

8. Find the equation of the set of points in the plane the ratio of distances of which from the point A (3, 0) and from the straight line $x = 12$ is $\lambda = 1/2$.

9. Find the equation of the path of the point M which, when moving over the plane, remains twice nearer to the point A (1, 0) than to the straight line $x = 4$.

10. Find the equation of the set of points in the plane each of which is found twice as near to the straight line $x = 2$ as to the point A (8, 0).

11. Find the equation of the path of the point $M(x, y)$ which, when moving over the plane remains three times nearer to the straight line $x = 1$ than to the point A (9, 0).

12. Write the equation of the set of points in the plane equidistant from the x-axis and the point A (0, —2).

13. Write the equation of the set of points in the plane equidistant from the y-plane and the point A (3, 0).

14. Write the equation of the set of points in the plane equidistant from: (1) the straight line $x = 4$ and the point A (—2, 3); (2) the straight line $y = -2$ and the point A (—3, 4).

15. Find the equation of the path of a moving point in the plane if the square of its distance from the point A (2, —1) is equal to the square of its distance from the x-axis.

16. Find the equation of the path of a moving point in the plane if the square of its distance from the point A (—3, 4) is equal to the doubled square of its distance from the x-axis.

Sec. 19.2. The Circle

The *circle* is defined as the set of all points in the plane at a given distance (called the radius) from a fixed point in this plane, called the centre.

The equation of a circle with centre at the origin and radius r has the form

$$x^2 + y^2 = r^2. \tag{19.1}$$

The equation of a circle with centre at point O_1 (a, b) and radius r has the form

$$(x - a)^2 + (y - b)^2 = r^2. \tag{19.2}$$

The equation of a circle in general form is written as follows

$$Ax^2 + Ay^2 + Bx + Cy + D = 0, \tag{19.3}$$

where A, B, C, and D are constant coefficients.

17. Write the equation of the circle with centre at the point (5, —7) and passing through the point (2, —3).

○ We find the radius of the circle as the distance from the centre to the given point: $r = \sqrt{(2-5)^2 + [-3-(-7)]^2} = 5$. Now, we substitute into Equation (19.2) the coordinates of the centre and the found value of the radius: $(x - 5)^2 + (y + 7)^2 = 25$. ●

18. Find the equation of the circle passing through the points A (3, 1), B (—2, 6), and C (—5, —3).

○ Let O_1 (a, b) be the centre of the desired circle; then $|O_1A| = |O_1B| = |O_1C|$ as radii of the same circle. We have $|O_1A| = \sqrt{(a-3)^2 + (b-1)^2}$; $|O_1B| = \sqrt{(a+2)^2 + (b-6)^2}$; $|O_1C| = \sqrt{(a+5)^2 + (b+3)^2}$. Let us form a system of equations with respect to the unknowns a and b and solve this system:

$$\begin{cases} \sqrt{(a-3)^2+(b-1)^2} = \sqrt{(a+2)^2+(b-6)^2} \\ \sqrt{(a-3)^2+(b-1)^2} = \sqrt{(a+5)^2+(b+3)^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} a-b+3=0, \\ 2a+b+3=0 \end{cases} \Leftrightarrow (a=-2,\ b=1);\ O_1(-2;\ 1).$$

We find $r = |O_1A| = \sqrt{(-2-3)^2 + (1-1)^2} = 5$. Consequently, the sought-for equation of the circle has the form $(x+2)^2 + (y-1)^2 = 25$. ●

19. Write the equation of the circle touching the axis of abscissas at the point A (3, 0) and having the radius equal to 6.

○ Let O_1 (a, b) be the centre of the circle (Fig. 134). The abscissas of the point of tangency and the centre of the circle are equal to each other ($a = 3$). We then find the ordinate b of the centre of the circle:

$$(3-3)^2 + (0-b)^2 = 6^2; \quad b = \pm 6,$$

that is, there are two centres: O_1 (3, 6) and O_2 (3, —6).

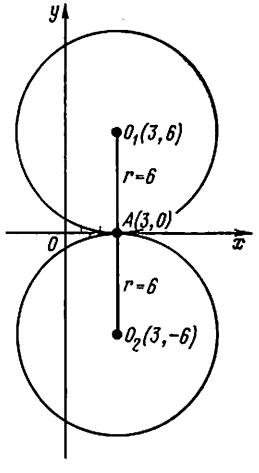


Fig. 134

Hence, we obtain the equations of two circles satisfying the given conditions: $(x-3)^2 + (y-6)^2 = 36$ and $(x-3)^2 + (y+6)^2 = 36$. ●

20. Find the equation of the circle touching the axis of ordinates and passing through the points A (4, 5) and B (18, —9).

○ Let O_1 (a, b) be the centre of the required circle (Fig. 135). We draw the radius to the point of tangency C (0, b). The radius of the circle $r = a$. We then form and solve the system of equations

$$\begin{cases} \sqrt{(a-4)^2 + (b-5)^2} = a, \\ \sqrt{(a-18)^2 + (b+9)^2} = a \end{cases} \Leftrightarrow \begin{cases} b^2 - 10b - 8a + 41 = 0, \\ b^2 + 18b - 36a + 405 = 0, \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{l} (a = 34, \ b = 21), \\ (a = 10, \ b = -3). \end{array}\right.$$

Consequently, there are two centres O_1 (34, 21) and O_2 (10, —3) and two values of the radius: $r_1 = 34$ and $r_2 = 10$, that is, two circles satisfy the conditions of the problem: $(x-34)^2 + (y-21)^2 = 34^2$ and $(x-10)^2 + (y+3)^2 = 10^2$. ●

21. Find the equation of the circle touching the coordinate axes and passing through the point A (18, —4).

○ The centre of the sought-for circle touching the coordinate axes and passing through a point belonging to Quadrant IV has the coordinates O_1 $(a, -a)$, where $a > 0$. The radius of the circle $r = a$ (Fig. 136). Consequently,

$$\sqrt{(a-18)^2 + (-a+4)^2} = a \Leftrightarrow (a^2 - 44a + 340 = 0) \Leftrightarrow \left[\begin{array}{l} a = 10, \\ a = 34. \end{array}\right.$$

Thus, there are two centres O_1 (34, —34) and O_2 (10, —10) and two values of the radius: $r_1 = 34$ and $r_2 = 10$, that is, two circles satisfy the conditions of the problem: $(x - 34)^2 + (y + 34)^2 = 34^2$ and $(x - 10)^2 + (y + 10)^2 = 10^2$. ●

22. Find the equation of the circle passing through the points A (5, 7) and B (—2, 4) if its centre lies on the straight line $4x + 3y - 18 = 0$.

○ Let O_1 (a, b) be the centre of the sought-for circle, and $|O_1A| = |O_1B|$ the radii of this circle. Thus, we get

$$\sqrt{(a-5)^2+(b-7)^2}=\sqrt{(a+2)^2+(b-4)^2} \Leftrightarrow 7a+3b-27=0.$$

The centre of the desired circle lies on the straight line $4x + 3y - 18 = 0$, consequently, its coordinates satisfy this equation, that is,

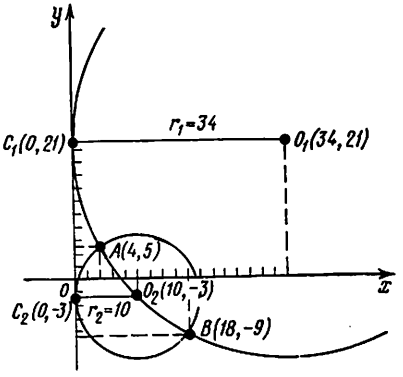


Fig. 135

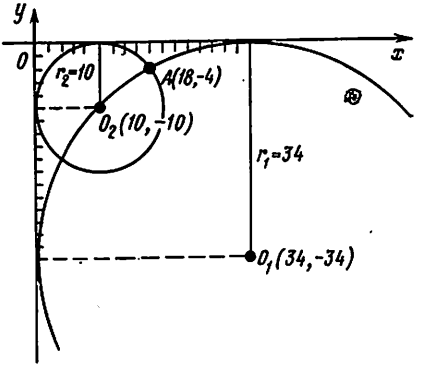


Fig. 136

$4a + 3b - 18 = 0$. We then find the point of intersection of the two lines:

$$\begin{cases} 7a+3b-27=0, \\ 4a+3b-18=0. \end{cases} \Leftrightarrow (a=3,\ b=2); \quad O_1(3,\ 2).$$

Further, we find $r = |O_1A| = \sqrt{(5-3)^2 + (7-2)^2} = \sqrt{29}$. Thus, the required equation has the form: $(x - 3)^2 + (y - 2)^2 = 29$. ●

23. The centre of a circle is found at the point O_1 (—3, 1). Find the equation of the circle if it touches the straight line $4x + 3y - 16 = 0$.

○ Since the slope of the tangent line $k_1 = -4/3$, the slope of the straight line O_1A, perpendicular to the tangent line (Fig. 137) $k_2 = 3/4$. Therefore, the equation of the line O_1A has the form:

$$y-1=\frac{3}{4}(x+3) \text{ or } 3x-4y+13=0.$$

We solve the system of equations

$$\begin{cases} 4x+3y-16=0, \\ 3x-4y+13=0 \end{cases} \Leftrightarrow (x=1,\ y=4);\ A(1,\ 4).$$

Now, we find $r = |O_1A| = \sqrt{(-3-1)^2+(1-4)^2} = 5$. Consequently, the desired equation has the form $(x+3)^2+(y-1)^2 = 25$. ●

24. Find the coordinates of the centre and the radius of the circle $x^2+y^2-8x-10y-8=0$.

○ We rewrite the given equation as follows: $x^2-8x+y^2-10y=8$. Completing the binomials x^2-8x and y^2-10y to perfect squares, we get

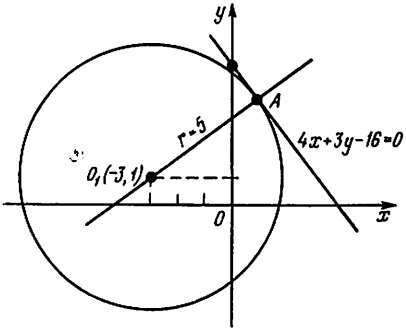


Fig. 137

$$x^2-2\cdot 4x+4^2+y^2-2\cdot 5y+5^2=8+4^2+5^2$$

or

$$(x-4)^2+(y-5)^2=49,$$

whence $a=4$, $b=5$, $r=7$, that is, the centre of the circle is the point (4, 5), and the radius is equal to 7. ●

25. Check whether the indicated points belong to the given circles: (1) A (2, 4), B (7, 1), and C (0, 2) to the circle $x^2+y^2-2x-4y=0$; (2) K (—4, 3) and L (5, 0) to the circle $x^2+y^2-4x-2y-35=0$.

26. Write the equation of the circle: (1) with centre at the origin and radius equal to $\sqrt{3}$; (2) with centre at the point (—2, —5) and radius equal to 3. Construct these circles.

27. Write the equation of the circle: (1) with centre at the point (—1, 4) and passing through the point (3, 5); (2) with centre at the point (—3, 0) and passing through the point (2, 4).

28. Find the equation of the circle if the end points of its diameter have the following coordinates: (1) (0, 3) and (6, —7); (2) (—2, 3) and (2, 5).

29. Find the equation of the circle whose diameter is the segment of the indicated line enclosed between the coordinate axes: (1) $4x+3y-24=0$; (2) $5x-4y+40=0$.

30. Find the equation of the circle passing through the origin and having the centre at the point: (1) (—2, 3); (2) (3, —5).

31. Find the coordinates of the points of intersection of the circle $x^2+y^2+4x+y-12=0$ with the coordinate axes.

32. Find the coordinates of the points of intersection of the circle $x^2+y^2-8x-2y-8=0$ and the straight line $4x+3y-19=0$.

33. Find the equation of the circle passing through the indicated points: (1) (2, 8), (4, —6) and (—12, —6); (2) (—2, —6), (—3, 1), and (4, 2).

34. Find the equation of the circle circumscribed about the triangle whose sides lie on the indicated lines: (1) $x - y + 4 = 0$, $3x + y - 16 = 0$ and $x + 2y - 2 = 0$; (2) $2x - y + 2 = 0$, $x - 3y - 14 = 0$ and $x + y - 2 = 0$; (3) $4x - 3y - 17 = 0$, $7x + y - 61 = 0$ and $x - 7y - 73 = 0$.

35. Write the equation of the circle touching the coordinate axes at the point A (0, 4) and having the radius equal to 5.

36. Find the equation of the circle touching the axis of abscissas and passing through the points A (7, 8) and B (6, 9).

37. Find the equation of the circle touching the coordinate axes and passing through the point A (8, 9).

38. Find the equation of the circle passing through the points (—3, 4) and (—3, —4) and having the centre at the origin.

39. Find the equation of the circle passing through the points A (8, 5) and B (—1, —4) and having the centre on the axis of abscissas.

40. Find the equation of the circle passing through the points A (3, 7) and B (5, —1) and having the centre on the axis of ordinates.

41. Write the equation of the circle: (1) passing through the points A (—8, 3) and B (2, —7) if its centre lies on the straight line $x + 4y + 16 = 0$; (2) passing through the points M (3, 2) and N (—1, —6) if its centre lies on the straight line intersecting the coordinate axes at the points A (2, 0) and B (0, —4).

42. The centre of a circle is found at the point (—1, —4). Find the equation of the circle if it touches the straight line intersecting the coordinate axes at the points A (9/4, 0) and B (0, 3).

43. Find the equation of the circle passing through the origin and intersecting the coordinate axes at the points A (6, 0) and B (0, 4).

44. Find the equation of the circle passing through the origin and intercepting the segments $a = -4$ and $b = -3$ on the x- and y-axes, respectively.

45. Find the coordinates of the centre and the radius of the indicated circle: (1) $x^2 + y^2 + 6x - 10y + 13 = 0$; (2) $x^2 + y^2 + 12y - 13 = 0$; (3) $4x^2 + 4y^2 - 4x + 20y - 23 = 0$; (4) $x^2 + y^2 - 4x - 10y + 29 = 0$; (5) $x^2 + y^2 + 6x + 14y + 81 = 0$.

46. Find the distance between the centres of the indicated circles: (1) $x^2 + y^2 - 10x + 16y + 80 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$; (2) $x^2 + y^2 + 4x - 12y + 36 = 0$ and $x^2 + y^2 - 8x + 10y + 5 = 0$.

47. Find the equation of the straight line passing through the centres of the indicated circles (1) $x^2 + y^2 - 8x - 4y + 11 = 0$ and $x^2 + y^2 + 4x + 12y + 4 = 0$; (2) $x^2 + y^2 + 4x - 6y - 23 = 0$ and $x^2 + y^2 - 10x - 14y + 58 = 0$.

48. In the circle $x^2 + y^2 + 6x - 4y - 12 = 0$ a diameter makes an angle of 60° with the axis of abscissas. Find the equation of this diameter.

49. A diameter of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ makes an angle of 135° with the positive direction of the x-axis. Find the equation of this diameter.

50. Given the circle $x^2 + y^2 - 8x - 2y + 4 = 0$. Find the equation of the diameter perpendicular to the chord $x - 5y - 12 = 0$.

51. Given the circle $x^2 + y^2 + 4x - 6y = 0$. Find the equation of the diameter perpendicular to the chord $2x - 3y + 13 = 0$.

52. Write the equation of the radius: (1) drawn to the point A (5, —6) belonging to the circle $x^2 + y^2 - 6x + 2y - 19 = 0$; (2) drawn to the point A (6, 3) belonging to the circle $x^2 + y^2 - 6x - 9 = 0$.

53. Find the equation of the tangent line: (1) drawn to the point A (4, —5) belonging to the circle $x^2 + y^2 - 2x + 2y - 23 = 0$; (2) drawn to the point A (—2, 4) belonging to the circle $x^2 + y^2 - 10y + 20 = 0$.

54. Find the equation of the common chord of two intersecting circles: $x^2 + y^2 + 2x + 2y - 23 = 0$ and $x^2 + y^2 - 26x - 2y + 45 = 0$.

55. Write the equation of the circle: (1) passing through the point A (4, —7) and concentric with the circle $x^2 + y^2 + 4x - 2y - 11 = 0$; (2) passing through the point A (5, 6) and concentric with the circle $x^2 + y^2 - 2x + 6y + 1 = 0$.

Sec. 19.3. The Ellipse

An *ellipse* is defined as the set of all points in a plane such that the sum of the two distances from each point to two fixed points (called the foci—plural of focus) is constant ($2a$); the latter is greater than the distance between the foci (or focal length) ($2c$).

The equation of the ellipse whose foci lie on the x-axis has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b), \tag{19.4}$$

where a is the length of the semimajor axis; b is the length of the semiminor axis (Fig. 138).

The parameters a, b, and c are related by the formula

$$a^2 - b^2 = c^2. \tag{19.5}$$

The ratio of the focal length $2c$ to the major axis $2a$ is the eccentricity of the ellipse:

$$e = c/a = \sqrt{a^2 - b^2}/a < 1. \tag{19.6}$$

If the foci of an ellipse lie on the y-axis (as in Fig. 139), then its equation has the form:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b). \tag{19.7}$$

In all the problems on the ellipse it is supposed that the axes of symmetry of the ellipse coincide with the coordinate axes.

56. Find the equation of the ellipse if its vertices are at the points A_1 (—6, 0) and A_2 (6, 0), and the foci at the points F_1 (—4, 0) and F_2 (4, 0).

○ It is obvious that $a = 6$ and $c = 4$. By Formula (19.5), we find $b^2 = 6^2 - 4^2 = 20$. Substituting the values of a^2 and b^2 into Equation (19.4), we get $x^2/36 + y^2/20 = 1$. ●

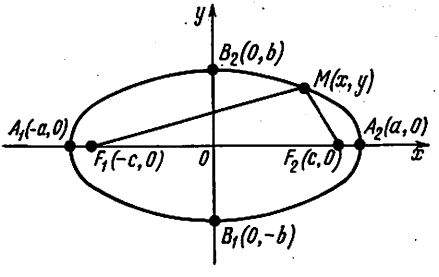


Fig. 138

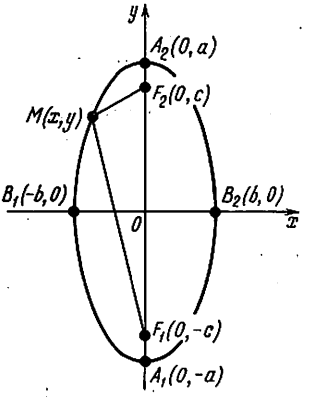


Fig. 139

57. Find the equation of the ellipse if its vertices are at the points B_1 (—8, 0) and B_2 (8, 0) and the foci at the points F_1 (0, —6) and F_2 (0, 6).

○ It follows from the hypothesis that the foci lie on the y-axis; then $b = 8$ and $c = 6$. By Formula (19.5), we have $a^2 = 8^2 + 6^2 = 100$. Substituting the values of a^2 and b^2 into Equation (19.7), we get $x^2/64 + y^2/100 = 1$. ●

58. Write the equation of the ellipse if the distance between its foci equals 6 (the foci lie on the x-axis) and the length of the major axis is equal to 10.

○ By the hypothesis, we have $a = 5$ and $c = 3$. By Formula (19.5), we find $b^2 = 5^2 - 3^2 = 16$. Substituting the values of a^2 and b^2 into Equation (19.4), we obtain $x^2/25 + y^2/16 = 1$. ●

59. Given the ellipse $x^2/49 + y^2/16 = 1$. Find the coordinates of its vertices and the lengths of its axes.

○ From the equation of the ellipse, we have $a^2 = 49$, $a = \pm 7$; $b^2 = 16$, $b = \pm 4$. Hence we find the coordinates of the vertices: A (± 7, 0) and B (0, ± 4). The lengths of the axes are: $2a = 2 \cdot 7 = 14$ and $2b = 2 \cdot 4 = 8$. ●

60. Given the ellipse $x^2/100 + y^2/36 = 1$. Find the coordinates of its foci and the distance between them.

○ From the equation of the ellipse, we have $a^2 = 100$ and $b^2 =$

36. By Formula (19.5), we find $c = \pm\sqrt{100 - 36} = \pm 8$. Hence, the coordinates of the foci are (—8, 0) and (8, 0), and the distance between them $2c = 2 \cdot 8 = 16$. ●

61. Given the ellipse $x^2/100 + y^2/51 = 1$. Compute its eccentricity.

○ From the equation of the ellipse, we have: $a^2 = 100$ and $b^2 = 51$. By Formula (19.5), we find $c = \sqrt{100 - 51} = 7$. The eccentricity of the ellipse is found by Formula (19.6): $e = 7/10$. ●

62. Write the equation of the ellipse whose foci are found at the points (—4, 0) and (4, 0) and the eccentricity $e = 0.8$.

○ By the hypothesis, we have: $c = 4$, $e = c/a = 0.8$. Substituting the value of c into the latter equality, we get $a = 5$. By Formula (19.5), we find $b^2 = 5^2 - 4^2 = 9$. Consequently, the desired equation has the form $x^2/25 + y^2/9 = 1$. ●

63. Find the equation of the ellipse whose foci are on the x-axis if its major axis is equal to 14 and the eccentricity $e = 2/3$.

○ From the hypothesis, we have: $a = 7$, $e = c/a = 2/3$. Substituting the value of a into the latter relationship, we get $c = 14/3$. Further, we find $b^2 = 7^2 - (14/3)^2 + 245/9$. Thus, the sought-for equation has the form

$$\frac{x^2}{49} + \frac{y^2}{245/9} = 1 \text{ or } \frac{x^2}{49} + \frac{9y^2}{245} = 1. \bullet$$

64. Find the equation of the ellipse with its foci on the x-axis if it passes through the points $A(\sqrt{3}, \sqrt{6})$ and $B(3, \sqrt{2})$.

○ In order to set up the equation of the ellipse, it is necessary to find the parameters a and b. Substituting the coordinates of the given points into the equation of the ellipse (19.4), we get the system of equations

$$\begin{cases} 3/a^2 + 6/b^2 = 1, \\ 9/a^2 + 2/b^2 = 1 \end{cases} \Leftrightarrow (a^2 = 12,\ b^2 = 8).$$

Thus, the required equation has the form $x^2/12 + y^2/8 = 1$. ●

65. Find the equation of the ellipse: (1) with foci on the x-axis if $2a = 8$ and $2b = 6$; (2) with foci on the y-axis if $2a = 10$, $2b = 4$.

66. Find the equation of the ellipse if: (1) its vertices are situated at the points (—5, 0) and (5, 0) and the foci at the points (—3, 0) and (3, 0); (2) its vertices are found at the points (0, —8) and (0, 8) and the foci at the points (—5, 0) and (5, 0); (3) its vertices are found at the points (0, —4) and (0, 4) and the foci at the points (0, —2) and (0, 2).

67. Find the equation of the ellipse if: (1) the distance between the foci is equal to 10 (the foci lie on the x-axis) and the major axis equals 12; (2) the foci lie at the points (—2, 0) and (2, 0), and the minor axis is equal to 8.

68. Find the equation of the ellipse whose foci are situated at the points $(0, -\sqrt{3})$ and $(0, \sqrt{3})$, and the major axis is equal to $4\sqrt{7}$.

69. Find the coordinates of the vertices and the lengths of the axes of the ellipse: (1) $x^2/25 + y^2/9 = 1$; (2) $x^2/16 + y^2/81 = 1$.

70. Find the coordinates of the foci and the focal length of the ellipse: (1) $x^2/12 + y^2/3 = 1$; (2) $x^2/10 + y^2/26 = 1$.

71. Compute the eccentricity of the ellipse: (1) $x^2/25 + y^2/9 = 1$; (2) $x^2/7 + y^2/16 = 1$.

72. Find the equation of the ellipse whose foci are at the points $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$, and the eccentricity $e = 1/3$.

73. Find the equation of the ellipse with foci on the x-axis if the distance between the foci is equal to 12 and the eccentricity $e = 0.6$.

74. Find the equation of the ellipse with foci on the x-axis if: (1) the major axis is equal to 10 and the eccentricity $e = 0.6$; (2) the minor axis is 16 and the eccentricity $e = 0.6$.

75. Write the equation of the ellipse with foci on the x-axis if: (1) the sum of the semiaxes is equal to 8 and the focal length to 8; (2) the sum of the semiaxes is equal to 25, and the coordinates of the foci are $(-5, 0)$ and $(5, 0)$.

76. Write the equation of the ellipse with foci on the x-axes if it passes through: (1) the point $(-6, 4)$ and the minor axis is equal to 10; (2) the point $(12, -12)$ and the major axis is equal to 40.

77. Find the equation of the ellipse with foci on the x-axis if it passes through the indicated points: (1) A $(6, 4)$ and B $(8, 3)$; (2) A $(\sqrt{2}, 2)$ and B $(2, \sqrt{3})$.

78. Find the coordinates of the points of intersection of: (1) the ellipse $x^2/225 + y^2/25 = 1$ and the straight line $x + 3y - 21 = 0$; (2) the ellipse $x^2/25 + y^2/9 = 1$ and the straight line $3x + 5y - 21 = 0$.

79. Find: (1) the length of the segment of the straight line $x + 4y - 28 = 0$ enclosed inside the ellipse $x^2/400 + y^2/25 = 1$; (2) the length of the segment of the straight line $x - 2y - 2 = 0$ enclosed inside the ellipse $x^2/100 + y^2/25 = 1$.

Sec. 19.4. The Hyperbola

A *hyperbola* is defined as the set of all points in a plane such that the difference between the two distances from each point to two fixed points, the foci, is the same ($2a$) and is less than the distance between the foci ($2c$).

The equation of the hyperbola whose foci lie on the x-axis has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{19.8}$$

where a is the length of the transverse (real) axis; b the length of the conjugate axis (Fig. 140).

The parameters a, b, and c are related by the formula

$$b^2 = c^2 - a^2. \tag{19.9}$$

The ratio of the focal distance to the length of its transverse axis is called the eccentricity of the hyperbola

$$e = c/a = \sqrt{a^2 + b^2}/a > 1. \tag{19.10}$$

A hyperbola has two asymptotes whose equations are

$$y = \pm (b/a)\, x. \tag{19.11}$$

A hyperbola with its transverse and conjugate axes equal in length is said

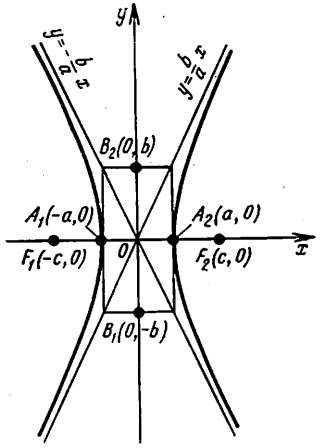


Fig. 140

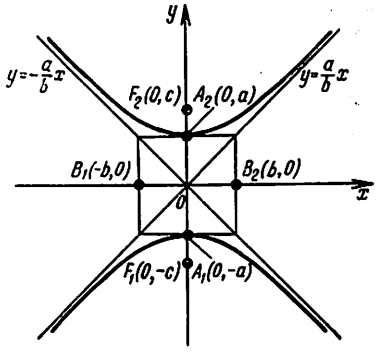


Fig. 141

to be a rectangular (or equilateral) hyperbola. Its equation is written as follows

$$x^2 - y^2 = a^2, \tag{19.12}$$

and the equations of its asymptotes are

$$y = \pm x. \tag{19.13}$$

If the foci of a hyperbola lie on the y-axis (Fig. 141), then its equation has the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1, \tag{19.14}$$

and the equations of the asymptotes to such a hyperbola are

$$y = \pm (a/b)\, x. \tag{19.15}$$

Formulas (19.9) and (19.10) for a hyperbola with its foci on the y-axis remain unchanged.

The hyperbolas represented by Formulas (19.8) and (19.14) are termed conjugate hyperbolas.

The equation of the rectangular hyperbola with its foci on the y-axis has the form

$$y^2 - x^2 = a^2. \tag{19.16}$$

In all the problems on the hyperbola it is supposed that the axes of symmetry of the hyperbola coincide with the coordinate axes.

80. Write the equation of the hyperbola if its vertices lie at the points A_1 (—3, 0) and A_2 (3, 0) and the foci are at the points F_1 (—5, 0) and F_2 (5, 0).

○ It follows from the hypothesis that $a = 3$ and $c = 5$. By Formula (19.9), we find $b^2 = 5^2 - 3^2 = 16$. Substituting the values of a^2 and b^2 into Equation (19.8), we get $x^2/9 - y^2/16 = 1$. ●

81. Given the equation of a hyperbola $x^2/81 - y^2/144 = 1$. Find the coordinates of its vertices and foci.

○ From the equation of the hyperbola we have $a^2 = 81$, $a = \pm 9$. By Formula (19.9), we find $c^2 = 81 + 144 = 225$, $c = \pm 15$. Consequently, the vertices of the hyperbola lie at the points (—9, 0) and (9, 0), its foci being the points (—15, 0) and (15, 0). ●

82. Given the equation of a hyperbola $x^2/25 - y^2/11 = 1$. Find its eccentricity.

○ From the given equation we have: $a^2 = 25$, $b^2 = 11$. The eccentricity is computed by Formula (19.10): $e = \sqrt{25 + 11}/5 = 6/5$. ●

83. Given the equation of a hyperbola $x^2/144 - y^2/256 = 1$. Find the equations of its asymptotes.

○ From the given equation we find $a = 12$, $b = 16$. Substituting the values of a and b into (19.11), we get $y = \pm(16/12)\,x$ or $y = \pm(4/3)\,x$. ●

84. Find the equation of the hyperbola from the coordinates of its foci (—20, 0) and (20, 0) and the eccentricity $e = 5/3$.

○ From the conditions of the problem, we have: $c = 20$, $e = c/a = 5/3$. Substituting the value of c into this equality, we get $20/a = 5/3$, that is, $a = 12$. Further, by Formula (19.9), we find $b^2 = 20^2 - 12^2 = 256$. Substituting the values of a^2 and b^2 into (19.8), we obtain $x^2/144 - y^2/256 = 1$. ●

85. Find the equation of the hyperbola with foci on the x-axis if the length of its real axis is equal to 12, the eccentricity being equal to 4/3.

○ By the hypothesis, $2a = 12$, whence $a = 6$. Using Equality (19.10), we have $4/3 = c/6$, that is, $c = 8$. Further, we find $b^2 = 8^2 - 6^2 = 28$. Thus, the desired equation has the form $x^2/36 - y^2/28 = 1$. ●

86. Find the equation of the hyperbola with foci lying on the x-axis if it passes through the points (6, 3) and $(5\sqrt{2}, -4)$.

○ To set up the equation of the hyperbola under consideration, it is necessary to find a^2 and b^2. Substituting the coordinates of the given points into (19.8), we get the system of equations

$$\begin{cases} 6^2/a^2 - 3^2/b^2 = 1, \\ (5\sqrt{2})^2/a^2 - (-4)^2/b^2 = 1 \end{cases} \Leftrightarrow (a^2 = 18,\ b^2 = 9).$$

Substituting now the values of a^2 and b^2 into Equation (19.8), we get $x^2/18 - y^2/9 = 1$. ●

87. Find the equation of the hyperbola if its asymptotes are given by the equations $y = \pm(\sqrt{6}/3)\,x$ and it passes through the point $(6, -4)$.

○ By the hypothesis, $b/a = \pm\sqrt{6}/3$. Substituting the coordinates of the given point into the equation (19.8), we solve the system of equations

$$\begin{cases} 6^2/a^2 - (-4)^2/b^2 = 1, \\ b/a = \pm\sqrt{6}/3, \end{cases} \Leftrightarrow (a^2 = 12,\ b^2 = 8).$$

Substituting now the values of a^2 and b^2 into (19.8), we get $x^2/12 - y^2/8 = 1$. ●

88. Find the vertices, foci, eccentricity, and asymptotes of the hyperbola $x^2/36 - y^2/64 = -1$.

○ The equation of the given hyperbola has the form of (19.14), that is, its foci lie on the y-axis. From the equation we obtain $a^2 = 64$, $a = \pm 8$ and $b^2 = 36$, $b = \pm 6$. The vertices of the hyperbola are situated at the points $A_1\,(0, -8)$ and $A_2\,(0, 8)$. By Formula (19.9), we have $c^2 = 64 + 36 = 100$, $c = \pm 10$; consequently, the foci lie at the points $F_1\,(0, -10)$, and $F_2\,(0, 10)$. The eccentricity is computed by Formula (19.10): $e = 5/4$. The asymptotes are found by Formula (19.15); $y = \pm(4/3)\,x$. ●

89. Write the equation of the hyperbola whose foci lie on the x-axis if its real axis is equal to 24, its conjugate axis being equal to 40.

90. Find the equation of the hyperbola if its vertices are found at the points $(-3, 0)$ and $(3, 0)$, and the foci at the points $(-3\sqrt{5}, 0)$ and $(3\sqrt{5}, 0)$.

91. Find the equation of the hyperbola with foci on the x-axis if the length of its real axis is equal to 12, and the distance between the foci is equal to 20.

92. Given the equation of a hyperbola: $x^2/36 - y^2/14 = 0$. Find the coordinates of its vertices and foci.

93. Find the lengths of the axes of the indicated hyperbola: (1) $x^2/25 - y^2/49 = 1$; (2) $x^2/12 - y^2/81 = 1$.

94. Given the equation of a hyperbola: $x^2/14 - y^2/22 = 1$. Find the coordinates of its foci and the focal length.

95. Find the distance between the foci of the indicated hyperbola: (1) $x^2/28 - y^2/36 = 1$; (2) $x^2/12 - y^2/24 = 1$.

96. Find the eccentricity of the following hyperbolas: (1) $x^2/9 - y^2/7 = 1$; (2) $x^2/25 - y^2/24 = 1$.

97. Find the equations of the asymptotes to the given hyperbola: (1) $x^2/64 - y^2/36 = 1$; (2) $x^2/9 - y^2/8 = 1$.

98. Find the equation of the hyperbola given the coordinates of its foci and the eccentricity: (1) $(\pm 2\sqrt{2}, 0)$, $e = 2$; (2) $(\pm 3\sqrt{3}, 0)$, $e = \sqrt{6}/2$.

99. Find the equation of the hyperbola with foci on the x-axis if: (1) the length of its real axis is equal to 6, and the eccentricity to 5/3; (2) the length of its conjugate axis is equal to 8, the eccentricity being equal to $3\sqrt{5}/5$.

100. Find the equation of the hyperbola whose foci lie on the x-axis if: (1) the sum of its semitransverse and semiconjugate axes is equal to 7 and the distance between the foci is equal to 10; (2) the difference between its semitransverse and semiconjugate axes is equal to 4, and the focal length is equal to 40.

101. Find the equation of the hyperbola whose foci lie on the x-axis if: (1) the length of its transverse axis is equal to 16, and the hyperbola passes through the point (—10, —3); (2) the length of its conjugate axis is equal to 12, and the hyperbola passes through the point (20, 8).

102. Find the equation of the hyperbola with foci on the x-axis if it passes through the points: (1) $(-6, -\sqrt{7})$ and $(6\sqrt{2}, 4)$; (2) $(-8, 2\sqrt{2})$ and (6, —1).

103. Write the equation of the hyperbola given the coordinates of its foci and the equations of its asymptotes: (1) $(\pm 5, 0)$, $y = \pm(4/3)x$; (2) $(\pm 3, 0)$, $y = \pm\sqrt{2}x$; (3) $(\pm 8, 0)$, $y = \pm\sqrt{3}x$.

104. Find the equation of the hyperbola given the equations of its asymptotes and the coordinates of the point it passes through: (1) $y = \pm(\sqrt{3}/3)x$, $(9, 3\sqrt{2})$; (2) $y = \pm(\sqrt{2}/2)x$, (—4, —2); (3) $y = \pm(\sqrt{3}/2)x$, $(4\sqrt{3}, 3\sqrt{3})$.

105. Find the vertices, foci, eccentricity, and the asymptotes of the hyperbola $x^2/9 - y^2/16 = -1$.

106. Find the equation of the rectangular hyperbola with foci on the x-axis if the hyperbola passes through the point: (1) A (—5, 4); (2) B (8, 2).

107. Find the equation of the rectangular hyperbola whose foci lie on the y-axis if the hyperbola passes through the point $(\sqrt{3}, -\sqrt{5})$.

Sec. 19.5. Parabola with Vertex at the Origin

A *parabola* is defined as the set of all points in a plane such that the distance of each point from a fixed point, the focus, is the same as its distance from a fixed line called the directrix.

The equation of the parabola with the vertex at the origin whose axis of symmetry lies on the x-axis, and whose branches are directed to the right (Fig. 142 (*a*)) has the form

$$y^2 = 2px, \tag{19.17}$$

where $p > 0$ (parameter of the parabola) is the distance from the focus to the directrix. The equation of the directrix is $x = -p/2$.

The equation of the parabola with the vertex at the origin whose axis of symmetry lies on the x-axis, and whose branches are directed leftward

(Fig. 142 (*b*)) has the form

$$y^2 = -2px \ (p > 0). \tag{19.18}$$

The equation of its directrix is $x = p/2$.

The equation of the parabola with the vertex at the origin, whose axis of symmetry lies on the y-axis, and which opens upward (Fig. 143 (*a*)) has the form

$$x^2 = 2py \ (p > 0). \tag{19.19}$$

The equation of its directrix is $y = -p/2$.

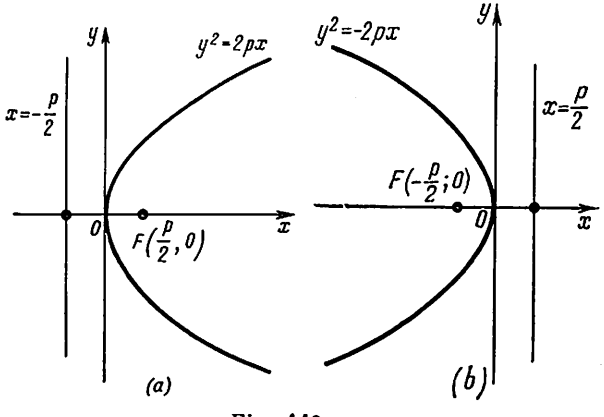


Fig. 142

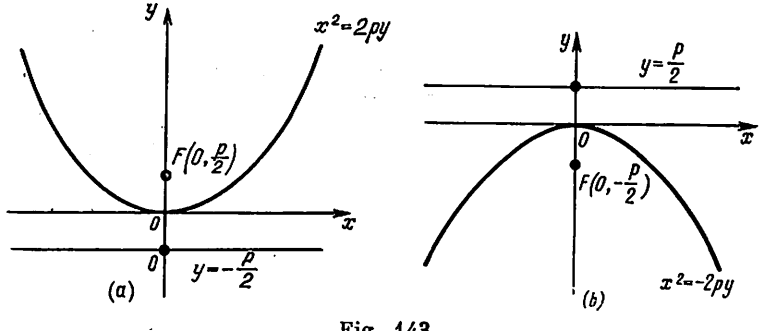


Fig. 143

The equation of the parabola with the vertex at the origin, whose axis of symmetry lies on the y-axis, and which opens downward (Fig. 143 (*b*)) has the form

$$x^2 = -2py \ (p > 0). \tag{19.20}$$

The equation of its directrix is $y = p/2$.

In all the problems of this section it is supposed that the axis of symmetry lies on one of the coordinate axes.

108. Find the equation of the parabola with the vertex at the origin if its focus is found at the point F (3, 0).

○ The focus of the parabola lies on the positive semiaxis Ox, consequently, the equation of the parabola has the form (19.17).

Since the coordinates of the focus $(p/2, 0)$, we have $p/2 = 3$, whence $p = 6$. Substituting the value of p into Equation (19.17), we get $y^2 = 12x$. ●

109. Set up the equation of the parabola with vertex at the origin if its directrix is represented by the straight line $x = -4$.

○ The distance of the directrix from the origin is equal to $p/2$; consequently, $p/2 = 4$, that is, $p = 8$. The equation of this parabola has the form (19.17), since the abscissa of the directrix is negative. Substituting the value of the parameter p into Equation (19.17), we get $y^2 = 16x$. ●

110. Find the equation of the parabola with the vertex at the origin symmetric with respect to the y-axis and passing through the point A (4, 2).

○ The sought-for parabola is symmetric with respect to the y-axis and passes through the point A (4, 2); consequently, its equation has the form (19.19). Substituting the coordinates of the point A into this equation, we find $p = 4$. On substituting the value of p into (19.19), we get $x^2 = 8y$. ●

111. Given the equation of the parabola: $y^2 = 6x$. Find the equation of its directrix.

○ From the equation of the parabola $y^2 = 6x$ we have $2p = 6$, that is, $p = 3$. Consequently, the equation of the directrix is $x = -3/2$ or $2x + 3 = 0$. ●

112. Given the equation of a parabola: $y^2 = 12x$. Compute the coordinates of its focus.

○ The distance from the origin to the focus is equal to $p/2$. From the equation of the parabola $y^2 = 12x$, we have $2p = 12$, whence $p = 6$ and $p/2 = 3$. Consequently, F (3, 0). ●

113. Find the coordinates of the focus of the parabola with the vertex at the origin if the equation of its directrix is $x = -3$.

○ The distance from the origin to the directrix is equal to the distance from the origin to the focus and amounts to $p/2$. From the equation of the directrix $x = -3$ it follows that $p/2 = 3$. To the equation of the directrix $x = -p/2$ there corresponds the parabola $y^2 = 2px$ whose focus if F (3, 0). ●

114. Given the parabola $y^2 = 12x$. Find the length of the chord passing through the focus of the parabola perpendicular to its axis.

○ The chord passes through the focus of the parabola perpendicular to its axis, therefore the abscissas of the points of intersection of the chord with the parabola are equal to the abscissa of the focus (Fig. 144). From the equation of the parabola we find the coordinates of its focus: $y^2 = 12x$, $2p = 12$, $p/2 = 3$; F (3, 0).

To compute the ordinates of the points of intersection of the chord and the parabola, we substitute the value $x = 3$ into the equation of the parabola: $y^2 = 12 \cdot 3 = 36$, whence $y_{1,2} = \pm 6$. Consequently, M_1 (3, 6) and M_2 (3, −6) are the desired points of intersection. The length of the chord M_1M_2 is equal to $2 \mid FM_1 \mid = 2 \cdot 6 = 12$. ●

115. Find the equation of the parabola with vertex at the origin if its focus is at the point: (1) F (5, 0); (2) F (—4, 0); (3) F (0, 2); (4) F (0, —3).

116. Find the equation of the parabola with vertex at the origin if its directrix is the straight line: (1) $x = -2$; (2) $x = 3$; (3) $y = -4$; (4) $y = 1$.

117. Write the equation of the parabola with vertex at the origin symmetric with respect to the x-axis and passing through the point: (1) (5, —3); (2) (—4, 2); (3) (—2, 2).

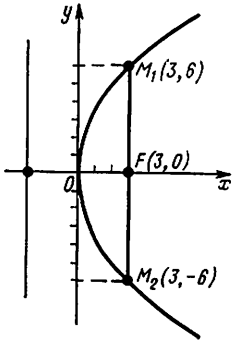


Fig. 144

118. Write the equation of the parabola with vertex at the origin symmetric with respect to the y-axis and passing through the point: (1) (2, —3); (2) (—3, 1).

119. Find the equation of the directrix of the indicated parabola: (1) $y^2 = 8x$; (2) $y^2 = -9x$; (3) $x^2 = 4y$; (4) $x^2 = -10y$.

120. Given the equation of a parabola, compute the coordinates of its focus: (1) $y^2 = 6x$; (2) $y^2 = -4x$; (3) $x^2 = 14y$; (4) $x^2 = -5y$.

121. Find the coordinates of the focus of the parabola with vertex at the origin if its directrix is given by the equation: (1) $x = 2$; (2) $x = -5$; (3) $y = 4$; (4) $y = -6$.

122. Given the parabola $y^2 = 20x$. Find the length of the chord passing through its focus perpendicular to its axis.

123. Find the points of intersection: (1) the parabola $y^2 = 16x$ and the straight line $2x - y + 2 = 0$; (2) the parabola $y^2 = 4x$ and the straight line $2x - 3y + 4 = 0$.

124. Find the points of intersection of the indicated parabolas: (1) $y = x^2$ and $x = y^2$; (2) $y^2 = 9x$ and $x^2 = 9y$.

Sec. 19.6. Parabola with Displaced Vertex

The equation of the parabola with the vertex at the point (a, b), the axis of symmetry parallel to the x-axis, and which opens rightward (Fig. 145 (a)) has the form

$$(y - b)^2 = 2p(x - a). \tag{19.21}$$

The equation of the parabola with the vertex at the point (a, b), the axis of symmetry parallel to the x-axis, and which opens leftward (Fig. 145 (b)) has the form

$$(y - b)^2 = -2p(x - a). \tag{19.22}$$

The equation of the parabola with the vertex at the point (a, b), the axis of symmetry parallel to the y-axis, and which opens upward (Fig. 146 (a)) has the form

$$(x - a)^2 = 2p(y - b). \tag{19.23}$$

The equation of the parabola with the vertex at the point (a, b), the axis of symmetry parallel to the y-axis, and which opens downward (Fig. 146 (b)) has

the form

$$(x - a)^2 = -2p\,(y - b). \qquad (19.24)$$

In each of the above equations the parameter of the parabola $p > 0$ is the distance from the focus of the parabola to its directrix.

125. Find the equation of the parabola having the vertex A (1, 2) and passing through the point M (4, 8) if the axis of symmetry is parallel to the x-axis.

○ According to the conditions of the problem, the equation of the sought-for parabola has the form (19.21), since the point M (4, 8)

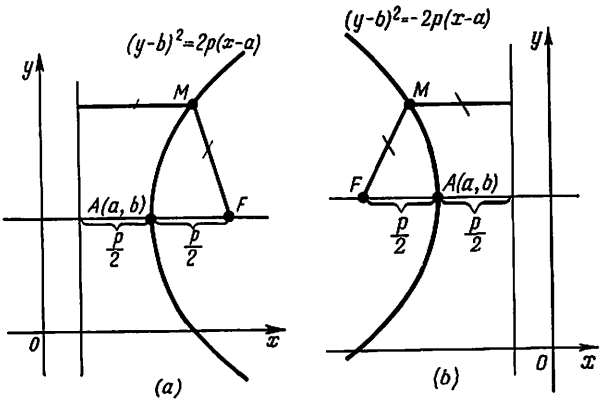


Fig. 145

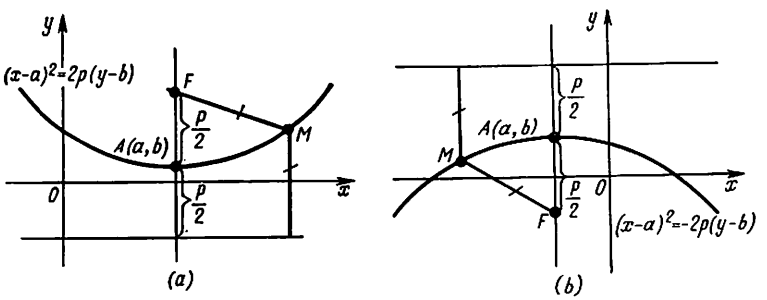


Fig. 146

is situated to the right of the vertex of the parabola, and, hence, the parabola opens rightward. To compute the parameter p, let us substitute into Equation (19.21) the coordinates of the vertex A and the point M: $(8 - 2)^2 = 2p\,(4 - 1)$, whence $p = 6$. Substituting now the found value $p = 6$ and the coordinates of the vertex A into (19.21), we get the desired equation: $(y - 2)^2 = 12\,(x - 1)$. ●

126. Find the equation of the parabola if its vertex is found at the point A (2, 3), and its focus at the point F (6, 3).

○ According to the conditions of the problem, the equation of the sought-for parabola has the form (19.21), since the axis of the parabola is parallel to the x-axis (the ordinates of the vertex and the focus are equal to each other, and, consequently, lie on a straight line parallel to the x-axis), and the parabola opens rightward (its focus is situated to the right of the vertex). Since the distance between the vertex and the focus of the parabola is equal to $p/2$, we have $p/2 = 6 - 2 = 4$, whence $p = 8$. Substituting into Equation (19.21) the coordinates of the vertex A and the found value of p, we get $(y - 3)^2 = 16(x - 2)$. ●

127. Find the equation of the parabola whose vertex lies at the point A (4, 6) and whose directrix is represented by the straight line $x = -2$.

○ By the hypothesis, the equation of the sought-for parabola has the form (19.21), since its directrix is perpendicular to the x-axis and, consequently, the axis of the parabola is parallel to the x-axis, and the parabola opens rightward (the directrix is situated to the left of the vertex). Since the distance from the directrix to the vertex of the parabola is equal to $p/2$, the quantity $p/2$ is equal to the sum of the absolute values of the abscissas of the directrix and the vertex of the parabola, that is, $p/2 = |-2| + 4 = 6$, whence $p = 12$. Substituting the coordinates of the vertex A and the found value of p into equation (19.21), we obtain $(y - 6)^2 = 24(x - 4)$. ●

128. Given the equation of a parabola: $x^2 - 8x - 4y + 28 = 0$. Find the coordinates of its vertex.

○ We reduce this equation to the form (19.23); to this end, we carry out the following transformations:

$$x^2 - 8x = 4y - 28; \quad x^2 - 2 \cdot 4x + 4^2 = 4y - 28 + 4^2;$$

$$(x - 4)^2 = 4(y - 3),$$

whence $a = 4$, $b = 3$, $2p = 4$; A (4, 3). ●

129. Compute the coordinates of the focus of the parabola $y^2 + 4y - 24x + 76 = 0$.

○ Transform the equation of the parabola to (19.21):

$$y^2 + 4y = 24x - 76; \quad y^2 + 2 \cdot 2y + 2^2 = 24x - 76 + 2^2;$$

$$(y + 2)^2 = 24(x - 3),$$

whence A (3, −2), $2p = 24$, $p = 12$.

The distance from the vertex of the parabola to its focus is $p/2 = 12/2 = 6$. The abscissa of the focus is equal to $3 + p/2 = 3 + 6 = 9$. The focus lies to the right of the vertex of the parabola, since the parabola opens rightward; and the ordinate of the focus is equal to the ordinate of the vertex, since the axis of the parabola is parallel to the x-axis (Fig. 147); then F (9, −2). ●

130. Given the parabola $y^2 - 6y - 8x - 7 = 0$. Find the equation of its axis.

○ The axis of the given parabola passes through its vertex parallel to the x-axis. We compute the coordinates of the vertex of the parabola by transforming the equation of the parabola to the form (19.21):

$$y^2 - 6y = 8x + 7; \quad y^2 - 2 \cdot 3y + 3^2 = 8x + 7 + 3^2;$$

$$(y - 3)^2 = 8(x + 2),$$

whence $a = -2$, $b = 3$; $A(-2, 3)$. Since the axis of the parabola passes through the vertex $A(-2, 3)$, its equation is $y = 3$. ●

131. Given the parabola $y^2 - 4y - 20x + 24 = 0$. Find the equation of its directrix.

○ The directrix of the parabola passes at a distance of $p/2$ from its vertex perpendicular to the axis of the parabola. From the equation of the parabola we find p:

$$y^2 - 4y = 20x - 24; \quad y^2 - 2 \cdot 2y + 4 = 20x - 24 + 4;$$

$$(y - 2)^2 = 20(x - 1),$$

whence $a = 1$, $b = 2$; $A(1, 2)$; $2p = 20$, $p/2 = 5$.

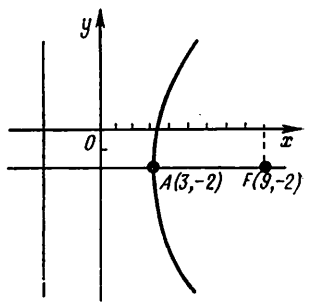


Fig. 147

The axis of symmetry of the parabola is parallel to the x-axis, and the parabola opens rightward, consequently, the directrix passes on the left of the vertex. It also passes on the left of the origin, since the distance from the vertex to the y-axis is equal to 1, and the distance from the vertex to the directrix, to 5. The abscissa of the directrix is equal to the difference $p/2 - 1 = 5 - 1 = 4$ taken with the minus sign; therefore the equation of the directrix is $x = -4$. ●

132. Construct the parabola $x^2 - 2x - y - 8 = 0$.

○ First Method. We find the vertex of the parabola by transforming the equation $y = x^2 - 2x - 8$ to the form (19.21):

$x^2 - 2x = y + 8$; $x^2 - 2x + 1 = y + 8 + 1$; $(x - 1)^2 = y + 9$; whence $a = 1$, $b = -9$; $A(1, -9)$.

We now find the points of intersection of the parabola with the x- and y-axes: $(-2, 0)$, $(4, 0)$, and $(0, -8)$. Thus, we obtain a number of characteristic points: $(-2, 0)$, $(0, -8)$, $(1, -9)$, and $(4, 0)$ by which we construct the desired parabola symmetric with respect to the axis $x = 1$ (Fig. 148).

Second Method. (It is used when the parabola intersects the x-axis). Setting $y = 0$, we get the equation $x^2 - 2x - 8 = 0$, whose roots are $x_1 = -2$ and $x_2 = 4$. The abscissa of the vertex of the parabola is equal to one-half the sum of the abscissas of the intersection points of the parabola with the x-axis; $x_v = (x_1 + x_2)/2 = (-2 + 4)/2 = 1$. The ordinate of the vertex is found by substituting the value of the

abscissa of the vertex into the given equation: $y(1) = 1^2 - 2 \cdot 1 - 8 = -9$; $A(1, -9)$.

Additional points are found by the first method. ●

133. Find the equation of the parabola with the axis of symmetry parallel to the x-axis if the parabola passes through the point M and has the vertex A: (1) $M(1, 3)$, $A(-4, -2)$; (2) $M(0, 0)$, $A(-2, -4)$; (3) $M(-3, -3)$, $A(3, -1)$.

134. Write the equation of the parabola with the axis of symmetry parallel to the y-axis if the parabola passes through the point M and has the vertex A: (1) $M(-6, -8)$, $A(2, 4)$; (2) $M(0, 0)$, $A(5, -5)$; (3) $M(0, 0)$, $A(3, 5)$.

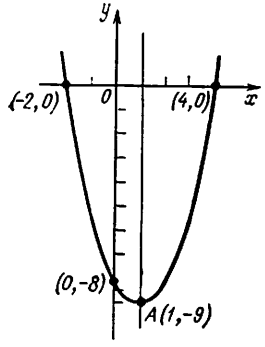


Fig. 148

135. Write the equation of the parabola with vertex A and focus F: (1) $A(4, 6)$, $F(-2, 6)$; (2) $A(3, -2)$, $F(3, 0)$; (3) $A(-1, 1)$, $F(-1, -4)$.

136. Find the equation of the parabola given the coordinates of its vertex A and the equation of its directrix: (1) $A(1, -3)$, $x = 5$; (2) $A(-2, 4)$, $y = -2$; (3) $A(-3, 5)$, $y = 7$.

137. Find the equation of the parabola symmetric with respect to the x-axis if the coordinates of its vertex A and the equation of the directrix are known: (1) $A(3, 0)$, $x = 0$; (2) $A(-4, 0)$, $x = 2$.

138. Find the equation of the parabola symmetric with respect to the y-axis given the coordinates of its vertex A and the equation of the directrix: (1) $A(0, 2)$, $y = 0$; (2) $A(0, -2)$, $y = -5$; (3) $A(0, -3)$, $y = 0$.

139. Write the equation of the parabola given the coordinates of its focus F and the equation of the directrix: (1) $F(-6, -1)$, $x = 2$; (2) $F(0, 0)$, $x = -4$; (3) $F(2, 2)$, $y = -4$; (4) $F(0, 0)$, $y = 4$.

140. Find the coordinates of the vertex of the indicated parabola: (1) $x^2 - 6x - 6y - 21 = 0$; (2) $x^2 + 8x + 5y + 21 = 0$; (3) $y^2 + 6y + 3x + 15 = 0$; (4) $y^2 - 6y - 12x + 33 = 0$.

141. Compute the coordinates of the focus of the parabola: (1) $y^2 - 8y - 8x - 8 = 0$; (2) $y^2 - 12x - 36 = 0$, (3) $x^2 + 10x + 8y + 41 = 0$; (4) $x^2 - 6y - 9 = 0$.

142. Find the equation of the axis of the parabola: (1) $y^2 - 10y - 10x + 5 = 0$; (2) $x^2 + 16x - 18y + 100 = 0$.

143. Find the equation of the directrix of the parabola: (1) $y^2 - 2y - 10x + 11 = 0$; (2) $y^2 + 8y + 8x + 32 = 0$; (3) $x^2 - 6x + 2y + 7 = 0$.

144. Construct the parabola: (1) $x^2 + 2x - y - 8 = 0$; (2) $x^2 + 8x + 4y = 0$; (3) $y^2 - 4x + 2y = 0$.

Sec. 19.7. A Tangent Line and a Normal to a Curve

Let on the curve $y = f(x)$ there be given a point $M_0(x_0, y_0)$ for which $y_0 = f(x_0)$ (Fig. 149).

The value of the derivative of the function $y = f(x)$ for $x = x_0$ is equal to the slope of the tangent line drawn to the curve $y = f(x)$ at its point with abscissa x_0, that is,

$$k = y'(x_0) = f'(x_0) = \tan \alpha,$$

where α is the angle between the tangent line to the curve at the point $M_0(x_0, y_0)$

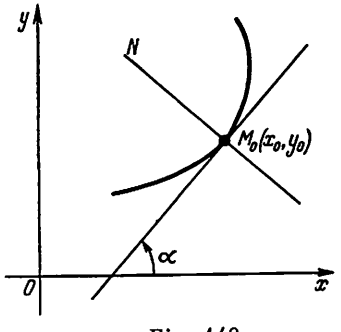


Fig. 149

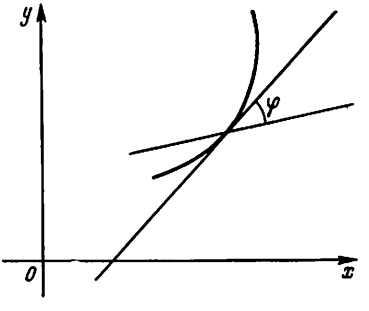


Fig. 150

and the positive direction of the x-axis.

The equation of the tangent line to the curve $y = f(x)$ at the point $M_0(x_0, y_0)$ has the form

$$y - y_0 = f'(x_0)(x - x_0). \quad (19.25)$$

The normal to the curve $y = f(x)$ at its given point $M_0(x_0, y_0)$ is defined as the perpendicular to the tangent line drawn through the point of tangency $M_0(x_0, y_0)$ (Fig. 149).

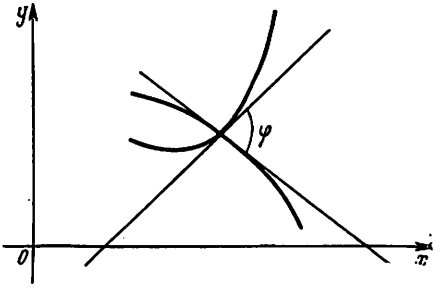


Fig. 151

The equation of the normal to the curve $y = f(x)$ at the point $M_0(x_0, y_0)$ has the form

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0). \quad (19.26)$$

The direction of a curve at each of its points is determined by the direction of the tangent line to this curve at this point, therefore, to find the angle of inclination of the curve at a given point, it is necessary to compute the angle between the tangent line drawn at this point and the x-axis.

The angle between a straight line intersecting a curve is defined as the angle between this line and the tangent line to the curve drawn through the point of their intersection (Fig. 150).

The angle between two intersecting curves is the angle between the tangents to these curves drawn at the point of their intersection (Fig. 151).

145. Find the angle of inclination of the parabola $y = x^2 - x + 1$ to the x-axis at the point $x = -1$.

○ To compute the angle of inclination of the curve at the given point, let us determine the angle made by the tangent drawn at this point with the x-axis.

We find the derivative of the function $y = x^2 - x + 1$ for $x = -1$:

$$y' = 2x - 1; \quad y'(-1) = 2 \cdot (-1) - 1 = -3.$$

We finally compute the angle of inclination of the tangent to the x-axis:

$$k = \tan \alpha = y'(-1) = -3; \quad \tan \alpha = -3; \quad \alpha \approx 108.4^\circ. \quad \bullet$$

146. Find the angle of inclinaton to the x-axis of the tangent line drawn to the curve $y = \sin x$ at the point $x = \pi/3$.

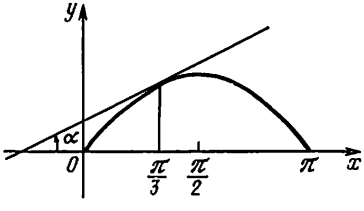


Fig. 152

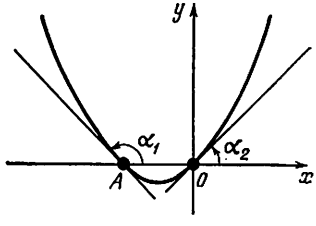


Fig. 153

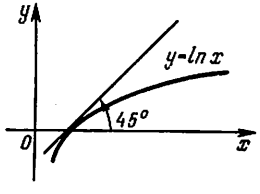


Fig. 154

○ We find the derivative of the function $y = \sin x$ for $x = \pi/3$:

$$y' = \cos x, \quad y'(\pi/3) = \cos(\pi/3) = 1/2.$$

The tangent of the angle of inclination of the tangent line at the point $x = \pi/3$ is equal to $1/2$, that is, $k = \tan \alpha = 1/2$, whence $\alpha \arctan(1/2) \approx 26.6^\circ$ (Fig. 152). ●

147. At what angles does the parabola $y = x^2 + x$ intersect the x-axis?

○ Let us find the points of intersection of the parabola $y = x^2 + x$ with the x-axis. To this end, we have to solve the system of equations

$$\begin{cases} y = x^2 + x, \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} (-1, 0), \\ (0, 0). \end{cases}$$

Hence, the parabola intersects the x-axis at the points A $(-1, 0)$ and O $(0, 0)$ (Fig. 153). We then find the slopes of the tangent lines to the parabola at these points:

$$y'' = (x^2 + x)' = 2x + 1; \quad k(-1) = 2 \cdot (-1) + 1 = -1;$$
$$k(0) = 2 \cdot 0 + 1 = 1.$$

We, finally, compute the angles α_1 and α_2 made by the tangent lines at the points of intersection of the parabola with the x-axis: $\tan \alpha_1 = -1$, $\alpha_1 = 135°$; $\tan \alpha_2 = 1$, $\alpha_2 = 45°$. ●

148. Find the angle formed by the curve $y = \ln x$ at the point of intersection with the x-axis.

○ Find the point of intersection of the curve $y = \ln x$ with the x-axis. At this point $\ln x = 0$, whence $x = 1$ (Fig. 154).

Compute the slope of the tangent at the point $x = 1$:

$$y' = (\ln x)' = 1/x; \quad y'(1) = 1.$$

Find the angle made by the tangent line at the point of intersection of the curve $y = \ln x$ with the x-axis: $\tan \alpha = 1$, $\alpha = 45°$. ●

149. Lines tangent and normal to the parabola $y = 3x^2 - x$ are drawn at the point $x = -1$. Find their equations.

○ To find the equation of the tangent, let us find the ordinate of the point M through which the tangent passes and its slope.

We then find the ordinate of the point of tangency by substituting the value $x = -1$ into the equation of the parabola:

$$y(-1) = 3 \cdot (-1)^2 - (-1) = 4; \quad M(-1, 4).$$

Compute the slopes of the tangent and the normal:

$$k_{\text{tan.}} = y' = (3x^2 - x)' = 6x - 1; \quad y'(-1) = 6 \cdot (-1) - 1 = 7;$$

$$k_{\text{norm.}} = 1/7.$$

Substituting the coordinates of the point M and the values of $k_{\text{tan.}}$ and $k_{\text{norm.}}$ into (19.25) and (19.26), we get

$$y - 4 = -7(x + 1) \quad \text{or} \quad 7x + y + 3 = 0$$

(equation of the tangent) and

$$y - 4 = \frac{1}{7}(x + 1) \quad \text{or} \quad x - 7y + 29 = 0$$

(equation of the normal). ●

150. Find the equations of the tangent and normal to the ellipse $x^2/27 + y^2/24 = 1$ at the point $(-3, -4)$.

○ Differentiate the equation of the ellipse with respect to x, considering y as a function of x:

$$\frac{2x}{27} + \frac{2yy'}{24} = 0, \quad 8x + 9yy' = 0, \quad \text{whence} \quad y' = -\frac{8x}{9y}.$$

Find the slopes of the tangent and normal at the point $(-3, -4)$:

$$k_{\text{tan.}} = y'(-3) = -\frac{8 \cdot (-3)}{9 \cdot (-4)} = -\frac{2}{3}; \quad k_{\text{norm.}} = \frac{3}{2}.$$

Hence we obtain

$$y + 4 = -(2/3)(x + 3) \quad \text{or} \quad 2x + 3y + 18 = 0$$

(the equation of the tangent) and

$$y + 4 = (3/2)(x + 3) \quad \text{or} \quad 3x - 2y + 1 = 0$$

(the equation of the normal). ●

151. Find the coordinates of the point at which the tangent to the parabola $y = x^2 - x - 12$ makes an angle of 45° with the x-axis.

○ Find the tangent of the angle of inclination of the tangent line drawn to the x-axis at the sought-for point: $\tan\alpha = y' =$

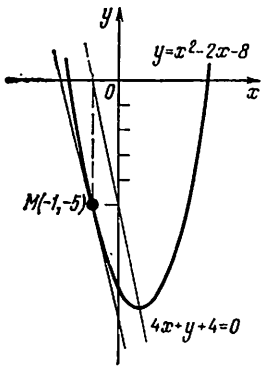


Fig. 155

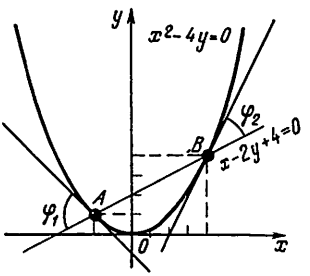


Fig. 156

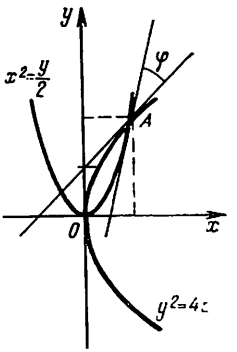


Fig. 157

$(x^2 - x - 12)' = 2x - 1$. By the hypothesis, the angle α is equal to 45°, consequently, $\tan 45° = 2x - 1$ or $1 = 2x - 1$, whence $x = 1$.

Finally, determine the ordinate of the desired point: $y(1) = 1^2 - 1 - 12 = -12$; $M(1, -12)$. ●

152. On the parabola $y = x^2 - 2x - 8$ find the point M at which the tangent line is parallel to the straight line $4x + y + 4 = 0$.

○ Determine the slope of the tangent line to the parabola: $k = y' = (x^2 - 2x - 8)' = 2x - 2$. Find the slope of the given line: $4x + y + 4 = 0$; $y = -4x - 4$; $k = -4$.

The tangent to the parabola is parallel to the given straight line, consequently, their slopes are equal to each other: $2x - 2 = -4$, whence the abscissa of the point of tangency $x = -1$. The ordinate of the point of tangency M is computed from the equation of the parabola: $y(1) = (-1)^2 - 2\cdot(-1) - 8 = -5$; $M(-1, -5)$ (Fig. 155). ●

153. Compute the acute angles formed at the point of intersection of the parabola $x^2 - 4y = 0$ and the straight line $x - 2y + 4 = 0$.

○ Let us find the points of intersection of the parabola and the straight line; to this end, let us solve the system of equations

$$\begin{cases} x^2 - 4y = 0, \\ x - 2y + 4 = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} (-2,\ 1), \\ (4,\ 4). \end{array}\right.$$

Thus, the points of intersection are: A $(-2, 1)$ and B $(4, 4)$ (Fig. 156).

We then find the slope of the given line: $x - 2y + 4 = 0$; $y = (1/2)\,x + 2$, $k = 1/2$, and compute the slopes of the tangents at the points A $(-2, 1)$ and B $(4, 4)$. Writing the equation of the parabola in the form $y = (1/4)\,x^2$, we find $k = y' = (1/2)\,x$. The slope of the tangent line at the point A is $k\,(-2) = (1/2)\cdot(-2) = -1$; the slope of the tangent line at the point B is equal to $k\,(4) = (1/2)\cdot 4 = 2$.

The angle between the parabola and the straight line at the point A is found as the angle between two straight lines with the slopes $k_1 = -1$ (the slope of the tangent line at the point A) and $k_2 = 1/2$ (the slope of the straight line):

$$\tan\varphi_1 = \frac{k_2 - k_1}{1 + k_2 k_1} = \frac{1/2 - (-1)}{1 + (1/2)\cdot(-1)} = 3, \quad \varphi_1 \approx 71.6^\circ.$$

Analogously, we find the angle between the parabola and the straight line at the point B:

$$\tan\varphi_2 = \frac{2 - 1/2}{1 + 2\cdot(1/2)} = \frac{3}{4}; \quad \varphi_2 \approx 36.9^\circ. \bullet$$

154. Compute the acute angles formed at the points of intersection of the two parabolas: $y^2 = 4x$ and $x^2 = y/2$.

○ Find the points of intersection of the given parabolas; for this purpose, solve the system of equations

$$\begin{cases} y^2 = 4x, \\ x^2 = y/2 \end{cases} \Leftrightarrow \left[\begin{array}{l} (0, 0), \\ (1, 2). \end{array}\right.$$

Consequently, the parabolas intersect at the points O $(0, 0)$ and A $(1, 2)$ (Fig. 157).

The angle between two intersecting parabolas is found as the angle between the tangent lines to them drawn at the point of intersection. Let us compute the slopes of the tangent lines to the parabolas at the points of their intersection. At the point $(0, 0)$ the tangents to the parabolas are represented by the x- and y-axes, consequently, at this point the parabolas intersect at right angles. To find the slope of the tangent line to the parabola $y^2 = 4x$ at the point A, let us rewrite its equation in the form $y = 2\sqrt{x}$ (the radical is taken with the plus sign, since the parabolas intersect in Quadrant I): $k = y' = 1/\sqrt{x}$; $k = y'\,(1) = 1$. We now compute the slope of the tangent line to the parabola $x^2 = y/2$ at the point A; writing its equation in the form $y = 2x^2$, we have $k = y' = 4x$; $k = y'\,(1) = 4\cdot 1 = 4$.

Finally, we find the angle φ between the tangent lines, knowing their slopes $k_1 = 1$ and $k_2 = 4$: $\tan\varphi = \dfrac{k_2 - k_1}{1 + k_2 k_1} = \dfrac{4 - 1}{1 + 4\cdot 1} = \dfrac{3}{5} = 0.6$; $\varphi \approx 31^\circ$. ●

155. Find the slope of the tangent drawn to the given parabola at the indicated point: (1) $y = -x^2 + x$ at the point $x = -2$; (2) $y = x^2 - 3x + 2$ at the point $x = 3$.

156. Find the angle of inclination to the x-axis of the tangent line drawn to the curve: (1) $y = x^2 - 2x$ at the point $x = 2$; (2) $y = x^3$ at the point $x = -2$; (3) $y = \sin x$ at the point $x = 2\pi/3$; (4) $y = \tan x$ at the points $x = \pi/3$ and $x = \pi/4$.

157. At what angles does the parabola $y = x^2 + 2x - 8$ intersect the x-axis?

158. Find the angle made by the curve $y = \sin x$ when intersecting the x-axis at the point: (1) $x = 0$; (2) $x = \pi$.

159. At what angle does: (1) the curve $y = \log x$ intersect the x-axis; (2) the curve $y = e^{\sqrt{3x}}$ intersect the y-axis?

160. Find the equations of the tangent and normal to: (1) the parabola $y = x^2 - 7x + 10$ at the point $x = 4$; (2) the curve $y = 2x^3$ at the point $x = -1$.

161. Find the equations of the tangent and normal to: (1) the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$; (2) the ellipse $x^2/100 + y^2/25 = 1$ at the point $(-8, 3)$; (3) the hyperbola $x^2/16 - y^2/64 = 1$ at the point $(-5, 6)$; (4) the parabola $y^2 = 8x$ at the point $(2, -4)$.

162. Find the equations of the tangent and normal to the curve: (1) $y = \sin 3x$ at the point $(\pi/3, 0)$; (2) $y = \sin (x/3)$ at the point $(\pi, \sqrt{3}/2)$; (3) $y = \cos 3x$ at the point $(\pi/6, 0)$.

163. Find the coordinates of the point at which the tangent line to the parabola $y = x^2 + 3x - 10$ makes an angle of 135° with the x-axis.

164. Find the coordinates of the point at which the tangent line to the curve $y = \sin x$ $(0 < x < \pi/2)$ makes an angle $\arctan (\sqrt{3}/2)$ with the x-axis.

165. On the parabola $y = -x^2 + 7x - 10$ find the point at which the tangent is parallel to the straight line $x + y - 1 = 0$.

166. At what point is the tangent line to the parabola $y = -x^2 + 4$ perpendicular to the straight line $x - 2y + 2 = 0$?

167. Compute the acute angle formed at the points of intersection of the parabola $y^2 - x = 0$ and the straight line $x + y - 6 = 0$.

168. Compute the acute angles formed by the intersecting parabolas: (1) $y = x^2$ and $x = y^2$; (2) $y^2 = 4x$ and $2x^2 = 27y$.

169. Compute the acute angle formed at the point of intersection of the curve $y = \log x$ and the straight line $y = 1$.

Sec. 19.8. Miscellaneous Problems

170. Through the centres of the circles $x^2 + y^2 - 12x - 6y + 29 = 0$ and $x^2 + y^2 + 4x + 6y + 4 = 0$ a straight line is drawn to intersect the x-axis. Compute the angle made by this line with the x-axis.

171. Find the angle between the straight lines passing through the centre of the circle $x^2 + y^2 - 4x - 16y + 32 = 0$ and through the foci of the ellipse $x^2/36 + y^2/20 = 1$.

172. Compute the angles at which the major and minor axes of the ellipse $x^2/36 + y^2/16 = 1$ are seen from the centre of the circle $x^2 + y^2 - 6x - 12y + 36 = 0$.

173. The circle $x^2 + y^2 + 2x - 6y - 40 = 0$ is cut by the straight line $3x - y + 16 = 0$ whose segment enclosed by the circle serves as the side of the rectangle inscribed in the circle. Write the equations of the sides of this rectangle.

174. Find the points of intersection of the ellipse $x^2/8 + y^2/2 = 1$ and the circle $x^2 + y^2 = 5$.

175. Inscribed in the circle $x^2 + y^2 = 4$ is a regular triangle one of the vertices of which has the coordinates (0, 2). Compute the coordinates of two other vertices of the triangle.

176. Find the equations of the straight lines passing through the foci of the ellipses $x^2/25 + y^2/16 = 1$ and $x^2/24 + y^2/49 = 1$.

177. Compute the area of the square inscribed in the ellipse $x^2/36 + y^2/9 = 1$.

178. Compute the area of the rectangle inscribed in the ellipse $x^2/16 + y^2/12 = 1$ so that two of its opposite sides pass through the foci of the ellipse.

179. Write the equation of the hyperbola whose vertices are found in the foci of the ellipse $x^2/20 + y^2/8 = 1$, and the foci at the vertices of the ellipse.

180. Find the distance from the vertex of the hyperbola $x^2/25 - 4y^2/25 = 1$ to the asymptote.

181. Find the equation of the ellipse whose foci coincide with those of the rectangular hyperbola $x^2 - y^2 = 18$ if the ellipse passes through the point $(5\sqrt{2}, 4\sqrt{2})$.

182. Find the points of intersection of two parabolas having a common vertex at the origin and the foci at the points F_1 (3, 0) and F_2 (0, 3/8).

183. The circle $x^2 + y^2 = 20$ intersects the parabola $x^2 = 8y$. Find the equation of their common chord.

184. From a point O a body was thrown at an acute angle to horizon which, on having described a parabolic arc, fell to earth at a distance of 40 m from the point O. Find the parameter of the parabolic path if the maximum height reached by the body is equal to 25 m (air resistance may be neglected).

QUIZ No. 1

Form I

(1) Write the equation of the radius drawn to the point A (—3, 1) belonging to the circle $x^2 + y^2 - 4x + 2y - 24 = 0$.

Form II

(1) Write the equation of the tangent drawn to the point A (—2, 1) belonging to the circle $x^2 + y^2 - 2x + 4y - 13 = 0$.

(2) Find the equation of the ellipse with foci on the x-axis if the focal length is 20, the eccentricity being equal to 5/6.

(3) Given the hyperbola $x^2/81 - y^2/63 = 1$. Find its eccentricity.

(4) Given the parabola $y^2 - 2y + 16x + 65 = 0$. Find the equation of its axis.

(5) Given the parabola $x^2 + 6x - 12y - 3 = 0$. Find the equation of its directrix.

(2) Given the ellipse $x^2/625 + y^2/400 = 1$. Find its eccentricity.

(3) Write the equation of the hyperbola with foci on the x-axis knowing the distance between the foci $2c = 90$ and the equations of its asymptotes $y = \pm(4/3)\,x$.

(4) Given the parabola $x^2 + 6x + 20y - 51 = 0$. Find the equation of its axis.

(5) Given the parabola $y^2 + 8y + 28x + 72 = 0$. Find the equation of its directrix.

Chapter 20
STRAIGHT LINES AND PLANES IN SPACE

Sec. 20.1. Parallelism of Straight Lines and Planes

I. Simplest Problems on Constructing Sections of Polyhedra

A *section* of a polyhedron is defined as the portion of the cutting plane bounded by the lines of intersection of this plane with the surface of the polyhedron. When constructing a section of a polyhedron, it is necessary to find: (1) the position of the cutting plane; (2) the line of intersection of the cutting plane with the surface of the polyhedron.

1. The tetrahedron $SABC$ is cut by the plane passing through three points K, L, M lying, respectively, on the edges SA, SB, and AC (the straight lines KL and AB are not parallel; Fig. 158).

○ Let α denote the plane passing through the points K, L, and M. The planes α and SAB have two common points K and L; therefore they intersect along the straight line KL. The line segment KL is the intersection of the face SAB and the plane α. The line segment KM is constructed in a similar manner.

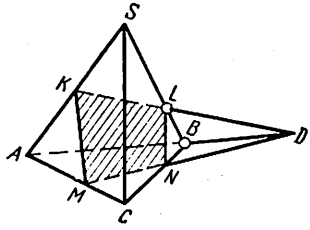


Fig. 158

The plane containing the face ABC and the cutting plane α have a common point M. To construct the line of intersection of these planes, it suffices to find one more point in common. Such point is the point D of intersection of the straight lines KL and AB [$D \in \alpha$ and $D \in (ABC)$]. Drawing the line MD, we obtain the point N on the edge BC. The line segments MN and NL are two other sides of the section. Thus, the quadrilateral $MKLN$ is the desired section. ●

2. Given the tetrahedron $SABC$ and points M and N, where $M \in [SC]$, $N \in [AB]$. Construct the intersection of the planes ABM and SCN.

3. Given the cube $ABCDA_1B_1C_1D_1$, and $K \in [AA_1]$, $L \in [CC_1]$, $M \in [DC]$. Construct the section of the cube by the plane passing through the points K, L, and M.

4. In the cube $ABCDA_1B_1C_1D_1$ construct the section by the plane passing through: (1) the vertices A_1, B, and D; (2) the midpoints of the edges emanating from one vertex; (3) the diagonal of the base BD and the vertex A_1; (4) three points lying on the edges AA_1, CC_1, and BC.

5. In the tetrahedron $SABC$ construct the section by a plane passing through three points lying on the edges SA, AC, and BC.

6. In the tetrahedron $SABC$ construct the section by the plane passing through the vertex S and the points $M \in [AB]$ and $N \in [AC]$.

7. Construct the section of the tetrahedron $SABC$ by the plane passing through the vertex A and the midpoints of the edges BS and CS. Find the area of the section if the edge of the tetrahedron is equal to a.

8. Construct the section of the tetrahedron $SABC$ by the plane passing through the edge SA and the median point of the face ACB. Find the area of the section if the edge of the tetrahedron is equal to a.

II. Skew Lines

To find the distance between the skew lines a and b, it is sufficient to pass through the line b a plane α parallel to a and from any point A on the line a to draw to the plane α the perpendicular AA_1 (Fig. 159).

9. On the model of a cube indicate its edges lying on skew lines.

10. Through a given point draw a straight line to form skew lines with a given line.

11. How many pairs of edges, lying on skew lines, has a tetrahedron?

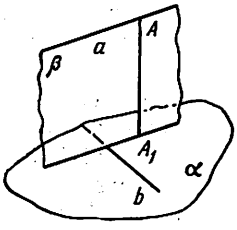


Fig. 159

12. Given: a and b forming skew lines, $\{A_1, A_2\} \in a$, $\{B_1, B_2\} \in b$. Prove that (A_1B_1) and (A_2B_2), (A_1B_2) and (A_2B_1) are skew lines.

III. Parallel Lines

13. The end points of a given line segment 50 cm long are 30 cm and 44 cm distant from a plane. Find the projection of this segment on the plane.

14. A line segment 15 cm long cuts a plane, its end points are 3 cm and 6 cm distant from the plane. Find the projection of its line segment on the plane.

15. A line segment cuts a plane; its end points are 3 cm and 12 cm distant from the plane. Find the distance of the midpoint of this segment from the plane.

IV. Parallelism of a Straight Line and a Plane

16. Given a tetrahedron $SABC$, in which $M \in [AS]$ and $N \in [AC]$. Construct the section of the tetrahedron by the plane passing through the points M and N parallel to the straight line AB.

○ *Analysis.* Suppose that the section is constructed (Fig. 160). The intersection of the cutting plane with the face ASC is obtained by joining the points M and N. By the hypothesis, the straight line AB is parallel to the cutting plane, therefore the faces ABC and ABS intersect the cutting plane along the line segments PN and MQ which are parallel to the line AB.

Construction. (1) $[NP] \parallel (AB)$, $P \in [BC]$; (2) $[MQ] \parallel (AB)$, $Q \in [SB]$. The quadrilateral $NMQP$ is the desired section.

Proof. The plane $NMQP$ is parallel to the line AB, since MQ is parallel to AB.

Investigation. The problem has only one solution, since the edge BC intersects the cutting plane only at the point P, and only one plane passes through the points M, N, P.

We have made use of the general scheme for solving problems on construction (analysis, construction, proof, and investigation). ●

17. (1) Given a point A and a straight line a ($A \notin a$). Through the point A draw a straight line parallel to a given plane.

(2) Given a point A and a straight line a ($A \notin a$). Through the point A draw a plane parallel to the straight line a.

18. Given $a \parallel b$. Through the straight line a pass a plane parallel to the straight line b.

19. Through a given point draw a line segment so that its projection on a given plane be equal to the length of the line segment.

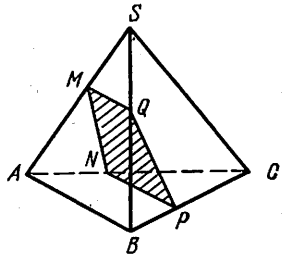


Fig. 160

20. In the cube $ABCDA_1B_1C_1D_1$ construct the section through: (1) the edges AB and D_1C_1; (2) the edge AD and the midpoint of the edge BB_1; (3) the midpoints of the edges AD and DC parallel to the edge DD_1; (4) the midpoint of the edge CC_1 parallel to the edges AB and A_1D_1.

21. In the tetrahedron $SABC$ construct the section through the midpoint of the edge AC parallel to the edges AB and CS.

22. Construct the section of the tetrahedron $SABC$ by the plane passing through the midpoints of the edges BS and CS and the interior point D of the edge AC.

23. Given the parallelepiped $ABCDA_1B_1C_1D_1$. Construct the point of intersection of the straight line AC_1 with the plane passing through the edges DC and A_1B_1.

V. Parallel Planes

24. Through a point of the face ASB of the tetrahedron $SABC$ construct the section parallel to: (1) the plane containing the face ABC, (2) the plane containing the face ASB.

25. Through a point on a lateral face of a prism draw the section parallel to: (1) the plane containing the base of the prism; (2) the plane of the diagonal section of the prism.

26. Two straight lines drawn from the point S cut three parallel planes at the points A_1, A_2, A_3 and B_1, B_2, B_3, respectively. It is

known that $|A_1A_2| = 4$ cm, $|B_2B_3| = 9$ cm, $|A_2A_3| = |B_1B_2|$. Compute $|A_1A_3|$ and $|B_1B_3|$.

27. Drawn in the tetrahedron $SABC$ are sections $A_1B_1C_1$ and $A_2B_2C_2$ whose planes are parallel to the face ABC. It is known that $|SB_1| = |A_1A_2| = 6$ cm, $|C_1C_2| = |B_2B| = 12$ cm, $|SA_1| = 4$ cm. Compute $|SA|$, $|SB|$, $|SC|$.

Sec. 20.2. Perpendicularity in Space. Dihedral and Polyhedral Angles

I. Perpendicularity of a Straight Line and a Plane

28. Given a parallelepiped $ABCDA_1B_1C_1D_1$ in which $M \in [AB]$. Construct the section of the parallelepiped by the plane passing through the point M perpendicular to the straight line AB.

29. Let $ABCD$ be a parallelogram, O the point of intersection of its diagonals; $|SA| = |SC|$, $|SB| = |SD|$, where S is a point not in the plane of the parallelogram. Prove that SO is perpendicular to the plane $ABCD$.

II. The Distance from a Point to a Plane

30. The diagonal of a cube is equal to m. Find the distance from a vertex of the cube to the plane of the opposite face.

31. The side of a regular triangle is equal to a; a point M is situated outside of the plane of the triangle at a distance l from all of its vertices. Find the distance from M to the plane of the triangle.

32. The legs of a right triangle are equal to a and b. A point M is found at a distance h from the plane of the triangle and is equidistant from all of its vertices. Find this distance.

III. Common Perpendicular of Skew Lines

A line segment having its end points on two skew lines and perpendicular to both of them is called the common perpendicular of skew lines.

The length of the common perpendicular of skew lines determines the distance between these lines.

33. Given a right parallelepiped $ABCDA_1B_1C_1D_1$. Indicate the common perpendicular to the straight lines: (1) AB and DD_1; (2) BC and AA_1; (3) A_1D_1 and DC.

34. In the cube $ABCDA_1B_1C_1D_1$ find the distance between the lines BB_1 and AC if the edge of the cube is equal to a.

IV. Theorem on Three Perpendiculars

35. From the vertex of the right angle in a right triangle, with legs 15 cm and 20 cm long, a perpendicular 16 cm long is drawn to the plane of the triangle. Find the distance from the end points of the perpendicular to the hypotenuse.

36. The sides of a triangle are equal to 51 cm, 30 cm, and 27 cm. From the vertex of the smaller angle of the triangle a perpendicular 10 cm long is drawn to its plane. Find the distance from the end points of the perpendicular to the opposite side of the triangle.

37. The diagonals of a rhombus are 60 cm and 80 cm long. At the point of intersection of the diagonals, a perpendicular 45 cm long is erected to the plane of the rhombus. Find the distance from this point to the side of the rhombus.

38. A point M is situated at a distance of 11 cm from either side of the isosceles trapezoid with bases 16 cm and 30 cm long. Compute the distance from the point M to the plane containing the trapezoid.

39. The point M is equidistant from all the vertices of a regular triangle a on a side and is situated at a distance h from the plane of the triangle. Find the distance from the point M to the side of the triangle.

V. The Angle Between an Inclined Line and a Plane

40. In the right parallelepiped $ABCDA_1B_1C_1D_1$ find the angles of inclination of the diagonal AC_1 to the planes of the faces having a common vertex A if $|AB| = |BC| = a$, $|AA_1| = 2a$.

41. From a point situated at a distance a from a plane two straight lines are drawn inclined to the plane at an angle of 45°, their projections making an angle of 120° between each other. Compute the distance between the end points of the inclined lines.

42. In an isosceles right triangle one of the legs makes an angle of 45° with the plane containing the other leg. Prove that the hypotenuse makes an angle of 30° with this plane.

43. The inclined line AB makes an angle of 45° with the plane α and the straight line AC lying in the plane α makes an angle of 45° with the projection of the inclined line AB. Prove that $\widehat{BAC} = 60°$.

44. Prove that if in a regular triangular pyramid the altitude is equal to the side of the base, then the lateral edges make an angle of 60° with the plane of the base.

VI. Dihedral Angles

45. Given on a face of the dihedral angle of 60° is a point situated at a distance m from the edge. Find the distance from this point to the other face.

46. Inside a dihedral angle of 120° a point M is given situated at a distance m from either face. Find the distance from this point to the edge of the dihedral angle.

47. Two isosceles triangles have a common base, their planes making an angle of 60°. The common base is equal to 12 cm, the lateral side of one of the triangles is equal to 10 cm, the lateral sides of the other being mutually perpendicular. Find the distance between the vertices of the triangles.

48. A line segment whose length is equal to a rests with its end points against the faces of a right dihedral angle making an angle α with either of them. Compute the projection of this line segment on the edge of the dihedral angle.

VII. Mutually Perpendicular Planes

49. The end points of the line segment AB lie on two mutually perpendicular planes. The lengths of the perpendiculars dropped from the points A and B on the line of intersection of the planes are respectively equal to a and b, the distance between their feet being equal to c. Compute the length of the line segment AB and the lengths of its projections on the given planes.

50. In a regular hexagonal pyramid the side of the base is equal to 4 cm; the lateral edge being equal to 8 cm. Through the midpoints of two adjacent sides of the base a plane is passed perpendicular to the base. Compute the area of the section.

VIII. The Area of a Projection of a Plane Figure

51. The area of a plane polygon is equal to 150 cm^2. Find the area of the projection of this polygon on the plane making an angle of 60° with the plane of the polygon.

52. Given the triangle with the sides $a = 13$ cm, $b = 14$ cm, $c = 15$ cm. Through the side BC a plane α is passed at an angle of 30° to the plane of the triangle ABC. Find the area of the projection of this triangle on the plane α.

53. Find the area of a plane polygon if the area of its projection is equal to 20 m^2, and the dihedral angle between the plane of the polygon and the plane of its projection is equal to 45°.

54. Find the area of the projection of a circle on the plane making an angle of 30° with the plane of the circle. The radius of the circle is equal to 2 m.

IX. Trihedral Angles

55. (1) Prove that if in a trihedral angle two plane angles are right angles, then the dihedral angles opposite to them are also right angles.

(2) Prove that if in a trihedral angle two dihedral angles are right angles, then the opposite plane angles are also right angles.

56. Each plane angle of a trihedral angle is equal to 60°. On one of its edges, from the vertex, a line segment is laid off whose length is equal to a, and from the end point of the line segment a perpendicular is dropped on the opposite face. Find the length of this perpendicular.

57. Each of the plane angles of a trihedral angle is equal to α. Compute the angle between the edge and the opposite face.

Sec. 20.3. Miscellaneous Problems

58. In the cube $ABCDA_1B_1C_1D_1$ draw a section through the midpoints of the edges A_1D_1 and D_1C_1 and the vertex A. Compute the area of this section if the edge of the cube is equal to a.

59. From the point O, lying outside of two parallel planes α and β three rays are drawn intersecting the planes α and β at the respective points A, B, C and A_1, B_1, C_1 ($OA < OA_1$). Compute the perimeter of the triangle $A_1B_1C_1$ if $|OA| = m$, $|AA_1| = n$, $|AB| = c$, $|AC| = b$, $|BC| = a$.

60. The point M lies outside of the plane of the right triangle ABC ($\widehat{C} = 90°$); $[MA] \perp [AC]$, $[MC] \perp [CB]$. Prove that $[MA]$ is perpendicular to the plane of the triangle ABC.

61. The smaller base of a trapezoid lies in the plane α which is 10 cm distant from the greater base of the trapezoid; the ratio of the trapezoid bases is 3 : 5. Find the distance of the intersection point of the diagonals of the trapezoid from the plane α.

62. In the triangle ABC the angle B is equal to 90° and $|BC| = a$. From the vertex A a perpendicular AD is drawn to the plane of the triangle. Find the distance from the point D to the leg BC if $|DC| = m$.

63. In a triangle whose sides are equal to 10, 17, and 21 cm, from the vertex of the greater angle a 15 cm perpendicular is drawn to its plane. Compute the distance from the end point of this perpendicular, lying outside of the plane of the triangle, to the greater side of the triangle.

64. The legs of a right triangle are equal to 18 cm and 32 cm. From the point D, bisecting the hypotenuse, the perpendicular DE 12 cm long is dropped to the plane of the triangle. Compute the distance from the point E to each leg.

65. Through a vertex of a square a straight line is drawn inclined to its plane and making an angle α with each of its sides emanating from this vertex. Find the angle between this inclined line and the diagonal of the square.

66. Passed through the side of a rhombus is a plane making angles α and 2α with its diagonals. Compute the acute angle of the rhombus.

67. The base of an isosceles triangle is equal to a, and the vertex angle to α; through the base of the triangle a plane is passed which makes an angle β with either lateral side. Find the distance of this plane from the vertex of the triangle.

68. Line segments enclosed between parallel planes are to each other as 2 : 3, and they make with the planes the angles whose ratio is equal to 2. Compute these angles.

69. Two congruent squares have a common side; their planes form a dihedral angle equal to α. From a common vertex diagonals are drawn in each of the squares. Compute the angle between these diagonals.

70. In one face of an acute dihedral angle a straight line is drawn at an angle of 30° to the other face and at an angle of 45° to the edge. Compute the dihedral angle.

71. Two isosceles triangles have a common base, and their planes form an angle of 60°. The common base is equal to 16 cm, the lateral side of one triangle is equal to 17 cm, and the lateral sides of the other are mutually perpendicular. Compute the distance between the vertices of the triangles.

72. In one of the faces of a dihedral angle, equal to α, a straight line is drawn making an angle β with the edge of the dihedral angle. Find the angle of inclination of this line to the other face.

73. In a trihedral angle each of the plane angles is equal to 60°. Through the point A taken on one of the edges of the angle at a distance a from its vertex a plane is passed perpendicular to this edge and intersecting two other edges at the points B and C. Find the perimeter of the triangle ABC.

74. In a trihedral angle two plane angles are equal to 45° (each), and the third angle measures 60°. Compute the dihedral angle opposite to the third plane angle.

75. In a trihedral angle each of the plane angles is equal to α. Find the dihedral angles.

QUIZ No. 1

Form I

(1) At the centre of the circle circumscribed about a right triangle with an acute angle of 30° a perpendicular is erected to its plane whose length is equal to 6 cm. The end point of the perpendicular lying outside of the plane of the triangle is 10 cm distant from the greater leg. Compute the hypotenuse of the triangle.

(2) Two isosceles triangles ABC and ACD have a common base AC, the dihedral angle AC is equal to 60°, and the angle made by the side BC with the plane ADC is equal to 45°. The side BC is equal to 6 cm. Compute the area of the triangle ABC.

Form II

(1) Given in a plane is a right triangle whose hypotenuse is equal to 12 cm. Given in space is a point 10 cm distant from each vertex of the triangle. Compute the distance of the given point from the plane.

(2) The base AC of the isosceles triangle ABC lies in the plane α, and the vertex B is $3\sqrt{2}$ cm distant from the plane α. Compute the area of the triangle ABC if $|AC| = 18$ cm and the plane of the triangle ABC is inclined to the plane α at an angle of 45°.

Chapter 21
VECTORS IN SPACE

Sec. 21.1. Basic Notions. Rectangular Coordinate System in Space

1. Vectors in Space. In space, the same as in the plane, the vector is defined as a directed straight-line segment. The basic notions for vectors in space are defined in the same manner: the modulus of a vector, the direction of a vector, equality of vectors.

The vectors lying on straight lines parallel to one and the same plane or lying in this plane are said to be *coplanar*.

Three vectors, among which there is at least one null vector are regarded to be coplanar.

Any vector **d** in space can be decomposed into three given noncoplanar vectors **a**, **b**, and **c**:

$$\mathbf{d} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}. \tag{21.1}$$

2. Rectangular Coordinate System in Space. Let there be given in space a triple of pairwise perpendicular unit vectors **i**, **j** and **k**, laid off from some

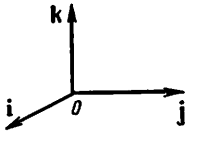


Fig. 161

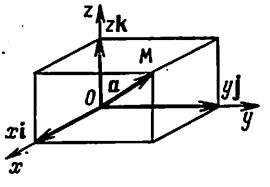


Fig. 162

origin (point O in Fig. 161). Such a triple of vectors is called a *rectangular basis* in space. The combination of the origin O and a rectangular basis (**i**, **j**, **k**) is called a *rectangular coordinate system in space*.

The decomposition of the vector **a** in the basis (**i**, **j**, **k**) has the form

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{21.2}$$

The coordinates of the point M (Fig. 162), that is, the numbers x, y, z in a given coordinate system, are called the *coordinates of the vector* $\overrightarrow{OM} = \mathbf{a}$.

If $\overrightarrow{OM} = \mathbf{a} = (x, y, z)$, then we write $M(x, y, z)$. The number x is termed the *abscissa*, y the *ordinate*, and z the *z-coordinate* of the point M or the vector $\overrightarrow{OM} = \mathbf{a}$. The initial points of the vectors located at the point O are defined as the origin of coordinates or, simply, the *origin*. The axes defined by the unit vectors **i**, **f**, **k**, are called the *coordinate axes*, and the planes passing through each two coordinate axes the *coordinate planes*. A space in which a coordinate system is specified is called a *coordinate space*.

The coordinate planes separate space into eight compartments, called octants.

Points lying on the coordinate planes have one of the coordinates equal to zero. Points lying on the coordinate axes have two coordinates equal to zero. The origin has all the three coordinates equal to zero.

The signs of the coordinates of points in space are indicated in the following table:

Coordinates	Octant							
	I	II	III	IV	V	VI	VII	VIII
Abscissa	+	−	−	+	+	−	−	+
Ordinate	+	+	−	−	+	+	−	−
z-coordinate	+	+	+	+	−	−	−	−

If all the coordinates of the vector a are different from zero, then this vector can be represented as the diagonal of a right parallelepiped the numerical values of the lengths of the edges of which are equal to $|x|$, $|y|$, $|z|$ (Fig. 162).

In a given rectangular basis (i, j, k) each triple of numbers (x, y, z) define a unique vector for which these numbers serve as coordinates.

By the definition of a rectangular basis, we have

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1.$$

If $A\ (x_A, y_A, z_A)$ is the initial point of the vector a and $B\ (x_B, y_B, z_B)$ its terminal point, then the vector $\mathbf{a} = \overrightarrow{AB}$ has the coordinates equal to the differences of the appropriate coordinates of the points B and A:

$$\mathbf{a} = \overrightarrow{AB} = (x_B - x_A,\ y_B - y_A,\ z_B - z_A), \tag{21.3}$$

and is written in the form

$$\mathbf{a} = \overrightarrow{AB} = (x_B - x_A)\,\mathbf{i} + (y_B - y_A)\,\mathbf{j} + (z_B - z_A)\,\mathbf{k}. \tag{21.4}$$

3. Rules for Performing Operations on Vectors Given by Their Coordinates. If in the basis (i, j, k) we are given $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$, then:

the coordinates of the sum of two (or more) vectors are equal to the sums of the respective coordinates of the summands, that is, $\mathbf{a} + \mathbf{b} = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$;

the coordinates of the difference of two vectors are equal to the differences of the appropriate coordinates of these vectors, that is, $\mathbf{a} - \mathbf{b} = (x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2)$;

the coordinates of the product of a vector by a number are equal to the products of the appropriate coordinates of the given vector by this number: $m\mathbf{a} = (mx_1,\ my_1,\ mz_1)$.

4. Condition of Collinearity of Two Vectors. The condition of collinearity of two vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ has the form

$$x_1 = mx_2, \quad y_1 = my_2, \quad z_1 = mz_2. \tag{21.5}$$

If $m > 0$, then the vectors a and b are in the same direction; if $m < 0$, then they have opposite directions.

5. The Length of a Vector. The length of a vector a (the distance between two points) is computed by the formula

$$|\mathbf{a}| = |\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}. \tag{21.6}$$

The length of the radius vector a is computed by the formula

$$|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}. \tag{21.7}$$

6. Division of a Line Segment in a Given Ratio. If the line segment AB is divided by a point C in the ratio $|AC| : |CB| = \lambda$, then the coordinates of the point C are found by the formulas

$$x_C = \frac{x_A + \lambda x_B}{1+\lambda}; \quad y_C = \frac{y_A + \lambda y_B}{1+\lambda}; \quad z_C = \frac{z_A + \lambda z_B}{1+\lambda} \tag{21.8}$$

For $\lambda = 1$ we obtain the formulas for finding the coordinates of the midpoint of the segment

$$x_C = \frac{x_A + x_B}{2}, \quad y_C = \frac{y_A + y_B}{2}, \quad z_C = \frac{z_A + z_B}{2}. \tag{21.9}$$

7. Direction Cosines of a Vector. The angles made by the radius vector a with the coordinate axes are computed by the formulas

$$\cos\alpha = \frac{x}{|a|} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \cos\beta = \frac{y}{|a|} = \frac{y}{\sqrt{x^2+y^2+z^2}},$$

$$\cos\gamma = \frac{z}{|a|} = \frac{z}{\sqrt{x^2+y^2+z^2}}. \tag{21.10}$$

The cosines of the angles computed by these formulas are called the *directed cosines* of the vector a.

The direction cosines of a vector are related by the following formula:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{21.11}$$

1. The line segment AB, where A (7, 2, —3), B (—5, 0, 4), is divided by the point C in the ratio $\lambda = |AC| : |CB| = 1 : 5$. Find the coordinates of the point C.

○ Substituting the values $x_A = 7$, $y_A = 2$, $z_A = -3$, $x_B = -5$, $y_B = 0$, $z_B = 4$, and $\lambda = 1/5$ into Relationship (21.8), we get

$$x_C = \frac{7 + (1/5)(-5)}{1+1/5} = 5, \quad y_C = \frac{2 + (1/5)\cdot 0}{1+1/5} = \frac{5}{3}$$

$$z_C = \frac{-3 + (1/5)\cdot 4}{1+1/5} = -\frac{11}{6}.$$

Thus, C (5, 5/3, —11/6). ●

2. Find the cosines of the angles made by the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ with the basis vectors.

○ Using Formula (21.6), we find the length of the vector a:

$$|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3.$$

By Formulas (21.10), we find the cosines of the angles made by the given vector with the basis vectors: $\cos\alpha = 1/3$, $\cos\beta = -2/3$, $\cos\gamma = 2/3$. ●

3. Given a parallelepiped $ABCDA_1B_1C_1D_1$. Lay off: (1) the vector $\overrightarrow{CD}$ from the point A; (2) the vector $\overrightarrow{AB}$ from the point B_1; (3) the vector AA_1 from the point C.

4. Given a tetrahedron $ABCD$. Find the sum of the vectors: (1) $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$; (2) $\overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$; (3) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$.

5. Given a parallelepiped $ABCDA_1B_1C_1D_1$. Find the sum of the vectors: (1) $\overrightarrow{BC} + \overrightarrow{CC_1} + \overrightarrow{C_1B_1}$; (2) $\overrightarrow{CB} + \overrightarrow{B_1A_1} + \overrightarrow{AD_1} + \overrightarrow{D_1C_1}$; (3) $\overrightarrow{AC_1} + \overrightarrow{D_1A} + \overrightarrow{BD_1} + \overrightarrow{D_1D}$; (4) $\overrightarrow{D_1C} + \overrightarrow{AA_1} + \overrightarrow{CB} + \overrightarrow{C_1C}$.

6. Given a prism $ABCA_1B_1C_1$. Find the sum of the vectors: (1) $\overrightarrow{AB} + \overrightarrow{BB_1} + \overrightarrow{B_1C}$; (2) $\overrightarrow{AC_1} + \overrightarrow{C_1B} + \overrightarrow{BA_1}$; (3) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CC_1} + \overrightarrow{C_1B_1} + \overrightarrow{B_1A_1}$.

7. Let M be the midpoint of the line segment AB, and let O be an arbitrary point in space. Prove that the equality $\overrightarrow{OM} = (\overrightarrow{OA} + \overrightarrow{OB})/2$ is fulfilled.

8. Given a tetrahedron $ABCD$. From the point B lay off a vector opposite to the vector: (1) $\overrightarrow{AD}$; (2) $\overrightarrow{CD}$; (3) $\overrightarrow{AB}$; (4) $\overrightarrow{AC}$.

9. A point O is taken outside of the plane of the triangle ABC. From the point O as origin lay off the following vectors: (1) $\overrightarrow{OB} - \overrightarrow{OA}$; (2) $-\overrightarrow{OC} - \overrightarrow{OB}$; (3) $\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}$.

10. Given a tetrahedron $ABCD$. Prove that $\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}$.

11. Given a parallelogram $ABCD$ and an arbitrary point O outside of it. Prove that $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$.

12. Let M be the median point of the triangle ABC, and let O be an arbitrary point in space. Prove that the equality $\overrightarrow{OM} = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})/3$ is fulfilled.

13. Given a parallelepiped $ABCDA_1B_1C_1D_1$. Indicate which of the following triples of vectors are coplanar: (1) $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{DD_1}$; (2) $\overrightarrow{AA_1}$, $\overrightarrow{B_1B}$, $\overrightarrow{CC_1}$; (3) $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CC_1}$; (4) $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{AA_1}$; (5) $\overrightarrow{AB}$, $\overrightarrow{A_1D_1}$, $\overrightarrow{CC_1}$; (6) $\overrightarrow{B_1B}$, $\overrightarrow{DD_1}$, $\overrightarrow{AA_1}$; (7) $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{A_1B_1}$.

14. Name three ordered pairs of the vertices of the tetrahedron $ABCD$ that specify collinear vectors and two triples of ordered pairs specifying coplanar and noncoplanar vectors.

15. Given a parallelepiped $ABCDA_1B_1C_1D_1$. Decompose into the vectors $\mathbf{p} = \overrightarrow{AB}$, $\mathbf{q} = \overrightarrow{AD}$, and $\mathbf{r} = \overrightarrow{AA_1}$ the vectors: (1) $\overrightarrow{AD_1}$; (2) $\overrightarrow{AC_1}$; (3) $\overrightarrow{AM}$, where M is the midpoint of $[BB_1]$; (4) $\overrightarrow{AN}$, where N is the midpoint of $[B_1C]$; (5) $\overrightarrow{AP}$, where $P \in [D_1C_1]$ and $|D_1P| : |PC_1| = 3 : 4$.

16. Given a tetrahedron $ABCD$. The medians of the face ABC intersect at the point M. Decompose the vector $\overrightarrow{DA}$ into the vectors $\overrightarrow{DB}$, $\overrightarrow{DC}$, $\overrightarrow{DM}$.

17. Given a parallelogram $ABCD$ and a point M outside of it. Decompose into the vectors $\overrightarrow{MA} = \mathbf{a}$, $\overrightarrow{MB} = \mathbf{b}$, $\overrightarrow{MC} = \mathbf{c}$ the vectors: (1) $\overrightarrow{MO}$, where $O = (AC) \cap (BD)$; (2) $\overrightarrow{MD}$; (3) $\overrightarrow{MN}$, where N is the midpoint of $[AD]$.

18. Construct the following points: A (2, 3, 4); B (—2, —3, —4); C (—2, —3, 4); D (2, —3, 4); E (—2, 3, 4); F (2, 3, —4); G (0, 0, 2); H (3, 0, —4).

19. Write the coordinates of the following vectors: (1) $3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$; (2) $2\mathbf{i} - \mathbf{k}$; (3) $0.5\mathbf{i} + \sqrt{2}\mathbf{j}$; (4) $3\mathbf{k}$; (5) $-4\mathbf{i}$; (6) $\mathbf{0}$.

20. Given the vectors: (1) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$; (2) $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Write their coordinates.

21. Construct the following vectors: (1) $\mathbf{a} = (2, 3, 4)$; (2) $\mathbf{b} = (2, -3, -4)$; (3) $\mathbf{c} = (3, -1, -4)$; (4) $\mathbf{d} = (-5, -4, 3)$.

22. Construct the vector $\overrightarrow{AB}$ if: (1) A (2, —3, 4) and B (—3, 2, —5); (2) A (0, —2, 3) and B (5, 0, —4).

23. Knowing the coordinates of the points A (4, —3, 2) and B (—2, 4, —3), M (0, 5, 1) and N (—4, 0, —3), find the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{MN}$.

24. Knowing the coordinates of the vectors $\mathbf{a} = (2, 3, -4)$, $\mathbf{b} = (-1, 2, 1)$ and $\mathbf{c} = (3, 0, 2)$, find the coordinates of the vectors (1) $\mathbf{a} + \mathbf{b}$; (2) $\mathbf{a} + \mathbf{c}$; (3) $\mathbf{a} + \mathbf{b} - \mathbf{c}$; (4) $3\mathbf{a}$; (5) $-\mathbf{a} + 2\mathbf{c}$; (6) $2\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$.

25. Making use of the criterion of collinearity of two vectors, check whether the indicated pairs of vectors are collinear: (1) $\mathbf{a} = (2/5, -1/3, 4/5)$ and $\mathbf{b} = (3/5, -1/2, 6/5)$; (2) $\mathbf{c} = (-6, 1/3, 3)$ and $\mathbf{d} = (-2, 1/9, -1/3)$.

26. For what values of n and p are the vectors $\mathbf{a} = (-3, n, 4)$ and $\mathbf{b} = (-2, 4, p)$ collinear?

27. Compute the length of the indicated vector: (1) $\mathbf{a} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$; (2) $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; (3) $\mathbf{c} = \mathbf{i} - \mathbf{k}$; (4) $\mathbf{d} = -3\mathbf{k}$.

28. Compute the length of the vector $\mathbf{a} + \mathbf{b}$ if: (1) $\mathbf{a} = (-1, 2, 1)$, $\mathbf{b} = (-2, 2, -1)$; (2) $\mathbf{a} = (1, -2, 3)$, $\mathbf{b} = (-1, 2, -3)$.

29. Compute the length of the vector $3\mathbf{a} + 2\mathbf{b}$ if $\mathbf{a} = (2, 0, 0)$, $\mathbf{b} = (1, 1, -1)$.

30. Compute the length of the vector $\overrightarrow{AB}$ if A (5, 3, 1) and B (4, 5, —1).

31. Find the perimeter of the triangle formed by the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ if A (8, 0, 6), B (8, —4, 6), C (6, —2, 5).

32. The line segment AB is given by the coordinates of its end points A (4, 2, —3) and B (6, —4, —1). Find the coordinates of the point C bisecting this line segment.

33. The line segment AB is given by the coordinates of its end points A (3, —2, —5) and B (7, 6, —1). Find the coordinates of the

point C dividing it in the ratio $\lambda = |AC| : |CB| = 1 : 3$

34. Find the median point of the triangle if its vertices lie at the points A (7, —4, 5), B (—1, 8, —2), and C (—12, —1, 6).

35. Find the cosines of the angles made with the basis vectors by the following vectors: (1) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; (2) $\mathbf{b} = (4, 3, 0)$; (3) $\mathbf{c} = -\mathbf{j} - 3\mathbf{k}$; (4) $\mathbf{d} = 3\mathbf{i}$.

Sec. 21.2. Scalar Product of Vectors in Space

The scalar product of the vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ given by their coordinates is found by the formula

$$\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2. \tag{21.12}$$

The angle between the vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ is computed by the formula

$$\cos(\widehat{\mathbf{a}, \mathbf{b}}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2}}. \tag{21.13}$$

The condition of perpendicularity of two vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ has the form

$$x_1x_2 + y_1y_2 + z_1z_2 = 0. \tag{21.14}$$

36. Find the scalar product of the vectors $\mathbf{a} = (4, -3, 1)$ and $\mathbf{b} = (5, -2, -3)$.

○ By Formula (21.12), we find $\mathbf{a} \cdot \mathbf{b} = 4 \cdot 5 + (-3) \cdot (-2) + 1 \cdot (-3) = 23$. ●

37. Given the vectors $\mathbf{a} = -4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Find the angle between them.

○ By Formula (21.13), we get

$$\cos\varphi = \frac{-4 \cdot (-2) + (-3) \cdot 3 + 5 \cdot 1}{\sqrt{(-4)^2 + (-3)^2 + 5^2} \cdot \sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{2\sqrt{7}}{35};$$

$$\varphi = \arccos\frac{2\sqrt{7}}{35}. \quad ●$$

38. Given the vertices of a triangle: A (—1, 4, 1), B (3, 4, —2), and C (5, 2, —1). Find $\widehat{ABC}$.

○ Find the coordinates of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$; we have $\overrightarrow{BA} = (-4, 0, 3)$, $\overrightarrow{BC} = (2; -2, 1)$. The angle $\widehat{ABC}$ is equal to the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, therefore

$$\cos(\widehat{\overrightarrow{BA}, \overrightarrow{BC}}) = \frac{(-4) \cdot 2 + 0 \cdot (-2) + 3 \cdot 1}{\sqrt{(-4)^2 + 0 + 3^2} \cdot \sqrt{2^2 + (-2)^2 + 1^2}} = -\frac{1}{3}. \quad ●$$

39. Find the scalar product of the following pairs of vectors: (1) $\mathbf{a} = (3, -2, 1)$ and $\mathbf{b} = (4, -7, -3)$; (2) $\mathbf{c} = (2/3, -5/6, 1/4)$ and $\mathbf{d} = (3/2, 6/5, 4/3)$.

40. Given the vectors: $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, and $\mathbf{c} = 4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$. Find the scalar product of the sum of the first two vectors by the third vector.

41. Given the cube $ABCDA_1B_1C_1D_1$. Find the angles between the indicated vectors: (1) $\overrightarrow{AD}$ and $\overrightarrow{BB_1}$; (2) $\overrightarrow{BC}$ and $\overrightarrow{D_1C_1}$; (3) $\overrightarrow{AD_1}$ and $\overrightarrow{BA_1}$; (4) $\overrightarrow{BD}$ and $\overrightarrow{DC_1}$; (5) $\overrightarrow{CB_1}$ and $\overrightarrow{AA_1}$.

42. Find the angle between the vectors: (1) $\mathbf{a} = 3\mathbf{i} - 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 12\mathbf{k}$; (2) $\mathbf{a} = (-2, 2, -1)$ and $\mathbf{b} = (-6, 3, 6)$, (3) $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ if $\mathbf{a} = (1, -1, 2)$ and $\mathbf{b} = (0, 2, 1)$.

43. In the triangle ABC, where A (1, 1, 5), B (−2, 0, 7), and C (−3, −2, 5), find $\widehat{ACB}$.

44. Check whether the indicated pairs of vectors are mutually perpendicular: (1) $\mathbf{a} = (3, 0, -6)$ and $\mathbf{b} = (4, 7, 2)$; (2) $\mathbf{c} = (-3, 2, 5)$, and $\mathbf{d} = (6, -3, 1)$.

45. Given the triangle: A (2, 4, 5), B (−3, 2, 2), C (−1, 0, 3). Show that $\overrightarrow{CA}$ is perpendicular to $\overrightarrow{BC}$.

46. Given the vectors: $\mathbf{a} = (-2, y, 1)$ and $\mathbf{b} = (3, -1, 2)$. Find the coordinate y if it is known that $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

Sec. 21.3. The Vector Product of Two Vectors

The *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a third vector $\mathbf{c}$ which satisfies the following conditions (Fig. 163):

(1) the modulus of the vector $\mathbf{c}$ is equal to the product of the moduli of the vectors $\mathbf{a}$ and $\mathbf{b}$ by the sine of the angle between them, that is,

$$|\mathbf{c}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin \widehat{(\mathbf{a}, \mathbf{b})}; \qquad (21.15)$$

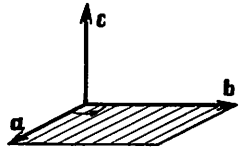


Fig. 163

(2) the vector $\mathbf{c}$ is perpendicular to the plane defined by the vectors $\mathbf{a}$ and $\mathbf{b}$.

(3) The vector $\mathbf{c}$ is directed so that the shortest possible rotation of the vector $\mathbf{a}$ towards the vector $\mathbf{b}$ is seen from the terminus of the vector $\mathbf{c}$ as occurring anticlockwise (that is, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a right-handed ordered trihedral or a right-handed system).

The vector product of $\mathbf{a}$ by $\mathbf{b}$ is symbolized $\mathbf{a} \times \mathbf{b}$.

The modulus of the vector product is numerically equal to the area of the parallelogram S constructed on the vectors $\mathbf{a}$ and $\mathbf{b}$.

The vector product is expressed by the formula

$$\mathbf{a} \times \mathbf{b} = S\mathbf{e}, \qquad (21.16)$$

where $\mathbf{e}$ is the unit vector of the direction $\mathbf{a} \times \mathbf{b}$.

Note the basic properties of the vector product.

1°. *The vector product* $\mathbf{a} \times \mathbf{b}$ *is equal to zero if the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are collinear or one of them is a null vector.*

2°. *If the factors are interchanged, then the vector product is multiplied by* -1 (reverses sign): $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (the vector product does not possess the commutative property).

3°. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (*distributive law*).

4°. $(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b})$ (*associative law*).

The vector product of the vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ given by their coordinates is computed in the following way:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}$$

$$= (y_1 z_2 - z_1 y_2)\mathbf{i} + (z_1 x_2 - x_1 z_2)\mathbf{j} + (x_1 y_2 - y_1 x_2)\mathbf{k}. \qquad (21.17)$$

The physical meaning of a vector product consists in the following. If $\mathbf{F}$ is a force and $\mathbf{r}$ is the radius vector of the point of its application with the origin at the point O, then the moment of the force $\mathbf{F}$ relative to the point O is a vector equal to the vector product of $\mathbf{r}$ by $\mathbf{F}$, that is, $m_0(\mathbf{F}) = \mathbf{r} \times \mathbf{F}$.

47. Find the following vector products: (1) $\mathbf{i} \times \mathbf{j}$; (2) $\mathbf{j} \times \mathbf{k}$; (3) $\mathbf{k} \times \mathbf{i}$; (4) $\mathbf{j} \times \mathbf{i}$; (5) $\mathbf{k} \times \mathbf{j}$; (6) $\mathbf{i} \times \mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors of the right-handed coordinate system.

○ (1) The vector $\mathbf{i} \times \mathbf{j}$ is collinear with $\mathbf{k}$ and is a unit vector, since the area of the parallelogram (square) constructed on the vectors $\mathbf{i}$ and $\mathbf{j}$ is equal to unity. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right-handed ordered trihedral, therefore $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Analogously, we find: (2) $\mathbf{j} \times \mathbf{k} = \mathbf{i}$; (3) $\mathbf{k} \times \mathbf{i} = \mathbf{j}$; (4) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ ($\mathbf{j}, \mathbf{i}$ and $\mathbf{k}$ form a right-handed trihedral); (5) $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$; (6) $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. ●

48. Given: $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$, $(\widehat{\mathbf{a}, \mathbf{b}}) = 30°$. Find $\mathbf{a} \times \mathbf{b}$.

○ By Formula (21.15), we find the modulus of the vector product: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin 30° = 4 \cdot 5 \cdot (1/2) = 10$.

By Formula (21.16), we have $\mathbf{a} \times \mathbf{b} = 10\mathbf{e}$, where $\mathbf{e}$ is the unit vector of the direction of $\mathbf{a} \times \mathbf{b}$. ●

49. Find the vector product of the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

○ By Formula (21.17), we obtain

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 5 \\ -1 & 3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}. \quad ●$$

50. Find the area of the parallelogram constructed on the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

○ The modulus of the vector product of two vectors is equal to the area of the parallelogram constructed on these vectors. By

Formula (21.17), we find

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 3 & 2 & 2\end{vmatrix}=\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}\mathbf{i}+\begin{vmatrix}2 & 2\\ 2 & 3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\mathbf{k}=-2\mathbf{i}+2\mathbf{j}+\mathbf{k}.$$

Since $|\mathbf{a}\times\mathbf{b}|=\sqrt{(-2)^2+2^2+1^2}=3$, the sought-for area $S=3$ (square units). ●

51. Given: the force $\mathbf{F}$ (2, 3, —1) and the point of its application A (—1, —1, 3). Find the moment of the force relative to the origin and the angles made by the moment with the coordinate axes.

○ Find the vector product of the radius vector $\mathbf{r}$ = (—1, —1, 3) of the point of application of the force by the force $\mathbf{F}$ = (2, 3, —1):

$$m_0(\mathbf{F})=\mathbf{r}\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 3\\ 2 & 3 & -1\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 3\\ 3 & -1\end{vmatrix}\mathbf{i}+\begin{vmatrix}3 & -1\\ -1 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ 2 & 3\end{vmatrix}\mathbf{k}=-8\mathbf{i}+5\mathbf{j}-\mathbf{k}.$$

Find the modulus of the moment:

$$|m_0(\mathbf{F})|=\sqrt{(-8)^2+5^2+(-1)^2}=\sqrt{90}\approx 9.49.$$

The direction cosines of the vector $m_0(\mathbf{F})$ are: $\cos\alpha=-8/9.49\approx -0.843$; $\cos\beta=5/9.49\approx 0.527$; $\cos\gamma=-1/9.49\approx -0.105$. The angles made by the moment of the force with the coordinate axes are: $\alpha=147.5°$, $\beta=58.2°$, $\gamma=96°$.

Computing check: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=(-0.843)^2+0.527^2+(-0.105)^2=1$. ●

52. Given: $|\mathbf{a}|=4$, $|\mathbf{b}|=6$; $\widehat{(\mathbf{a},\mathbf{b})}=\varphi$. Find $\mathbf{a}\times\mathbf{b}$ if (1) $\varphi=0$; (2) $\varphi=90°$; (3) $\varphi=150°$.

53. Find the vector product of the vectors $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

54. Find the area of the parallelogram constructed on the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

55. Find the area of the triangle given the coordinates of its vertices: A (2, —3, 4), B (1, 2, —1), and C (3, —2, 1).

56. Given the force $\mathbf{F}$ = (—2, —3, 2) and the point of its application A (—1, 3, —1). Find the moment of the force relative to the origin and the angles made by the moment with the coordinate axes.

Sec. 21.4. Miscellaneous Problems

57. Prove that the quadrilateral with the vertices A (1, 4, 3), B (2, 3, 5), C (2, 5, 1) and D (3, 4, 3) is a parallelogram.

58. On the axis of ordinates find the point equidistant from the points A (1, —3, 7) and B (5, 7, —5).

59. Prove that the quadrilateral with the vertices A (3, —1, 2), B (1, 2, —1), C (—1, 1, —3), and D (3, —5, 3) is a trapezoid.

60. The vertices of a triangle lie at the points A (10, —2, 8), B (8, 0, 7), and C (10, 2, 8). Compute the perimeter of the triangle.

61. The line segment AB, whose end points are A (—6, 1, 12) and B (9, 4, —9), is divided into three equal parts. Find the coordinates of the division points.

62. Given two vectors: $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Compute the coordinates of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

63. The vector $\overrightarrow{AB} = \mathbf{a}$ is given by the coordinates of its initial and terminal points: A (—4, 1, 3) and B (2, —5, 6). Compute the cosines of the angles made by the vector $\mathbf{a}$ with the basis vectors.

64. Given the vectors $\mathbf{a} = (2, 2, -1)$ and $\mathbf{b} = (-3, 6, -6)$. Compute the cosine of the angle between them.

65. A parallelogram is constructed on the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -\mathbf{j} + \mathbf{k}$. Compute the acute angle between its diagonals.

QUIZ No. 1

Form I

(1) The vector $\overrightarrow{AB} = \mathbf{a}$ is given by the coordinates of its initial and terminal points: A (2, 4, —3) and B (6, —3, 1). Compute its length and the cosines of the angles made by the vector with the basis vectors.

(2) Given the vectors $\mathbf{a} = (2, -4, 5)$ and $\mathbf{b} = (4, -3, 5)$. Compute the cosine of the angle between them.

(3) Find the vector product of the vectors $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Form II

(1) Given a triangle whose vertices lie at the points A (—2, —4, 0), B (—2, —1, 4) and C (—2, 3, 1). Compute its interior angle at the vertex A.

(2) Given three vectors: $\mathbf{a} = 12\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$; $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. Compute the projection of the vector $\mathbf{b} + \mathbf{c}$ on the vector $\mathbf{a}$.

(3) Compute the area of the parallelogram constructed on the vectors $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Chapter 22
EQUATIONS OF A STRAIGHT LINE AND A PLANE IN SPACE

Sec. 22.1. Plane

1. The Equations of a Plane. Let P be a plane, $M_0\,(x_0,\ y_0,\ z_0)$ a point belonging to this place (Fig. 164), and $\mathbf{n} = (A,\ B,\ C)$ a nonvanishing vector perpendicular to the plane P (it is called the *normal vector to the plane P*). If $M\,(x,\ y,\ z)$ is an arbitrary point on the plane P, different from M_0, then the vector $\overrightarrow{M_0M} = (x - x_0,\ y - y_0,\ z - z_0)$ is perpendicular to the vector $\mathbf{n} = A,\ B,\ C)$, that is, the scalar product of these vectors is equal to zero: $\mathbf{n} \cdot \overrightarrow{M_0M} =$ (0 or

$$\mathbf{n}\,(\mathbf{r} - \mathbf{r}_0) = 0. \tag{22.1}$$

Equation (22.1) is called the *equation of a plane in vector form.* If it is written in coordinate form, then the following equation is obtained

$$A\,(x - x_0) + B\,(y - y_0) + C\,(z - z_0) = 0, \tag{22.2}$$

which is termed the *equation of a plane passing through a given point in a given direction.*

Equation (22.2) can be rewritten as follows:

$$Ax + By + Cz + D = 0, \tag{22.3}$$

where $D = -(Ax_0 + By_0 + Cz_0)$. Equation (22.3) is called the *general equation of a plane.* Note that since the normal vector $\mathbf{n}$ is nonvanishing, the coefficients A, B, and C of the general equation of a plane are not all zero.

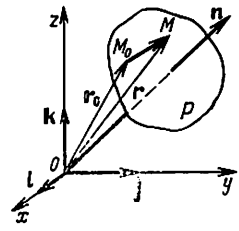


Fig. 164

Consider the particular cases of the general equation of a plane.

1. Let $D = 0$. Then the equation of a plane has the form $Ax + By + Cz = 0$; the plane passes through the origin.

2. Let $C = 0$. Then the equation of a plane has the form $Ax + By + D = 0$; since the normal vector $\mathbf{n} = (A,\ B,\ 0)$ is perpendicular to the z-axis, the plane under consideration is parallel to the z-axis.

Analogously, for $A = 0$ and $B = 0$ the planes $By + Cz + D = 0$ and $Ax + Cz + D = 0$ are obtained which are parallel to the x- and y-axes, respectively.

3. Let $C = D = 0$. Then the equation of a plane has the form $Ax + By = 0$; this time the plane passes through the origin and is parallel to the z-axis, that is, passes through the z-axis.

Analogously, for $A = D = 0$ and $B = D = 0$ the planes $By + Cz = 0$ and $Ax + Cz = 0$ are obtained passing through the x- and y-axes, respectively.

4. Let $C = B = 0$. Then the equation of a plane has the form $Ax + D = 0$; the plane is parallel to the y- and z-axes, that is, perpendicular to the x-axis.

Analogously, for $A = B = 0$ and $A = C = 0$ the planes $Cz + D = 0$ and $By + D = 0$ are obtained which are perpendicular to the z- and y-axes, respectively.

5. Let $B = C = D = 0$. Then the equation of a plane has the form $Ax = 0$, that is, $x = 0$; the plane coincides with the y, z-plane.

Analogously, for $A = C = D = 0$ and $A = B = D = 0$ we obtain the planes $y = 0$ and $z = 0$ coinciding with the x, z- and x, y-coordinate planes, respectively.

2. The Angle Between Two Planes. The angle between two planes is equal to the angle between their normal vectors. Let two intersecting planes $A_1x +$

$B_1y + C_1z + D_1 = 0$ and $A_2y + B_2y + C_2z + D_2 = 0$ have normal vectors $\mathbf{n}_1 = (A_1, B_1, C_1)$ and $\mathbf{n}_2 = (A_2, B_2, C_2)$, respectively. Then the angle between these planes is computed by the formula

$$\cos\varphi = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \quad (22.4)$$

3. Conditions of Parallelism and Perpendicularity of Two Planes. For two planes to be parallel, it is necessary that their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ be collinear, that is, $\mathbf{n}_1 = \lambda\mathbf{n}_2$, where $\lambda \neq 0$. If none of the coordinates of the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is equal to zero, then from the last equality it follows that

$$A_1/A_2 = B_1/B_2 = C_1/C_2 \quad (22.5)$$

that is, the coefficients of the corresponding coordinates are proportional.

In order for the planes to be perpendicular, their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ must also be perpendicular, that is, their scalar product is equal to zero: $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. Hence it follows that

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \quad (22.6)$$

1. Find the equation of the plane passing through the point M_0 (—3, 0, 2) perpendicular to the vector $\mathbf{n} = (2, 3, 5)$.

○ Here $A = 2$, $B = 3$, $C = 5$. Substituting the values of the coefficients A, B, C and the coordinates of the point M into Equation (22.2), we get

$$2(x + 3) + 3(y - 0) + 5(z - 2) = 0 \quad \text{or}$$

$$2x + 3y + 5z - 4 = 0. \quad \bullet$$

2. Write the equation of the plane perpendicular to the x-axis and passing through the point M_0 (2, —1, 3).

○ The equation of a plane perpendicular to the x-axis has the form $Ax + D = 0$. Substituting the coordinates of the point M_0 into this equation, we find $D = -2A$. Substituting now the value of D into the equation $Ax + D = 0$, we get $Ax - 2A = 0$, that is, $x - 2 = 0$. ●

3. Write the equation of the plane passing through the x-axis and the point M (3, 2, 4).

○ The equation of the desired plane has the form $By + Cz = 0$. Substituting into this equation the coordinates of the point M, we obtain $2B + 4C = 0$, that is, $B = -2C$. Substituting now the value of B into the equation $By + Cz = 0$, we find $-2Cy + Cz = 0$, that is, $2y - z = 0$. ●

4. Set up the equation of the plane parallel to the z-axis and passing through the points M_1 (3, —1, 2) and M_2 (—2, 3, 4).

○ Since the sought-for plane is parallel to the z-axis and passes through the points M_1 (3, —1, 2) and M_2 (—2, 3, 4), its normal vector $\mathbf{n} = (A, B, C)$ may be represented by the vector perpendicular to the vectors $\overrightarrow{M_1M_2} = (-5, 4, 2)$ and $\mathbf{n}_1 = (0, 0, 1)$ (the unit vector of the z-axis). On the other hand, it is known that the vector product of two vectors is a vector perpendicular to the vectors enter-

ing the vector product; therefore for $\mathbf{n}$ we may take the vector product $\overrightarrow{M_1M_2}$ and $\mathbf{n}_1$. Consequently,

$$\mathbf{n} = \overrightarrow{M_1M_2} \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -5 & 2 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -5 & 4 \\ 0 & 0 \end{vmatrix} = 4\mathbf{i} + 5\mathbf{j}.$$

It remains to take advantage of the equation of the plane passing through the given point M_1 (3, —1, 2) perpendicular to the vector $\mathbf{n}$ = (4, 5, 0). We have

$$4(x - 3) + 5(y + 1) = 0 \quad \text{or} \quad 4x + 5y - 7 = 0. \quad \bullet$$

5. Find the equation of the plane passing through the point M_0 (2, —1, 3) parallel to the vectors $\mathbf{a}$ = (3, 0, —1) and $\mathbf{b}$ = (—3, 2, 2).

○ It is obvious that for the normal vector $\mathbf{n} = (A, B, C)$ of the required plane we may take the vector product of $\mathbf{a}$ by $\mathbf{b}$:

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -1 \\ -3 & 2 & 2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ -3 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 0 \\ -3 & 2 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}.$$

Using now Equation (22.2) for $A = 2$, $B = -3$, $C = 6$, $x_0 = 2$, $y_0 = -1$, $z_0 = 3$, we get

$$2(x - 2) - 3(y + 1) + 6(z - 3) = 0 \quad \text{or}$$

$$2x - 3y + 6z - 25 = 0. \quad \bullet$$

6. Write the equation of the plane passing through the poin M (—2, 3, 4) parallel to the plane $x + 2y - 3z + 4 = 0$.

○ Since the desired plane is parallel to the plane $x + 2y - 3z + 4 = 0$, we may take, as its normal vector, the normal vector $\mathbf{n}$ = (1, 2, —3) of the given plane. Using now the equation of the plane passing through a given point in a given direction, we get

$$(x + 2) + 2(y - 3) - 3(z - 4) = 0 \text{ or } x + 2y - 3z + 8 = 0. \bullet$$

7. Find the equation of the plane passing through the points M_1 (—2, —3, 1) and M_2 (1, 4, —2) perpendicular to the plane $2x + 3y - z + 4 = 0$.

○ We take the vector product $\overrightarrow{M_1M_2}$ = (3, 7, —3) and $\mathbf{n}_1$ = (2, 3, —1) as the normal vector $\mathbf{n}$ of the sought-for plane (compare

with the solutions of Problems 4 and 5). Thus,

$$\mathbf{n}=\overrightarrow{M_1M_2}\times\mathbf{n}_1=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 7 & -3\\ 2 & 3 & -1\end{vmatrix}$$

$$=\mathbf{i}\begin{vmatrix}7 & -3\\ 3 & -1\end{vmatrix}-\mathbf{j}\begin{vmatrix}3 & -3\\ 2 & -1\end{vmatrix}+\mathbf{k}\begin{vmatrix}3 & 7\\ 2 & 3\end{vmatrix}=2\mathbf{i}-3\mathbf{j}-5\mathbf{k}.$$

It remains to take advantage of the equation of the plane passing through the given point M_1 $(-2, -3, 1)$ perpendicular to the vector $\mathbf{n} = (2, -3, -5)$:

$$2(x+2)-3(y+3)-5(z-1)=0 \quad \text{or} \quad 2x-3y-5z=0.$$

Note that, when setting up the equation of the desired plane, we might take the point M_2 instead of the point M_1. ●

8. Find the angle between the planes $2x - 3y + 4z - 1 = 0$ and $3x - 4y - z + 3 = 0$.

○ To compute the angle φ between the given planes, we make use of Formula (22.4). We have $A_1 = 2$, $B_1 = -3$, $C_1 = 4$ and $A_2 = 3$, $B_2 = -4$, $C_2 = -1$. Consequently,

$$\cos\varphi\frac{|2\cdot 3+(-3)\cdot(-4)+4\cdot(-1)|}{\sqrt{2^2+(-3)^2+4^2}\cdot\sqrt{3^2+(-4)^2+(-1)^2}}=\frac{14}{\sqrt{29\cdot 26}}=0.51,$$

$$\varphi=59.3^\circ. \bullet$$

9. Find the distance from the point A $(2, 3, 4)$ to the plane $4x + 3y + 12z - 5 = 0$.

○ Let $|AB|$ be the distance from the point A to the given plane. The vector $\overrightarrow{AB}$ is collinear with the vector $\mathbf{n} = (4, 3, 12)$, therefore $\overrightarrow{AB} = \lambda\mathbf{n}$. Denoting the coordinates of the point B by x_1, y_1, z_1, we find $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_1 - 2, y_1 - 3, z_1 - 4)$.

It follows from the equality $\overrightarrow{AB} = \lambda\mathbf{n}$, that $x_1 - 2 = 4\lambda$, $y_1 - 3 = 3\lambda$, $z_1 - 4 = 12\lambda$. Using the formula for the distance between two points, we have

$$|\overrightarrow{AB}|=\sqrt{(4\lambda)^2+(3\lambda)^2+(12\lambda)^2}=13|\lambda|.$$

Since the coordinates of the point B (x_1, y_1, z_1) satisfy the equation of the plane, we have

$$4(2+4\lambda)+3(3+3\lambda)+12(4+12\lambda)-5=0, \quad \text{i.e.}$$

$$\lambda=-60/169.$$

Consequently, $|AB| = 13\cdot|-60/169| = 60/13$. ●

10. Find the distance between the two parallel planes:

$$2x - 3y + 6z + 28 = 0 \quad \text{and} \quad 2x - 3y + 6z - 14 = 0.$$

○ To find the required distance, it is necessary to take a point on one of the planes and to determine the distance from this point to the other plane. Setting $y = 0$, $z = 0$ in the equation of the first of the given planes, we have $2x + 28 = 0$, that is, $x = -14$; thus we have obtained the point M (—14, 0, 0).

Now, in the same way as in Problem 9, we find the distance from the given point M (—14, 0, 0) to the given plane $2x - 3y + 6z - 14 = 0$. This distance is equal to 6 (make sure independently). ●

11. Given four points: A (3, —2, —1), B (0, 0, 2), C (—3, 1, 0), D (—4, —2, 2.5). Indicate which of them belong to the plane $2x - 3y + 4z - 8 = 0$.

12. Write the equation of the plane passing through the point M_0 (3, 4, 5) perpendicular to the vector $\mathbf{n} = (-1, -3, 2)$.

13. Find the equation of the plane passing through the point M_0 (2, —3, —1) perpendicular to the vector $\overrightarrow{M_1M_2}$, where M_1 (3, 4, 1) and M_2 (1, —2, —3).

14. Given two points: A (3, —2, 4) and B (1, 4, 2). Write the equation of the plane passing through the point A perpendicular to the vector $\overrightarrow{AB}$.

15. Write the equation of the plane perpendicular to the z-axis and passing through the point M_0 (—2, —3, —1).

16. Find the equation of the plane: (1) parallel to the x, y-plane and passing through the point M_0 (2, —2, 3); (2) parallel to the x, z-plane and passing through the point M_0 (—3, —2, —4).

17. Write the equation of the plane: (1) passing through the z-axis and the point M (1, 1, 1); (2) passing through the y-axis and the point M (—2, —3, —4).

18. Write the equation of the plane: (1) parallel to the y-axis and passing through the points M_1 (1, —2, —1) and M_2 (3, 2, —4); (2) parallel to the x-axis and passing through the points M_1 (—4, 2, 5) and M_2 (—5, —1, 3).

19. Find the equation of the plane: (1) passing through the point M_0 (—4, —3, 1) parallel to the vectors $\mathbf{a} = (5, 2, -3)$ and $\mathbf{b} = (1, 4, -2)$; (2) passing through the point M (—1, —2, 3) parallel to the plane $2x - 3y + z - 1 = 0$.

20. Find the equation of the plane: (1) passing through the points A (1, —4, —3) and B (4, —2, —1) perpendicular to the plane $x - y - 3z + 7 = 0$; (2) passing through the points M_1 (2, —1, —3) and M_2 (—3, 4, 1) perpendicular to the plane $x - y - 3z + 2 = 0$.

21. Find the equation of the plane passing through the point M (—1, —1, 2) perpendicular to the planes $x + 2y - 2z + 4 = 0$ and $x - 2y + z - 4 = 0$.

22. Find the angle between the planes $x - y + z + 1 = 0$ and $2x + 3y - z - 3 = 0$.

23. Find the distance: (1) from the point A (1, —2, 1) to the plane

$10x - 2y + 11z - 10 = 0$; (2) from the point A (2, 3, —2) to the plane $6x - 7y - 6z - 124 = 0$.

24. Find the distance between the parallel planes $x - y + 2z - 4 = 0$ and $x - y + 2z + 10 = 0$.

Sec. 22.2. A Straight Line in Space

1. Equations of a Straight Line in Space. The equation of a straight line passing through a given point M_0 (x_0, y_0, z_0) parallel to the vector $\mathbf{q} = (m, n, p)$ has the form

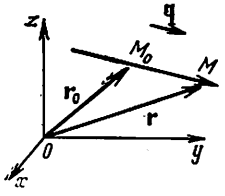


Fig. 165

$$\mathbf{r} = r_0 + t\mathbf{q} \tag{22.7}$$

and is called the *vector-parametric equation of a straight line.* Here $\mathbf{r}$ is the radius vector of an arbitrary point M (x, y, z) of the straight line (Fig. 165); $\mathbf{r}_0$ the radius vector of the point M_0 (x_0, y_0, z_0), and t the parameter taking on all possible values. The vector $\mathbf{q}$ is called the *direction vector of the straight line*, and its coordinates (that is, the numbers m, n, p) the *direction numbers of the straight line.*

Passing to the coordinates of the vectors in Equation (22.7), we get the *parametric equation of a straight line*:

$$x = x_0 + tm, \quad y = y_0 + nt, \quad z = z_0 + pt. \tag{22.8}$$

Eliminating the parameter t from Equations (22.8), we get the *canonical equations of a straight line*:

$$(x - x_0)/m = (y - y_0)/n = (z - z_0)/p. \tag{22.9}$$

The *two-point form of the equations of a straight line* (a straight line passing through the points M_1 (x_1, y_1, z_1) and M_2 (x_2, y_2, z_2) is

$$(x - x_1)/(x_2 - x_1) = (y - y_1)/(y_2 - y_1) = (z - z_1)/(z_2 - z_1). \tag{22.10}$$

A straight line in space can be considered as the line of intersection of two planes, that is, a straight line is defined by the system of two simultaneous linear equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases} \tag{22.11}$$

Equations (22.11) are called the *general equations of a straight line.*

2. Direction Cosines of a Straight Line. The direction cosines of the vector $\mathbf{q} = (m, n, p)$ are defined as the *direction cosines of a straight line.* Since the vector $-\mathbf{q} = (-m, -n, -p)$ may also be taken as the direction vector, the straight line thus has two triples of direction cosines:

$$\cos\alpha = \pm \frac{m}{\sqrt{m^2+n^2+p^2}}, \quad \cos\beta = \pm \frac{n}{\sqrt{m^2+n^2+p^2}},$$

$$\cos\gamma = \pm \frac{p}{\sqrt{m^2+n^2+p^2}}. \tag{22.12}$$

3. Conditions of Parallelism and Perpendicularity of Two Straight Lines in Space. The Angle Between Two Straight Lines. Let there be given two straight lines: $(x - x_1)/m_1 = (y - y_1)/n_1 = (z - z_1)/p_1$ and $(x - x_2)/m_2 = (y - y_2)/n_2 = (z - z_2)/p_2$. Then the condition of parallelism of these lines is

written in the form

$$m_1/m_2 = n_1/n_2 = p_1/p_2, \tag{22.13}$$

the condition of perpendicularity in the form

$$m_1m_2 + n_1n_2 + p_1p_2 = 0, \tag{22.14}$$

and the angle φ between them is computed by the formula

$$\cos\varphi = \pm \frac{m_1m_2+n_1n_2+p_1p_2}{\sqrt{m_1^2+n_1^2+p_1^2}\cdot\sqrt{m_2^2+n_2^2+p_2^2}}. \tag{22.15}$$

25. Write the equations of the straight line passing through the point M_0 (2, 1, 3) parallel to the vector $\mathbf{q} = (4, -5, -6)$.

○ Using Equalities (22.9), we find the canonical equations of the straight line:

$$(x - 2)/4 = (y - 1)/(-5) = (z - 3)/(-6).$$

Writing these equations in the form of a system, we get the general equations of the straight line:

$$\begin{cases} (x-2)/4 = (y-1)/(-5), \\ (y-1)/5 = (z-3)/6, \end{cases} \quad \text{or} \quad \begin{cases} 5x + 4y - 14 = 0, \\ 6y - 5z + 9 = 0. \end{cases} \bullet$$

26. Find the equations of the straight line parallel to the x-axis and passing through the point M (1, 1, 1).

○ The direction vector $\mathbf{q}$ of the straight line is collinear with the x-axis; consequently, its projections on the y- and z-axes are equal to zero. The vector $\mathbf{q}$ may have any of two possible directions and any length. Let us set $|\mathbf{q}| = 1$ and choose the direction coinciding with the positive direction of the x-axis; then $\mathbf{q} = (1, 0, 0)$. We then set up the canonical equations of the desired line: $(x - 1)/1 = (y - 1)/0 = (z - 1)/0$. The general equations of the line have the form

$$\begin{cases} y - 1 = 0, \\ z - 1 = 0. \end{cases} \bullet$$

27. Find the equations of the straight line passing through the point A (−1, 2, 1) parallel to the straight line $(x - 3)/2 = (y - 2)/3 = (z + 2)/1$.

○ Since the sought-for line is parallel to the given line, we may take the direction vector $\mathbf{q} = (2, 3, 1)$ of the given line for its direction vector. Using now Equalities (22.9), we get the canonical equations of the desired line: $(x + 1)/2 = (y - 2)/3 = (z - 1)/1$. ●

28. Find the parametric equations of the straight line passing through the origin and the point A (2, −3, −2).

○ We take the vector $\mathbf{q} = (2, -3, -2)$ for the direction vector. Using Formulas (22.8), we obtain $x = 2 + 2t$, $y = -3 - 3t$, $z = -2 - 2t$. ●

29. Write the equations of the straight line passing through the points A (1, —2, —1) and B (3, 0, 4).

○ By Formulas (22.10), we get

$$(x-1)/(3-1)=(y+2)/(0+2)=(z+1)/(4+1)$$

or $(x-1)/2=(y+2)/2=(z+1)/5$. ●

30. Compute the angles made by the straight line $(x-2)/3=(y+3)/2=(z-1)/6$ with the coordinate axes.

○ By Formulas (22.12), we get

$$\cos\alpha=\pm\frac{3}{\sqrt{3^2+2^2+6^2}}=\pm\frac{3}{7},\quad \cos\beta=\pm\frac{2}{7},\quad \cos\gamma=\pm\frac{6}{7}.$$ ●

31. Compute the acute angle between the two straight lines: $(x-3)/2=(y-1)/1=(z+4)/2$ and $(x+1)/12=(y+3)/3=(z-2)/4$.

○ Setting in Equality (22.15) $m_1=2$, $n_1=1$, $p_1=2$ and $m_2=12$, $n_2=3$, $p_2=4$, we find

$$\cos\varphi=\frac{2\cdot 12+1\cdot 3+2\cdot 4}{\sqrt{2^2+1^2+2^2}\cdot\sqrt{12^2+3^2+4^2}}=0.897;\quad \varphi=26.2^\circ.$$ ●

32. Write the equations of the straight line: (1) passing through the point M_0 (3, 0, —2) parallel to the vector $\mathbf{q}=(2, 1, 1)$; (2) passing through the point M_0 (1, 0, —2) parallel to the vector $\mathbf{q}=(2, 1, 0)$.

33. Find the equations of the straight line passing through the point M (2, —1, 3) parallel to the z-axis.

34. How is the straight line situated relative to the coordinate axes if it has the direction vector: (a) $\mathbf{q}=(0, 0, 1)$; (b) $\mathbf{q}=(0, 1, 0)$; (c) $\mathbf{q}=(1, 0, 0)$?

35. Write the equations of the straight line passing through the point A (2, —3, —1) parallel to the line $(x-4)/4=(y+1)/3=(z+3)/2$.

36. Set up the parametric equation of the straight line passing through the origin and point M (1, 4, —3).

37. Write the equations of the straight line passing through the points A (—2, —1, —3) and B (0, 2, 1).

38. Compute the angles made by the straight line $(x-1)/4=(y-4)/3=(z+2)/12$ with the coordinate axes.

39. Prove that the straight lines $(x-1)/(-2)=(y+2)/3=z/(-4)$ and $(x+2)/5=(y-1)/6=(z-5)/2$ are mutually perpendicular.

40. Compute the acute angle between the two straight lines: $(x-1)/3=(y+4)/(-2)=(z-2)/4$ and $(x+3)/2=(y-1)/3=(z+1)/(-2)$.

Sec. 22.3. A Plane and a Straight Line

The angle φ between the straight line

$$(x - a)/m = (y - b)/n = (z - c)/p \qquad (*)$$

and the plane

$$Ax + By + Cz + D = 0 \qquad (**)$$

is computed by the formula

$$\sin\varphi = \frac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{m^2 + n^2 + p^2}}. \qquad (22.16)$$

The condition of parallelism of the straight line (*) and the plane (**) is written in the form

$$Am + Bn + Cp = 0, \qquad (22.17)$$

and the condition of perpendicularity in the form

$$A/m = B/n = C/p. \qquad (22.18)$$

The conditions at which the line (*) belongs to the plane (**) have the form

$$\begin{cases} Aa + Bb + Cc + D = 0, \\ Am + Bn + Cp = 0. \end{cases} \qquad (22.19)$$

The equation of a sheaf of planes passing through a given straight line

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

has the form

$$A_1x + B_1y + C_1z + D_1 + \lambda (A_2x + B_2y + C_2z + D_2) = 0, \qquad (22.20)$$

where λ is any real number.

41. Compute the angle between the line $(x - 2)/3 = (y + 1)/4 = (z - 3)/2$ and plane $x + 2y - 3z + 4 = 0$.

○ Take advantage of Formula (22.16). Since $A = 1$, $B = 2$, $C = -3$, $m = 3$, $n = 4$, $p = 2$, we have

$$\sin\varphi = \frac{|1 \cdot 3 + 2 \cdot 4 + (-3) \cdot 2|}{\sqrt{1^2 + 2^2 + (-3)^2} \cdot \sqrt{3^2 + 4^2 + 2^2}} = 0.248; \quad \varphi = 14.4°. \bullet$$

42. Write the equation of the plane passing through the point M $(-1, 2, -3)$ perpendicular to the straight line $(x + 2)/4 = (y - 1)/3 = (z + 3)/2$.

○ It is obvious that, as the normal vector $\mathbf{n}$ of the desired plane, we may take the direction vector $\mathbf{q} = (4, 3, 2)$ of the given line which is parallel to $\mathbf{n}$. It remains to take advantage of the equation of the plane passing through the given point M perpendicular to the vector $\mathbf{q}$:

$$4(x + 1) + 3(y - 2) + 2(z + 3) = 0 \quad \text{or}$$

$$4x + 3y + 2z + 4 = 0. \bullet$$

43. Through the point M (1, 3, 2) draw a straight line perpendicular to the plane $x - 2y + 2z - 3 = 0$. Compute the direction cosines of this line.

○ We take the normal vector $\mathbf{n} = (1, -2, 2)$ of the given plane for the direction vector of the sought-for line. It is obvious that the two vectors are parallel to each other. Knowing the point M (1, 3, 2) through which the desired line passes and the direction vector of this line, we write its canonical equations: $(x - 1)/1 = (y - 3)/(-2) = (z - 2)/2$. The direction cosines of the required line are found by Formulas (22.12):

$$\cos\alpha = \pm\frac{1}{\sqrt{1^2+(-2)^2+2^2}} = \pm\frac{1}{3}, \quad \cos\beta = \pm\frac{2}{3}, \quad \cos\gamma = \pm\frac{2}{3}. \bullet$$

44. Find the point at which the straight line $(x - 2)/4 = (y - 3)/2 = (z + 1)/5$ pierces the plane $x + 2y - 3z - 4 = 0$.

○ Let us write the equations of the straight line in parametric form. Setting $(x - 2)/4 = (y - 3)/2 = (z + 1)/5 = t$, we get: $x = 2 + 4t$; $y = 3 + 2t$; $z = -1 + 5t$. Substituting the found values of x, y, z into the equation of the plane, we have

$$2 + 4t + 2(3 + 2t) - 3(-1 + 5t) - 4 = 0,$$

whence $t = 1$. Substituting then the value $t = 1$ into the parametric equations of the straight line, we get: $x = 6$, $y = 5$, $z = 4$. Thus, (6, 5, 4) is the required point of intersection of the straight line and the plane. ●

45. Make sure that the straight line $(x - 2)/4 = (y + 4)/3 = (z - 1)/(-2)$ is parallel to the plane $5x - 2y + 7z + 3 = 0$.

○ Using the condition (22.17) of parallelism of a straight line and a plane, we obtain $5\cdot 4 + (-2)\cdot 3 + 7\cdot(-2) = 0$, that is, the straight line is parallel to the plane. ●

46. Check whether the straight line $(x - 3)/4 = (y - 1)/2 = (z + 2)/3$ lies in the plane $2x - y - 2z - 9 = 0$.

○ Using Conditions (22.19) for $a = 3$, $b = 1$, $c = -2$, $m = 4$, $n = 2$, $p = 3$, $A = 2$, $B = -1$, $C = -2$, $D = -9$, we find $2\cdot 3 + (-1)\cdot 1 + (-2)\cdot(-2) - 9 = 0$; $2\cdot 4 + (-1)\cdot 2 + (-2)\cdot 3 = 0$. Consequently, the line lies in the plane. ●

47. Write the equation of the plane passing through the straight line

$$\begin{cases} 2x + y - 3z + 4 = 0, \\ x - y + z + 3 = 0 \end{cases}$$

and the point M (2, 1, −1).

○ Using Equality (22.20), we write the equation of the sheaf of planes passing through the given line:

$$2x + y - 3z + 4 + \lambda(x - y + z + 3) = 0. \qquad (*)$$

Since the coordinates of the point M must satisfy the equation of the plane, substituting $x = 2$, $y = 1$, $z = -1$ into Relationship (*), we have

$$2 \cdot 2 + 1 - 3(-1) + 4 + \lambda(2 - 1 - 1 + 3) = 0 \quad \text{or}$$

$$12 + 3\lambda = 0,$$

whence $\lambda = -4$. Substituting now the found value of λ into (*), we get $2x - 5y + 7z + 8 = 0$. ●

48. Compute the angle between the straight line $(x + 4)/3 = (y - 1)/2 = (z - 3)/4$ and the plane $2x - 3y - 2z + 5 = 0$.

49. Write the equation of the plane passing through the point M (2, —1, —4) perpendicular to the straight line $(x - 3)/2 = (y + 2)/4 = (z + 5)/3$.

50. Through the point (1, 2, 4) a straight line is drawn perpendicular to the plane $2x + 3y + z - 4 = 0$. Compute the direction cosines of this line.

51. Find the equation of the perpendicular to the plane $4x - 5y - z - 3 = 0$ passing through the point M (—1, 1, —2).

52. Find the point of intersection of the straight line $(x + 3)/2 = (y - 1)/3 = (z + 5)/2$ with the plane $2x + 3y + z - 22 = 0$.

53. Write the equations of the perpendicular to the plane $x - 3y + 2z - 26 = 0$ passing through the point (—2, 2, —4). Find the coordinates of the foot of this perpendicular.

54. Check whether the straight line $(x - 1)/(-2) = (y - 4)/(-3) = (z + 1)/3$ is parallel to the plane $3x - 5y - 3z - 4 = 0$.

55. Check whether the straight line $(x - 1)/2 = (y + 3)/(-1) = (z - 4)/5$ lies in the plane $3x - 4y - 2z - 7 = 0$.

56. Find the equation of the plane passing through the straight line

$$\begin{cases} 2x - 4y + 5z + 5 = 0, \\ 2x - y + 2z - 1 = 0 \end{cases}$$

and the point M (3, 2, 1).

Sec. 22.4. Miscellaneous Problems

57. Find the canonical and parametric equations of the straighr line passing through the point M (—2, 2, —3) parallel to the vectot $\mathbf{q} = (2, -4, 5)$.

58. Write the canonical and parametric equations of the straight line passing through the point M (1, 2, 3) if the direction vector $\mathbf{q}$ of the line makes the angles $\alpha = 2\pi/3$, $\beta = \pi/3$, $\gamma = \pi/4$ with the x-, y-, and z-axes, respectively.

59. Compute the angle between the straight lines $(x - 3)/1 = (y + 2)/(-1) = z/\sqrt{2}$ and $(x + 2)/1 = (y - 3)/1 = (z + 5)/\sqrt{2}$.

60. Write the equations of the plane passing through the z-axis and the point A (1, —2, 1).

61. Find the equation of the plane if the point M (2, —1, 2) serves as the foot of the perpendicular dropped from the origin on this plane.

62. Find the projection of the point M (5, 2, —1) on the plane $2x - y + 3z + 23 = 0$.

63. Compute the angle between the straight line $(x - 1)/4 = y/12 = (z - 1)/(-3)$ and the plane $6x - 3y - 2z = 0$.

64. Find the point at which the straight line $(x - 12)/4 = (y - 9)/3 = (z - 1)/1$ pierces the plane $3x + 5y - z - 2 = 0$.

QUIZ No. 1

Form I

(1) Write the canonical and parametric equations of the straight line passing through the point A (2, —3, 4) parallel to the vector $\mathbf{q} = (-1, 4, -2)$.

(2) Write the equations of the plane passing through the points A (1, 2, 1) and B (0, 3, 4) perpendicular to the plane $x + 2y - z = 0$.

(3) Compute the angle between the straight line $(x - 5)/2 = (y + 1)/(-2) = z/(-1)$ and the plane $2x + y - 2z + 5 = 0$.

Form II

(1) Write the canonical and parametric equations of the straight line passing through the points A (1, 3, —5) and B (4, —1, 2).

(2) Write the equation of the plane passing through the point M (—2, 1, 2) parallel to two vectors $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$.

(3) Find the point of intersection of the straight line $(x - 1)/3 = (y + 2)/4 = (z - 3)/(-2)$ and the plane $2x - y + 3z - 1 = 0$.

Chapter 23
POLYHEDRA AND AREAS OF THEIR SURFACES

Sec. 23.1. The Prism

1. How many faces, vertices, edges and lateral edges have triangular, quadrangular, and hexagonal prisms?

2. How many plane (flat), dihedral, and trihedral angles have a triangular, quadrangular, hexagonal, and n-gonal prisms?

3. How many diagonals have triangular, quadrangular, hexagonal, and n-gonal prisms?

4. (1) How many diagonal sections can be drawn through one lateral edge in triangular, quadrangular, hexagonal, and n-gonal prisms?

(2) How many diagonal sections can be drawn through all the lateral edges in quadrangular, hexagonal, and n-gonal prisms?

5. What is the sum of all plane angles in triangular, quadrangular, hexagonal, and n-gonal prisms?

6. Prove that a section perpendicular to a lateral edge of a prism is perpendicular to each of its lateral faces.

7. Prove that the number of edges of a prism is multiple of 3.

8. What is the sum of all the dihedral angles formed by the lateral faces of right prisms: (a) triangular, (b) quadrangular, (c) hexagonal, (d) n-gonal?

9. Prove that the angles of inclination of all lateral edges of a prism to the plane of its base are equal.

10. Prove that in an oblique triangular prism the distance from a lateral edge to the opposite face is equal to the altitude of the triangle which serves as a perpendicular section of the prism.

11. In a regular hexagonal prism the side of the base equals m, and the lateral faces are squares. Find the diagonals of the prism and the areas of the diagonal sections.

12. In a regular triangular prism each edge is equal to a. A plane is passed through a side of the base and the midpoint of the axis. Compute the area of the section.

13. In a regular quadrangular prism the diagonal of the base is equal to m, and the diagonal of a lateral face is n. Compute the diagonal of the prism.

14. In a regular quadrangular prism the side of the base is equal to a and altitude to h. Compute the shortest distance from the side of the base to the diagonal of the prism which does not intersect this side.

○ AB and DB_1 are skew lines. The orthogonal projection of two skew lines on the plane perpendicular to one of them is represented by a straight line and a point. The distance between this line and the point is equal to the distance between the skew lines (Fig. 166).

Projecting the straight lines AB and DB_1 on the face AA_1D_1D, we get the point $A = \mathrm{pr}_{AA_1D_1D}AB$ and the line segment $A_1D = \mathrm{pr}_{AA_1D_1D}DB_1$. The distance between AB and DB_1 is equal to the distance from the point A to A_1D, that

is, to the altitude AK of the right triangle AA_1D:

$$|AK| = \frac{|AD| \cdot |AA_1|}{|A_1D|} = \frac{ah}{\sqrt{a^2+h^2}}. \bullet$$

15. In a regular quadrangular prism the diagonal is inclined to a lateral face at an angle of 30°. Compute the angle of its inclination to the base.

16. In a regular triangular prism the side of the base is equal to 12 cm, and a lateral edge to $10\sqrt{3}$ cm. Compute the area of the section passing through a lateral edge perpendicular to the opposite face.

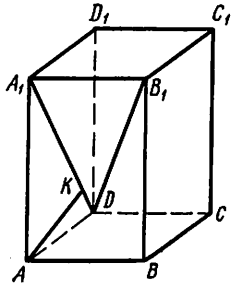


Fig. 166

17. In a right triangular prism the sides of the base are equal to 13 cm, 20 cm, and 21 cm; the altitude of the prism is equal to 25 cm. Compute the area of the section passed through a lateral edge and the smaller altitude of the base.

18. The base of a right prism is a rhombus; the diagonals and altitude of the prism are respectively equal to 8 cm, 5 cm, and 2 cm. Compute the side of its base.

19. Compute the length of the diagonal of a rectangular parallelepiped having the following dimensions: (1) 12, 16, 21; (2) 2, 4, 6.

20. Compute the diagonals of a right parallelepiped each edge of which is equal to a, and the base angle to 60°.

21. In a right parallelepiped all the diagonals are equal. Prove that the given parallelepiped is rectangular.

22. Find the relation between the edge of a cube a and its diagonal d.

23. Find the relation between the diagonal of a cube d and the diagonal of its face d_1.

24. Compute the angle between the diagonal of a cube and its base.

25. Compute the acute angle between the diagonals of a cube.

26. Of what length must be the wire for making the frame of a cube with all the diagonals if the edge of the cube is equal to 10 cm?

27. The edge of a cube is equal to a. Find the shortest distance from a diagonal to a nonintersecting edge.

28. Find the distance from a vertex of a cube to its diagonal.

29. Find the diagonal of a rectangular parallelepiped if the diagonals of its faces are equal to 11, 19, and 20.

30. In a right parallelepiped the diagonals form angles of 45° and 60° with the plane of the base. The sides of the base are equal to 17 cm and 31 cm. Compute the diagonals of this parallelepiped.

31. Compute the areas of diagonal sections of a rectangular par-

allelepiped in which the sides of the base are equal to a and b, the altitude being equal to h.

32. The faces of a parallelepiped are congruent rhombi with the side a and angle 60°. Compute the areas of its diagonal sections.

33. Is it possible to cut a cube by a plane so that to get in section a triangle: (1) scalene; (2) isosceles; (3) equilateral; (4) right-angled?

34. The edge of a cube measures a. Compute the area of the diagonal section.

35. Given a cube $ABCDA_1B_1C_1D_1$. A plane is passed through the midpoints of three of its edges A_1B_1, A_1D_1, and CD. Prove that the section thus obtained is a regular hexagon. Find the area of the section if the edge of the cube is equal to a.

36. The edge of a cube is equal to a. Find the area of the section of the cube by the plane passed through the midpoints of the edges emanating from one vertex.

37. Compute the perimeter and area of the section passing through the end points of three edges emanating from a vertex of a cube. The edge of the cube is equal to a.

38. In an oblique triangular prism the distances between the lateral edges are equal to 20 cm, 34 cm, and 42 cm. Find the distance between the larger lateral face and the opposite lateral edge.

39. The base of an oblique parallelepiped is a rhombus $ABCD$ in which $\widehat{BAD} = 60°$; the lateral edges are inclined to the plane of the base at an angle of 60°, and the plane AA_1C_1C is perpendicular to the plane of the base. Prove that the area of the section BB_1D_1D is to that of the section AA_1C_1C as 2 : 3.

40. The base of an oblique parallelepiped is a square, a on a side, one of the vertices of the other base is projected into the centre of this square; the altitude of the parallelepiped is equal to H. Find the angle of inclination of the lateral side to the plane of the base.

41. The base of an oblique parallelepiped is a rhombus. One of the lateral edges makes equal angles with adjacent sides of the base. Prove that the vertex of the parallelepiped lying on this edge is projected on the plane of the base into a point which lies on the diagonal of the base.

Sec. 23.2. Surface Area of a Prism

The area of the lateral surface of a prism is equal to the product of the perimeter of the perpendicular section of the prism by the length of its lateral edge:

$$S_{\text{lat}} = Pl. \tag{23.1}$$

42. The surface area of a cube is equal to S. Find its edge.

43. Compute the surface area of a cube given: (1) its diagonal l; (2) the area Q of its diagonal section.

44. The areas of total surfaces of two cubes are equal to S and Q. What is the ratio of the edges of these cubes.

45. Compute the area of the lateral surface of a rectangular parallelepiped whose altitude is equal to h, the area of the base to S, and the area of the diagonal section to Q.

46. In a rectangular parallelepiped the dimensions are as 3 : 6 : 22, and its diagonal is equal to 23 cm. Compute the area of the total surface of the parallelepiped.

47. In a rectangular parallelepiped the area of the lateral surface amounts to two thirds of the area of its total surface. Prove that the parallelepiped is a cube.

48. The sides of the base of a rectangular parallelepiped are equal to 6 cm and 8 cm, and the area of the diagonal section to 180 cm^2. Compute the area of the total surface of the parallelepiped.

49. In a rectangular parallelepiped a lateral edge is equal to 12 cm, the area of the diagonal section to 312 cm^2, and the area of the base to 240 cm^2. Compute the sides of the base.

50. In a right parallelepiped the sides of the base are equal to 3 cm and 8 cm and make an angle of 60°. The longer diagonal of the parallelepiped is equal to 49 cm. Find the area of the lateral surface of the parallelepiped.

51. The base of a right parallelepiped is a rhombus with diagonals 12 cm and 16 cm; the diagonal of a lateral face is equal to 26 cm. Compute the area of the total surface of the parallelepiped.

52. The diagonals of a right parallelepiped make angles of 30° and 45° with the plane of the base, and the sides of the base are equal to 6 cm and 8 cm. Compute the area of the lateral surface of the parallelepiped.

53. Given the side of the base a and the lateral edge l, compute the area of the total surface of a regular prism: (1) triangular; (2) quadrangular; (3) hexagonal.

54. The greatest diagonal of a regular hexagonal prism is equal to 13 cm; the area of its lateral surface is equal to 180 cm^2. Find the area of its base.

55. In a regular quadrangular prism the diagonal is equal to l and makes an angle α with a lateral face. Compute the area of the lateral surface of the prism.

56. In a regular hexagonal prism the diagonals are equal to 17 cm and 15 cm. Compute the area of the lateral surface of the prism.

57. The area of the greatest section of a regular hexagonal prism is equal to S. Find the area of the lateral surface of the prism.

58. The base of a right prism is the trapezoid $ABCD$ in which the parallel sides BC and AD are respectively equal to 26 cm and 40 cm and the nonparallel sides AB and CD to 15 cm and 13 cm, respectively. The area of the section AA_1C_1C is equal to 370 cm^2. Compute the area of the lateral surface of the prism.

59. The base of a right prism is a rhombus. The diagonals of the prism are equal to 10 cm and $6\sqrt{2}$ cm; its altitude being 6 cm. Compute the area of the lateral surface of the prism.

60. In a right triangular prism the sides of the base are as 17 : 15 : 8, and a lateral edge is equal to 20 cm. The area of the total surface of this prism equals 2080 cm^2. Find the area of its lateral surface.

61. The base of a right prism is an isosceles trapezoid whose lateral side is 26 cm, and the bases are 22 cm and 42 cm; the area of its diagonal section is equal to 400 cm^2. Find the total surface area of the prism.

62. In a right triangular prism the sides of the base are 34 cm, 50 cm, and 52 cm. The area of the section passed through a lateral edge and the greatest altitude of the base is 480 cm^2. Compute the area of the lateral surface of the prism.

63. The base of an oblique prism is a regular triangle with side a; lateral edge is equal to b; one of the lateral edges makes an angle of 45° with adjacent sides. Find the area of the lateral surface of this prism.

64. In an oblique triangular prism the distances between lateral edges are equal to 10 cm, 10 cm, and 12 cm, and the lateral edge is equal to 15 cm. Compute the area of the lateral surface of the prism.

65. The base of an oblique parallelepiped is a rectangle with sides a and b. The lateral edge is equal to c and makes with the sides of the base angles of 60°. Find the area of the lateral surface of the parallelepiped.

Sec. 23.3. The Pyramid. A Frustum of a Pyramid

66. Given the side of the base a and lateral edge b, find the altitude of a regular pyramid; (1) triangular; (2) quadrangular; (3) hexagonal.

67. Given the side of the base a and altitude h, find the slant height of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal.

68. Given the side of the base a and altitude h, find the lateral edge of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal.

69. Prove that nonintersecting edges of a regular triangular pyramid (tetrahedron) are mutually perpendicular.

70. In a regular hexagonal pyramid the dihedral angle at the side of the base is equal to 60°, this side being equal to a. Find the altitude and slant height of the pyramid.

71. In a regular triangular pyramid a lateral edge is equal to m and makes an angle α with the plane of the base. Compute the side of the base of the pyramid.

72. The altitude of a regular quadrangular pyramid is equal to h; the lateral faces of the pyramid are inclined to the plane of the base at an angle α. Find the length of the lateral edge of the pyramid.

73. In a regular quadrangular pyramid the angle between the altitude and lateral edge is equal to α; the side of the base is a. Compute the length of the lateral edge of the pyramid.

74. In a regular triangular pyramid the angle between the altitude and lateral edge is equal to α; the side of the base of the pyramid is equal to a. Compute the altitude of the pyramid.

75. Given the side of the base a and lateral edge m of a regular triangular pyramid, compute the area of the section passed through a lateral edge and the altitude of the pyramid.

76. In a regular triangular pyramid $SABC$ each lateral edge is equal to m and is inclined to the plane of the base at an angle α. Through the midpoints of the edges AB and BC a plane is passed parallel to the edge SB. Compute the area of the section thus obtained.

77. Find the area of the diagonal section of a regular quadrangular pyramid if the side of the base is equal to a, and a lateral edge makes an angle α with the plane of the base.

78. Find the area of the section passed in a regular quadrangular pyramid through the diagonal of the base parallel to a lateral edge. The side of the base of the pyramid is equal to a, and the lateral edge is inclined to the plane of the base at an angle α.

79. The dihedral angle at the base of a regular triangular pyramid is equal to φ. Find the plane angle at the vertex of the pyramid.

80. In a regular triangular pyramid the side of the base is equal to 9 cm, and the lateral edge to 18 cm. Find the area of the section passed through the midpoints of two adjacent lateral edges perpendicular to the base of the pyramid.

81. Prove that if the lateral edges (or the angles of inclination of the lateral edges) are equal, then the vertex of the pyramid is projected into the centre of the circle circumscribed about the base.

82. Prove that the plane passing through the altitude of a regular pyramid and slant height of a lateral face is perpendicular to the plane of the lateral face.

83. Prove that if in a regular triangular pyramid the altitude is equal to the side of the base, then the lateral edges make an angle of 60° with the plane of the base.

84. The base of a pyramid is a rectangle with sides 12 cm and 16 cm; each lateral edge of the pyramid is equal to 26 cm. Find the altitude of the pyramid.

85. The base of a pyramid is a triangle with sides 20 cm, 21 cm, and 29 cm. The lateral faces of the pyramid make angles of 45° with the plane of the base. Find the altitude of the pyramid.

86. The base of a pyramid is a right triangle. Two lateral faces passing through one of the legs and hypotenuse of the triangle are perpendicular to the plane of the base. Prove that all the lateral faces of this pyramid are right triangles.

87. The base of a pyramid is a parallelogram whose sides are equal

to 3 cm and 7 cm, and one of the diagonals to 6 cm. The altitude of the pyramid is equal to 4 cm and passes through the point of intersection of the diagonals of the base. Find the lateral edges of the pyramid.

88. The base of a pyramid is a rhombus whose side and one of the diagonals are equal to 4 cm. The altitude of the pyramid is equal to the longer diagonal of the base and passes through the vertex of the acute angle of the rhombus. Compute the area of the section passing through the shorter diagonal of the base perpendicular to the greater lateral edge of the pyramid.

89. Three successive angles of the base of a quadrangular pyramid are to one another as 2 : 3 : 4; the lateral edges of the pyramid make equal angles with the plane of the base. Find the plane angles at the base.

90. The base of a pyramid is an isosceles triangle whose base is equal to 12 cm and the lateral side to 10 cm. The lateral faces make with the base equal dihedral angles 45° in size. Compute the altitude of this pyramid.

91. Through the midpoint of the altitude of a pyramid a section is passed parallel to the base. Find the area of the section if the area of the base is equal to S.

92. In what ratio does a section parallel to the base divide the altitude of a pyramid if the area of the section is equal to: (1) 1/2 the area of the base; (2) 1/4 the area of the base; (3) m/n the area of the base?

93. The altitude of a pyramid is divided into four equal parts, and through the division points planes are passed parallel to the base. The area of the base is equal to 400 cm^2. Compute the areas of the sections thus obtained.

94. The altitude of a pyramid is equal to 36 cm; the area of the base amounts to 400 cm^2. At what distance from the base is the section parallel to the base and having an area of 100 cm^2 situated?

95. In a pyramid, a section parallel to the base divides the altitude in the ratio 2 : 5 (reckoning from the vertex of the pyramid); the area of the section is less than that of the base of the pyramid by 189 cm^2. Find the area of the section.

96. Compute the altitude of the frustum of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal if the sides of the lower and upper bases are respectively equal to a and b, and the lateral edge is equal to c.

97. How many diagonals is it possible to draw in the frustum of an n-gonal pyramid?

98. In a frustum of a regular quadrangular pyramid the sides of the greater and lesser bases are respectively equal to a and b, and the lateral edges make an angle of 60° with the base. Find the altitude of the frustum.

99. The sides of the bases in the frustum of a regular quadrangular

pyramid are equal to 10 dm and 2 dm, its altitude being equal to 2 dm. Find the lateral edge of the frustum.

100. In the frustum of a regular quadrangular pyramid the altitude is equal to 36 cm, the slant height to 45 cm, and the sides of the bases are to each other as 1 : 4. Find these sides.

101. In the frustum of a regular quadrangular pyramid the altitude is equal to 7 cm, and the sides of the bases are equal to 3 cm and 5 cm. Find the diagonal of this frustum.

102. Prove that if B_1 and B_2 are the areas of the bases of a truncated pyramid, and M is the area of its midsection, then $\sqrt{M} = 0.5\,(\sqrt{B_1} + \sqrt{B_2})$.

103. In the frustum of a regular triangular pyramid the sides of the bases are equal to a and b $(a > b)$, and the lateral edges make an angle of 45° with the base. Compute the area of the section passed through a lateral edge and the altitude of the base.

Sec. 23.4. Surface Areas of Pyramids and Frustums of Pyramids

The lateral area of a regular pyramid is equal to half the product of the perimeter of the base and the slant height:

$$S = \frac{1}{2} Ph. \tag{23.2}$$

The lateral area of a frustum of a regular pyramid is equal to half the product of the slant height and the sum of the perimeters of the bases

$$S = \frac{1}{2} (P_1 + P_2)h \tag{23.3}$$

104. Given the side of the base a and altitude h, compute the total area of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal.

105. Compute the lateral area of a regular triangular pyramid if its height is equal to 9 cm, and the slant height is 18 cm.

106. In a regular quadrangular pyramid the lateral area is equal to 240 cm^2, and the total surface area is 384 cm^2. Compute the side of the base and the altitude of the pyramid.

107. The side of the base of a regular triangular pyramid is equal to a, and the lateral edges make an angle of 30° with the plane of the base. Compute the total surface area of the pyramid.

108. Given the side of the base a, compute the lateral area of a regular quadrangular pyramid in which the area of the diagonal section is equal to the area of the base.

109. Compute the side of the base of a regular quadrangular pyramid, given its height h and the lateral area M.

110. In a regular triangular pyramid the slant height equal to 6 cm

makes an angle of 60° with the plane of the base. Compute the total surface area of the pyramid.

111. The areas of the base and diagonal section of a regular quadrangular pyramid are equal to S square units each. Find the area of the lateral surface of the pyramid.

112. The centre of the upper base of a cube and the midpoints of the sides of its lower base serve as vertices of the pyramid inscribed in this cube. Compute its lateral area if the edge of the cube is equal to a.

113. In a regular hexagonal pyramid the slant height is equal to 15 cm, and the altitude is 12 cm. Compute the total surface area of the pyramid.

114. Compute the lateral area of a regular triangular pyramid, given its edge b and height h.

115. In a regular hexagonal pyramid the slant height is equal to k, and the apothem of the base is equal to r. Compute the total surface area of the pyramid.

116. The base of a pyramid is a square 16 cm on a side, and two of its lateral faces are perpendicular to the plane of the base. Compute the total surface area of the pyramid if its altitude is equal to 12 cm.

117. The base of a pyramid is a right triangle whose legs are equal to 3 cm and 4 cm. Each lateral face is inclined to the plane of the base at an angle of 60°. Find the total surface area of the pyramid.

118. The base of a pyramid is a rhombus with diagonals equal to 6 m and 8 m. The altitude of the pyramid is equal to 1 m. Compute the total area of this pyramid if all the dihedral angles at the base are equal.

119. The base of a pyramid is a triangle with sides equal to 6 cm, 10 cm, and 14 cm. The planes of the lateral faces are inclined to the base at an angle of 60°. Compute the total surface area of the pyramid.

120. Given the sides of the bases a and b ($a > b$) and altitude h, find the lateral area of the frustum of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal; (4) n-gonal.

121. In the frustum of a regular quadrangular pyramid the sides of the bases are equal to 24 cm and 8 cm, and the height is 15 cm. Find the total surface area.

Sec. 23.5. Miscellaneous Problems

122. In a rectangular parallelepiped the sides of the bases are to each other as 7 : 24, and the area of the diagonal section is equal to 50 cm^2. Compute the lateral area of the parallelepiped.

123. In a right triangular prism the sides of the base are equal to 10 cm, 17 cm, and 21 cm. The area of the section passed through a

lateral edge and the smallest altitude equals 72 cm^2. Compute the lateral area of the prism.

124. In a right parallelepiped the sides of the base are equal to 17 cm and 28 cm; one of the diagonals of the base is equal to 25 cm; the sum of the areas of the diagonal sections is to the area of the base as 16 : 15. Compute the areas of the diagonal sections.

125. In a right triangular prism the sides of the base are to one another as 9 : 10 : 17, and one of the lateral edges is equal to 16 cm. The total surface area of this prism is 1440 cm^2. Find the area of its lateral surface.

126. The base of a right prism is a rhombus with side a and angle 60°. The section through the greater diagonal of one base and the vertex of the obtuse angle of the other base is a right triangle. Find the total surface area of the prism.

127. The diagonal of a regular quadrangular prism is equal to 9 cm, and the area of its total surface is 144 cm^2. Find the area of its lateral surface.

128. The ratios of the sides of the base of a right triangular prism are 3 : 4 : 5. One of the lateral edges is equal to 10 cm, and the total area is 672 cm^2. Compute the area of the lateral surface of the prism.

129. The base of a right prism is an isosceles trapezoid. The area of the diagonal section and the areas of the parallel lateral faces are respectively equal to 320 cm^2, 176 cm^2, and 336 cm^2. Find the area of the lateral surface of the prism.

130. In a right triangular prism the sides of the base are to one another as 17 : 15 : 8, and a lateral edge is equal to 16 cm. The total surface area of this prism is 1760 cm^2. Find the sides of its base.

131. The base of a prism is a regular triangle circumscribed about a circle of radius r, while the lateral faces are squares. Compute the total surface area of the prism.

132. The base of a prism is a regular triangle with side equal to 6 cm; the lateral edges of the prism are equal to 5 cm each. One of the vertices of the upper base is projected into the centre of the lower base. Compute the area of the lateral surface of the prism.

133. The base of a parallelepiped is a square with side a. The altitude of the parallelepiped is also equal to a. The projection of one of the lateral edges on the plane of the lower base coincides with one-half of its diagonal. Compute the area of the lateral surface of the parallelepiped.

134. Prove that a quadrangular prism whose diagonal sections are perpendicular to the plane of the base is a right prism.

135. Prove that if all the edges of a quadrangular pyramid are equal, then such pyramid is regular.

136. The base of a pyramid is a right triangle. The lateral edges are equal. Prove that the lateral face passing through the hypotenuse is perpendicular to the plane of the base.

137. The base of a pyramid is an isosceles trapezoid. The lateral

faces of the pyramid are equally inclined to the base. Prove that the lateral side of the trapezoid is equal to its median.

138. The base of a pyramid is a square with side a. Two lateral faces having a common edge are perpendicular to the base and two others are inclined to the base at an angle of 60°. Compute the total surface area of the pyramid.

QUIZ No. 1

Form I

(1) The base of a right parallelepiped is a parallelogram with sides 3 cm and 5 cm, the angle between them being equal to 60°; the area of the greater diagonal section is 63 cm^2. Compute the total area of the parallelepiped.

(2) In a frustum of a regular quadrangular pyramid the areas of the bases are 25 cm^2 and 9 cm^2, and the lateral edges make an angle of 45° with the plane of the lower base. Compute the lateral area of this pyramid.

Form II

(1) The base of a right prism is an isosceles trapezoid whose bases are 11 cm and 21 cm, and the lateral side is 13 cm; the area of the diagonal section is 180 cm^2. Compute the total area of the prism.

(2) The base of a pyramid is a scalene triangle with sides equal to 6 cm, 10 cm, and 14 cm. Each dihedral angle at the base is equal to 30°. Compute the lateral area of the pyramid.

Chapter 24
SOLIDS OF REVOLUTION

Sec. 24.1. The Cylinder

1. The radius of the base of a cylinder is 3 cm, the altitude 8 cm. Find the length of the diagonal of the axial section and the acute angle at which it is inclined to the plane of the base.

2. The diagonal of the axial section of a cylinder is equal to 26 cm, the altitude of the cylinder is 24 cm. Find the area of its base.

3. The radius of the base of a cylinder is 13 cm, the altitude of the cylinder is equal to 20 cm. Find the area of the section passed parallel to the axis of the cylinder at a distance of 5 cm from it.

4. In a cylinder, a plane is passed parallel to its axis cutting off the circle of the base an arc of 60°. The altitude of the cylinder is equal to 15 cm. The distance of the cutting plane from the axis of the cylinder is 3 cm. Compute the area of the section.

5. The diagonal of the axial section of an equilateral cylinder (whose axial section is a square) is equal to a. Find the area of the total surface of the hexagonal prism inscribed in this cylinder.

6. The area of the base of a cylinder is to the area of the axial section as $\pi : 2$. Find the acute angle between the diagonals of the axial section.

7. All the vertices of an isosceles triangle, whose base is equal to 12 cm, and the altitude is 4 cm, lie on a cylindrical surface whose axis is perpendicular to the base of the triangle and makes an angle of 30° with its plane. Find the radius of the cylindrical surface.

8. All the vertices of a square 1 on a side lie on the cylindrical surface whose axis is perpendicular to a side of the square and makes an angle α with its plane. Compute the radius of the cylindrical surface.

9. Circumscribed about a cylinder of radius r, whose axial section is a square, is a regular triangular prism. Compute the area of its lateral surface.

10. The base of a right prism is a right triangle with legs 6 cm and 8 cm, and a lateral edge is 10 cm. Compute the areas of the axial sections of the cylinders inscribed in, and circumscribed about, the prism.

11. In a regular triangular prism each lateral edge is equal to a; the line segment joining the midpoint of the lateral edge to the centre of the base makes an angle α with the plane of the base. Compute the area of the axial section of the cylinder inscribed in the prism.

Sec. 24.2. The Cone. A Frustum of a Cone

12. The radius of the base of a cone is 5 cm, the altitude of the cone is equal to 12 cm. Find the area of the axial section, the length of the generatrix, and the angle of its inclination to the plane of the base.

13. The area of the axial section of a cone is equal to 48 cm^2. Its generatrix makes an angle α with the plane of the base. Find the area of the base of the cone.

14. The radius of the base of a cone is 6 cm, its altitude is equal to 12 cm. Find the area of the section passed parallel to the axis of the cone at a distance of 2 cm from it.

15. A plane is passed in a cone parallel to its axis and cutting off the circle of the base an arc of 120°. The altitude of the cone is equal to 12 cm, the distance of the cutting plane from the axis of the cone is 3 cm. Compute the area of the section.

16. The radius of the base of a cone is R. Compute the area of the parallel section dividing the altitude of the cone in the ratio $m : n$ (from vertex to base).

17. In an equilateral cone (whose axial section is a regular triangle) the radius of the base is equal to R. Find the area of the section passed through two elements the angle between which is equal to 60°.

18. The altitude of a cone is H. The angle between the altitude and generatrix is equal to 30°. Compute the area of the section passed through two elements the angle between which is equal to 60°.

19. The radius of the base of a cone is R, the altitude of the cone is H. Compute the area of the section passed through two elements if it cuts off an arc of 90° on the circle of the base.

20. Through the vertex of a cone, at an angle of 60° to the plane of the base a plane is passed cutting off an arc of 90°. The altitude of the cone is equal to H. Compute the area of the section.

21. Given in a cone are the radius of the base R and the altitude H. Find the edge of the inscribed cube.

22. Given in a cone are the radius of the base R and the altitude H. Inscribed in it is a regular triangular prism whose lateral faces are squares. Compute the edge of this prism.

23. Prove that the angle at the vertex of the axial section of a cone is right, acute, or obtuse according as the altitude of the cone is equal to, greater than, or less than the radius of the base.

24. In a regular triangular pyramid the length of the side of the base is a, the dihedral angle at the base is α. Find the area of an axial section of the inscribed cone.

25. The lateral surface of a cone with the altitude H and radius of the base R is developed on a plane. Compute the angle of the sector thus obtained.

26. The directrix of a cone is L, the radius of the base is R. Com-

pute the angle α in the development of the lateral surface of the cone (consider separately the case of an equilateral cone).

27. A semicircle is rolled up into a conical surface. Find the angle between the directrix and altitude of the cone.

28. A sector whose radius is L and the angle is α is rolled up into a conical surface. Find the radius of the base of the cone.

29. The radii of the bases of a frustum of a cone are R and r, and the altitude is equal to H. Find the directrix.

30. The radii of the bases of a frustum of a cone are R and r; the directrix is inclined to the base at an angle α. Find the altitude.

31. The radii of the bases of a frustum of a cone are equal to 18 cm and 30 cm; the directrix is equal to 20 cm. Find the distance from the centre of the smaller base to the circumference of the greater base.

32. The radii of the bases of a frustum of a cone are R and r; the directrix is equal to l. Find the area of the axial section.

33. The areas of the bases of a frustum of a cone are M_1 and M_2. Find the area of the midsection parallel to the bases.

34. The areas of the bases of a frustum of a cone are equal to 32 cm^2 and 2 cm^2. The altitude is divided into three equal parts, and through the division points planes are passed parallel to the bases. Find the areas of the sections thus obtained.

35. The radii of the bases of a frustum of a cone are equal to 10 cm and 6 cm. The altitude is divided into four equal parts, and through the division points planes are passed parallel to the bases. Find the areas of the sections thus obtained.

Sec. 24.3. The Sphere and the Ball

The equation of a sphere with centre at the point O (a, b, c) and radius R has the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2.$$

36. Given the sphere $x^2 + y^2 + z^2 = 450$. Find the coordinates of the points at which the sphere is cut by the straight line passing through the origin and the point A (4, 5, 3).

○ The equation of the straight line passing through the origin and the point A (4, 5, 3) has the form

$$(x - 4)/4 = (y - 5)/5 = (z - 3)/3.$$

Let us write these equations in parametric form:

$$(x - 4)/4 = t; \quad (y - 5)/5 = t; \quad (z - 3)/3 = t,$$

whence

$$x = 4 + 4t; \quad y = 5 + 5t; \quad z = 3 + 3t. \qquad (*)$$

Substituting the values of x, y, and z into the equation of the sphere, we get

$$((4 + 4t)^2 + (5 + 5t)^2 + (3 + 3t)^2 = 450)$$
$$\Leftrightarrow (t^2 + 2t - 8 = 0) \Leftrightarrow (t_1 = -4;\ t_2 = 2).$$

Substituting now the found values of t into Relationship (*), we get two points of the intersection of the sphere and the straight line: (—12, —15, —9) and (12, 15, 9). ●

37. Find the distance from the point A (6, —3, —2) to the sphere $x^2 + y^2 + z^2 = 9$.

○ Let us find the distance from the origin to the point A (6, —3, —2); we have $|OA| = \sqrt{6^2 + (-3)^2 + (-2)^2} = 7$. Since the radius of the sphere is equal to 3, the distance from the point A to the sphere is equal to $7 - 3 = 4$. ●

38. Find the coordinates of the points of intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the coordinate axes.

39. Write the equation of the sphere with centre S and radius R if: (1) S (2, 3, 4), $R = 5$; (2) S (—3, 0, 4), $R = \sqrt{2}$.

40. Given the points A (—1, 3, 2), B (0, 3, 1), C (2, —2, 0), D (—4, 2, 2). Which of the points belong to the sphere with centre S (—2, 1, 0) and radius 3?

41. Find the centre and radius of the sphere: (1) $x^2 + y^2 + z^2 = 16$; (2) $(x - 1)^2 + (y + 3)^2 + (z - 5)^2 = 36$; (3) $x^2 - 6x + y^2 + 8y + z^2 - 4z + 4 = 0$; (4) $x^2 + 10x + y^2 + 2y + z^2 + 6z - 1 = 0$.

42. A sphere passes through the point A (—3, 4, —2), and its centre lies at the origin. Write the equation of the sphere.

43. A sphere is centred at the point C (2, —1, 3) and passes through the origin. Find the equation of the sphere.

44. The point A (4, —2, 3) lies on a sphere with centre C (2, —3, —1). Write the equation of the sphere.

45. Given the sphere $x^2 + y^2 + z^2 = 200$. Find the coordinates of the points of intersection of the sphere and the straight line passing through the origin and the point A (5, —3, 4).

46. Find the distance: (1) from the point A (1, —2, 2) to the sphere $x^2 + y^2 + z^2 = 16$; (2) from the point A (2, 4, 3) to the sphere $(x + 1)^2 + (y + 2)^2 + (z - 1)^2 = 4$.

47. The radius of a sphere is equal to 70 cm. A point is found on a tangent plane at a distance of 24 cm from the point of tangency. Find its shortest distance from the surface of the sphere.

48. The sphere $x^2 + y^2 + z^2 = 9$ is cut by a plane. Find the coordinates of the centre O and the radius r of the section if the equation of this plane is known: (1) $z = 1$; (2) $y = 2$.

49. What figure is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane: (1) $x = 2$; (2) $x - y = 1$?

50. A sphere whose radius is R is cut by a plane at a distance a from the centre. Compute the area of the section.

51. Compute the ratio of the area of the section passed at a distance m from the centre of a sphere to the area of the great circle. The radius of the sphere is equal to R.

52. The radius of a sphere is R. Through the end point of the radius a plane is passed at an angle α to it. Compute the area of the section.

53. The radius of a sphere is R. A point and a circle are given on its surface. The point lies at a distance a from all the points of the circle (on a straight line). Find the radius of this circle.

Sec. 24.4. Inscribed and Circumscribed Spheres

A polyhedron is said to be *inscribed in a sphere* (a ball) if all the vertices of the polygon belong to the sphere.

A sphere is said to be *inscribed in a polyhedron* if all the faces of the polyhedron touch the sphere.

A sphere is said to be *circumscribed about a cylinder* if the sections of the sphere serve as the bases of the cylinder. In this case the cylinder is said to be *inscribed in the sphere.*

A sphere is said to be *inscribed in a cylinder* if the bases of the cylinder and all of its elements touch the sphere. In this case the cylinder is said to be *circumscribed about the sphere.*

A sphere can be inscribed in a cylinder only if the latter is equilateral (that is, its axial section is a square).

A sphere is said to be *circumscribed about a cone* if its base is a section of the sphere, and the vertex of the cone lies on the sphere. In this case, the cone is said to be *inscribed in the sphere.*

A sphere is said to be *inscribed in a cone* if its base and all of its elements touch the sphere. In this case, the cone is said to be *circumscribed about the sphere.*

A sphere can be inscribed in any cone.

A sphere can be inscribed in a frustum of a cone only if the length of the generatrix is equal to the sum of the lengths of the radii of the base.

54. The edge of a cube is a. Find the radii of the spheres inscribed in the cube and circumscribed about it.

55. The dimensions of a rectangular parallelepiped are 4 cm, 6 cm, and 12 cm. Compute the length of the radius of the circumscribed sphere.

56. The altitude of a regular hexagonal prism is equal to 8 cm, and the diagonal of a lateral face is 13 cm. Find the length of the radius of the circumscribed ball.

57. A regular hexagonal prism is circumscribed about a ball of radius R. Find the area of its total surface.

58. In a regular quadrangular pyramid the altitude is equal to h, and each of the lateral edges is b. Find the length of the radius of the circumscribed sphere.

59. The edge of a regular tetrahedron is equal to a. Compute the radii of the circumscribed and inscribed balls.

60. In a regular pyramid the lateral edges, equal to b (each), make an angle α with the base. Find the length of the radius of the circumscribed ball.

61. Inscribed in a ball of radius R is a regular quadrangular pyramid whose lateral edge makes an angle α with the base. Compute the length of the lateral edge.

62. Find the radius of the ball inscribed in a regular quadrangular pyramid whose altitude is equal to h, and the dihedral angle at the base is α.

63. The altitude of a regular triangular pyramid is h, and the lateral edges are mutually perpendicular. Compute the radius of the circumscribed sphere.

64. In a frustum of a regular triangular pyramid the altitude is equal to 17 cm, and the radii of the circles circumscribed about the bases are equal to 5 cm, and 12 cm. Find the length of the radius of the circumscribed sphere.

65. In a frustum of a regular hexagonal pyramid the sides of the bases are equal to 3 cm and 4 cm, the altitude is 7 cm. Compute the length of the radius of the circumscribed sphere.

66. A sphere is circumscribed about a cylinder with radius of 2 cm and generatrix equal to 3 cm. Find its radius.

67. A sphere is inscribed in an equilateral cylinder with radius of 5 cm. Compute the area of the section perpendicular to the axis of the cylinder passed at a distance of 3 cm from the centre of the sphere.

68. The altitude of a cone is equal to h, and its generatrix is l. Find the radius of the inscribed sphere.

69. A sphere is inscribed in a cone in which the radius of the base is r and the generatrix is l. Compute the length of the line segment along which the sphere touches the lateral surface of the cone.

70. The altitude of a cone is equal to 20 cm, and the generatrix to 25 cm. Compute the radius of the inscribed hemisphere whose base lies on the base of the cone.

71. In a cone, the generatrix, whose length is l, makes an angle α with the base. Compute the length of the radius of: (1) the circumscribed sphere; (2) the inscribed sphere.

72. A cone with altitude h is inscribed in a sphere of radius R. Find the area of the axial section of the cone.

73. The area of the axial section of a cone is equal to S; the generatrix makes an angle α with the base of the cone. Find the radius of the sphere inscribed in the cone.

74. About a sphere of radius R a frustum of a cone is circumscribed whose generatrix makes an angle α with the base. Compute the area of the axial section of the cone.

75. In a frustum of a cone, the radii of the bases are equal to 3 cm and 4 cm; the altitude of the frustum is 7 cm. Compute the radius of the circumscribed sphere.

76. The radii of the bases of the frustum of a cone circumscribed about a sphere are R and r. Find the radius of the sphere.

77. The radius of a spherical sector is R, and the arc in the axial section is equal to 60°. Compute the radius of the inscribed sphere and the length of the circle along which they touch each other.

78. Compute the radius of the sphere inscribed in a spherical sector if the central angle of the sector is equal to α and the radius of this sector is R.

79. The radius of a spherical segment is r, and the arc in the axial section is equal to α. Compute the radius of the sphere inscribed in the segment.

80. Inscribed in a spherical sector are two mutually touching balls whose radii are 10 cm and 30 cm. Compute the radius of the given sector.

Sec. 24.5. Miscellaneous Problems

81. In a cylinder a plane is passed parallel to the axis and cutting off an arc of 120° from the circle of the base. Find the area of the section if the axis is 10 cm long, and its distance from the cutting plane is equal to 2 cm.

82. The edge of a cube is a; the diagonal of the cube serves as the axis of a cylindrical surface which touches: (1) an edge of the cube; (2) a diagonal of a face of the cube. Find the radius of the cylindrical surface.

83. The altitude of a cylinder is equal to 15 cm, and the radius of its base to 10 cm. The end points of a given line segment having length of $3\sqrt{41}$ cm lie on the circles of both bases. Find the shortest distance from this segment to the axis of the cylinder.

84. The radius of the base of a cone is R, and the generatrix is inclined at an angle α to the plane of the base. In this cone, a plane is passed through its vertex at an angle φ to the altitude. Compute the area of the section thus obtained.

85. The base of a right prism is an isosceles trapezoid with acute angle α. A cone is inscribed in the prism so that its base is inscribed in the base of the prism, and the vertex of the cone is found in the plane of the other base. Find the total surface area of the prism, knowing that the diameter of the base of the cone is d, and the angle of inclination of the generatrix of the cone to the plane of its base is φ.

86. In a cone a section is passed through its vertex at an angle of 30° to the altitude of the cone. Compute the area of the section if the altitude of the cone is equal to $3\sqrt{3}$ cm, and the radius of the base is equal to 5 cm.

87. The base of a pyramid is a right triangle whose legs are equal to 80 cm and 60 cm. A cone is inscribed in this pyramid. Find the

angle between the generatrix and the plane of the base of the cone if its altitude is equal to 10 cm.

88. The diagonals of the axial section of a frustum of a cone are mutually perpendicular; the altitude is H. Find the area of the section of the frustum passed through the midpoint of the altitude parallel to the bases.

89. The diagonals of a rhombus are equal to 30 cm and 40 cm. A sphere touches all the sides of the rhombus. Find the distance from the centre of the sphere to the plane of the rhombus if the radius of the sphere is equal to 13 cm.

90. The radius of a ball is equal to 6 cm. Through the end point of the radius a plane is passed at an angle of 30° to it. Compute the area of the section thus obtained.

91. On a sphere, whose radius is equal to 10 cm, a rhombus is placed so that each of its sides, equal to 12.5 cm, touches the sphere. The plane of the rhombus is 8 cm distant from the centre of the sphere. Compute the area of the rhombus.

92. The radii of two balls are equal to 16 cm and 20 cm, and the distance between their centres is equal to 25 cm. Find the length of the circle along which their surfaces intersect.

93. The radius of a spherical segment is r, the arc in the axial section is α. Compute: (1) the length of its base; (2) the altitude.

QUIZ No. 1

Form I

(1) Inscribed in a cone is an equilateral cylinder. Find the altitude of the cylinder if the altitude of the cone is H and the angle at the vertex of the axial section is α.

(2) Circumscribed about a ball is a right prism whose base is a rhombus with diagonals 6 cm and 8 cm. Find the total surface area of the prism.

Form II

(1) The section of a cylinder by a plane parallel to its altitude is a square. The cutting plane is located at a distance d from the axis of the cylinder and cuts off an arc α from the circle of the base. Compute the area of the section.

(2) The side of the base of a regular quadrangular pyramid is a, and the dihedral angle at the base is α. Compute the radius of the sphere inscribed in the pyramid.

Chapter 25

VOLUMES OF POLYHEDRA AND SOLIDS OF REVOLUTION

Sec. 25.1. Volume of a Parallelepiped and a Prism

The volume of a rectangular parallelepiped is equal to the product of its three dimensions:

$$V = abc. \tag{25.1}$$

The volume of a prism is equal to the product of the area of its base and the altitude:

$$V = S_{\text{base}} H. \tag{25.2}$$

1. Compute the volume of a cube, given: (1) its diagonal l; (2) its surface area S.

2. Compute the volume of a cube if the area of its diagonal section is equal to M.

3. Find the edge of the cube whose volume is equal to the sum of the volumes of the cubes with edges m and n.

4. The dimensions of a rectangular parallelepiped are equal to 6 cm, 16 cm, and 18 cm. Find the edge of an equivalent cube.

5. The areas of faces of a rectangular parallelepiped are equal to S_1, S_2, and S_3. Find its volume.

6. The dimensions of a rectangular parallelepiped are to one another as 2 : 7 : 26; the diagonal of the parallelepiped is equal to 81 cm. Find the volume of the parallelepiped.

7. The diagonals of the faces of a rectangular parallelepiped are equal to 7 cm, 8 cm, and 9 cm. Find the volume of the parallelepiped.

8. The diagonal of a rectangular parallelepiped is equal to l and makes an angle α with one face and an angle β with the other. Compute the volume of the parallelepiped.

9. The base of a right parallelepiped is a parallelogram with sides 8 cm and 32 cm and acute angle $\alpha = 60°$. The greater diagonal of the parallelepiped is equal to 40 cm. Compute the volume of the parallelepiped.

10. The sides of the base of a right parallelepiped are equal to 25 cm and 39 cm, and the areas of its diagonal sections are equal to 204 cm^2 and 336 cm^2. Find the volume of the parallelepiped.

11. The sides of the base of a right parallelepiped are equal to 17 cm and 25 cm, one of the diagonals of the base is equal to 26 cm. The shorter diagonal of the parallelepiped makes an angle of 30° with the plane of the base. Compute the volume of the parallelepiped.

12. The base of a right parallelepiped is a rhombus the ratio of whose diagonals is 5 : 16. The diagonals of the parallelepiped are equal to 26 cm and 40 cm. Find the volume of the parallelepiped.

13. Compute the volume of a right parallelepiped whose base is a rhombus with side a and angle α, and the diagonal of a lateral face makes an angle β with the lateral edge.

14. The base of a right parallelepiped is a rhombus with side a and acute angle α. The greater diagonal of the parallelepiped makes an angle β with the plane of its base. Compute the volume of the parallelepiped.

15. The side of the base of a regular prism is a, each lateral edge is b. Find the volume of the prism: (1) triangular; (2) quadrangular; (3) hexagonal.

16. The side of the base of a regular triangular prism is a; the lateral area is equivalent to the sum of the areas of the bases. Compute the volume of this prism.

17. The base of a right prism is a trapezoid whose area is 306 cm^2. The areas of the parallel lateral faces are equal to 40 cm^2 and 30 cm^2, and the areas of two other lateral faces are equal to 75 cm^2 and 205 cm^2. Find the volume of the prism.

18. In a right quadrangular prism the area of the base is M, the areas of the diagonal sections are equal to P and Q, the dihedral angle between them being equal to α. Find the volume of the prism.

19. The base of an oblique parallelepiped is a rectangle with sides 4 cm and 6 cm, each lateral edge is equal to 2 cm and makes an angle of 60° with each of the adjacent sides of the base. Compute the volume of the parallelepiped.

20. Each edge of an oblique triangular prism is equal to a, one of the lateral edges makes an angle α with each adjacent side of the base. Compute the volume of the prism.

21. Construct the section of an oblique parallelepiped by a plane which passes through a given point of the side of the base and divides the parallelepiped into two prisms having equal volumes.

22. The base of an oblique triangular prism is an isosceles triangle whose altitude is h, and the base angle is φ. Each lateral edge, equal to a, is inclined to the plane of the base at an angle β. Find the volume of the prism.

23. The base of an oblique parallelepiped is a parallelogram with sides a and b and angle φ between them. Each lateral edge is equal to l and is inclined to the plane of the base at an angle α. Find the volume of the parallelepiped.

Sec. 25.2. Volume of a Pyramid

The volume of any pyramid is equal to one-third the product of its base and altitude:

$$V=\frac{1}{3}SH. \tag{25.3}$$

24. Given the side of the base a and lateral edge b, compute the volume of a regular pyramid: (1) triangular; (2) quadrangular; (3) hexagonal.

25. The slant height of a regular triangular pyramid is equal to k, and altitude to h. Find the volume of the pyramid.

26. The area of the base of a regular quadrangular pyramid and the area of its lateral surface are respectively equal to S and Q. Find the volume of the pyramid.

27. Compute the volume of a regular tetrahedron whose edge is a.

28. The centres of the faces of a regular tetrahedron are the vertices of a new regular tetrahedron. Find the ratio of their volumes.

29. Compute the volume of a regular quadrangular pyramid in which the area of a lateral face is S, and the plane angle at the vertex is α.

30. Find the volume of a regular triangular pyramid if the side of the base is equal to a, and the lateral faces make an angle α with the plane of the base.

31. The altitude of a regular quadrangular pyramid makes an angle α with each lateral face. Compute the volume of the pyramid if the area of the lateral surface is equal to S.

32. The base of a pyramid is a right triangle with leg a and adjacent angle of 30°. The lateral edges are inclined to the plane of the base at an angle of 60°. Find the volume of the pyramid.

33. The base of a pyramid is a rhombus with side equal to 15 cm, each face of the pyramid is inclined to the base at an angle of 45°. Find the volume of the pyramid if the area of its lateral surface is equal to 300 cm^2.

34. The base of a pyramid is an isosceles trapezoid whose parallel sides are equal to 3 cm and 5 cm, and the lateral side is equal to 7 cm. The altitude of the pyramid passes through the point of intersection of the diagonals of the base, and the greater lateral edge is equal to 10 cm. Find the volume of the pyramid.

35. The base of a pyramid is a rectangle whose area is equal to 1 m^2. Two lateral faces are perpendicular to the base, and two others are inclined to it at angles 30° and 60°. Compute the volume of the pyramid.

36. The base of a pyramid is a rectangle. Each lateral edge is equal to a and makes angles α and β with adjacent sides. Compute the volume of the pyramid.

37. The base of a pyramid is a rhombus with side a and acute angle α. The lateral faces of the pyramid are inclined to the plane of the base at an angle β. Compute the volume of the pyramid.

Sec. 25.3. Volume of a Frustum of a Pyramid

The volume of a frustum of a pyramid is computed by the formula

$$V = \frac{1}{3}(S_1 + \sqrt{S_1 S_2} + S_2)\,H, \tag{25.4}$$

where S_1 and S_2 denote the areas of the bases of a frustum of a pyramid, and H its altitude.

38. Given a lateral side l and the sides of the base a and b, find the volume of the frustum of a regular pyramid: (1) triangular, (2) quadrangular, (3) hexagonal.

39. The sides of the bases of the frustum of a regular pyramid are equal to 4 cm and 8 cm, and the diagonal is 11 cm. Compute the volume of the frustum.

40. In a frustum of a regular hexagonal pyramid, the slant height is equal to 10 cm and the altitude to 8 cm. The sum of the lengths of two sides of its upper and lower bases is equal to $8\sqrt{3}$ cm. Find the volume of the pyramid.

41. The lateral edges of the frustum of a regular triangular pyramid are inclined to the plane of the greater base at an angle α, the sides of the bases are equal to a and b $(a > b)$. Compute the volume of the pyramid.

42. The lateral edge of the frustum of a regular quadrangular pyramid is inclined to the side of the greater base at an angle φ. The sides of the bases are equal to a and b $(a > b)$. Compute the volume of the frustum.

43. The sides of one base of a frustum of a pyramid are equal to 27 cm, 29 cm, and 52 cm; the perimeter of the other base is equal to 72 cm; the altitude of the frustum is 10 cm. Compute the volume of the frustum.

44. The greater base of a frustum of a pyramid is a right triangle with legs a and b. Each lateral edge is inclined to the plane of this base at an angle α. The perimeter of the smaller base is $\frac{1}{n}$ the perimeter of the greater base. Compute the volume of the frustum.

Sec. 25.4. Investigations for Extremum in Problems on Volumes of Polyhedra

45. Among all right parallelepipeds with a given area of total surface S and a square base, find the one which has the greatest possible volume.

○ Let x be the side of the base of a parallelepiped and let y be its altitude. Then the total area $S = 2x^2 + 4xy$, whence $y = (S - 2x^2)/(4x)$. Consequently, the volume of the parallelepiped

$$V = x^2y = x^2\frac{S-2x^2}{4x} = \frac{1}{4}Sx - \frac{1}{2}x^3 \quad (0 < 2x^2 < S,\ 0 < x < \sqrt{S/2}).$$

Investigate the function for extremum with the aid of the second derivative:

$$V' = \frac{1}{4}S - \frac{3}{2}x^2,\ \frac{1}{4}S - \frac{3}{2}x^2 = 0,\quad x = \sqrt{S/6} = \sqrt{6S}/6,\ V'' = -3x,$$

$$V''(\sqrt{6S}/6) < 0.$$

The second derivative for $x = \sqrt{6S}/6$ is negative; hence, for this value of the argument the function has a maximum.

We now find $y = \dfrac{S - 2\,(\sqrt{6S}/6)^2}{4(\sqrt{6S}/6)} = \sqrt{6S}/6$, that is, the altitude of the parallelepiped and the side of its base are equal to each other. Thus, of all parallelepipeds satisfying the conditions of the problem, a cube has the greatest volume. ●

46. From a rectangular sheet of tin-plate with sides 80 cm and 50 cm it is required to make an open box of maximum volume by cutting away equal squares at the corners and bending the sheet in order to form lateral walls. What must be the length of the side of the cut-away squares (Fig. 167)?

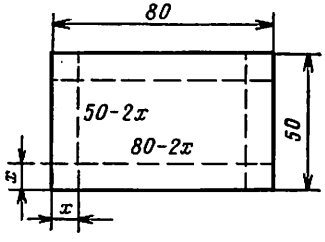


Fig. 167

47. Among all right parallelepipeds with a given volume V and a square base find the one which has the minimum total area.

48. Find the dimensions of an open (without cover) box with a square bottom having the smallest area of the total surface at a given volume V.

49. Compute the dimensions of an open box with a square bottom having the greatest volume if the total surface area of the lateral walls and bottom is equal to S.

Sec. 25.5. Volumes of Solids of Revolution

The volume of a cylinder is equal to the product of the area of its base and altitude:

$$V = \pi R^2 H. \tag{25.5}$$

The volume of a cone is equal to one-third the product of the area of its base and altitude:

$$V = \frac{1}{3}\pi R^2 H. \tag{25.6}$$

The volume of a frustum of a cone is computed by the formula

$$V = \frac{1}{3}(R_1^2 + R_1 R_2 + R_2^2)\,H, \tag{25.7}$$

where R_1 and R_2 denote the radii of the circles of its bases, and H the altitude.

The volume of a ball of radius R is computed by the formula

$$V = \frac{4}{3}\pi R^3. \tag{25.8}$$

The volume of a spherical sector is computed by the formula

$$V = \frac{2}{3}\pi R^2 H, \tag{25.9}$$

where R denotes the radius of the ball, and H the altitude of the spherical sector.

The volume of a spherical segment is computed by the formula

$$V=\frac{1}{3}\pi H^2(3R-H), \tag{25.10}$$

where R denotes the radius of the ball, and H the altitude of the spherical segment.

50. The diagonal of a rectangle makes an angle α with one of its sides. Compute the ratio of the volumes of the cylinders generated by revolving the rectangle about each of the adjacent sides.

51. The diagonal d of an axial section of a cylinder is inclined to the plane of the base at an angle α. Compute the volume of the cylinder.

52. The area of an axial section of an equilateral cylinder is equal to S. Find the volume of the cylinder.

53. A cylinder of the greatest volume is ground from a cube. What is the percentage of the material ground down.

54. A right triangle with leg a and adjacent angle α rotates about the hypotenuse. Find the volume of the solid of revolution thus obtained.

55. The development of the lateral surface of a cone is a circular sector of radius R and central angle equal to α radians. Find the volume of the cone.

56. In a right trapezoid the bases are equal to a and b $(b > a)$. Find the ratio of the volumes of the solids generated by rotating the trapezoid about the bases.

57. A rhombus with side a and acute angle α rotates about the axis drawn through the vertex of the acute angle perpendicular to the side of the rhombus. Compute the volume of the solid of revolution obtained.

58. A ball of radius r is inscribed in a cone. The radius drawn to the point of tangency makes an angle α with the altitude of the cone. Find the volume of the cone.

59. The radii of the bases of a frustum of a cone are equal to R and r, the generatrix is inclined to the base at an angle α. Find the volume of the frustum.

60. In a frustum of a cone the radii of the bases are equal to 27 cm and 11 cm; the generatrix is to the altitude as 17 : 15. Find the volume of the frustum.

61. The frustum of a cone with radii of the bases 6 cm and 9 cm and altitude 12 cm is cut by two planes parallel to the bases which divide the altitude into three equal parts. Find the volume of the midportion of the cone.

62. The bases of an isosceles trapezoid are equal to 11 cm and 21 cm; its lateral side is equal to 13 cm. Compute the volume of the solid generated by rotating this trapezoid about its axis.

63. A rhombus with side a and acute angle α rotates about the axis drawn through the vertex of the acute angle perpendicular to the side of the rhombus. Find the volume of the obtained solid of revolution.

64. In the frustum of a cone the ratio of the areas of the bases is equal to 4, and the generatrix of length l is inclined to the plane of the base at an angle α. Compute the volume of the frustum.

65. The diameters of three balls are equal to 6 cm, 8 cm, and 10 cm. Find the diameter of the ball whose volume is equal to the sum of the volumes of these three balls.

66. A ball of the greatest volume is ground from a cube. What is the percentage of the material ground down?

67. Find the ratio of the volumes of the balls inscribed in, and circumscribed about a cube.

68. Find the ratios of the volumes of a cylinder, ball, and cone if the diameters of the bases of the cylinder and cone, and their altitudes are equal to the diameter of the ball.

69. The outer radius of a hollow ball is equal to 9 cm, the walls are 3 cm thick. Find the volume enclosed between the walls.

70. A plane perpendicular to the diameter of a ball divides the diameter into two parts equal to 3 cm and 9 cm. Find the volumes of the respective parts of the ball.

71. Two equal balls are arranged so that the centre of one ball is found on the surface of the other. What is the ratio of the volumes of their common portion and the entire ball?

72. The radius of the base of a spherical segment is equal to 8 cm, the arc of the axial section measures 60°. Compute the volume of the segment.

73. The radius of the base of a spherical segment is 8 cm, its altitude is 4 cm. Compute the volume of the segment.

74. The radius of a ball is R, the angle in the axial section of a spherical sector is equal to 120°. Compute the volume of the spherical sector.

75. The radius of the base of a spherical sector is 60 cm, the radius of the ball is 75 cm. Compute the volume of the spherical sector.

76. The radius of a ball is R, the arc in the axial section of a spherical sector is α. Find the volume of the spherical sector.

77. The arc in an axial section of a spherical sector is α, the altitude of the sector is h. Find the volume of the spherical sector.

78. The radii of parallel sections of a ball are equal to 20 cm and 24 cm and the radius of the ball is 25 cm. Compute the volume of the portion enclosed between these sections (consider two cases).

79. In a ball whose radius is equal to 65 cm two parallel planes are passed on one side of the centre at distances of 19 cm and 25 cm from it. Compute the volume of the portion of the ball enclosed between these planes.

Sec. 25.6. Investigations for Extremum in Problems on Volumes of Solids of Revolution

80. Among all cylinders inscribed in a given cone with radius of the base R and altitude H find the one having the greatest volume.

○ Let r denote the radius of the base of the sought-for cylinder, and let h denote its altitude (Fig. 168). From the similarity of the

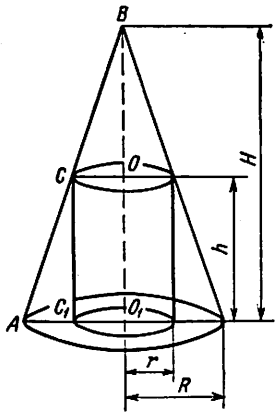


Fig. 168

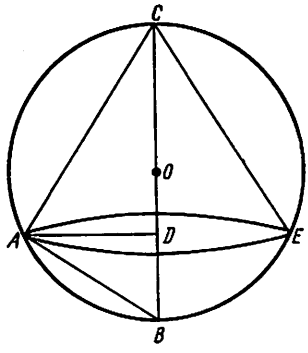


Fig. 169

triangles ABO_1 and ACC_1, we have $R/H = (R - r)/h$, whence $h = H(R - r)/R$. Substituting the value of h into the formula for the volume of a cylinder $V = \pi r^2 h$, we get

$$V = \pi r^2 \frac{H(R-r)}{R} = \frac{\pi H}{R}(Rr^2 - r^3); \quad 0 < r < R.$$

Investigate the function for extremum with the aid of the second derivative:

$$V' = \frac{\pi H}{R}(2Rr - 3r^2); \quad \frac{\pi H}{R}(2Rr - 3r^2) = 0; \quad 2Rr - 3r^2 = 0;$$

$$r(2R - 3r) = 0;$$

$$r_1 = 0, \quad r_2 = \frac{2}{3}R; \quad V'' = \frac{\pi H}{R}(2R - 6r); \quad V''(2R/3) = -2\pi H.$$

The second derivative for $r = 2R/3$ is negative, consequently, for $r = 2R/3$ the function has a maximum.

We now find the altitude of the desired cylinder:

$$h = \frac{H(R - 2R/3)}{R} = \frac{1}{3}H.$$

Thus, the cone with radius of the base $r = 2R/3$ and altitude $h = H/3$ has the greatest volume. ●

81. Among all cones inscribed in a ball of radius R, find the one having the greatest volume.

○ Let $|AD| = r$ be the radius of the base of the sought-for cone, and let $|DC| = h$ be its altitude (Fig. 169). From the triangle BAC, by the theorem on metric relations in a right triangle, we have $|AD|^2 = |BD| \cdot |DC|$ or $r^2 = (2R - h)h$. Substituting the value of r^2 into the formula for the volume of a cone $V = \pi r^2 h/3$, we get

$$V = \frac{1}{3}\pi(2R-h)\cdot h\cdot h = \frac{1}{3}\pi(2Rh^2 - h^3); \quad 0 < h < 2R.$$

Investigate this function for extremum with the aid of the second derivative:

$$V' = \frac{1}{3}\pi(4Rh - 3h^2); \quad \frac{1}{3}\pi(4Rh - 3h^2) = 0; \quad 4Rh - 3h^2 = 0;$$

$$h(4R - 3h) = 0;$$

$$h_1 = 0; \quad 4R - 3h = 0; \quad h_2 = \frac{4}{3}R; \quad V'' = \frac{1}{3}\pi(4R - 6h) = \frac{2}{3}\pi(2R - 3h);$$

$$V''(4R/3) = \frac{2}{3}\pi\left(2R - 3\cdot\frac{4}{3}R\right) = \frac{2}{3}\pi(-2\pi R) = -\frac{4\pi R}{3}.$$

The second derivative for $h = 4R/3$ is negative, consequently, for this value of the argument the function has a maximum.

Let us now find the value of r for $h = 4R/3$:

$$r^2 = \left(2R - \frac{4}{3}R\right)\cdot\frac{4}{3}R = \frac{2R}{3}\cdot\frac{4}{3}R = \frac{8}{9}R^2, \quad \text{i.e.} \quad r = \frac{2\sqrt{2}R}{3}.$$

Thus, the cone with radius of the base $2R\sqrt{2}/3$ and altitude $4R/3$ has the greatest volume. ●

82. Among all cylinders inscribed in a ball of radius R find the one having the greatest volume.

83. An open circular cylindrical chute is made from a strip of tin-plate a cm wide. At what central angle α will the volume of the chute be the greatest?

84. Among all cones with a given generatrix l find the one having the greatest volume.

85. A sector is cut away from a paper circle of radius R, and a conical funnel is glued from the remaining portion of the circle. What angle must the cut-away sector have to get the funnel of the greatest volume? Find the radius of the base and the altitude of the funnel.

Sec. 25.7. Computing Volumes of Solids of Revolution with the Aid of Definite Integrals

The volume of the solid generated by rotating about the x-axis the curvilinear trapezoid bounded by the continuous curve $y = f(x)$ $(a \leqslant x \leqslant b)$, the x-axis and the straight lines $x = a$ and $x = b$ (Fig. 170) is computed by the formula

$$V = \pi \int_a^b y^2\, dx. \tag{25.11}$$

Analogously, the volume of the solid generated by rotating about the y-axis the curvilinear trapezoid bounded by the continuous curve $x = \varphi(y)$ $(c \leqslant y \leqslant d)$, the y-axis, and the straight lines $y = c$ and $y = d$ is found by the formula

$$V = \pi \int_c^d x^2\, dy. \tag{25.12}$$

In Problems 86 through 92 we shall compute the volumes of the solids generated by rotating the regions (areas) bounded by the indicated lines:

86. $y^2 = 4x$, $y = 0$ and $x = 4$ about the x-axis.

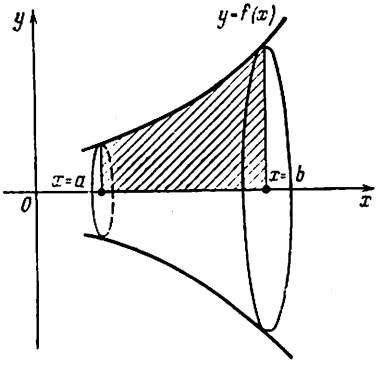


Fig. 170

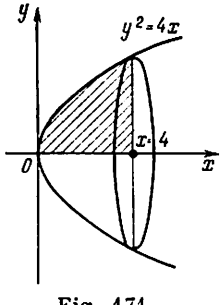


Fig. 171

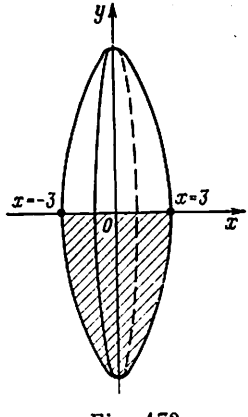


Fig. 172

○ We first carry out the construction shown in Fig. 171. The solid of revolution is a paraboloid; the limits of integration are $a = 0$ and $b = 4$. By Formula (25.11), we obtain

$$V = \pi \int_0^4 4x\, dx = 2\pi x^2 \Big|_0^4 = 32\pi \text{ (cubic units)}. \bullet$$

87. $y = x^2 - 9$ and $y = 0$ about the x-axis.

○ We first carry out the construction shown in Fig. 172. By virtue of the symmetry of the solid with respect to the y-axis, we take the limits of integration from 0 to 3, and then double the result. By For-

mula (25.11), we find

$$V/2=\pi\int_0^3 (x^2-9)^2\,dx=\pi\int_0^3 (x^4-18x^2+81)\,dx$$

$$=\pi\left[\frac{x^5}{5}-6x^3+81x\right]_0^3=129.6\pi;$$

$$V=2\cdot 129.6\pi=259.2\pi \text{ (cubic units).} \bullet$$

88. $x-2y+6=0$, $y=0$, and $x=2$ about the x-axis.

○ We carry out the construction shown in Fig. 173. The straight line $x-2y+6=0$ intersects the x-axis at the point A $(-6, 0)$,

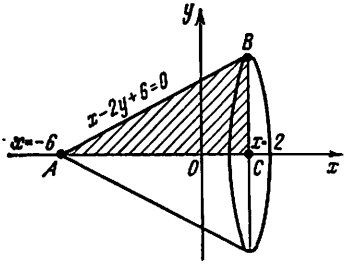


Fig. 173

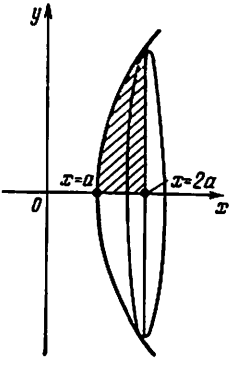


Fig. 174

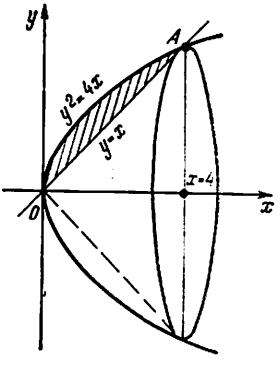


Fig. 175

the limits of integration are $a=-6$ and $b=2$. We then compute the volume of the cone generated by rotating the triangle ABC in which the side AB is expressed by the equation $y=(1/2)x+3$:

$$V=\pi\int_{-6}^{2}\left(\frac{1}{2}x+3\right)^2 dx=\pi\int_{-6}^{2}\left(\frac{1}{4}x^2+3x+9\right)dx$$

$$=\pi\left[\frac{x^3}{12}+\frac{3x^2}{2}+9x\right]_{-6}^{2}=42\frac{2}{3}\pi \text{ (cubic units).} \bullet$$

89. $x^2/a^2+y^2/b^2=1$, $y=0$ about the x-axis.

○ The solid of revolution is an ellipsoid. The given ellipse intersects the x-axis at the points $x_1=-a$ and $x_2=a$. From the equation of the ellipse we have $y^2=(b^2/a^2)(a^2-x^2)$. The ellipse is symmetric with respect to the y-axis, therefore we may take the limits of integration from 0 to a and then double the obtained result:

$$V/2=\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx=\pi\left[b^2x-\frac{b^2x^3}{3a^2}\right]_0^a=\frac{2}{3}ab^2\pi;$$

$$V=2\cdot\frac{2}{3}ab^2\pi=\frac{4}{3}ab^2\pi \text{ (cubic units).} \bullet$$

90. $x^2/a^2 - y^2/b^2 = 1$, $y = 0$, $x = a$, $x = 2a$ about the x-axis.

○ The solid of revolution is a hyperboloid. From the equation of hyperbola we have $y^2 = (b^2/a^2)(x^2 - a^2)$ (Fig. 174). Consequently

$$V = \pi \int_a^{2a} \left(\frac{b^2x^2}{a^2} - b^2\right) dx = \pi \left[\frac{b^2x^3}{3a^2} - b^2x\right]_a^{2a}$$

$$= \pi \left[\left(\frac{8ab^2}{3} - 2ab^2\right) - \left(\frac{ab^2}{3} - ab^2\right)\right]$$

$$= \pi \frac{8ab^2 - 6ab^2 - ab^2 + 3ab^2}{3} = \frac{4\pi ab^2}{3} \text{ (cubic units).} \bullet$$

91. $y^2 = 4x$ and $y = x$ about the x-axis.

○ Let us first construct the solid (Fig. 175). Solving the system $\begin{cases} y^2 = 4x \\ y = x \end{cases}$ we find the points of intersection of the parabola and the straight line: O (0,0) and A (4, 4); consequently, the limits of integration are: $a = 0$ and $b = 4$. The sought-for volume represents the difference between the volume V_1 of the paraboloid generated by rotating the curve $y^2 = 4x$ and the volume V_2 of the cone generated by rotating the straight line $y = x$:

$$V_1 = \pi \int_0^4 4x\,dx = \pi 2x^2 \Big|_0^4 = 32\pi;$$

$$V_2 = \pi \int_0^4 x^2\,dx = \pi \frac{x^3}{3}\Big|_0^4 = \frac{64\pi}{3} = 21\frac{1}{3}\pi;$$

$$V = V_1 - V_2 = 32\pi - 21\frac{1}{3}\pi = 10\frac{2}{3}\pi$$

(cubic units). ●

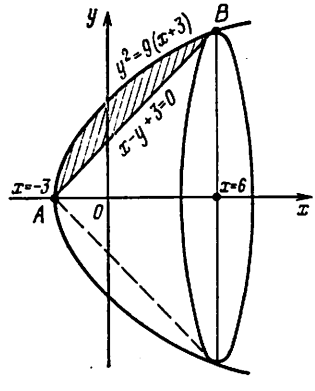


Fig. 176

92. $y^2 = 9(x + 3)$ and $x - y + 3 = 0$ about the x-axis.

○ We carry out the construction of the solid (Fig. 176). Solving the system $\begin{cases} y^2 = 9(x+3), \\ x - y + 3 = 0, \end{cases}$ we find the points of intersection of the parabola and the straight line: A (−3, 0) and B (6, 9); the limits of integration are: $a = -3$ and $b = 6$. The sought-for volume is equal to the difference between the volume V_1 of the paraboloid formed by rotating the curve $y^2 = 9(x + 3)$ and the volume V_2 of the cone

generated by rotating the straight line $y = x + 3$:

$$V_1 = \pi \int_{-3}^{6} 9\,(x+3)\,dx = 9\pi \left[\frac{x^2}{2} + 3x\right]_{-3}^{6} = 364.5\pi;$$

$$V_2 = \pi \int_{-3}^{6} (x+3)^2\,dx = \frac{\pi}{3}(x+3)^3 \Big|_{-3}^{6} = 243\pi;$$

$$V = V_1 - V_2 = 364.5\pi - 243\pi = 121.5\pi \text{ (cubic units).} \bullet$$

In Problems 93 through 101 compute the volumes of the solids generated by rotating the regions bounded by the given lines:

93. (1) $y^2 = x$, $y = 0$, $x = 1$, and $x = 2$; (2) $y^2 = 2x$, $y = 0$, $x = 2$, and $x = 4$; (3) $y^2 = 6x$, $y = 0$, $x = 1$, and $x = 3$; (4) $y^2 = 2(x+2)$, $y = 0$, and $x = 0$ (in all cases about the x-axis).

94. (1) $y = x^2 - 1$ and $y = 0$, (2) $y = 3x - x^2$ and $y = 0$, (3) $y = -x^2 - x$ and $y = 0$ (in all cases about the x-axis).

95. $x + 2y - 4 = 0$, $y = 0$, $x = 0$ about the x-axis.

96. $x^2 + y^2 = r^2$, $y = 0$ about the x-axis.

97. $x^2/a^2 + y^2/b^2 = 1$, $x = 0$ about the y-axis.

98. (1) $x^2 - y^2 = 4$, $y = 0$, $x = 2$, $x = 4$ about the x-axis; (2) $x^2/9 - y^2/4 = 1$, $y = 0$, $x = 3$, $x = 6$ about the x-axis.

99. (1) $y = \sin x$, $x = 0$, $x = \pi$ and $y = 0$ about the x-axis; (2) $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.

100. $y^2 = 9x$ and $y = 3x$ about the x-axis.

101. (1) $y^2 = 4(x+2)$ and $x - y + 2 = 0$ about the x-axis; (2) $y^2 = 4(x-2)$, $y = 0$, $x = 3$, and $x = 6$ about the x-axis; (3) $y = -x^2 + 5x$, $y = 0$, $x = 0$, and $x = 3$ about the x-axis; (4) $y^2 = 4(x+2)$ and $x - y + 2 = 0$ about the x-axis.

Sec. 25.8. Miscellaneous Problems

102. In a right triangular prism the sides of the base are equal to 25 cm, 24 cm, and 36 cm, and its total surface area is equal to 1620 cm^2. Find the volume of the prism.

103. In a right parallelepiped the sides of the base are 17 cm and 28 cm, and the greater diagonal of the base is equal to 39 cm. The smaller diagonal of the parallelepiped makes an angle of 30° with the plane of the base. Compute the volume of the parallelepiped.

104. The diagonal of a rectangular parallelepiped makes an angle φ with the smaller lateral face. A section is passed through the greater sides of the bases of the parallelepiped. The perimeter of this section is equal to m, and its plane makes an angle with the plane of the base. Find the volume of the parallelepiped.

105. Compute the volume of a regular triangular pyramid in which the side of the base is a and the lateral edges are mutually perpendicular.

106. Given the edge a of a regular octahedron, compute its surface area and volume.

107. The side of the base of a regular hexagonal pyramid is equal to a, and the dihedral angle at the base is equal to 45°. Compute the volume of the pyramid.

108. Find the volume of the pyramid whose base is a triangle with angles α and β if the altitude h of the pyramid makes an angle φ with each lateral edge.

109. The base of a pyramid is a parallelogram whose diagonals intersect at an angle α. The altitude of the pyramid passes through the point of intersection of the diagonals of the base and is equal to H. The unequal lateral edges make angles β and γ with the base. Compute the volume of the pyramid.

110. In a frustum of a regular triangular pyramid the radii of the circles inscribed in the bases are equal to 4 cm and 2.5 cm, and the lateral edges are inclined to the plane of the base at an angle of 30°. Compute the volume of this pyramid.

111. A cylinder is cut by a plane parallel to the axis. This plane cuts off an arc α from the circle of the base. The diagonal of the section is equal to l and makes an angle φ with the base. Compute the volume of the cylinder.

112. Inscribed in the base of a cone is a square whose side is equal to a. The plane passing through the vertex of the cone and one of the sides of this square cuts in section a triangle whose angle at the vertex is equal to α. Find the volume of the cone.

113. The axial section of a cone is a triangle with the angle at the vertex equal to α. The radius of the circle circumscribed about this triangle is equal to R. Compute the volume of the cone.

114. The frustum of a cone whose altitude is 12 cm, and the radii of the bases are 6 cm and 9 cm is cut by two planes parallel to the bases and dividing the altitude into three equal parts. Compute the volume of the portion enclosed between the cutting planes.

115. Circumscribed about a ball of radius R is a frustum of a cone one of whose bases is twice the area of the other. Find the volume of the frustum.

116. A ball is inscribed in a cone. The generatrix of the cone is inclined to the plane of the base at an angle of 60°, and the radius of the ball is equal to R. Compute the volume of the cone.

117. A cone is inscribed in a ball of radius R. The angle at the vertex of the axial section of the cone is equal to 60°. Find the volume of the cone.

118. Inscribed in a ball of radius R is a cylinder. The diagonal of the axial section of the cylinder is inclined to the base at an angle of 30°. Find the volume of the cylinder.

119. A cylinder is inscribed in a ball. The radius of the ball is R, the radius of the base of the cylinder is r. Find the volume of the cylinder.

120. A cylinder is inscribed in a cone. The diagonals of the axial section of the cylinder are parallel to two generatrices of the cone. The generatrix of the cone is equal to l and makes an angle α with the plane of the base. Find the volume of the solid bounded by the base of the cone and the lateral surfaces of the cylinder and cone.

121. A right triangle with hypotenuse equal to $2a$ and acute angle of 30° rotates about the axis passing through the vertex of the right angle parallel to the hypotenuse. Compute the volume of the solid thus obtained.

QUIZ No. 1

Form I

(1) The base of a pyramid is a right triangle one of whose legs is equal to 6 cm, the opposite angle is equal to 60°, and each lateral edge is equal to 4 cm. Find the volume of the pyramid.

(2) An isosceles triangle whose base is a and base angle α rotates about a lateral side. Find the volume of the solid of revolution.

(3) Compute the volume of the solid generated by rotating about the x-axis the region bounded by the lines: $y^2 - x + 1 = 0$, $x - 2 = 0$, $y = 0$.

Form II

(1) The base of a right parallelepiped is a parallelogram one of whose diagonals is equal to 17 cm and the sides are equal to 9 cm and 10 cm. The total surface area of the parallelepiped is 334 cm^2. Find its volume.

(2) A rhombus with side a and acute angle α rotates about the straight line drawn through the vertex of the acute angle perpendicular to its side. Compute the volume of the solid of revolution.

(3) Compute the volume of the solid generated by rotating about the x-axis a region bounded by the lines: $y = -x^2 + 2x$ and $y = 0$.

Chapter 26
SURFACE AREAS OF SOLIDS OF REVOLUTION

Sec. 26.1. Areas of the Lateral and Total Surfaces of a Cylinder

The area of the lateral surface of a cylinder is equal to the product of the circumference of the base and the altitude of the cylinder:

$$S = 2\pi RH. \tag{26.1}$$

The area of the total surface of a cylinder is computed by the formula

$$S = 2\pi RH + 2\pi R^2. \tag{26.2}$$

1. The area of an axial section of a cylinder is equal to S. Find the area of the lateral surface of the cylinder.
2. The generatrix of an equilateral cylinder is equal to l. Compute the area of the lateral surface of this cylinder.
3. The area of the lateral surface of a cylinder is half the area of its total surface. The diagonal of an axial section is equal to 5 cm. Find the total surface area of the cylinder.
4. In the base of a cylinder a chord of length a subtends an arc φ. The altitude of the cylinder is equal to H. Find the total surface area of the cylinder.
5. In a cylinder, through the midpoint of the radius of the base a section is passed perpendicular to it. The section is a square whose area is 16 cm^2. Compute the area of the lateral surface of the cylinder.
6. The total surface area of a cylinder is 1170π cm^2, and the radius of its base is equal to 15 cm. At what distance from the axis of the cylinder does the section having the shape of a square pass?
7. A square with side a rotates about an outer axis which is parallel to its side. The axis passes at a distance a from the square. Compute the total surface area of the obtained solid of revolution.

Sec. 26.2. Areas of the Lateral and Total Surfaces of a Cone

The area of the lateral surface of a cone is equal to half the product of the circumference of the base and the length of the generatrix:

$$S = \pi RL. \tag{26.3}$$

The area of the total surface of a cone is computed by the formula

$$S = \pi RL + \pi R^2. \tag{26.4}$$

8. Compute the area of the lateral surface of a cone if the area of its base is equal to M, and the area of an axial section is equal to N.
9. The area of an axial section of a cone is equal to Q. Compute the area of its total surface.
10. The generatrix of a cone makes an angle α with the plane of the base. The chord drawn in the base of the cone and equal to a

is seen from the centre of the base at an angle φ. Find the area of the total surface of the cone.

11. The radius of the base of a cone is equal to R. Through two elements of the cone, making an angle β, a plane is passed intersecting the base of the cone along a chord subtended by an arc α ($\alpha < 180°$). Compute the total surface area of the cone.

12. A right triangle whose legs are 3 cm and 4 cm rotates about the axis parallel to the hypotenuse and passing through the vertex of the right angle. Compute the surface area of the solid of revolution thus generated.

13. An equilateral cylinder is inscribed in an equilateral cone. Find the area of the lateral surface of the cone if the area of the lateral surface of the cylinder is equal to S.

14. Inscribed in a cone is a regular n-gonal pyramid in which each plane angle at the vertex is equal to α. Compute the area of the lateral surface of the cone if the radius of its base is r.

15. A triangle with sides 8 cm and 5 cm and the angle between them of 60° rotates about the axis drawn through the vertex of this angle perpendicular to the shorter side. Find the area of the surface of revolution thus generated.

16. A cone is inscribed in a regular quadrangular pyramid. The side of the base of the pyramid is equal to a, the dihedral angle at it is equal to α. Compute the total surface area of the cone.

17. In a regular pyramid each lateral edge is equal to m and makes an angle α with the plane of the base. Compute the total surface of the cone circumscribed about the pyramid.

Sec. 26.3. Areas of the Lateral and Total Surfaces of a Frustum of a Cone

The area of the lateral surface of a frustum of a cone is equal to the product of the half-sum of the circumferences of its bases and the length of the generatrix:

$$S = \pi (R_1 + R_2) L. \tag{26.5}$$

The area of the total surface of a frustum of a cone is computed by the formula

$$S = \pi (R_1 + R_2) L + \pi (R_1^2 + R_2^2). \tag{26.6}$$

18. The radii of the bases of a frustum of a cone are R and r, and the generatrix is inclined to the plane of the base at an angle of 60°. Compute the area of the lateral surface of the frustum.

19. Circumscribed about a ball of radius r is a frustum of a cone in which the generatrix makes an angle α with the base. Find the area of the lateral surface of this frustum.

20. Compute the altitude of a frustum of a cone if the area of its lateral surface is equivalent to the sum of the areas of the bases, and the radii of the bases are equal to R and r.

21. In a frustum of a cone the diagonals of an axial section are mutually perpendicular, and the generatrix makes with the plane of the lower base an angle α and is equal to l. Find the total surface area of the frustum.

22. The ratios of the lower and upper bases and the lateral surface in a frustum of a cone are $m : n : p$. Find the angle between the generatrix and the plane of the lower base.

23. The generatrix of a frustum of a cone is equal to l and is inclined to the base at an angle α. The ratio of the radii of the bases is $m : n$ $(m > n)$. Compute the area of the lateral surface of the frustum.

24. Circumscribed about a sphere of radius 6 cm is a frustum of a cone in which the ratio of the radii of the bases is 4 : 9. Compute the area of the lateral surface of the frustum.

25. A sphere of radius r is inscribed in a frustum of a cone. From the centre of the sphere the diameter of the greater base is seen at an angle α. Find the area of the lateral surface of the frustum.

Sec. 26.4. Surface Areas of a Sphere and Its Parts

The surface area of a sphere of radius R is computed by the formula

$$S = 4\pi R^2. \tag{26.7}$$

The surface area of a spherical zone and also of a spherical segment is equal to the product of the altitude and the circumference of the great circle of the sphere

$$S = 2\pi RH, \quad S = 2\pi RH. \tag{26.8}$$

26. Compute the area of the sphere inscribed in: (1) a cube whose surface area is equal to S; (2) a regular tetrahedron whose surface area is equal to S; (3) an equilateral cylinder the diagonal of whose axial section is equal to m; (4) an equilateral cone whose surface area is equal to S.

27. Compute the ratio of the surface areas of two spheres one of which is inscribed, and the other circumscribed, about: (1) a cube; (2) a regular tetrahedron; (3) an equilateral cylinder; (4) an equilateral cone.

28. The volume of a ball is V. Compute its surface area.

29. The radii of the bases of a frustum of a cone are equal to 24 cm and 15 cm, the altitude is equal to 27 cm. Find the area of the circumscribed sphere.

30. The radii of the bases of a spherical zone are equal to 10 cm and 12 cm, its altitude is 11 cm. Compute the surface area of the spherical zone.

31. A spherical sector of radius R and vertex angle α is revolving about the diameter of a circle nonintersecting the arc of the sector and making an angle β with the nearest of its radii. Compute the area of the spherical zone corresponding to the arc of the sector.

32. Find the total surface area of a spherical sector if the arc of its axial section measures 120°, and the radius of the ball is equal to R.

33. The area of a spherical segment together with the area of its base is equal to S. Find the altitude of the segment if the radius of the ball is equal to R.

34. A circular segment with arc of 120° and area Q rotates about its altitude. Compute the total area of the obtained solid of revolution.

Sec. 26.5. Investigations for Extremum in Problems on Surface Areas of Solids of Revolution

35. Among all cylinders inscribed in a given cone with radius of the base R and altitude H find the one having the greatest area of lateral surface.

○ Let r denote the radius of the base of the sought-for cylinder, and let h denote its altitude (see Fig. 168). From the similarity of the triangles ABO_1 and CBO we have $R/r = H/(H-h)$, whence $r = R(H-h)/H$. Substituting the value of r into the formula for the lateral surface area of a cylinder $S = 2\pi rh$, we get

$$S = 2\pi h \frac{R(H-h)}{H} \quad \text{or} \quad S = \frac{2\pi R}{H}(Hh - h^2) \quad (0 < h < H).$$

Investigate the function for extremum with the aid of the second derivative

$$S' = \frac{2\pi R}{H}(H-2h); \quad \frac{2\pi R}{H}(H-2h) = 0, \quad H-2h = 0;$$

$$h = \frac{H}{2}; \quad S'' = -\frac{4\pi R}{H}$$

The second derivative is negative, consequently, the function has a maximum for $h = H/2$.

Find the radius of the base of the desired cylinder: $r = \frac{R(H-H/2)}{H} = \frac{R}{2}$. ●

36. Among all cones with a given lateral area S find the one having the greatest volume.

○ The volume of a cone is $V = \pi R^2 H/3$, but $H = \sqrt{l^2 - R^2}$. From the formula for the lateral area of a cone $S = \pi Rl$ we find $l = S/(\pi R)$, then

$$H = \sqrt{\frac{S^2}{\pi^2 R^2} - R^2} = \frac{1}{\pi R}\sqrt{S^2 - \pi^2 R^4} \quad (S^2 - \pi^2 R^4 > 0, \; R < \sqrt{S/\pi}).$$

Substituting the value of H into the formula for the volume of a cone, we get

$$V=\frac{1}{3}\pi R^2\cdot\frac{1}{\pi R}\sqrt{S^2-\pi^2R^4}=\frac{1}{3}R\sqrt{S^2-\pi^2R^4}.$$

Investigate this function for extremum with the aid of the first derivative:

$$V'=\frac{1}{3}\left(\sqrt{S^2-\pi^2R^4}-\frac{2\pi^2R^4}{\sqrt{S^2-\pi^2R^4}}\right)=\frac{S^2-3\pi^2R^4}{3\sqrt{S^2-\pi^2R^4}};$$

$$S^2-3\pi^2R^4=0,\quad R^4=S^2/(3\pi^2),\quad R=\sqrt{S/(\pi\sqrt{3})}.$$

If $R<\sqrt{S/(\pi\sqrt{3})}$, then $V'>0$, and if $R>\sqrt{S/(\pi\sqrt{3})}$, then $V'<0$.

Consequently for $R=\sqrt{S/(\pi\sqrt{3})}$ the function has a maximum. Find the altitude of the desired cone:

$$H=\frac{1}{\pi\sqrt{S/(\pi\sqrt{3})}}\sqrt{S^2-\pi_2\frac{S^2}{3\pi^2}}$$

$$=\frac{1}{\pi\sqrt{S/(\pi\sqrt{3})}}S\sqrt{\frac{2}{3}}=\sqrt{\frac{2S}{\pi\sqrt{3}}}.\ \bullet$$

37. Among all cylinders inscribed in a given cone (R and H are given) find the one having the greatest total area.

38. Among all cylinders of a given volume V find the one having the smallest total area.

39. Find the radius of the base and the altitude of a cylindrical tank having the smallest area (without cover) at a given volume V.

40. Among all cylinders inscribed in a ball of radius R find the one having the greatest lateral area.

41. Find the radius of the base and the altitude of a cylindrical tank (without cover) having the greatest volume at a given surface area S.

42. Among all cylinders with a given total area S find the one having the greatest volume.

43. Among all cones inscribed in a ball of radius R find the one having the greatest lateral area.

Sec. 26.6. Computing Surface Areas of Solids of Revolution with the Aid of the Definite Integral

Rotating the arc $\overset{\smile}{AB}$ of a plane curve $y=f(x)$ about the x-axis, we generate a surface of revolution (Fig. 177).

The area differential dS of this surface is equal to the area of the lateral surface of a frustum of a circular cone with the radii of the bases y and $y+dy$

and generatrix dl:

$$dS = \frac{2\pi y + 2\pi (y+dy)}{2}\, dl = \pi (2y+dy)\, dl \approx 2\pi y\, dl$$

(the term dy may be neglected because of its smallness as compared with $2y$).

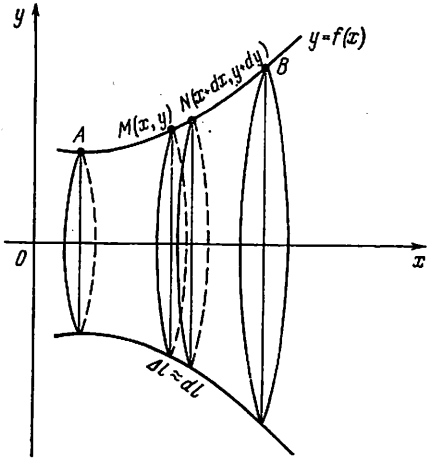


Fig. 177

The area of the surface generated by revolving the arc $\overset{\smile}{AB}$ about the x-axis is computed by the formula

$$S = \int_a^b dS = 2\pi \int_a^b y\, dl = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx, \qquad (26.9)$$

where a and b are the values of the independent variable x at the points A and B, respectively.

Analogously, when revolving the arc $\overset{\smile}{AB}$ about the y-axis, we have $dS \approx 2\pi x\, dl$, whence

$$S = 2\pi \int_c^d x\, dl = 2\pi \int_c^d x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \qquad (26.10)$$

where c and d are the values of the independent variable y at the points A and B, respectively.

44. Find the area of the ball generated by rotating the circle $x^2 + y^2 = r^2$ about the x-axis.

⊙ Differentiating the equation of the circle $x^2+y^2=r^2$, we get $2x+2y\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{x}{y}$. We find the differential of the arc:

$$dl = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+\left(-\frac{x}{y}\right)^2}\, dx = \sqrt{\frac{y^2+x^2}{y^2}}\, dx = \frac{r\, dx}{y}.$$

Substituting the value of the differential dl into Formula (26.9) and taking the limits of integration from $-r$ to r, we get

$$S = 2\pi \int_{-r}^{r} y \cdot \frac{r\,dx}{y} = 2\pi r \int_{-r}^{r} dx = 2\pi r x \Big|_{-r}^{r} = 4\pi r^2. \quad \bullet$$

45. Find the area of the surface generated by rotating about the x-axis the arc of the circle $(x-4)^2 + y^2 = 36$ contained between the points $A\ (2,\ 4\sqrt{2})$ and $B\ (4,\ 6)$.

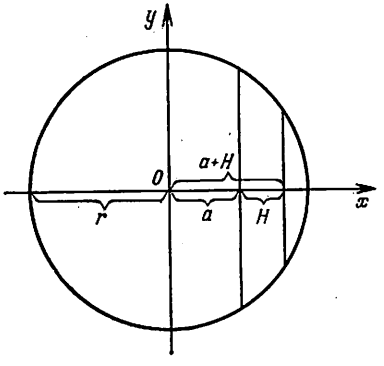


Fig. 178

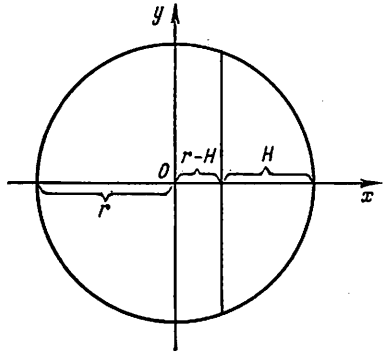


Fig. 179

○ Differentiating the equation of the circle with respect to x, we obtain

$$2(x-4) + 2y\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{x-4}{y}, \quad y\frac{dy}{dx} = -(x-4).$$

Consequently,

$$S = 2\pi \int_2^4 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_2^4 \sqrt{y^2 + \left(y\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_2^4 \sqrt{36 - (x-4)^2 + (x-4)^2}\,dx = 12\pi \int_2^4 dx = 12\pi x \Big|_2^4 = 24\pi. \quad \bullet$$

46. Find the area of the spherical zone with altitude H generated by rotating an arc of the circle $x^2 + y^2 = r^2$ about the x-axis (Fig. 178).

47. Find the area of the spherical segment having altitude H generated by rotating an arc of the circle $x^2 + y^2 = r^2$ about the x-axis (Fig. 179).

48. Find the area of the spherical zone generated by rotating about the x-axis the arc of the circle $x^2 + y^2 = 16$ enclosed between the points $A\ (2,\ 2\sqrt{3})$ and $B\ (3,\ \sqrt{7})$.

49. Find the area of the spherical zone generated by rotating about the y-axis the arc of the circle $x^2 + y^2 = 25$ enclosed between the points A (4, —3) and B (3, 4).

50. Find the area of the surface generated by rotating about the y-axis the arc of the circle $x^2 + (y - 2)^2 = 25$ enclosed between the points A $(2\sqrt{6}, 1)$ and B (4, 5).

51. Compute the area of the surface generated by rotating about the x-axis the arc of the parabola $y^2 = 4x$ bounded by the points O (0, 0) and A $(3, 2\sqrt{3})$.

52. Compute the area of the surface generated by rotating about the x-axis the arc of the parabola $y^2 = 9x$ bounded by the points (0, 0) and (4, 6).

Sec. 26.7. Miscellaneous Problems

53. A cylinder is inscribed in a ball. The radius of the ball is equal to 29 cm, the area of the lateral surface of the cylinder equals 1680π cm^2. Find the volume of the cylinder.

54. A cylinder is inscribed in a ball. The radius of the base of the cylinder is to its altitude as 2 : 3. The area of the ball is equal to 225π. Compute the total surface area of the cylinder.

55. A ball is circumscribed about an equilateral cylinder. Find the ratios of their areas and volumes.

56. A cylinder is circumscribed about a ball. Find the ratios of their areas and volumes.

57. The altitude of a cylinder is equal to 5 cm. With an increase in the altitude by 4 cm its volume will increase by 36π cm^3. Compute the lateral area of the cylinder.

58. A cone and a cylinder have a common base, whose area is equal to 4π cm^2, and a common altitude. The ratio of the lateral area of the cone to that of the cylinder is 5 : 6. Compute the generatrices of the given solids.

59. An isosceles triangle with base a and base angle α rotates about a lateral side. Compute the surface area and the volume of the solid thus generated.

60. The sides of a parallelogram are equal to m and n; the angle between them is α. Find the surface area of the solid generated by rotating the parallelogram about the side n.

61. A cone rolls over a plane rotating about its fixed vertex. The altitude of the cone is equal to h, the generatrix is equal to l. Find the area of the surface described by the altitude of the cone.

62. A rhombus with side a and acute angle of 60° rotates about the axis drawn through the vertex of this angle perpendicular to the side. Find the surface area of the obtained solid of revolution.

63. The altitude of a frustum of a cone is equal to h, the angle between the diagonals of an axial section (facing the base) is equal

to α, and the generatrix makes an acute angle β with the base. Compute the area of the lateral surface.

64. The radius of a sphere is equal to 25 cm. The sphere is separated by a section of 49π cm^2 into two parts. Find the areas of the spherical parts.

65. The area of the lateral surface of a spherical segment is 0.05 the surface area of the sphere. Find the altitude of the segment if the radius of the sphere is equal to 100 cm.

66. Prove that the surface area of the solid generated by rotating a square about its side is equal to the surface area of the ball whose radius is equal to the side of the square.

67. A biconvex glass consists of two equal spherical segments; the area of their common base is equal to S, and the arc in the axial section of each segment is equal to α. Compute the surface area of this glass.

68. Find the surface area of the sphere if the total surface area of the cylinder circumscribed about it is equal to S.

69. Compute the ratio of the total surface area of a cone to the area of the sphere inscribed in it if the generatrix of the cone is inclined to the plane of the base at an angle α.

QUIZ No. 1

Form I

(1) A rectangle whose area is equal to S rotates about the axis passing through its vertex parallel to the diagonal. Compute the surface area of the solid of revolution if the angle between the diagonals is equal to α.

(2) In a cone, the generatrix is equal to l and makes an angle α with the altitude of the cone. Compute the area of the circumscribed sphere.

Form II

(1) A rhombus with side a and acute angle α rotates about a straight line drawn through the vertex of the acute angle perpendicular to its side.

(2) In a cone the generatrix is equal to l and makes an angle α with the altitude of the cone. Compute the area of the inscribed sphere.

ANSWERS

Chapter 1

3. (1) 0.2; (2) 0.4; (3) 0.3; (4) 0.42. 4. $385.5 < x < 386.5$. 5. 0.5. 6. $10.61 < 10.63 < 10.65$ (A). 7. $1.0077 < 1.0082 < 1.0087$; $63.29 < 63.44 < 63.59$. 8. $24.2 < 24.5 < 24.8$ (cm^2). 13. (1) 2; (2) 4.6 and 3; (3) 7; (4) 1, 2, 7, and 8; (5) 3. 14. All the three digits are correct. 15. All digits are correct in a rigorous sense. 16. (1) $28 \cdot 10^1 \pm 10$; (2) $89 \cdot 10^2 \pm 100$; (3) $53 \cdot 10^4 \pm 100$; (4) $57.4 \cdot 10^2 \pm 10$. 17. (1) 0.3 ± 0.08; (2) 2.06 ± 0.007; (3) 14 ± 1; (4) 24.7 ± 0.1. 18. (1) $1.24 \cdot 10^4 \pm 50$; (2) $1.58 \cdot 10^4 \pm 100$; (3) $8.7 \cdot 10^3 \pm 50$; (4) $8.1 \cdot 10^2 \pm 10$. 19. (1) $\Delta a = 0.01$; (2) $\Delta a = 0.1$; (3) $\Delta a = 0.01$; (4) $\Delta a = 0.001$. 20. (1) $0.01 \cdot 10^2$; (2) $0.0001 \cdot 10^2$; (3) $0.001 \cdot 10^{-4}$; (4) $0.01 \cdot 10^{-3}$. 25. 0.05%; 26. (1) 2.1%; 1.4%; (2) 1.8%; 0.44%; (3) 1.3%; 0.14%. 27. 0.3%. 28. 2%. 29. 1, 2 and 8.

Chapter 2

2. 24.5. 3. $a = 5.55$; $\Delta a = 0.001$; $\varepsilon_a = 0.00018$. 4. $a = 6.61$; $\Delta a = 0.0015$; $\varepsilon_a = 0.00023$. 5. $R = 18.8$ Ohms; $\Delta R = 0.06$; $\varepsilon_R = 0.3\%$. 7. 6.07. 8. $a = 1.37$; $\Delta a = 0.001$; $\varepsilon_a = 0.1\%$. 11. 1.5; 0.3%. 12. 39 ± 0.2 (cm). 13. 300; one significant digit. 16. 0.3%. 17. 0.94; 9 and 4 are correct digits. 18. 2.51 ± 0.005. 22. 0.4%. 23. 0.06%. 26. $\Delta R = \Delta H = 0.004$ cm. 27. To within 0.01 m. 34. (1) $B = 27.4°$, $a = 23.8$, $b = 12.2$; (2) $A = 41.1°$, $a = 42.3$, $b = 48.4$; (3) $B = 61.5°$, $a = 298$, $b = 549$; (4) $A = 47.3°$, $a = 0.430$, $b = 0.397$. 35. (1) $B = 37.7°$, $b = 275$, $c = 450$; (2) $A = 53.6°$, $b = 6.32$, $c = 10.6$; (3) $A = 17.4°$; $a = 0.297$, $c = 0.993$; (4) $B = 33.2°$, $b = 31.4$, $c = 57.3$. 36. (1) $A = 41.3°$, $B = 48.7°$, $b = 28.3$; (2) $A = 41.8°$, $B = 48.2°$, $a = 20.4$; (3) $A = 53.3°$, $B = 36.7°$, $a = 150$; (4) $A = 46.9°$, $B = 43.1°$, $b = 0.428$. 37. (1) $A = 51.6°$, $B = 38.4°$, $c = 1.05$; (2) $A = 77°$, $B = 13°$, $c = 431$; (3) $A = 31.6°$, $B = 58.4°$, $C = 2.79$; (4) $A = 37.2°$, $B = 52.8°$, $c = 41$. 38. (1) $A = 95.6°$, $a = 83.4$; (2) $B = 77.7°$, $a = 30.7$; (3) $A = 23.6°$, $b = 308$; (4) $B = 21.8°$, $b = 3.04$; (5) $A = 103.6°$, $a = 1080$, $b = 686$; (6) $B = 57.7°$, $a = 53.8$, $b = 50.4$; (7) $A = 50.8°$, $B = 64.6°$; (8) $A = 130.6°$, $B = 24.7°$, $a = 210$; (9) $A = 21.8°$, $B = 79.1°$, $b = 8.41$; (10) $A = 20.8°$, $B = 79.6°$, $a = 12.8$; (11) $B = 61.9°$, $a = 59.9$, $b = 63.5$; (12) $A = 80.8°$, $B = 49.6°$, $b = 33.1$. 44. (1) $c = 71.4$, $A = 67.4°$, $B = 47.8°$; (2) $a = 669$, $B = 47.2°$, $C = 84.5°$; (3) $b = 6.96$, $A = 46.6°$, $C = 69.7°$; (4) $c = 689$, $A = 30°$, $B = 37.8°$. 45. (1) $B = 39.3°$, $a = 981$, $c = 959$; (2) $A = 22.2°$, $a = 23.9$, $b = 57.2$; (3) $B = 76.6°$, $b = 7.99$, $c = 7.4$; (4) $C = 76.3°$, $a = 111$, $b = 236$. 46. (1) $A = 59.7°$, $B = 55.7°$, $C = 64.6°$; (2) $A = 126.3°$, $B = 20.5°$, $C = 33.2°$; (3) $A = 130.5°$, $B = 32.6°$, $C = 16.9°$; (4) $A = 44.4°$, $B = 53.2°$, $C = 82.4°$. 47. (1) $B = 59.7°$, $C = 52.1°$, $c = 218$; (2) Case I: $B = 83.4°$, $C = 33.9°$, $c = 22.9$; Case II: $B = 96.6°$, $C = 20.7°$, $c = 14.5$; (3) $B = 33.5°$, $C = 104°$, $c = 12.6$; (4) Case I: $A = 82.2°$, $C = 49.6°$, $a = 829$; Case II: $A = 1.5°$, $C = 130.3°$, $a = 21.2$. 48. 3.7%. 49. 0.035. 50. 1.2%. 51. 1%. 52. 364,000 cm^3; 3 and 6 are correct digits. 53. 7.9 ± 0.03 (cm); 7 and 9 are correct digits. 54. To within 0.2 cm.

Quiz. No 1. Form I. (1) $a = 4.38$; $\varepsilon_a = 0.02\%$; (2) 3600; 3 and 6 are correct digits; (3) 63.4 ± 0.1; (4) 0.1%; (5) to within 0.02 m. Form II. (1) $a = 0.67$; $\varepsilon_a = 0.15\%$; (2) 3800; 3 and 8 are correct digits; (3) 2.85 ± 0.004; (4) 0.02%; (5) to within 0.05.

Chapter 3

4. (1), (2) Yes; (3) no. 5. (1) $\{0\}$; (2) $\{2/3\}$; (3) $\{0.25\}$; (4) $\varnothing$. 6. (1) $\{10/19\}$; (2) $\{-3\}$; (3) $\{3\}$; (4) $\{-21\}$. 7. (1) $\{2\}$. (2) $\varnothing$; (3) $\{2.5\}$; (4) $\{-1\}$; (5) $\varnothing$; (6) $\{x \mid x \neq -4,\ x \neq 4\}$. 8. 35 litres. 9. Copper —86.1 kg, zinc—36.9 kg. 10. 240 kg. 11. 637.5 and 212.5. 12. 85. 13. 6 cm. 15. (1) $(-1, +\infty)$; (2) $[-2, +\infty)$, (3) $(-\infty, 3]$; (4) $(2, +\infty)$. 16. (1) $(0.96; +\infty)$; (2) $[56, +\infty)$; (3) $(-\infty, -3.5)$; (4) $(-\infty, 2.2]$. 17. (1) $[-2, +\infty)$; (2) $(-\infty, -3)$; (3) $[1, +\infty)$; (4) $(0.5, +\infty)$; 18. (1) $(-\infty, +\infty)$; (2) $(-\infty, +\infty)$; (3) $(-\infty, +\infty)$. 19. (1) $\varnothing$; (2) $\varnothing$. 23. (1) $(1, 4)$; (2) $(-\infty, -3)$; (3) $\varnothing$; (4) $(-2, +\infty)$. 24. (1) $(1.5, +\infty)$; (2) $(-\infty, -3.5)$; (3) $\varnothing$; (4) $(0.25, 0.5)$. 25. (1) $(-\infty, 1) \cup (3, +\infty)$; (2) $(-\infty, 10]$; (3) $(-\infty, +\infty)$; (4) $[-7, +\infty)$. 26. (1) $(-\infty, -2) \cup (5, +\infty)$; (2) $(-8, +\infty)$; (3) $(-\infty, 1.2) \cup (7, +\infty)$; (4) $(-\infty, +\infty)$. 27. (1) $(-\infty, -3/2) \cup (-1/3, +\infty)$; (2) $(-\infty, 2/3) \cup [5, +\infty)$; (3) $(-\infty, 1/3) \cup (4, +\infty)$; (4) $[2.4)$; 28. (1) $(-2/3, -1/2)$; (2) $[1/2; 4/3]$; (3) $(-\infty, 4) \cup [5, +\infty)$; (4) $(-\infty, 3) \cup (4, +\infty)$. 29. (1) $(-\infty, -2/3) \cup [-0.25; +\infty)$; (2) $(2.5; 11)$; (3) $(-\infty, -2] \cup (-0.5, +\infty)$; (4) $(2.5, 12)$; (5) $[-2, -1.5)$. 33. (1) 4; (2) 1; (3) 1; (4) 2/3. 34. (1) $(-2, 6)$; (2) $(-9, -7)$; (3) $(-12, -2)$; (4) $(a - \varepsilon;\ a + \varepsilon)$; (5) $[2.4]$. 35. (1) $(-\infty, -5) \cup (-1, +\infty)$; (2) $(-\infty, -3) \cup (13, +\infty)$; (3) $(-\infty, 1) \cup (3, +\infty)$; (4) $(-\infty, -3] \cup [-1; +\infty)$; (5) $(-\infty; 0.5) \cup (1.5, +\infty)$. 38. (1) $\{(3; 4)\}$; (2) $\varnothing$; (3) an infinite set of solutions. (4) $\{(2; 3)\}$; (5) $\{(-3, -5)\}$; (6) $\{(-2, 4)\}$. 39. (1) $\{(-2, 2)\}$; (2) $\{(3, 1/2)\}$; (3) $\{(-2, 1)\}$; (4) $\{(6, 0)\}$; (5) $\{(5, -2)\}$; (6) $\{(0, -6.2)\}$. 40. $a = 3$. 41. $a = 1.5$. 42. 74. 43. 16 and 12 m. 44. 12 and 15 roubles. 45. 150 and 80 km. 46. 60° and 70°. 47. 200 and 150. 50. (1) 87; (2) —6; (3) 0; (4) 42. 51. (1) $\{(8, 4, 2)\}$; (2) $\{(9, 6, 7)\}$; (3) $\{(3, -2, 5)\}$; (4) $\{(1, -2, 3)\}$; (5) $\{(1, 2, 3)\}$; (6) $\{(1, 1, 1)\}$; (7) $\{(3, 5, 6)\}$; (8) an infinite set of solutions. 54. (1) $\{-\sqrt{2}, \sqrt{2}\}$; (2) $\{0, 1\}$; (3) $\{0, 3\}$; (4) $\{-5, 5\}$; (5) $\varnothing$; (6) $\{0, 0.4\}$; (7) $\{-5, 0\}$; (8) $\{-5, 0\}$. 55. (1) $\{2, 4\}$; (2) $\{-5, -4\}$; (3) $\{-4, 3\}$; (4) $\{2/3, 2\}$; (5) $\{-0.75, -0.25\}$; (6) $\{-0.8, 6\}$; (7) $\{2\}$; (8) $\varnothing$. 56. (1) $\{-3, 8\}$; (2) $\{15.8, 18\}$; (3) $\{-0.7, 10\}$; (4) $\{-0.5, 2\}$. 61. (1) $\{1, 3\}$; (2) $\{2, 5\}$; (3) $\{-3, 5\}$; (4) $\{-3, 4\}$; (5) $\{-4, -2\}$; (6) $\{-5, 3\}$. 62. (1) $x^2 - 13x + 40 = 0$; (2) $x^2 + 3x - 10 = 0$; (3) $x^2 + 9x + 20 = 0$; (4) $15x^2 - 22x + 8 = 0$; (5) $32x^2 - 4x - 3 = 0$; (6) $30x^2 + 37x + 10 = 0$; (7) $4x^2 - 17x + 4 = 0$; (8) $3x^2 - 8x - 3 = 0$. 63. (1) The roots have opposite signs; the greater (in modulus) root is negative; (2) the roots have opposite signs; the greater (in modulus) root is positive; (3) (4), (6) both roots are positive; (5) both roots are negative. 64. (1) $(x - 3)(2x - 1)$; (2) $(2a + 3) \times (3a - 2)$; (3) $(y - 5)(3y + 4)$; (4) $(4x + 1)(3x + 1)$; (5) $(m - 3)(m + 1)$; (6) $(8x - 3)(9x - 5)$. 65. (1) $(x - 2)/(x + 3)$; (2) $(y + 3)/(2y + 5)$; (3) $(2a - 1)/(a + 5)$; (4) $(4 - a)/(3a + 1)$; (5) $(x + 2)/(4 - x)$; (6) $(7y - 1)/(5y + 2)$. 70. (1) $\{-5; 4\}$; (2) $\{2.5, 5\}$; (3) $\{2\}$; (4) $\{0.75, 1\}$; (5) $\{-3.5, -1\}$; (6) $\{-4, 9\}$; (7) $\{-3, 2/3\}$; (8) $\{5; 6\}$. 71. (1) $\{-2, -1, 1, 2\}$; (2) $\{-5, -3, 3, 5\}$; (3) $\{-2, -1/3, 1/3, 2\}$; (4) $\{-4, -1/2, 1/2, 4\}$. 72. (1) $(x - 2)(x + 2)(2x - 1) \times (2x + 1)$; (2) $(x - 2)(x + 2)(x - 11)(x + 11)$. 73. (1) $\{-3, 2, 5\}$; (2) $\{-2, 1, 3, 4\}$; (3) $\{-4, -2, 1, 3\}$; (4) $\{-1/2, 1, 3\}$; (5) $\{-3, -1, 0, 4\}$; (6) $\{-2, 1, 3\}$. 74. (1) $x^3 + 2x^2 - 5x - 6 = 0$; (2) $6x^3 - x^2 - 4x - 1 = 0$. 75. (1) $\{-3, 3\}$; (2) $\{-0.4\}$; (3) $\{-2.5, 2.5\}$; (4) $\{0.6\}$. 76. 23 and 24. 77. 0.6. 78. 24. 79. 9 and 12 cm. 80. 12 and 16 cm. 81. 8 and 12 cm. 82. 18. 83. 6 and 12 days. 84. 2 and 2.5 km per day. 85. 60 and 48 g. 86. 2 and 3 m. 87. 30 and 24 km/h. 88. 80 and 60 km/h. 89. 24 km/h. 92. (1) $(-2, 10)$; (2) $(-\infty, -9) \cup (3, +\infty)$; (3) $(-\infty, 2.5) \cup (4, +\infty)$; (4) $\varnothing$; (5) $(-1/3, 2)$; (6) $(-\infty, 0.5) \cup (0.5, +\infty)$; (7) $(-\infty, +\infty)$; (8) $(-\infty, 6) \cup (6, +\infty)$. 93. (1) $(-\infty, 2/3) \cup (3/2, +\infty)$; (2) $(-7/2, 2/3)$; (3) $(-\infty, 2/7) \cup (8/3, +\infty)$; (4) $(-8/5, 7/3)$. 96. (1) $\{-2, 0, 3\}$; (2) $\{4\}$; (3) $\{7\}$; (4) $\{4\}$; (5) $\{5\}$; (6) $\{3\}$; (7) $\{-5, 5\}$; (8) $\{-2, 2\}$. 97. (1) $\{1, 5\}$; (2) $\{1\}$; (3) $\varnothing$; (4) $\{-5, 4\}$; (5) $\varnothing$; (6) $\{17/16\}$; (7) $\{-0.5, 0.5\}$;

(8) {7}. 98. (1) {5, 6}; (2) {2}; (3) {1}; (4) {1}. 100. (1) (4, $+\infty$); (2) (1, $+\infty$); (3) (−4.5, 0); (4) ($-\infty$, −1) ∪ (1, 2.6); (5) [0.3]; (6) ($-\infty$, −2] ∪ [5, 74/13). 101. (1) (0, $+\infty$); (2) (27, $+\infty$); (3) [−2.2); (4) ($-\infty$, −2] ∪ (14, $+\infty$); (5) (0.5, $+\infty$); (6) [0, $+\infty$); (7) (0.25, 2]; (8) (2/3, 3]. 102. (1) [8, 12) ∪ (24, $+\infty$); (2) (0.3); (3) (7.75, 10]. 106. (1) {(1, 3); (19, −3)}; (2) {(3, 1); (−3, −1); (12, −3.5); (−12, 3.5)}; (3) {(3, 5); (−3, −5); (−8, 5); (8, −5)}; (4) {(3, −6); (−3, 6); (6, 3); (−6, −3)}; (5) {(8, 4); (4, 8)}; (6) {(2, 3), (−2, −3); (3, 2); (−3, −2)}; (7) {(3, 2); (−3, −2); (−2, 3); (2, −3)}; (8) {(3$\sqrt{2}$, $\sqrt{2}$); (−3$\sqrt{2}$, −$\sqrt{2}$); (3$\sqrt{2}$, −$\sqrt{2}$); (−3$\sqrt{2}$, $\sqrt{2}$)}; (9) {(3, 2); (−2, −3)}; (10) {(3, 2); (−3, −2); ($\sqrt{3}$/3, 5$\sqrt{3}$/3), (−$\sqrt{3}$/3, −5$\sqrt{3}$/3)}. 107. 4 and 36. 108. 12 and 35 cm. 109. 87 and 36 cm. 110. 15, 36, and 39 cm. 111. 45, 24, and 51 cm. 112. 68. 115. (1) $z_{max} = 17$ for $x_1 = 2$, $x_2 = 3$; (2) $z_{max} = 16$ for $x_1 = 2$, $x_2 = 4$. 116. $z_{max} = 27$ roubles, 6 articles of type I and 5 of type II. **Quiz No. 1.** Form I. (1) (1 − ∞, 2); (2) {(3, 2)}; (3) {−2, 1, 3}; (4) ($-\infty$, 0.8] ∪ [4, $+\infty$]; (5) {(−8, −4); (−8, 3); (7, −4); (7, 3)}. Form II. (1) (14.5, $+\infty$); (2) {(4, 3)}; (3) {−1, 1, 2}; (4) [−2/3, 5]; (5) {(−4, −3); (4, 3)}.

Chapter 4

4.(1) $F(0) = 4$; $F(-1) = 8$; $F(2) = 20$; (2) $s(0) = 8$; $s(2) = 0$; $s(-1) = 15$. 7. (1), (2), (3), ($-\infty$, $+\infty$). 8. (1) ($-\infty$, 1/2) ∪ (1/2 + ∞); (2) ($-\infty$, 4) ∪ (4, $+\infty$); (3) ($-\infty$, −2) ∪ (−2, $+\infty$). 9. ($-\infty$, −1) ∪ (−1, 1) ∪ (1, $+\infty$); (2) ($-\infty$, −3) ∪ (−3, 4) ∪ (4, $+\infty$); (3) ($-\infty$, −1/3) ∪ (−1/3, 2) ∪ (2, $+\infty$); (4) ($-\infty$, 4) ∪ (4,5) ∪ (5, $+\infty$). 10. (1) ($-\infty$, 1]; (2) ($-\infty$, 3]; (3) [4, $+\infty$). 11. (1) [0, 4]; (2) [5, $+\infty$). 12. (1) (3, 5]; (2) ($-\infty$, 1) ∪ (1,7]. 13. (1) ($-\infty$, −2] ∪ [4, $+\infty$); (2) ($-\infty$, −5] ∪ [−3, $+\infty$); (3) [−5, 2]. 14. (1) [8, 12); (2) (1/2, 2]. 19. (1) {6}; (2) {−2}; (3) {2}; (4) {1/16}; (5) {8}; (6) {1/125}. 20. (1) ($-\infty$, 3/2); (2) (5/4, $+\infty$); (3) (−8, 4). 27. (1) {3}; (2) {2, 6}; (3) {−1, 3}; (4) {9}; (5) {1/2, 1}. 28. (1) {4}; (2) {5}; (3) {3/2}; (4) {(log 13 + 2 log 9)/(2 log 13 − log 9). 29. {$\approx$ −0.4367}; (2) {3}; (3) {2}; (4) {0}. 30. (1) {0, $\log_7 5$}; (2) {1, 2}; (3) {2, $\log_3$ (2/3)}; (4) {0, 1}; (5) {$\log_{3/2}$ 0.5 (1 + $\sqrt{5}$)}; (6) {$\log_8 3$, $\log_9 8$}. 31. (1) {$\approx$ −0.77, 2, 4}; (2) {$\approx$1.7, −2}. 33. (1) {(2, 1)}; (2) {(2, 3); (3 $\log_2 3$, 2 $\log_3 2$)}; (3) {(1, 1); (5/3, (5/3)$\sqrt{5/3}$)}; (4) {(3, 2)}. 34. (1) {(−2, 0)}; (2) {(−2, 1.5)}; (3) {(3, 2)}; (4) {(2, 1)}. 35. (1) {(4, 1)}; (2) {(3, 2)}; 37. (1) (3, $+\infty$); (2) (3, $+\infty$); (3) ($-\infty$, 3) ∪ (5, $+\infty$). 38. (1) ($-\infty$, 3) ∪ (5, 6); (2) (2, 3) ∪ (4, $+\infty$). 41. (1) {15}; (2) {216}; (3) {13/21, 2}; (4) {5}. 42. (1) {100, 1000}; (2) {28/9, 12}. 43. (1) {1}; (2) {1, 4}; (3) {1}. 44. (1) {64}; (2) {1/25, $25^{2/3}$}. 46. (1) {(1, 10)}; (2) {(0.5, 0.25)}; (3) {(10, 0.1)}; (4) {(16, 10)}. 47. (1) {(25, 4); (4, 25)}; (2) {(6, 2)}; (3) {(3, 4); (−7$\sqrt{2}$/2, $\sqrt{2}$/2)}; (4) {(16, 4)}; (5) {(3, 27)}; (6) {(2, 6)}; (7) {(9, 25); (25, 9)}. 49. (1) ($-\infty$, −1/2) ∪ (2, $+\infty$); (2) (−2, 1) ∪ (3, 6); (3) (−6, −2); (4) (−1, 0) ∪ (2, $+\infty$); (5) (4/3, 3/2); (6) (0, $+\infty$); (7) (4, $+\infty$); (8) (−1.3) ∪ (7,11). 50. (1) ($-\infty$, 1) ∪ (1, $+\infty$); (2) ($-\infty$, 4) ∪ (4, $+\infty$); (3) ($-\infty$, −$\sqrt{3}$] ∪ [$\sqrt{3}$, $+\infty$); (4) ($-\infty$, −5) ∪ (−5, 4) ∪ (4, $+\infty$); (5) ($-\infty$, 2] ∪ [6, $+\infty$); (6) [5, $+\infty$); (7) ($-\infty$, −3) ∪ (3, $+\infty$); (8) (0, $+\infty$). 51. (1) {−1; 2}; (2) {4}; (3) {−2; 2}. 52. (1) {1}; (2) {4}; (3) {−1}; (4) {2}; (5) {1}; (6) {0.5, 1.5}; (7) {0, 1, −1}. 53. (1) (0, 1); (2) ($-\infty$, −1) ∪ (5, $+\infty$); (3) (3, 4) ∪ (4, 5) ∪ (6, $+\infty$). 54. (1) {2, 8}; (2) {2}; (3) {5}; (4) {6, 10}; (5) {5, 8}; (6) {3, 7}; (7) {4}; (8) {11}; (9) {7}; (10) {9}. 55. (1) {8}; (2) {4, 36}; (3) {0, 6}; (4) {10, 100,000}; (5) {10/3, 30}; (6) {10}; (7) {1/5, 25}; (8) {8}. 56. (1) (−13/20, −1/2) ∪ (−1/2, −7/20); (2) (5/3, 3); (3) (−4, 2) ∪ (2, 8); (4) (1/4, 7/2); (5) (4, $+\infty$); (6) $\log_3$ (28/27), $\log_3 4$. **Quiz No. 2.** Form I. (1) {3}; (2) {8}; (3) (0, 1); (4) (2, 7/2) ∪ (7/2, 5); (5) (2m + 1)/2. Form II. (1) {−2}; (2) {$\sqrt{3}$}; (3) (−1/3, 3); (4) (1/2, 1); (5) (a + 1)/(a + b).

Chapter 5

6. (1) 7, 9, 11, 13, 15; (2) 1, 1/3, 1/5, 1/7; 1/9; (3) —1, 1/2, —1/3; 1/4, —1/5; (4) 4, 4, 4, 4, 4; (5) —3.3, —3.3, —3; (6) 2, 4, 6, 8, 10; (7) 5/2, 17/4, 65/8, 257/16, 1025/32; (8) 8, 23, 46, 77, 116; (9) 0, 2/3, 4/5, 6/7, 8/9, and 1/3, 1/15, 1/35, 1/63, 1/99; (10) 1/3, 1/8, 1/15, 1/24, 1/35. 7. $x_n = 5n + 3$. 8. (1) $1/n^2$; (2) $6n - 5$; (3) 2^n; (4) $2n^2 - 1$. 10. Sequences (2), (4), (5), (6), (7), and (8). 18. (1) —3/2; (2) 0; (3) 0; (4) 1; (5) 1; (6) 2/3.

Chapter 6

5. (1) 25; (1) —1. 6. (1) —12; (2) 10; 7. (1) 54; (2) 9; (3) 8. 8. (1) —6; (2) 3. 9. (1) ∞; (2) ∞; 10. (1) 1/2; (2) 1. 11. (1) 1/6; (2) —6; 12. (1) 1/5; (2) 3. 13. (1) 2/3;(2) 3. 14. (1) 1; (2) 1/4. 15. (1) 6; (2) 1/2. 16. (1) 2/3; (2) 2/3. 17. (1) —1/6; (2) 1. 18. (1) ∞; (2) $-\infty$. 19. (1) 0; (2) 5. 20. (1) 3; (2) 1/2. 21. (1) 2; (2) 3; (3) 4. 22. (1) ∞; (2) ∞. 23. (1) —1/2; (2) 5/2. 29. (1) 1.099; (2) 1.386; (3) 1.609; (4) 2.303; (5) —1.204; (6) —0.223; (7) 1.758; (8) —1.427; (9) 2.747. 30. (1) 0.9031; (2) 1.6021; (3) 0.8508; 31. (1) 148.5; (2) 1.396; (3) 0.1353; 32. (1) 6.909; (2) —4.606; (3) 1.535; 33. (1) $e^2/^3$; (2) $e^{-5}/^4$; 34. (1) $e^{12/5}$; (2) $1/e$. 35. (1) $1/\sqrt{e}$; (2) e. 36. (1) 1; (2) 4/3. 37. (1) 4; (2) 3/4. 38. (1) $\sqrt{2/3}$; (2) 27. 39. (1) 0; (2) 2; (3) 1/2. 40. (1) ∞; (2) 1/2. 41. (1) $e^{4/9}$; (2) $e^{10/3}$. 42. (1) ek; (2) e^{-3}. **Quiz No. 1. Form I.** (1) 13/14; (2) 3; (3) 1/9; (4) 2/3; (5) e^{-3}. **Form II.** (1) 1/4; (2) $\sqrt{3}$; (3) 1/2; (4) 5; (5) e^{-3}. 45. $\Delta y = 2.25$. 46. (1) $\Delta y = 2x\,\Delta x + 2\,\Delta x + (\Delta x)^2 = 0.81$; (2) $\Delta y = 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 - 2.791$. 47. $\Delta y = -\dfrac{\Delta x}{x\,(x + \Delta x)} = -\dfrac{1}{6}$. 48. $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = 0.049$. 49. (1) $\Delta y = \sqrt{2\,(x + \Delta x)} - \sqrt{2x} \approx 0.135$; (2) $\Delta y = \sqrt[3]{x + \Delta x} - \sqrt[3]{x} = 0.06$. 52. Continuous on set R. 53. Continuous on set R. 54. The functions are continuous. 56. (1) $x = 1/2$; (2) $x = 0$; (3) $x = -1$, $x = 1$; (4) $x = 1$; (5) $x = -2$, $x = 5$; (6) $x = 2$. In all cases, discontinuity of second kind occurs. 59. (1) $x = 1$; (2) $x = 0$; (3) $y = 1$; (4) $y = 0$ and $y = 1$. 60. (1) $y = x$ for $x \to \infty$; (2) $y = -x$ for $x \to -\infty$ and $y = x$ for $x \to +\infty$; (3) $y = x$ for $x \to \infty$. 61. (1) $y = 0$, $x = -2$; (2) $y = 0$, $x = -5$ and $x = 5$; (3) $x = 0$ and $y = x + 6$; (4) $x = -1$, $x = 1$, $y = x$ and $y = -x$; (5) $x = -3$ and $y = x - 3$; (6) $y = 3$; (7) $x = 0$ and $y = x$; (8) $x = 4$ and $y = x - 1$. 65. (1) $(-\infty, -2) \cup (4,5)$; (2) $(-1,2) \cup (3, +\infty)$; (3) $(-\infty, -4] \cup [1,5)$. 66. (1) $[2,3]$; (2) $(-\infty, -4) \cup (-3, +\infty)$. 67. (1) $(-\infty, -2) \cup (-1.1] \cup [2.3]$; (2) $(-3, -2) \cup (2.3) \cup (4, +\infty)$; (3) $[-2, -1] \cup (1,3)$. 68. (1) $(-\infty, -3) \cup (1, +\infty)$; (2) $(1, 3/2) \cup (2, +\infty)$; (3) $(1,2) \cup (3, +\infty)$; (4) $(-1, 3)$. **Quiz No. 2. Form I.** (1) $(-2,3) \cup (5, +\infty)$; (2) $(-5, -3] \cup (1,2] \cup [4, +\infty)$; (3) $(1.2) \cup (4.5)$; (4) $\Delta y \approx -0.174$; (5) $x = 2$—a vertical asymptote, $y = 1$—a horizontal asymptote. **Form II.** (1) $(-\infty, -5) \cup (4, 6)$; (2) $(-\infty, -5) \cup [-2, 1] \cup [3, 4)$; (3) $(-\infty, 1) \cup (2, 3) \cup (4, +\infty)$; (4) $\Delta y \approx 0.01$; (5) $x = -2$—a vertical asymptote, $y = 0$—a horizontal asymptote.

Chapter 7

3. (1) 15; (2) 40 m/s. 4. (1) 100 m/s. (2) 24 m/s. 6. (1) —1; (2) —3; (3) 12. 7. (1) 1/3; (2) 2. 8. (1) 1/4; (2) —1/16; (3) 1/6. 15. (1) $4x^3$; (2) $6x^2$; (3) $-15x^6$; (4) $6/x^3$; (5) $(7/5)\,x^{2/5}$; (6) $6\sqrt{x}$; (7) $-3x^{-8/5}$; (8) $3\sqrt{x}$; (9) $-(3/4)\,x^{-7/4}$; (10) $-(2/3)\,x^{-5/3}$. 16. (1) $2/x^3$; (2) $-4x^{-7/3}$; (3) $-(9/2)\,x^{-5/2}$; (4) $(20/3)\,x^{7/3}$; (5) $(7/2)\,x^{5/2}$; (6) $3\sqrt{x}$; (7) $-5x^{-7/2}$; (8) $-x^{-7/6}$; (9) $-(3/4)\,x^{-7/4}$; (10) $(2/3)\,x^{-1/3}$; (11) $-(7/6)\,x^{-13/6}$; (12) $(2/3)\,t^{-1/3}$. 17. (1) —48; (2) —8/3; (3) 11/6. 18. (1) —20; (2) 21; (3) 3.5. 19. (1) $15x^{-6} - 60x^{-5} + 6x^{-4} - x^{-2}$; (2) $3x^{-1/4} + 2x^{-1/3} + 2x^{-1/2} + 3$. 20. (1) $0.75x^{-1/4} + x^{-4/3} - 4x^{-3} - x^{-2}$; (2) $x^{-1/2} - x^{-5/3} + x^{-3/2} + x^{-2}$.

21. (1) $6x^2 + 14x + 1$; (2) $24x^3 + 22x$; (3) $4x^3$. 22. (1) $2a/(x + a)^2$; (2) $4x/(2 - x^2)^2$; (3) $(2x^2 - 2)/(x^2 + x + 1)^2$; (4) $(x^2 - 1)/(x^2 + 1)^2$. 28. (1) $5(x^3 - 2x^2 + 5)^4 (3x^2 - 4x)$; (2) $18x^2 (x^3 - 1)^5$. 29. (1) $n(ax^2 + bx + c)^{n-1} \times (2ax + b)$; (2) $-8x(r^2 - x^2)^3$. 30. (1) $15x^2/(1 - x^3)^6$; (2) $-an/(ax + b)^{n+1}$. 31. (1) $6x^2 (x^4 + 1)^2 (x^4 + 2x - 1)/(x^3 + 1)^3$; (2) $2an(a + x)^{n-1}/(a - x)^{n+1}$. 32. (1) $(x - 2)/\sqrt{x^2 - 4x + 6}$; (2) $\sqrt{3}/2$; (3) $-x/\sqrt{r^2 - x^2}$; (4) $-bx/(a\sqrt{a^2 - x^2})$; (5) $bx/(a\sqrt{x^2 - a^2})$; (6) $\sqrt{2px}/(2x)$. 33. (1) $(2x^2 - 1)/\sqrt{x^2 - 1}$; (2) $t(5t - 2)/\sqrt{2t - 1}$; (3) $t(3t^2 - 1)/\sqrt{t^2 - 1}$; (4) $-5(1 - 2x)^{3/2}$. 34. (1) $-a/(2(ax + b)^{3/2})$; (2) $-x/(x^2 + 1)^{3/2}$; (3) $x/(1 - x^2)^{3/2}$; (4) $-2x^3/(x^4 - 1)^{3/2}$. 35. (1) $(3 - 2x)/(1 - 2x)^{3/2}$; (2) $-3/(x^2 - 1)^{3/2}$; (3) $4/(x^2 + 4)^{3/2}$; (4) $-9x^2(9 + x^2)^{-1/2}$. 36. (1) $x^2/\sqrt[3]{(x^3 - 1)^2}$; (2) $(3/4)\, a/(ax + b)^{1/4}$; (3) $3\sqrt{2x - 1}$; (4) 1. 42. (1) 32 m/s; 22 m/s^2; (2) 0.5 m/s; -0.25 m/s^2; (3) 17 m/s; 2 m/s^2. 43. (1) 7 m/s^2; (2) 11 m/s^2. 44. $t = 4$ s. 45. 3 deg/s. 46. 20,000 J. 47. 35 A/s. 50. (1) 1; (2) 2.5657; (3) $x(2 \ln x + 1)$; (4) $-\ln x$; (5) 26.53. 51. (1) $2/(x \ln^2 x)$; (2) $-1/(x \ln^2 x)$. 52. (1) $1/x$; (2) $4x/(2x^2 - 3)$. 53. (1) $2/(1 - x^2)$; (2) $-1/(2 + x)$. 54. (1) $0.4343/x$; (2) $0.8686/(2x + 1)$. 55. (1) $1/(2x - 1)$; (2) $x/(x^2 - a^2)$; (3) $0.4343x/(x^2 + 4)$; (4) $1/(x^2 - 1)$. 56. (1) $3(\ln^2 3x)/x$; (2) $4(2x + 1)^{-1} \ln(2x + 1)$; (3) $4x(x^2 - 1)^{-1} \ln(x^2 - 1)$. 60. (1) $e(x^{-1} + \ln x)$; (2) $xe^x(2 + x)$; (3) $-xe^x$; (4) $3^x e^x (1 + \ln 3)$; (5) $e^x (1 - \ln 2)/2^x$; (6) $5 + e$. 61. (1) $-7e^x/(e^x + 2)^2$; (2) $-1/e^x$. 62. (1) $3x^2 5^{x^3} \ln 5$; (2) $2^{\sqrt{x}} \ln 2/(2\sqrt{x})$; (3) $(3^{\ln x} \ln 3)/x$. 63. (1) $-2xe^{-x^2}$; (2) $e^{\sqrt{x}}/(2\sqrt{x})$; (3) $e^{\ln x}/x$. 64. (1) $e^{x/3} + e^{-x/3}$; (2) $4/(e^{-x} - e^x)^2$. 65. (1) $-2/(x - 1)^2$; (2) 1; (3) $(x^4 + 3x^2 - 2x)/(x^2 + 1)^2$; (4) $-3/4$. 66. (1) $(1/2)\, t^{-1/2} + (1/3)\, t^{-2/3}$; (2) $x^{-4/3} - x^{-3/2}$; (3) $-1/16$; (4) 4/9. 67. (1) 12 $(m + n)$; (2) $4(x - 1)/(x + 1)^3$; (3) $-9x^2/(3x - 1)^4$; (4) $18\sqrt{3}$. 68. (1) 1/8; (2) -2; (3) $a(1 - ax)^{-3/2}(1 + ax)^{-1/2}$; (4) $-4/15$. 69. (1) 17/18; (2) $\sqrt{3}/36$; (3) $(7 - 4\sqrt{3})/2$. 70. (1) $4/(16x^2 - 1)$; (2) $-1/\sqrt{1 + x^2}$; (3) $-1/\sqrt{x^2 - 1}$; (4) $2/\sqrt{x^2 - 1}$; (5) $a/(1 - a^2x^2)$; (6) $(2 \ln x)/x$. 71. (1) $1.5\sqrt{e^{3x}}$; (2) $e^{2x}(2x + 1)$.

Quiz No. 1. Form I. (1) 15; (2) $9\sqrt{3}/2$; (3) 1/3; (4) $2e^2$; (4) $2e^2$; (5) 20 m/s^2. Form II. (1) -3; (2) $7\sqrt{2}$; (3) 1/2; (4) $2\sqrt{e}$; (5) 30 m/s^3.

Chapter 8

3. (1) Decreases in $(-\infty, 3)$, increases in $(3, +\infty)$; (2) decreases in $(-\infty, 1)$, increases in $(1, +\infty)$; (3) increases in $(-\infty, 2)$, decreases in $(2, +\infty)$. 4. (1) Increases in $(-\infty, 0)$ and $(2, +\infty)$, decreases in $(0, 2)$; (2) decreases in $(-\infty, 0)$ and $(1, +\infty)$, increases in $(0, 1)$. 5. (1) Decreases in $(-\infty, 1)$, increases in $(1, +\infty)$; (2) decreases in $(-\infty, 2)$, increases in $(2, +\infty)$; (3) increases in $(-\infty, -1)$, decreases in $(-1, +\infty)$. 6. (1) Increases in $(-\infty, 1)$ and $(2, +\infty)$, decreases in $(1, 2)$; (2) decreases in $(-\infty, 2)$ and $(3, +\infty)$, increases in $(2, 3)$. 7. (1) Increases in $(-\infty, 0)$ and $(0, +\infty)$; (2) decreases in $(-\infty, 2/3)$ and $(2/3, +\infty)$. 8. (1) Decreases in $(-\infty, 0)$, increases in $(0, +\infty)$; (2) decreases in $(0, +\infty)$. 9. (1) Decreases in $(0, 1)$, increases in $(1, +\infty)$; (2) increases in $(0, 1)$; decreases in $(1, +\infty)$. 10. (1) Decreases in $(-\infty, +\infty)$; (2) decreases in $(-\infty, 0)$, increases in $(0, +\infty)$; (3) decreases in $(\infty, 0)$ and $(0, +\infty)$. 11. (1) Increases in $(0, 1/4)$, decreases in $(1/4, 1/2)$; (2) decreases in $(-\infty, 0)$, increases in $(3, +\infty)$. 15. (1) $f_{\min} = f(1/2) = -1/4$; (2) $f_{\min} = f(-3/2) = -9/4$. 16. (1) $f_{\max} = f(1) = 1$; (2) $f_{\max} = f(-1/2) = 1/4$. 17. (1) $f_{\min} = f(4) = -4$; (2) $f_{\min} = 25/4$; (3) $f_{\max} = f(1/4) = 9/8$. 19. (1) $f_{\min} = f(1/2) = -3/8$; (2) $f_{\max} = f(2) = 12$. 20. (1) $f_{\max} = f(-2) = 16/3$, $f_{\min} = f(2) = -16/3$; (2) $f_{\max} = f(0) = 0$; $f_{\min}/f(2) = -4/3$. 21. (1) $f_{\max} = f(1) = -3$, $f_{\min} = f(2) = -4$; (2) $f_{\max} = f(-1) = 15$, $f_{\min} = f(2) = -12$; (3) $f_{\max} = f(-2) = -6$; $f_{\min} = f(-1) = -7$. 22. (1) $f_{\max} = f(0) = 5$; (2) $f_{\max} = f(8) = 4$; $f_{\min} = f(0) = 0$. 23. (1) $f_{\max} = f(-0.4) \approx 1.95$, $f_{\min} = f(0) = 0$; (2) $f_{\max} = f(4) \approx 15.1$, $f_{\min} = f(0) = 0$; 24. (1) $f_{\min} = f(0) = 2$; (2) $f_{\max} = f(2) = 4e^{-2}$,

$f_{min} = f(0) = 0$. 25. (1) $f_{min} = f(2) = 2 - 2\ln 2$; (2) $f_{min} = f(1/e) = -1/e$. 27. (1) $f_{min} = f(0) = -3$; (2) $f_{min} = f(1) = -1$; (3) $f_{min} = f(5/4) = -9/8$; (4) $f_{max} = f(2) = 4$; (5) $f_{max} = f(1/2) = 25/4$. 28. (1) $f_{max} = f(1) = 16/3$; $f_{min} = f(3) = 4$; (2) $f_{max} = f(1) = 22/3$; $f_{min} = f(5) = -10/3$; (3) $f_{max} = f(1) = 1/2$, $f_{min} = f(2) = 0$; (4) $f_{min} = f(0) = -4$. 29. (1) $f_{max} = f(-1) = -2$; $f_{min} = f(1) = 2$; (2) $f_{min} = f(-1) = -3/2$; $f_{max} = f(1) = 3/2$. 31. (1) $f_{least} = f(3) = 4$, $f_{gr} = f(0) = f(6) = 13$; (2) $f_{least} = f(-2) = f(2) = 6$; $f_{gr} = f(0) = 8$. 32. (1) $f_{least} = f(3) = -9/2$, $f_{gr} = f(1) = 1/6$; (2) $f_{least} = f(0) = f(6) = 0$, $f_{gr} = f(4) = 32$. 33. (1) $f_{least} = f(-4) = -41$; $f_{gr} = f(-1) = 40$; (2) $f_{least} = f(2) = -10$, $f_{gr} = f(0) = 10$. 37. $a/2$ and $a/2$. 38. $\sqrt{a}$ and $\sqrt{a}$. 39. A square 12.5 cm on a side. 40. A square $p/2$ on a side. 41. A square $R\sqrt{2}$ on a side. 42. The ratio of the sides of the rectangle is equal to 1/2. 43. The ratio of the sides of the rectangle is equal to 1/4. 44. The base and altitude of the triangle are equal to $a/2$. 45. An equilateral triangle $a\sqrt{3}$ on a side. 46. $a/2$ and $h/2$. 47. $0.5ab/\sqrt{a^2 + b^2}$ and $0.5\sqrt{a^2 + b^2}$. 48. $m\sqrt{3}/4$ and $m/2$. 49. 12 m/s. 50. $v_0^2/(2g)$. 51. 19.6 m. 54. Convex downwards at the point $x_1 = -1$, convex upwards at the point $x_2 = 1$; (2) at the points $x_1 = -2$ and $x_2 = 1$—convex downwards. 55. (1) Convex upwards in $(-\infty, 0)$, convex downwards in $(0, +\infty)$; (2) convex downwards everywhere; (3) convex upwards everywhere; (4) convex downwards everywhere. 56. Convex upwards in $(-\infty, 2)$, convex downwards in $(2, +\infty)$; (2) convex upwards in $(-1, 2)$, convex downwards in $(-\infty, -1)$ and in $(2, +\infty)$. 58. (1) (0, 0); (2) (3; 2). 59. (1) (2, —20), (3, 35); (2) (1, —6), (3, —86); (3) (1, —3), (2, 6). 60. (1) $(2, 2/e^2)$; (2) $(1/\sqrt{2}; e^{-1/2})$; $(-1/\sqrt{2}, e^{-1/2})$. **Quiz No. 1.** Form I. (1) Decreases in $(-\infty, 0)$ and $(1, +\infty)$, increases in (0, 1); (2) $y_{least} = y(1) = -3/2$, $y_{greatest} = y(-2) = 3$; (3) (a) convex upwards in $(-\infty, -1)$, convex downwards in $(-1, +\infty)$, a point of inflection (—1, 2); (b) convex upwards in $(-\infty, 0)$, convex downwards in $(0, +\infty)$, a point of inflection (0, 0); (4) $v_{max} = 1$ m/s. Form II. (1) Decreases in $(-\infty, 1)$, increases in $(1, +\infty)$; (2) $y_{least} = y(1) = -17/3$, $y_{greatest} = y(-3) = 5$; (3) (a) convex upwards in $(-\infty, 4)$, convex downwards in $(4, +\infty)$, a point of inflection (4, 17); (b) convex upwards in $(-\infty, -1)$, convex downwards in $(-1, +\infty)$, a point of inflection (—1, 1); (4) $y_{max} = 14$ m/s.

Chapter 9

8. (1) $\pi/6$; (2) $\pi/4$; (3) $\pi/3$; (4) $\pi/2$; (5) $2\pi/3$; (6) $3\pi/4$; (7) $5\pi/6$; (8) π; (9) $7\pi/6$; (10) $5\pi/4$; (11) $3\pi/2$; (12) $11\pi/6$. 9. (1) 0.2670; (2) 1.2421; (3) 0.2667; (4) 2.0199; (5) 3.7554; (6) 5.4510; (7) 1.0021; (8) 1.5272. **10.** (1) 0.253; (2) 0.476; (3) 1.309; (4) 2.27; (5) 0.675; (6) 1.501. 11. (1) 25°; (2) 105°; (3) 110°; (4) 100°; (5) 99°; (6) 78°; (7) 330°; (8) 240°. 12. (1) 17°14′; (2) 32°49′; (3) 60°; (4) 77°05′; (5) 85°05′; (6) 100°. 13. 0.56 m. 14. 1.08 m. 15. 4.6 rad. 16. 0.75 m. 17. 0.2 m. 18. $\pi/2$ s. 19. 3 m/s. 20. 6 rad/s, —2 rad/s²; in 5 s. 21. $\pi/9$, $\pi/3$, $5\pi/9$. 22. $\pi/3$, $4\pi/9$, $5\pi/9$, $2\pi/3$. 23. 0.3 m. 24. $2\pi/15$. 25. $13\pi/18$. 26. Hour-hand $\pi/21{,}600$; minute-hand $\pi/1800$; second-hand $\pi/30$. 29. (1) in II; (2) in IV; (3) in II; (4) in II; (5) in II; (6) in III; (7) in I; (8) in III; (9) in I; (10) in I; (11) in II; (12) in II; (13) in I; (14) in IV; (15) in IV; (16) in II; (17) in II; (18) in III. 30. (1) $360° \cdot 6 + 40°$; (2) $360°(-2) + 170°$; (3) $360°(-2) - 20°$; (4) $360° \cdot 3 + 90°$; (5) $360°(-4) - 10°$; (6) $360° \cdot 1 + 160°$. 31. (1) $2\pi k$, $k \in Z$; (2) $\pi/2 + 2\pi k$, $k \in Z$; (3) $\pi + 2\pi k = \pi(2k + 1)$, $k \in Z$; (4) $-\pi/2 + 2\pi k$ or $3\pi/2 + 2\pi k$, $k \in Z$. 32. $\pi/6 + 2\pi k$; $\pi/3 + 2\pi k$, $\pi/2 + 2\pi k$, $2\pi/3 + 2\pi k$, $5\pi/6 + 2\pi k$, $\pi(2k + 1)$, $7\pi/6 + 2\pi k$, $4\pi/3 + 2\pi k$, $3\pi/2 + 2\pi k$, $5\pi/3 + 2\pi k$, $11\pi/6 + 2\pi k$, $2\pi k$, $k \in Z$. 33. (1) -2π, $-\pi$, 0, π, 2π; (2) $-15\pi/4$, $-7\pi/4$, $\pi/4$, $9\pi/4$, $17\pi/4$; (3) $-11\pi/3$, $-5\pi/3$, $\pi/3$, $7\pi/3$, $13\pi/3$; (4) $-11\pi/6$, $-7\pi/6$, $\pi/6$, $5\pi/6$, $13\pi/6$. 36. (1) In II; (2) in I; (3) at the point (0, —1); (4) at the point (0; —1). 37. (1) $\alpha = \pi/8 + \pi k$, $k \in Z$; (2) $\alpha = \pi/3 + 2\pi k/3$, $k \in Z$; (3) $\alpha = \pi/12 + \pi k/4$, $k \in Z$; (4) $\alpha = 40° + 180° k$, $k \in Z$. 38. (2), (4), (6) and (8). No. 39. (1), (4) and (6) Yes; (2) (3) and (5) only tangent

and cotangent. 40. Negative values can be attained by $\cos\alpha$, $\tan\alpha$ and $\cot\alpha$. 42. $|\tan\alpha| \geqslant |\sin\alpha|$. 43. $-1 \leqslant \sin\alpha \leqslant 1$; $-1 \leqslant \cos\alpha \leqslant 1$; $-\infty < \tan\alpha < +\infty$; $-\infty < \cot\alpha < +\infty$. 44. (1) $\alpha = 0$, $\alpha = \pi$, $\alpha = 2\pi$; (2) $\alpha = \pi/2$; (3) $\alpha = 3\pi/2$; (4) $\alpha = \pi/2$; $\alpha = 3\pi/2$; (5) $\alpha = 0$, $\alpha = 2\pi$; (6) $\alpha = \pi$. 45. (1) $\alpha = 0$, $\alpha = \pi$, $\alpha = 2\pi$; (2) $\alpha = \pi/4$, $\alpha = 5\pi/4$; (3) $\alpha = 3\pi/4$, $\alpha = 7\pi/4$; (4) $\alpha = \pi/2$, $\alpha = 3\pi/2$; (5) $\alpha = \pi/4$, $\alpha = 5\pi/4$; (6) $\alpha = 3\pi/4$, $\alpha = 7\pi/4$. 51. The sign "+" in (1), (2), (6), (7), (12); does not exist in (5); the sign "—" in the remaining problems. 52. The sign "+" in (1), (2), (3), (7), and (8), the sign "—" in the remaining problems. 53. The sign "+" in (1), (2), (5), (6), and (7); the sign "—" in the remaining problems. 54. (1) 1; (2) —2; (3) —1. 55. (1) in II or IV; (2) in III or IV; (3) in I or IV; (4) in I or II. 56. (1) I; (2) 0; (3) —1; (4) 0; (5) 0; (6) $\sqrt{2} - 2\sqrt{3}$; (7) —5; (8) —13; (9) 21; (10) $(a-b)/(a+b)$; (11) $a-b$; (12) 2; (13) $3\sqrt{3}$. 57. (1) 5; 3; $-5-\sqrt{3}$; —5; (2) $(3+\sqrt{3})/2$; (3) 4; (4) 3/4. 58. The sign "+" in (1), (2), (7), (10), and (12); the sign "—" in the remaining problems. 59. The sign "+" in (1) and (4); the sign "—" in the remaining problems. 60. (1) to I or III; (2) to I or II; (3) to I or III; (4) to I or II; (5) to I or IV; (6) to I or III; (7) to any quadrant; (8) to I or III. 61. (1) +; (2) +; (3) —. 62. (1) 0; (2) —3; (3) $(a-b)^2$; (4) $\sqrt{3}/4$; (5) 4; (6) $(4-\sqrt{2})/4$; (7) 3/4. 63. (1) $(a+b):(a-b)$; (2) a^2+b^2; (3) $-a^3+b^3$; (4) $(a+b)/a$. 72. (1) $\cos\alpha = 12/13$, $\tan\alpha = -5/12$, $\cot\alpha = -12/5$; (2) $\sin\alpha = 15/17$, $\tan\alpha = -15/8$, $\cot\alpha = -8/15$, (3) $\sin\alpha = -8/17$; $\cos\alpha = -15/17$; $\cot\alpha = 15/8$, (4) $\sin\alpha = -24/25$, $\cos\alpha = 7/25$, $\tan\alpha = -24/7$. 73. (1) $1/\cos^2\alpha$ ($\alpha \neq \pi/2 + \pi k$, $k \in Z$); (2) $\sin^2\alpha$; (3) 0; (4) $2/\sin\alpha$ ($\alpha \neq \pi k$, $k \in Z$), (5) 1; (6) 1; (7) $\cos\alpha - \sin\alpha$ (8) 1. 75. (1) 6/19; (2) 3/4. 76. (1) 2; (2) $\cos^2\alpha$ (3) 1; (4) 0 ($\alpha \neq \pi k$, $k \in Z$); (5) $2\cot\alpha$ ($\alpha \neq \pi k$, $k \in Z$); (6) $\cot\alpha$ ($\alpha \neq \pi k/2$, $k \in Z$); (7) $2/\cos^2\alpha$ ($\alpha \neq \pi/2 + \pi k$, $k \in Z$); (8) $-1/\cos\alpha$ ($\alpha \neq \pi/2 + \pi k$, $k \in Z$). 78. (1) 17/25; (2) 13/25; (3) 25/21; (4) 21/40. 79. (1) $\pm\sqrt{5}$; (2) $\pm 3\sqrt{5}$; (3) 7; (4) 18. **Quiz No. 1.** Form I. (1) $\sqrt{6}/6$; (2) —; (3) $\cos\alpha = 4/5$, $\tan\alpha = -3/4$, $\cot\alpha = -4/3$; (4) $\sin\alpha = -\sqrt{3}/2$, $\cos\alpha = -1/2$, $\tan\alpha = \sqrt{3}$; (5) $\alpha \neq \pi k/4$, $k \in Z$. Form II. (1) $5\sqrt{3}/2$; (2) —; (3) $\sin\alpha = 3/5$, $\tan\alpha = -3/4$, $\cot\alpha = -4/3$; (4) $\sin\alpha = -\sqrt{3}/2$, $\cos\alpha = -1/2$, $\cot\alpha = \sqrt{3}/3$; (5) $\alpha \neq \pi k/2$, $k \in Z$, $\alpha \neq -\pi/4 + \pi n$, $n \in Z$. 83. (1) $\sqrt{3}/2$; (2) 0; (3) 1; (4) $\sqrt{3}$; (5) 0.682. 84. (1) 1.5389; (2) 2; (3) $(\sqrt{3}-1)/2$; (4) —0.121; (5) 1. 85. (1) $\lambda = 2\pi/3$; (2) $\lambda = 8\pi$; (3) $\lambda = \pi/2$; (4) $\lambda = 5\pi$. 86. (1) 2π; (2) 24π; (3) 6π; (4) 16π. 87. (1) 5; (2) 7; (3) 1; (4) 0; (5) —1; (6) —1/2; (7) —1. 88. (1) $2\sin^2\alpha$; (2) 3; (3) $2\cos\alpha$; (4) 2. 92. Equalities (4), (7), (8) are not true. 93. (1) 0.4189; (2) 1.4835; (3) 0.1745; (4) 1.3963; (5) 0.2618; (6) 1.2217; (7) 0.4363; (8) 1.3788. 94. (1) $-\pi/3$; (2) $2\pi/3$; (3) $-\pi/6$; (4) $3\pi/4$; (5) $5\pi/6$; (6) —1.1275; (7) 2.6826; (8) —0.9721; (9) 2.5098; (10) 1.8710. 96. (1) $\pi/2$; (2) $\pi/2$; (3) $\pi/2$; (4) π; (5) $\pi/2$; (6) $\pi/2$; (7) 0; (8) $3\pi/2$. 113. (1) $\pi/6$; (2) $\pi/4$; (3) $-\pi/4$; (4) $-\pi/3$; (5) arcsin (3/4). 114. (1) $(-1)^k\pi/3 + \pi k$, $k \in Z$; (2) $(-1)^{k+1}\pi/6 + \pi k$, $k \in Z$; (3) $\pi/2 + 2\pi k$, $k \in Z$; (4) $(-1)^{k+1}\pi/4 + \pi k$, $k \in Z$. 115. (1) $(-1)^k$ arcsin (1/3) + πk; $k \in Z$; (2) $(-1)^{k+1}$ arcsin (2/3) + πk, $k \in Z$; (3) $(-1)^k$ arcsin 0.6 + πk, $k \in Z$. 116. (1) $\pm 2\pi/3 + 2\pi k$, $k \in Z$; (2) $\pm\pi/6 + 2\pi k$, $k \in Z$; (3) $\pm 3\pi/4 + 2\pi k$, $k \in Z$; (4) $\pm 5\pi/6 + 2\pi k$, $k \in Z$. 117. (1) $\pm$arccos (4/5) + $2\pi k$, $k \in Z$; (2) $\pm[\pi$ — arccos (4/5)] + $2\pi k$, $k \in Z$; (3) $\pm$arccos 0.6 + $2\pi k$, $k \in Z$. 118. (1) $\pi/6$; (2) $-\pi/3$; (3) arctan (1/2); (4) —arctan 0.7. 119. (1) πk, $k \in Z$; (2) $\pi/6 + \pi k$, $k \in Z$; (3) $-\pi/4 + \pi k$, $k \in Z$; (4) $-\pi/3 + \pi k$, $k \in Z$; (5) arctan $\sqrt{2} + \pi k$, $k \in Z$. 120. (1) —arctan (2/3) + πk, $k \in Z$; (2) arctan 2 + πk, $k \in Z$; (3) — arctan 1.5 + πk, $k \in Z$. 121. (1) $2\pi/3 + \pi k$, $k \in Z$; (2) $5\pi/6 + \pi k$, $k \in Z$; (3) $\pi/4 + \pi k$, $k \in Z$; (4) arccot (1/2) + πk, $k \in Z$. 122. (1) π — arccot 2 + πk, $k \in Z$; (2) arccot 0.8 + πk, $k \in Z$; (3) π — arccot (2/3) + πk, $k \in Z$. 132. (1) $\{(-1)^k\pi/4 + \pi k \mid k \in Z\}$; (2) $\{(-1)^{k+1}\pi/4 + \pi k \mid k \in Z\}$; (3) $\{(-1)^{k+1}\pi/3 + \pi k \mid k \in Z\}$; (4) $\{(-1)^k\pi/3 + \pi k, k \in Z\}$; (5) $\{(-1)^k$ arcsin (4/5) + $\pi k \mid k \in Z\}$. 133. (1) $\{\pm\pi/3 + 2\pi k \mid k \in Z\}$; (2) $\{\pm 3\pi/4 + 2\pi k \mid k \in Z\}$; (3) $\{\pm 5\pi/6 + 2\pi k \mid k \in Z\}$; (4) $\{\pm\pi/6 + 2\pi k \mid k \in Z\}$; (5) $\{\pm(\pi$ — arccos 0.3) + $2\pi k \mid k \in Z\}$. 134. (1) $\{-\pi/6 + \pi k \mid k \in Z\}$; (2) $\{\pi/4 + \pi k \mid k \in Z\}$; (3) $\{53° + 180° k$.

$k \in Z$}. 135. (1) $\{\pi/4 + \pi k \mid k \in Z\}$; (2) $\{5\pi/6 + \pi k \mid k \in Z\}$; (3) $\{2\pi/3 + \pi k \mid k \in Z\}$; (4) $\{26^\circ + 180^\circ k \mid k \in Z\}$. 136. (1) $\{2(-1)^k \arcsin(1/4) - \pi/3 + 2\pi k \mid k \in Z\}$; (2) $\{\pi/12 - 1/3 + \pi k/3 \mid k \in Z\}$; (3) $\{\pi/18 + \pi k/3 \mid k \in Z\}$; (4) $\{(-1)^k/4 + k \mid k \in Z\}$; (5) $\{\pm\sqrt{2\pi \mid k \mid} \mid k \in Z\}$. 137. (1) $\{\pi k \pm \pi/4 \mid k \in Z\}$; (2) $\{\pi k \pm \pi/2 \mid k \in Z\}$; (3) $\{\pi k \pm \pi/3 \mid k \in Z\}$; (4) $\{\pi k \mid k \in Z\}$. 138. (1) $\{\pi k \mid k \in Z\}$; (2) $\{\pi k \pm \pi/4 \mid k \in Z\}$; (3) $\{\pi k \pm \arccos(1/3) \mid k \in Z\}$; (4) $\{\pi k \pm \pi/2 \mid k \in Z\}$. 139. (1) $\{\pi k \pm \pi/4 \mid k \in Z\}$; (2) $\{\pi k \pm \pi/6 \mid k \in Z\}$; (3) $\{\pi k \pm \pi/6 \mid k \in Z\}$. 140. (1) $\{2\pi k, \pm\pi/3 + 2\pi k \mid k \in Z\}$; (2) $\{\pi(2k+1) \mid k \in Z\}$; (3) $\{\pi/4 + \pi k;\ \operatorname{arccot} 0.6 + \pi k \mid k \in Z\}$; (4) $\{\pm\pi/4 + \pi k \mid k \in Z\}$, (5) $\{\pm\pi/6 + \pi k \mid k \in Z\}$; (6) $\varnothing$; (7) $\{\pi(2k+1);\ \pm\pi/3 + 2\pi k \mid k \in Z\}$. 141. (1) $\{\pi k;\ k \in Z\}$; (2) $\{3\pi/4 + \pi k \mid k \in Z\}$; (3) $\{-3\pi/8 + \pi k/2 \mid k \in Z\}$; (4) $\{\pi k,\ -\pi/4 + \pi k \mid k \in Z\}$; (5) $\{\pi/4 + \pi k \mid k \in Z\}$; (6) $\{2\pi k \mid k \in Z\}$. 142. (1) $\{\pi k \mid k \in Z\}$; (2) $\{\pi/4 + \pi k;\ \pi k \mid k \in Z\}$; (3) $\varnothing$; (4) $\{-\pi/4 + \pi k,\ \pm\pi/3 + \pi k \mid k \in Z\}$; (5) $\arctan 1.5 + \pi k \mid k \in Z\}$. 143. (1) $\arctan 3 + \pi k$, $\arctan 7 + \pi k \mid k \in Z\}$; (2) $\{-\pi/4 + \pi k;\ \arctan(3/4) + \pi k \mid k \in Z\}$; (3) $\{\pi/4 + \pi k, \arctan 5 + \pi k \mid k \in Z\}$; (4) $\{\pi k;\ \arctan 2 + \pi k \mid k \in Z\}$. 148. (1) $(\pi(2k-1), 2\pi k)$, $k \in Z$; (2) $(2\pi k, \pi(2k+1))$, $k \in Z$; (3) $(-3\pi/2 + 2\pi k, \pi/2 + 2\pi k)$, $k \in Z$, (4) $(-5\pi/6 + 2\pi k, -\pi/6 + 2\pi k)$, $k \in Z$; (5) $(-\pi/6 + 2\pi k, 7\pi/6 + 2\pi k)$, $k \in Z$; (6) $(-\pi/3 + 2\pi k, 4\pi/3 + 2\pi k)$, $k \in Z$, (7) $(-\pi/6 + \pi k, \pi/6 + \pi k)$, $k \in Z$; (8) $(\pi/6 + 2\pi k, 5\pi/6 + 2\pi k)$, $k \in Z$. 149. (1) $(\pi/2 + 2\pi k, 3\pi/2 + 2\pi k)$, $k \in Z$; (2) $(-\pi/2 + 2\pi k, \pi/2 + 2\pi k)$, $k \in Z$, (3) $(\pi(2k-1), \pi(2k+1))$, $k \in Z$; (4) $(2\pi k, 2\pi(k+1))$, $k \in Z$; (5) $(\pi/3 + 2\pi k, 5\pi/3 + 2\pi k)$, $k \in Z$, (6) $(-\pi/3 + 2\pi k, \pi/3 + 2\pi k)$, $k \in Z$, (7) $(\pi/3 + \pi k, 2\pi/3 + \pi k)$, $k \in Z$. 150. (1) $(-\pi/2 + \pi k, -\pi/3 + \pi k)$, $k \in Z$; (2) $(-\pi/3 + \pi k, \pi/2 + \pi k)$, $k \in Z$; (3) $(-\pi/3 + \pi k, \pi/3 + \pi k)$, $k \in Z$. 151. (1) $(-\pi/4 + \pi k, \pi k)$, $k \in Z$; (2) $(\pi k, 3\pi/4 + \pi k)$, $k \in Z$; (3) $(\pi k, \pi/6 + \pi k)$, $k \in Z$. (4) $(\pi/4 + \pi k, 3\pi/4 + \pi k)$, $k \in Z$. 152. (1) $(-5\pi/12 + \pi k, -\pi/12 + \pi k)$, $k \in Z$; (2) $(-4\pi/3 + 4\pi k, 4\pi/3 + 4\pi k)$, $k \in Z$; (3) $(3\pi k, 3\pi/4 + 3\pi k)$, $k \in Z$, (4) $(-\pi/12 + \pi k/3, \pi/6 + \pi k/3)$, $k \in Z$; (5) $(-\pi/2 + 2\pi k, 3\pi/2 + 2\pi k)$, $k \in Z$; (6) $(-3\pi/4 + 2\pi k, -\pi/4 + 2\pi k)$, $k \in Z$; (7) $(-\pi/4 + 2\pi k, 5\pi/4 + 2\pi k)$, $k \in Z$; (8) $(5\pi/6 + 2\pi k, 7\pi/6 + 2\pi k)$, $k \in Z$; (9) $(-\pi/2 + \pi k, \pi/6 + \pi k)$, $k \in Z$; (10) $(\pi/4 + \pi k;\ \pi + \pi k)$, $k \in Z$. Quiz No. 2. Form I. (1) $((2k-1)2\pi, 4\pi k)$, $k \in Z$; (2) $\{\pi(2k+1) \mid k \in Z\}$; (3) $\{(-1)^k \pi/18 + \pi k/3 \mid k \in Z\}$; (4) $\{\pi/4 + \pi k, \pm\pi/3 + 2\pi k \mid k \in Z\}$; (5) $\pi/4 + \pi k$; $-\arctan(3/2) + \pi k \mid k \in Z$. Form II. (1) $(\pi k, 5\pi/6 + \pi k)$, $k \in Z$; (2) $\{(-1)^k \pi/3 - \pi/3 + 2\pi k \mid k \in Z\}$; (3) $\{\pi/4 + \pi k/2;\ \pi k \mid k \in Z\}$; (4) $\{-\pi/2 + 2\pi k, \pm\pi/3 + \pi k \mid k \in Z)\}$; (5) $\{\pi/4 + \pi k, -\arctan(5/3) + \pi k \mid k \in Z\}$. 155. (1) $\sqrt{2}/2$; (2) $-\sqrt{3}/2$; (3) $-\sqrt{2}/2$; (4) $\sqrt{2}/2$; (5) $-1/2$; (6) $-\sqrt{3}/2$. 156. (1) $\sqrt{3}/2$; (2) $-\sqrt{2}/2$; (3) $-\sqrt{2}/2$; (4) $\sqrt{2}/2$; (5) $-1/2$; (6) $-\sqrt{2}/2$. 157. (1) -0.375; (2) -0.743; (3) -0.375; (4) -0.899. 162. (1) $-\sqrt{3}/2$; (2) -1, (3) $\sqrt{3}/2$; (4) -0.8391; (5) $-\sqrt{3}/2$; (6) -0.9848; (7) 0.8391; (8) 2.747; (9) $-1/2$; (10) -0.1736; (11) 0.1736; (12) -0.364; (13) -1.4281; (14) -0.2588; (15) 0.4226; (16) -0.4663. 163. (1) $-\sqrt{2}/2$; (2) 1/2; (3) $\sqrt{3}$; (4) 1; (5) $\sqrt{2}/2$; (6) $-\sqrt{3}$; (7) $-\sqrt{3}$; (8) -0.6428; (9) -0.6428; (10) 2.747; (11) $-\sqrt{2}/2$; (12) 0.9397. 164. (1) -0.375; (2) -0.857; (3) -0.424. 165. (1) -1; (2) -3; (3) 0; (4) $\sqrt{3}$; (5) $-\sqrt{3}$; (6) $\sqrt{3} - 1$; (7) $-\sqrt{3}/2$. 166. (1); (2) 1. 168. (1) $\{2\pi k \mid k \in Z\}$; (2) $\{(-1)^k \pi/6 + \pi k \mid k \in Z\}$; (3) $\{\pi/2 + \pi k \mid k \in Z\}$; (4) $\{\pi k;\ (-1)^k \arcsin 0.4 + \pi k \mid k \in Z\}$; (5) $\{\pi/2 + \pi k;\ \pm\arccos(1/3) + 2\pi k \mid k \in Z\}$; (6) $\{\pi k;\ \arctan 2.4 + \pi k \mid k \in Z\}$; (7) $\{\pi/2 + \pi k;\ \operatorname{arccot} 1.5 + \pi k \mid k \in Z\}$. Quiz No. 3. Form I. (1) 0; (2) $2 + \sqrt{2}$; (4) $\{\arctan 2 + \pi k,\ \arctan 4 + \pi k \mid k \in Z\}$; (5) $\{\pi/4 + \pi k;\ -\arctan 2 + \pi k \mid k \in Z\}$. Form II. (1) 1; (2) 0; (4) $\arctan 3 + \pi k$; $\arctan 7 + \pi k \mid k \in Z\}$; (5) $\{\pi/4 + \pi k;\ \arctan 3 + \pi k \mid k \in Z\}$. 177. $\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4$, $\sin(5\pi/12) = (\sqrt{6} + \sqrt{2})/4$, $\sin(7\pi/12) = (\sqrt{6} + \sqrt{2})/4$; $\cos(\pi/12) = (\sqrt{6} + \sqrt{2})/4$, $\cos(5\pi/12) = (\sqrt{6} - \sqrt{2})/4$, $\cos(7\pi/12) = (\sqrt{2} - \sqrt{6})/4$. 178. (1) 0; 24/25; (2) $-13/85$, $-77/85$; (3) $-\sqrt{2}/10$, $-7\sqrt{2}/10$. 179. (1) 71/98;

(2) 1; (3) 3/11; (4) $-5/12$. 180. (1) $2-\sqrt{3}$; (2) $2+\sqrt{3}$; (3) $-2-\sqrt{3}$; (4) $\sqrt{3}$; (5) $2-\sqrt{3}$. 181. (1) $\cos(5\pi/12)$; (2) $\sin(\pi/12)$; (3) $2\cos\alpha\sin\beta$; (4) $-3/4$; (5) $\cos^2\alpha$; (6) $\tan\alpha$; (7) $\cot\alpha$. 183. (1) $\{\pi k \mid k\in Z\}$; (2) $\{\operatorname{arccot} 4\sqrt{3}+\pi k \mid k\in Z\}$; (3) $\{\arctan 3\sqrt{3}+\pi k \mid k\in Z\}$. 184. (1) $\cos(\alpha+\beta)=-3/5$; $\cos(\alpha-\beta)=117/125$}; (2) $\sin(\alpha-\beta)=3/5$, $\cos(\alpha+\beta)=4/5$. 185. (1) 1/5); (2) 5. 186. (1) $\sqrt{6}/2$; (2) $\sin^2\alpha$. 191. (1) $\{\pi/6\,(2k+1) \mid k\in Z\}$; (2) $\{\pi k/2 \mid k\in Z\}$; (3) $\{\pi k \mid k\in Z\}$. Quiz No. 4. Form I. (1) -1; (2) 140/221; (5) $\{\pi/6+\pi k/3, \pi/2+\pi k \mid k\in Z\}$. Form II. (1) 4/5; (2) $-171/221$; (5) $\{\pi/6+\pi k/3, \pi/2+\pi k \mid k\in Z\}$. 194. (1) 24/25, 7/25, 24/7; (2) $-120/169$, 119/169, 120/119; (3) $-24/25$, 7/25, $-24/7$. 195. (1) $-8/15$; (2) 120/169, 119/169, 120/119; (3) 119/169, 120/119. 196. (1) $-3/5$; (2) $-7/15$. 197. (1) $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$; (2) $\cos 4\alpha = \cos^4\alpha - 6\sin^2\alpha\cos^2\alpha\sin^4\alpha$; (3) $\tan 3\alpha = (3\tan\alpha - \tan^3\alpha)/(1-3\tan^2\alpha)$; (4) $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$; (5) $\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$. 198. (1) $-\sin(8x/3)$; (2) $-\sin 3x$; (3) $\sin 5x$; (4) $\sin x$. 200. (1) $\{\pi k;\ \pm\pi/3+2\pi k \mid k\in Z\}$; (2) $\{\pi/4+\pi k \mid k\in Z\}$; (3) $\{\pi k \mid k\in Z\}$. 207. (1) 4/5, 3/5, 4/3; (2) $3\sqrt{34}/34$, $-5\sqrt{34}/34$, $-3/5$; (3) $\sqrt{5}/5$, $2\sqrt{5}/5$, 1/2. 208. (1) $-2/\sqrt{13}$; (2) $-3/\sqrt{13}$. 209. (1) 3/5; (2) $-3/5$; (3) $-\sqrt{3}$; (4) 1/7. 210. (1) $\{2\pi k;\ (-1)^k\pi/3+2\pi k \mid k\in Z\}$; (2) $\{\pi(2k+1);\ (\pm 2\pi/3+4\pi k \mid k\in Z\}$; (3) $\{\pi(2k+1);\ \pi/2+2\pi k \mid k\in Z\}$. 211. (1) 24/25, $-7/25$, $-24/7$; (2) 1/2, $\sqrt{3}/2$, $\sqrt{3}/3$; (3) $3\sqrt{10}/10$, $\sqrt{10}/10$; 3. 212. (1) 1/2; (2) $\sqrt{3}/2$; (3) 1/8. 213. (1) 720/1681; (2) $-1/9$; (3) 175/288. 214. (1) $2\sin^2 20°$; (2) $\tan^2(\alpha/2)$; (3) $\tan^2(\pi/4-\alpha/2)$; (4) $\sin 2\alpha$; (5) $\sin 2\alpha$; (6) 1. 215. (1) 0.5 $\tan 2\alpha$; (2) 1; (3) $\cos 2\alpha$; (4) $2\cot\alpha$; (5) $\cot^2\alpha$; (6) 1. 219. (1) $\{(-1)^k\pi/12+\pi k/2 \mid k\in Z\}$; (2) $\{\pm\pi/3+\pi k \mid k\in Z\}$; (3) $\{(-1)^k\pi/12+\pi k \mid k\in Z\}$; (4) $\{-\pi/4+\pi k;$ $\operatorname{aretan}(2\pm\sqrt{3})+\pi n \mid k,\ n\in Z\}$; (5) $\{\pi k \mid k\in Z\}$; (6) $\{4\pi k;\ \pi(2k+1) \mid k\in Z\}$; (7) $\{2\pi k;\ \pi(1+4k) \mid k\in Z\}$; (8) $\{\pi(2k+1);\ \pi/3+2\pi k \mid k\in Z\}$. Quiz No. 5. Form I. (1) $\sin 2\alpha = -\sqrt{3}/2$, $\cos 2\alpha = 1/2$, $\tan 2\alpha = -\sqrt{3}$; (2) $\sqrt{5}/5$; $-2\sqrt{5}/5$ and $-1/2$; (5) $\{2\pi k;\ 2\pi/3+2\pi n \mid k,\ n\in Z\}$. Form II. (1) $\sin 2\alpha = 24/25$, $\cos 2\alpha = -7/25$, $\tan 2\alpha = -24/7$; (2) 1/2, $\sqrt{3}/2$ and $\sqrt{3}/3$; (5) $\{2\pi k \mid k\in Z\}$. 222. (1) $(1/2)\cos 2x + (1/2)\cos 12x$; (2) $(1/2)\cos 10x - (1/2)\cos 12x$; (3) $(1/2)\sin 3x + (1/2)\sin 7x$; (4) $(1/2)\sin 2\alpha - (1/2)\sin 2\beta$; (5) $(1/2)\cos\alpha + (1/2)\cos(3\alpha+2\beta)$; (6) $(1/4)\cos x + (1/4)\cos y + (1/4)\cos(x+y) + (1/4)$. 223. (1) $(1/2)\cos 2x$; (2) $\sqrt{3}+2\cos 2x$; (3) $\cos 2x + \sqrt{3}\sin 2x + \sqrt{3}$; (4) $2+\cos 2x - \sqrt{3}\sin 2x$. 224. (1) $(3/4)\cos x + (1/4)\cos 3x$; (2) $3/8 - (1/2)\cos 2x + (1/8)\cos 4x$; (3) $3/8 + (1/2)\cos 2x + (1/8)\cos 4x$; (4) $(5/8)\sin x - (5/16)\sin 3x + (1/16)\sin 5x$; (5) $(5/8)\cos x + (5/16)\cos 3x + (1/16)\cos 5x$. 228. (1) 1; (2) $2\sin\left(\frac{\pi}{4}-\frac{\alpha-\beta}{2}\right)\times\cos\left(\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)$; (3) $2\sqrt{2}\cos\alpha\sin\left(\frac{\pi}{4}-\alpha\right)$; (4) $\sqrt{3}\cos 10°$; (5) $\sin 10°/(\cos 15°\cos 25°)$; (6) $\sin 8\alpha\sin 2\alpha$. 229. (1) $2\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\beta\right)$; (2) $\cos 10°$. 230. (1) $4\sin 8°\sin 13°\sin 21°$; (2) $4\sin\frac{A+B}{2}\cos\left(\frac{A-B}{4}+\frac{\pi}{6}\right)\cdot\cos\left(\frac{A-B}{4}-\frac{\pi}{6}\right)$; (3) $4\cos\alpha\sin\left(15°-\frac{\alpha}{4}\right)\cos\left(15°+\frac{\alpha}{4}\right)$; (4) $4\sin\frac{A}{2}\cdot\sin\left(\frac{\pi}{4}-\frac{B}{2}\right)\sin\left(\frac{\pi}{4}+\frac{C}{2}\right)$; (5) $4\sin 31°\cos 5°\cos 1°$. 233. (1) $\{\pi/12+\pi k/6;\ \pi k \mid k\in Z\}$; (2) $\{\pi/2+\pi k \mid k\in Z\}$; (3) $\{\pi k \mid k\in Z\}$; (4) $\{45°+180°k \mid k\in Z\}$. 237. (1) $2\cos\left(\frac{\pi}{6}+\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{6}-\frac{\alpha}{2}\right)$; (2) $4\sin\left(\frac{\alpha}{2}+\frac{\pi}{6}\right)\sin\left(\frac{\alpha}{2}-\frac{\pi}{6}\right)$; (3) $4\sin\left(\frac{\pi}{12}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{12}+\frac{\alpha}{2}\right)$; (4) $4\sin\left(\frac{\pi}{12}+\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{12}-\frac{\alpha}{2}\right)$;

(5) $\dfrac{\sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right)}{\cos\alpha}$; (6) $\dfrac{\cos 2\alpha}{\cos^2\alpha}$; (7) $-\dfrac{\cos 2\alpha}{\sin^2\alpha}$; (8) $4\cos\left(\frac{\pi}{12}+\frac{\alpha}{2}\right)\times\cos\left(\frac{\pi}{12}-\frac{\alpha}{2}\right)$; (9) $\sin\left(\alpha-\frac{\pi}{3}\right)\cos\left(\alpha+\frac{\pi}{3}\right)$; (10) $2\sqrt{2}\sin\left(\frac{\pi}{8}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{8}+\frac{\alpha}{2}\right)$; (11) $\tan\left(\frac{\pi}{4}-\varphi\right)$, where $\varphi=\arctan\frac{b}{a}$. 238. (1) $2\sin\times\left(\alpha-\frac{\pi}{6}\right)$; (2) $5\sin(\alpha-36°52')$;

(3) $3\sin(\alpha+48°10')$. 239. $\{(-1)^k\,\pi/6+\pi/6+\pi k \mid k\in Z\}$; (2) $\{\pi/3+2\pi k \mid k\in Z\}$; (3) $\{(-1)^k\,\pi/3-\pi/3+\pi k \mid k\in Z\}$. 240. (1) $\sin\alpha+\sin 2\alpha+\sin 3\alpha$; (2) $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\sin 4\alpha$; (3) $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\cos 4\alpha$. 241. (1) 3/16; (2) 3. 242. (1) $4\sin 3\alpha\cos(\alpha/2)\cos 3\alpha/2$; (2) $4\sin 2\alpha\cos^2(\alpha/2)$. 243. (1) $\{\pi/6+\pi k \mid k\in Z\}$; (2) $\{\pm\pi/4+\pi k \mid k\in Z\}$. **Quiz No. 6. Form I.** (1) $\sin 1°$; (4) $\{\pi k/2 \mid k\in Z\}$; (5) $\{\pi k/2;\ (-1)^n\,\pi/6+\pi n \mid k,\ n\in Z\}$. **Form II.** (1) $\sqrt{3}\cos 25°$; (4) $\{-\pi/8+\pi k/2 \mid k\in Z\}$; (5) $\{-\pi/4+(2k+1)\,\pi;\ \pi/12+2\pi k/3 \mid k\in Z\}$. 249. (1) 2; (2) $\sqrt{2}/2$. 250. (1) $-1/2$; (2) -2. 251. (1) 3/2; (2) 0; (3) 1/16; (4) 27. 252. (1) 0; (2) ∞. 253. (1) 2; (2) 2; (3) 4; (4) 6; (5) 0. 254. (1) $-1/2$; (2) 0. 255. (1) 1; (2) 1. 256. (1) 2/7; (2) 5/3. 259. (1) 2/3; (2) $2x+\cos x$; (3) $\sin x+x\cos x$. 260. (1) $3\cos 3x$; (2) $4\cos(4x-1)$; (3) $2t\cos t^2$; (4) $\sqrt{2}/4$. 261. (1) $\sin 2x$; (2) $30\varphi\sin^2 5\varphi^2\cos 5\varphi^2$; (3) $-\cos x/\sin^2 x$; (4) $-3\cos 3x/\sin^2 3x$; (5) $-2\cos x/\sin^3 x$. 262. (1) $0.5\cos t/\sqrt{\sin t}$; (2) $\cot 2x\sqrt{\sin 2x}$; (3) $-1.5\cot 3x/\sqrt{\sin 3x}$; (4) $-1.5\cot x/\sqrt{\sin^3 x}$. 263. (1) $4\sqrt{2}-6$; (2) $4\sqrt{3}$; (3) $2\cos x+\sin x$; (4) $\sqrt{3}+1$. 264. (1) $\cos 2t$; (2) $\cos x-\cos 2x$; (3) $\cos x-x\sin x$; (4) $\cos 2x-\sin x$. 265. (1) $-3x^2\sin x^2$; (2) $(2/x^3)\sin(1/x^2)$; (3) $-3\sin x\cos^2 x$; (4) $2\sin 2x/\cos^2 2x$; (5) $2\sin x/\cos^3 x$. 266. (1) $-\tan 2x\sqrt{\cos 2x}$; (2) $x\tan x^2/\sqrt{\cos x^2}$. 267. (1) $4\sin 3x\cdot\sin x$; (2) $-\sin 2x/(1+\cos^2 x)^2$; (3) $-1/2$. 268. (1) $1/(1-\sin 2x)$; (2) $\tan^2 x$; (3) $(u\sin 2u+u^2)/\cos^2 u$; (4) 0. 269. (1) $a/\cos^2(ax+b)$; (2) $\dfrac{1}{3\cos^2(x/3)}$; (3) $2x/\cos^2 x^2$; (4) $\dfrac{1}{\sqrt{2x}\cos^2\sqrt{2x}}$; (5) $6\sin 3x/\cos^3 3x$. 270. (1) $\tan^2 x+2\sin^2 x$; (2) $-3\tan^2 x$; (3) 13.5; (4) $1/(2\cos^4(x/2))$. 271. (1) 1/4; (2) $-2/\sin^2 2x$. 272. (1) $-\cot^2 x$; (2) $1/\cos^2 x$; (3) -4. 273. (1) $-3x^2/\sin^2 x^3$; (2) $-3\cos^2 x/\sin^4 x$; (3) $1/(2\sin^4(x/2))$; (4) $4\sin 2x/\cos^3 2x$. 274. 6 m/s. 275. $2\sqrt{3}$ m/s. 278. (1) $\sqrt{2}$; (2) $2\sqrt{3}$; (3) $\arcsin x+\arccos x$. 279. (1) $3/\sqrt{1-9x^2}$; (2) $-1/\sqrt{a^2-x^2}$; (3) $2x/\sqrt{1-x^4}$; (4) $-a/\sqrt{1-a^2x^2}$. 280. (1) $1.5/\sqrt{3x(1-3x)}$; (2) $-0.5/\sqrt{(2-x)(x-1)}$; (3) $2a/(x^2+a^2)$. 281. (1) 1/4; (2) $\operatorname{arctan} x+\operatorname{arccot} x$. 282. (1) $2x/(1+x^4)$; (2) $-3/(1+9x^2)$; (3) $-a/(x^2+a^2)$; (4) $-a/(a^2+x^2)$. 283. (1) $0.5/\sqrt{x}(1+x)$; (2) $-0.5/\sqrt{x}(1+x)$. 284. (1) $-1/(1+x^2)$; (2) $-1/(1+x^2)$. 287. (1) $-\cos x$; (2) $2\sin x/\cos^3 x$; (3) $-1/\cos^2 x$; (4) $-4\cot 2x/\sin 2x$; (5) $e^{\cos t}(\sin^2 t-\cos t)$; (6) $e^{-\sin t}(\cos^2 t+\sin t)$. 288. (1) $v=\pi\sqrt{2}/8$, $a=-\pi^2\sqrt{2}/32$; (2) $v=\pi\sqrt{3}/6$, $a=\pi^2/18$. 289. $F=9m$. 292. $y=5\sin(6\pi t+0.8)$. 293. (1) $R=2$, $T=2\pi/3$, $\theta=\pi/3$; (2) $R=1/3$, $T=2$, $\theta=-\pi/4$; (3) $R=1$, $T=2\pi$, $\theta=-5$. 294. $-5\pi^2 m/18$. 295. (1) $13\sin\left(2t+\arctan\frac{5}{12}\right)$; (2) $17\sin\left(5x+\frac{\pi}{6}-\arctan\frac{15}{8}\right)$. 296. (1) 8; 0; (2) $2\sqrt{3}$; $\pi/6$; (3) 2; $\pi/4$. 299. (1) $x\neq\sqrt{\pi\mid k\mid}$, $k\in Z$; (2) $x\neq\pi k/2$, $k\in Z$; (3) $x\neq\pi k/2$, $k\in Z$; (4) $x\neq\pi k/3$, $k\in Z$; (5) $x\neq\pi/2+\pi k$, $k\in Z$. 300. (1) $x\neq\pi k/2$, $k\in Z$; (2) $x\neq\pi k$, $k\in Z$; (3) $x\neq-\pi/4+\pi k$, $k\in Z$; (4) $x\neq\pi k$; $x\neq\pi/2+2\pi n$; k, $n\in Z$; (5) $x\neq\pi k$, $x\neq\pi/4+\pi n$; k, $n\in Z$. 301. (1) $y_{gr}=1$ for $x=\pi/2+1+2\pi k$; $y_{sm}=-1$ for $x=-\pi/2+1+2\pi k$, $k\in Z$; (2) $y_{gr}=1$ for $x=-\pi/4+2\pi k$, $y_{sm}=-1$ for $x=3\pi/4+2\pi k$, $k\in Z$; (3) $y_{gr}=3$

for $x = \pi/8 + \pi k/2$, $y_{sm} = -3$ for $x = -\pi/8 + \pi k/2$, $k \in Z$; (4) $y_{gr} = 1/2$ for $x = 2\pi k/5$, $y_{sm} = -1/2$ for $x = \pi(2k+1)/5$, $k \in Z$. **302.** (1) $E(y) = [-1, 1]$; (2) $E(y) = [0, 1]$; (3) $E(y) = [-1, 1]$; (4) $E(y) = [0, 1]$. **303.** (1) $E(y) = [0, 2]$; (2) $E(y) = [2, 4]$; (3) $E(y) = [1, 3]$; (4) $E(y) = [4, 6]$. **304.** (1) No; (2) $y_{sm} = 0$; (3) $y_{sm} = 0$. **305.** (1) $y_{gr} = 5$, $y_{sm} = 3$; (2) $y_{gr} = 6$, $y_{sm} = 5$; (3) $y_{gr} = 5$, $y_{sm} = 2$. **306.** (1) $E(y) = [-\sqrt{2}, \sqrt{2}]$; (2) $E(y) = [-2\sqrt{3}, 2\sqrt{3}]$; (3) $E(y) = [-2, 2]$. **314.** $\alpha = \pi/6$. **315.** (1) $-\cos^3 x$; (2) $\cos 2x$. **316.** (1) $3/(\sin 6x - 1)$; (2) $8/\sin^2 4x$; (3) $-32 \cot 4x/\sin^2 4x$. **317.** (1) $-2/\sin 2x$; (2) $\sqrt{3}/3$; (3) $-2 \tan x$. **318.** (1) $-1/\sin x$; (2) $2 \cot(x - 1)$; (3) $8z/\sin 2z^2$. **319.** (1) $2 \cos 2t$; (2) $e^{\sin x}(\cos^2 x - \sin x)$; (3) $e^{\tan x}(1 - \sin 2x)$. **320.** (1) $1/\sqrt{1 - x^2}$; (2) $-1/(1 + z^2)$; (3) 0. **321.** (1) $e^x/\sqrt{1 - e^{2x}}$; (2) $2/(e^x + e^{-x})$; (3) $1/(2\sqrt{e^x - 1})$. **Quiz No. 7.** Form I. (1) 0; (2) -3; (3) -1; (4) $2\sqrt{2}$; (5) $\pm\pi/6 + \pi k$, $k \in Z$. Form II. (1) 8; (2) 2; (3) $-1/2$; (4) -2; (5) $\pi/4 + \pi k/2$, $k \in Z$.

Chapter 10

3. (1) $-10x(1-x^2)^4\,dx$; (2) $6ax(ax^2+b)\,dx$; (3) $-\dfrac{2x\,dx}{\sqrt{4-2x^2}}$; (4) $-\dfrac{dx}{\sqrt{(2x-1)^3}}$; (5) $-2 \tan x\,dx$; (6) $-\dfrac{dx}{2x}$; (7) $-\dfrac{2x\,dx}{\sqrt{1-x^4}}$; (8) $\dfrac{dx}{1+x^2}$.

4. (1) $-\dfrac{2\,dx^2}{\cos^2 x}$; (2) $-\dfrac{16 \cos 4x\,dx^2}{\sin^2 4x}$; (3) $9 \ln^2 a \cdot a^{3x}\,dx^2$; (4) $-\dfrac{x\,dx^2}{(1-x^2)^{3/2}}$; (5) $\dfrac{2(1-3x^4)\,dx^2}{(1+x^4)^2}$. **6.** $\approx 1\%$. **7.** $\approx 1.5\%$. **11.** (1) ≈ 0.023; (2) ≈ 0.13; (3) ≈ 0.001. **12.** 6 cm^3. **13.** (1) ≈ 7.07; (2) ≈ 19.18; (3) ≈ 2.83. **16.** (1) 82.08; (2) 1.036; (3) 985; (4) 1.05; (5) 0.9. **17.** (1) 1.004; (2) 5.016; (3) 4.984; (4) 10.05; (5) 9.975; (6) 1.003. **18.** (1) 1.01; (2) 0.1007; (3) 0.992. **19.** (1) 0.0122; (2) 0.0366; (3) 0.0209; (4) 0.0576. **30.** $\approx 1.2\%$. **31.** $\approx 0.1\%$. **32.** $\approx 1.6\%$. **33.** 0.04%. **34.** (1) $e^x(\sin x + \cos x)\,dx$; (2) $(ae)^x(\ln a + 1)\,dx$; (3) $\dfrac{e^x(2x+1)\,dx}{\sqrt{2x}}$; (4) $2(e^{2x} - e^{-2x})\,dx$; (5) $-\dfrac{e^x dx}{(e^x-1)^2}$; (6) $-\dfrac{dx}{e^x}$. **35.** (1) ≈ 0.02; (2) ≈ 0.001; (3) ≈ 0.04. **36.** (1) ≈ 1.001; (2) ≈ 112.4; (3) ≈ 0.5375. **37.** (1) $\approx 0.2\%$; (2) $\approx 2\%$. **38.** (1) $6 \cot 3x \cdot \Delta x$; (2) $\dfrac{4\Delta x}{\sin 2x}$. **39.** (1) $\sin 2x \cdot \Delta x$; (2) $\dfrac{\ln 3}{2}\dfrac{\Delta x}{\sqrt{x}}$. **Quiz No. 1.** Form I. (1) -0.02; (2) 0.3%; (3) 0.08; (4) 18.66; (5) 1.002. Form II. (1) 0.12; (2) 0.2%; (3) 0.002; (4) 87.6; (5) 1.14.

Chapter 11

9. (1) $a\varphi + C$; (2) $(m-1)y + C$; (3) $x^5/5 + C$; (4) $x^m/m + C$; (5) $x^{2-n}/(2-n) + C$; **10.** (1) $t^4 + C$; (2) $x^n + C$; (3) $u^4 - 2u^3 - 2u^2 + 3u + C$; (4) $x^4/3 - x^3/4 + 5x + C$. (5) $ax^4 - 2bx^3 - 2cx^2 + dx + C$. **11.** (1) $12x^5/5 - 4x^3 + 3x + C$; (2) $x^4/4 + x^5 + C$; (3) $x^6/6 - x^5/5 + C$; (4) $2x^4 - 4x^3 + 3x^2 - x + C$. **12.** (1) $u^2/6 - u/3 + C$; (2) $2\varphi/5 - \varphi^3/5 + C$. **13.** (1) $-1/x^3 - 2/x^4 + C$; (2) $-1/(3x^3) + 1/(2x^2) + 3/x + x + C$. **14.** (1) $-1/t + C$; (2) $x^n/n + C$; (3) $2u^{5/3} - 4u^{7/4} + C$; (4) $2x^2\sqrt{x} + C$; (5) $3\sqrt[3]{x} + C$. **15.** (1) $-2x^{-1/2} + 3x^{-2/3}/2 - 4x^{-3/4}/3 + C$; (2) $4x^{3/4}/3 + 3x^{2/3}/2 + 2x^{1/2} + C$. **16.** (1) $3\sqrt[3]{u} + C$. (2) $\sqrt{\varphi} + C$; (3) $3x\sqrt[3]{x}/4 + C$; (4) $-3/t - 4\sqrt{t} + 6\sqrt[3]{t^2} + C$. **17.** (1) $a \ln|\varphi| + C$; (2) $2 \ln|x + 3| + C$; (3) $-3 \ln|2 - \varphi| + C$; (4) $(1/3) \ln|x^3 + 1| + C$;

(5) $-(1/3)\ln|a^3-x^3|+C$. 18. (1) $5^x/\ln 5+C$; (2) $4^{2x}/(2\ln 4)+C$; (3) $5^{x^3}/(3\ln 5)+C$. 19. (1) e^x+x^2+C; (2) $3^x/\ln 3-e^x-x+C$; (3) $e^{5x}/5+C$; (4) $e^{2x^2}/4+C$; (5) $-e^{x^3}/3+C$. 20. (1) $-\cos x-5x+C$; (2) $-2\cos x+C$; (3) $-(1/6)\cos 6x+C$; (4) $-4\cos(x/4)+C$; (5) $-(1/2)\cos x^2+C$. 21. (1) $4x-3\sin x+C$; (2) $(1/4)\sin 4x+C$; (3) $6\sin(x/6)+C$; (4) $-1(1/3)\sin(2-3x)+C$; (5) $(1/3)\sin^3+C$. 22. (1) $(1/3)\ln|3\sin x-1|+C$; (2) $\ln(2-\cos x)+C$; (3) $(1/2)\ln(3+2\sin x)+C$. 23. (1) $(1/2)\tan x+C$; (2) $(1/5)\tan 5x+C$; (3) $(1/a)\tan(ax+b)+C$; (4) $-(1/3)\cot(3x+2)+C$; (5) $-(1/2)\cot x^2+C$. 24. (1) $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C$; (2) $\frac{1}{10}\ln\left|\frac{u-5}{u+5}\right|+C$; (3) $\ln|x+\sqrt{x^2+9}|+C$; (4) $\ln|x+\sqrt{x^2-16}|+C$. 25. (1) $2\arcsin x+C$; (2) $\arcsin(u\sqrt{3})+C$; (3) $(1/2)\arcsin(2x/\sqrt{5})+C$; (4) $(\sqrt{2}/2)\arcsin(\varphi\sqrt{2}/3)+C$; (5) $\arcsin(3x/4)+C$. 26. (1) $a\cdot\arctan x+C$; (2) $(1/5)\arctan(x/5)+C$; (3) $\arctan(z/3)+C$; (4) $(1/20)\arctan(5x/4)+C$. (5) $(1/\sqrt{6})\arctan(x\sqrt{3}/2)+C$. 33. (1) $y=x^4-x^2+3x+C$; (2) $y=x^2+6x+C$. 34. (1) $y=x^2-5x+4$; (2) $y=x^3-2x^2+5x$; (3) $y=3e^x+x^2+5$. 35. (1) $3\sin x-\cos x+3$; (2) $\sin x-e^x+x^2+4$; (3) $3\arctan x+2x$; (4) $-1/(2x^2)-x^3/3+1/2$. 36. (1) $y=-3x^2/2+C$; (2) $y=x^2/2+2x+C$. 37. (1) $y=C/x$, (2) $x^2-y^2=C$; (3) $x^2+y^2=C$. 38. $y=x^2/6$. 39. $y=x^3-x^2+4$. 40. $y=x^3+4$. 41. $y=e^{x+1}$. 46. (1) $s=t^3/3-4t^2+2t+C$; (2) $s=2t^2-t^3+C$. 47. $s=t^2-3t+6$. 48. $s=t^3+2t^2-t$. 49. $s=2\sin t+3$. 50. $s=v_0t-gt^2/2$. 51. $s=t^4+t^3+t+3$. 52. (1) $v=-3t^2+18t+24$, $s=-t^3+9t^2+24t+15$; (2) $a(2)=6m/s^2$, $v(2)=48m/s$; $s(2)=91\ m$; (3) $v_{\max}=v(3)=51\ m/s$. 58. (1) $-(7-2x)^4/8+C$; (2) $(5t-1)^5/25+C$; (3) $\frac{1}{3(4-3x)}+C$; (4) $-\frac{1}{10(5z+1)^2}+C$. 59. (1) $(3x+1)^{5/3}/5+C$; (2) $(2x-1)^{3/3}/3+C$; (3) $-(4-3t)^{5/3}/5+C$; (4) $-2/(3\times\sqrt{3x-1})+C$; (5) $\sqrt[3]{3x-5}+C$. 60. (1) $(x^2+3)^6/12+C$; (2) $(x^4-1)^3/3+C$; (3) $\frac{1}{3(1-2z^3)^2}+C$; (4) $-\frac{1}{80(5x^4+3)^4}+C$. 61. (1) $(4x^3+1)^{3/2}/18+C$; (2) $(x^4-1)^{5/2}/10+C$; (3) $(2\sin x-1)^{3/2}/3+C$; (4) $2(e^x+1)^{3/2}/3+C$. 62. (1) $(3x^4+2)^{5/2}/30+C$; (2) $-(1-3x^2)^{7/3}/14+C$. 63. (1) $\sqrt{x^2+1}+C$; (2) $3\sqrt[3]{5x^4+2}/20+C$; (3) $-2/(3\sqrt{x^3-1})+C$; (4) $-2\sqrt{1-\sin x}+C$; (5) $-\frac{1}{2(e^x+1)^2}+C$. 64. (1) $(1/3)\ln|1+z^3|+C$; (2) $(1/3)\ln(e^{3x}+1)+C$; (3) $(1/2)\ln|2\sin x+1|+C$. 65. (1) $-(1/3)\ln|\cos 3x|+C$; (2) $(1/k)\ln|\sin kx|+C$; (3) $2\ln|\sin(x/2)|+C$. 66. (1) $(1/2)\ln|\tan x|+C$; (2) $3\ln|\tan(x/6)|+C$; (3) $\frac{1}{3}\ln\left|\tan\left(\frac{\pi}{4}+\frac{3x}{2}\right)\right|+C$; (4) $2\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{4}\right)\right|+C$. 67. (1) $\ln|1+\ln t|+C$; (2) $-0.5(2-\ln t)^2+C$. 68. (1) $a^{x^4}/(4\ln a)+C$; (2) $0.5(ab)^{2x}:\ln(ab)+C$. 69. (1) $2e^{\sqrt{x}}+C$; (2) $-(1/2)e^{-x^2}+C$; (3) $e^{\sin x}+C$; (4) $-e^{1/x}+C$. 70. (1) $-(1/2)\cos(t^2-1)+C$; (2) $-2\cos(z/2)+C$; (3) $(1/4)\sin x^4+C$; (4) $2\sin\sqrt{x}+C$; (5) $(1/2)\sin(x^2+1)+C$. 71. (1) $2\tan\sqrt{x}+C$; (2) $(1/3)\times\tan x^3+C$; (3) $\cot(\pi/3-\varphi)+C$; (4) $\cot(1/x)+C$; (5) $-\cot\ln x+C$. 72. (1) $\arcsin e^{\varphi}+C$; (2) $\arcsin\ln z+C$. 73. (1) $-(1/a)\arctan((\cos x)/a)+C$; (2) $\arctan e^x+C$; (3) $(1/3)\arctan x^3+C$; (4) $\arctan\ln x+C$. 75. (1) $x\sin x+\cos x+C$; (2) $(x-1)\cos x-\sin x+C$. 76. (1) $-\ln x/(2x^2)-1/(4x^2)+C$; (2) $x(\ln^2 x-2\ln x+2)+C$. 77. (1) $e^x(x-1)+C$; (2) $-x\cot x+\ln|\sin x|+C$. 78. (1) $x\arcsin x+\sqrt{1-x^2}+C$; (2) $x\arctan x-(1/2)\times\ln(1+x^2)+C$. 79. (1) $0.5e^x(\sin x+\cos x)+C$; (2) $0.5e^x(\sin x-\cos x)+C$. 85. (1) $x/2-(1/4)\sin 2x+C$; (2) $3x/8-(1/4)\sin 2x+(1/32)\sin 4x+C$. 86. (1) $-\cot x-x+C$; (2) $-(1/3)\cot^3 x+\cot x+C$. 87. (1) $\sin x-(1/3)\sin^3 x+C$; (2) $-\cos x+(2/3)\cos^3 x-(1/5)\cos^5 x+C$; (3) $\sin x-(2/3)\sin^3 x+(1/5)\sin^5 x+C$; (4) $-(1/2)\cot^2 x-\ln|\sin x|+C$. 88. (1) $(1/4)\sin 2x-(1/8)\sin 4x+C$; (2) $(1/4)\sin 2x+(1/16)\sin 8x+C$; (3)

$-(1/2)\cos x - (1/14)\cos 7x + C$. 89. (1) $-(1/2)\cos 2x - (1/6)e^3x^2 + x + C$. 90. $-\ln(\cos x + e) + 3$. 91. $y = 2\sin(x/2) + 1$. 92. $s = -(1/2)\cos 2t + 1$. 93. (1) $v = -3t^2 + 24t - 6$; (2) $s = -t^3 + 12t^2 - 6t + 15$; (3) 6 m/s^2 39 m/s; 78 m; (4) 4 s. 94. $2\sqrt{x^3 - 1}/3) + C$. 95. $2\sqrt{(e^{\sin x} + 1)^3}/3 + C$. 96. $\ln(1 - \cos x) + C$. 97. $(1/3)\arctan((\sin x)/3) + C$. 98. $(1/8)x - (1/32)\sin 4x + C$. 99. $-(1/8)\cos 4x - (1/12)\cos 6x + C$. **Quiz No. 1.** Form I. (1) $2x^{3/2}/3 + 6x^{5/6}/5 + \ln x + C$; (2) $\arcsin(2x/3) - e^{-x} + C$; (3) $\ln|\tan x| + C$; (4) $y = x^2 - 4x - 4$; (5) $s = t^3 + 3t^2 - 4t - 4$. Form II. (1) $\ln x - 6x^{1/6} + 1/x + C$; (2) $\arcsin(x/\sqrt{3}) - e^{-x} + C$; (3) $(4/3)\sin^3 x - \sin x + C$; (4) $y = -\cos x + 1$; (5) $s = t^3 + 3t^2 - 5t$.

Chapter 12

5. (1) 8/3; (2) 15/4; (3) 48.4. 6. (1) 40; (2) $-5/4$. 7. (1) 3/2; (2) 1; (3) 16/3. 8. (1) 93/5; (2) 15/2; (3) 8/3; (4) 40/3. 9. (1) $e^2(e^4 - 1)/2$; (2) $(e^3 - 1)/3$. 10. (1) $\ln 2 = 0.6931$; (2) $\ln(3/2) = 0.4055$; (3) $\ln 2 = 0.6931$; (4) $\ln(5/4) = 0.2231$. 11. (1) 1/2; (2) $\sqrt{2}$; (3) 2. 12. (1) 1/2; (2) 1/2; (3) 2. 13. (1) 4; (2) $2\sqrt{3}/3$. 14. (1) 0; (2) 2. 15. (1) $\pi/6$; (2) $\pi/2$; (3) $\pi/12$. 16. (1) $\pi/12$; (2) $\pi/18$; (3) $\pi/60$. 18. (1) $-1/4$; (2) 7/64. 19. (1) 15/4; (2) 48.4; (3) 3. 20. (1) 81/8; (2) 8/9; (3) 52; (4) 3.1; (5) 1/3; (6) 15/64. 21. (1) 14/9; (2) $-2/3$. 22. (1) $\ln(5/4) = 0.2231$; (2) $\ln(3/2) = 0.4055$; (3) $-\ln(3/2) = -0.4055$. 23. (1) $\sqrt{e} - 1$; (2) $(e - 1)/2$; (3) $e^2 - 1$. 24. (1) $(\sqrt{3} - \sqrt{2})/4$; (2) 3/2; (3) $\sqrt{3}/6$. 25. (1) $(\sqrt{3} - 1)/4$; (2) $(\sqrt{2} - 1)/6$; (3) $2(\sqrt{3} - 1)$. 26. (1) $(\sqrt{3} - 1)/2$; (2) 1/3; (3) $2\sqrt{3}$. 27. (1) $2\sqrt{3}/9$; (2) $\sqrt{3} - 1$; (3) $2\sqrt{3}$. 28. (1) $\pi/4$; (2) $\pi\sqrt{3}/9$; (3) $\pi\sqrt{2}/24$. 29. (1) $\pi/36$; (2) $\pi\sqrt{2}/12$; (3) $\pi\sqrt{3}/36$. 31. (1) $\pi/2 - 1$; (2) $\pi/2 - 1$; (3) $\pi/4 - 1/2$. 32. (1) $e - 2$; (2) $1 - 2e^{-1}$. 34. (1) 0.719; (2) 0.694; (3) -6.14; (4) 0.758; (5) 1.37.

Chapter 13

12. (1) 7.5 sq. un.; (2) 12 sq. un. 13. (1) 4 sq. un.; (2) 14.75 sq. un. 14. (1) 9 sq. un.; (2) 35 sq. un. 15. (1) 6 sq. un.; (2) 25/3 sq. un.; (3) 6 sq. un. 16. $2\sqrt{3}$ sq. un. 17. (1) 36 sq. un. ; (2) 8 sq. un.; (3) 11/3 sq. un. 18. (1) 1.0986 sq. un.; (2) 1.3863 sq. un. 19. (1) 1 sq. un.; (2) 0.6931 sq. un.; (3) 0.5493 sq. un. 20. (1) 6 sq. un.; (2) 9 sq. un.; (3) 6 sq. un.; (4) 5 sq. un. 21. (1) 7 sq. un.; (2) 6 sq. un.; (3) 32/3 sq. un. 22. (1) 8 sq. un.; (2) 17 sq. un.; (3) 0.5 sq. un.; (4) 8 sq. un. 23. (1) 208/3 sq. un.; (2) 32 sq. un. 24. (1) 9π sq. un.; (2) πab sq. un.; (3) 12π sq. un. 25. (1) 3 sq. un.; (2) 4 sq. un. 26. (1) 45 sq. un.; (2) 36 sq. un.; (3) 4.5 sq. un.; (4) 32/3 sq. un. 27. (1) 18 sq. un.; (2) 4.5 sq. un.; (3) 40.5 sq. un.; (4) 18 sq. un. 28. (1) 36 sq. un.; (2) 16/3 sq. un. 29. (1) 8/3 sq. un.; (2) 1/3 sq. un. 30. (1) 1.5 sq. un.; (2) 4.5 sq. un. **Quiz No. 1.** Form I. (1) 4; (2) $3 - \sqrt{3}$; (3) $\pi\sqrt{3}/36$; (4) 125/6 sq. un.; (5) 36 sq. un. Form II. (1) 100/3; (2) $\sqrt{3} - 1$; (3) $\pi/18$; (4) 32/3 sq. un.; (5) 4.5 sq. un. 37. 270 m. 38. 7 m. 39. 108 m. 40. (1) 54 m; (2) 22 m; (3) 64 m. 41. 1100 m. 42. 4 s; 96 m. 43. 44.1 m. 49. 21.6 J. 50. 5 J. 51. 1.2 J. 52. 76.8 J. 53. 0.05 m. 54. 28,090 J. 57. 176,526 J. 58. $176{,}508\pi$ J. 59. 3269π J. 60. 2452π J. 61. 6538π J. 65. 156,912 N. 66. 68,649 N. 67. 235.4 N. 68. 7.1 N. 69. 167 N. 70. 2324 N. 75. 1.196 un. of length. 76. 9.29 un. of length. 77. 1.196 un. of length. 78. 56/27 un. of length. 79. 14/3 un. of length. 80. $(e^{1/2} - e^{-1/2})/2$ un. of length. **Quiz No. 2.** Form I. (1) 1 m; (2) 1.8 J; (3) 160 J; (4) $19{,}614\pi$ J; (5) 147,105 N. Form II. (1) 54 m; (2) 2.5 J; (3) 0.06 m; (4) 235,368 J; (5) 39,228 N.

Chapter 14

7. 1; $(1/2)+(\sqrt{3}/2)i$; $(-1/2)+(\sqrt{3}/2)i$; -1; $(-1/2)-(\sqrt{3}/2)i$; $(1/2)-(\sqrt{3}/2)i$. 8. (1) $-3-2i$; (2) $3+2i$; (3) $3-2i$. 9. (1) $x=3$, $y=1$; (2) $x=3$, $y=4$; (3) $x=2$, $y=3$. 10. (1) $x=2$, $y=4$ or $x=3$, $y=4$; (2) $x=1/4$, $y=0$. 11. $x=-1$. 12. (1) $r=3$, $\varphi=0$; (2) $r=3$, $\varphi=\pi$; (3) $r=3$, $\varphi=\pi/2$; (4) $r=3$; $\varphi=-\pi/2$; (5) $r=2\sqrt{2}$; $\varphi=-3\pi/4$; (6) $r=2$, $\varphi=\pi/3$; (7) $r=2$, $\varphi=-\pi/3$; (8) $r=2$; $\varphi=5\pi/6$. 13. (1) $\pi/2$ or $-\pi/2$; (2) π; (3) 0; (4) has no definite argument. 14. $-\varphi$ 15. (1) Circle $x^2+y^2=4$; (2) ray $y=-x$, $x\leqslant 0$. 16. (1) $3\pi/4+2\pi k$, $k\in\mathbf{Z}$; (2) $-\pi/6+2\pi k$, $k\in\mathbf{Z}$. 23. (1) $-7i$; (2) 12; (3) -12; (4) $1-i$; (5) $4-i$; (6) $-2-i$. 24. (1) -1; (2) i; (3) i; (4) -1; (5) 0; (6) $-i$. 25. (1) 20; (2) $6+10i$; (3) 25; (4) $25-19i$; (5) $-1-3i$; (6) $4+39i$; (7) $24-5i$; (8) $(2/3)+(7/9)i$; (9) $0.22+0.07i$. 26. (1) $-i$; (2) $(1/2)+(1/2)i$; (3) $-i$; (4) $-0.3-1.1i$; (5) $(18/13)-(1/13)i$; (6) $(1/68)-(21/68)i$; (7) $-(3/26)-(41/26)i$; (8) $(a^2-b^2)/(a^2+b^2)+i\cdot 2ab/(a^2+b^2)$; (9) $-a+bi$; (10) $(1/3)+(\sqrt{5}/3)i$; (11) $0.1+0.3i$; (12) $i\cdot 4ab/(a^2+b^2)$. 27. (1) $(m+ni)\times(m-ni)$; (2) $(2m+3ni)(2m-3ni)$; (3) $(a/3+bi/4)(a/3-bi/4)$; (4) $(\sqrt{m}+i\sqrt{n})(\sqrt{m}-i\sqrt{n})$; (5) $(\sqrt{2}+i\sqrt[4]{3})\cdot(\sqrt{2}-i\sqrt[4]{3})$; (6) $(1+i\sin\alpha)\times(1-i\sin\alpha)$; (7) $(1-i\sqrt{2})(1+i\sqrt{2})$. 28. (1) -64; (2) $256(1+i)$; (3) 1; (4) $(1/4)+(\sqrt{2}/2)i$; (5) $-i/2$; (6) $-(1/4)+(1/4)i$; (7) 1. 36. (1) $3[\cos(\pi/2+2\pi k)+i\sin(\pi/2+2\pi k)]$, $k\in\mathbf{Z}$, (2) $\sqrt{2}[\cos(3\pi/4+2\pi k)+i\sin(3\pi/4+2\pi k)]$, $k\in\mathbf{Z}$; (3) $2[\cos(-\pi/3+2\pi k)+i\sin(-\pi/3+2\pi k)]$, $k\in\mathbf{Z}$; (4) $2[\cos(-\pi/6+2\pi k)+i\sin(-\pi/6+2\pi k)]$, $k\in\mathbf{Z}$; (5) $\cos(-\pi/6+2\pi k)+i\sin(-\pi/6+2\pi k)]$, $k\in\mathbf{Z}$; (6) $5[\cos(126°\,52'+360°\,k)+i\sin(126°\,52'+360°\,k)]$, $k\in\mathbf{Z}$. 37. (1) $5i$; (2) $2-2i\sqrt{3}$; (3) -1; (4) $\sqrt{2}+i\sqrt{2}$; (5) 3. 38. (1) $3[\cos(\pi/3)+i\sin(\pi/3)]$; (2) $10[\cos(\pi/12)+i\sin(\pi/12)]$; (3) $\cos 7+i\sin 7$; (4) $\cos(\pi/6)+i\sin(\pi/6)$; (5) $8(\cos 45°+i\sin 45°)$; (6) $\cos(3\pi/2)+i\sin(3\pi/2)$. 39. (1) $(1/6)[\cos(11\pi/12)+i\sin(11\pi/12)]$; (2) $8[\cos(-\pi/3)+i\sin(-\pi/3)]$; (3) $6\sqrt{2}[\cos(7\pi/12)+i\sin(7\pi/12)]$; (4) $12\sqrt{6}[\cos(-7\pi/12)+i\sin(-7\pi/12)]$; (5) $5\sqrt{2}(\cos 60°+i\sin 60°)$; (6) $6\sqrt{3}[\cos(\pi/24)+i\sin(\pi/24)]$. 40. (1) $3[\cos(\pi/4)+i\sin(\pi/4)]$; (2) $\cos 60°+i\sin 60°$; (3) $\cos(-\pi/6)+i\sin(-\pi/6)$; (4) $\cos 270°+i\sin 270°$. 41. (1) -1; (2) -256; (3) $(1/2)-(\sqrt{3}/2)i$. 42. (1) -128; (2) 0. 43. (1) $(1/2)+(\sqrt{3}/2)i$, -1, $(1/2)-(\sqrt{3}/2)i$; (2) $(\sqrt{2}/2)+(\sqrt{2}/2)i$, $-(\sqrt{2}/2)+(\sqrt{2}/2)i$, $-(\sqrt{2}/2)-(\sqrt{2}/2)i$, $(\sqrt{2}/2)-(\sqrt{2}/2)i$; (3) $(\sqrt{3}/2)+(1/2)i$, $-(\sqrt{3}/2)+(1/2)i$, $-i$; (4) $\sqrt{2}$, $-\sqrt{2}$, $i\sqrt{2}$, $-i\sqrt{2}$, (5) $(\sqrt{6}/2)+(\sqrt{2}/2)i$, $-(\sqrt{2}/2)+(\sqrt{6}/2)i$, $-(\sqrt{6}/2)-(\sqrt{2}/2)i$, $(\sqrt{2}/2)-(\sqrt{6}/2)i$; (6) 1, $(1/2)+(\sqrt{3}/2)i$, $(-1/2)+(\sqrt{3}/2)i$, $(-1/2)-(\sqrt{3}/2)i$, $(1/2)-(\sqrt{3}/2)i$. 49. (1) $\cos 1+i\sin 1$; (2) -1; (3) $e(\cos 1+i\sin 1)$; (4) i; (5) $(1/2)+(\sqrt{3}/2)i$, (6) $e^4(\cos 3+i\sin 3)$; (7) $e^2(\cos 1-i\sin 1)$; (8) $e^{-2}(\cos 3+i\sin 3)$. 50. (1) $i(e^2-1)/(2e)$, (2) $\cos 1(e^2+1)/(2e)+i\sin 1(1-e^2)$: $(2e)$; (3) $\sin 1(e^2+1)/(2e)+i\cos 1\times(1-e^2)/(2e)$. 52. (1) $e^{0\cdot i}$; (2) $2e^{i\pi/6}$; (3) $2\sqrt{3}e^{i\pi 16}$; (4) $2\sqrt{2}e^{2\pi i/3}$. 53. (1) $4e^{5\pi i/12}$; (2) $e^{i\pi/12}$; (3) $16e^{i\pi}$; (4) $\sqrt{2}e^{i\pi/18}$, $\sqrt[3]{2}e^{13i\pi/18}$, $\sqrt[3]{2}e^{-11\pi i/18}$; (5) $\sqrt[4]{2}e^{i\pi/16}$, $\sqrt[4]{2}e^{9\pi i/16}$, $\sqrt[4]{2}e^{-15\pi i/16}$, $\sqrt[4]{2}e^{-7\pi i/16}$. 54. (1) $ab^2-a^2b-2abi\sqrt{ab}$; (2) $2+i(a^2-b^2)/(ab)$. 55. (1) $(2x+5i)(2x-5i)$; (2) $(a+2i\sqrt{3})(a-2i\sqrt{3})$; (3) $(\sqrt{5}+3i)(\sqrt{5}-3i)$; (4) $(\sqrt{2}+i\sqrt[4]{3})(\sqrt{2}-i\sqrt[4]{3})$. 56. (1) $3m+2in$; (2) $\sqrt{x}-i$; (3) $\sqrt{a}-i\sqrt{b}$. 57. (1) $1-i\sqrt{6}$; (2) i. 58. $\sqrt{2}$ and $\pi/4+2\pi k$, $k\in\mathbf{Z}$. 60. (1) $3-2i$; (2) $5i$. 61. (1) $2-2i\sqrt{3}$; $-(1/4)+(\sqrt{2}/2)i$. 65. (1) 0; (2) $-(1/2)+(1/4)i$; (3) 0. 66. (1) $5[\cos(\pi/12)+i\sin(\pi/12)]$, $5e^{i\pi/12}$; (2) $2[\cos(\pi/4)+i\sin(\pi/4)]$, $2e^{i\pi/4}$. 67. (1) $(\sqrt{2}/2)+(\sqrt{2}/2)i$; (2) -1. 69. (1) $\cos(\pi/8)+i\sin(\pi/8)$; $-\cos(3\pi/8)+i\sin(3\pi/8)$, $-\cos(\pi/8)-i\sin(\pi/8)$, $\cos(3\pi/8)-i\sin(3\pi/8)$;

(2) $\sqrt[4]{2}\,[\cos(\pi/8)+i\sin(\pi/8)]$, $-\sqrt[4]{2}\,[\cos(\pi/8)+i\sin(\pi/8)]$. **70.** (1) $\{2; -1+i\sqrt{3}; -1-i\sqrt{3}\}$; (2) $\{3/2, (-3/4)+(3\sqrt{3}/4)\,i; (-3/4)-(3\sqrt{3}/4)\,i\}$; (3) -5; $(5/2)+(5\sqrt{3}/2)\,i$; $(5/2)-(5\sqrt{3}/2)\,i\}$; (4) $\{-1/3; (1/6)+(\sqrt{3}/6)\,i, (1/6)-(\sqrt{3}/6)\,i\}$; (5) $\{(3\sqrt{2}/2)+(3\sqrt{2}/2)\,i; (3\sqrt{2}/2)-(3\sqrt{2}/2)\,i; (-3\sqrt{2}/2)+(3\sqrt{2}/2)\,i; (-3\sqrt{2}/2)-3(\sqrt{2}/2)\,i\}$; (6) $\{2; -2; 1+i\sqrt{3}; 1-i\sqrt{3}; -1+i\sqrt{3}\}$; $-1-i\sqrt{3}\}$. **71.** (1) $x^2+1=0$; (2) $x^2-6x+10=0$; (3) $x^2-2x+6=0$. **72.** $\{(1/2)+(\sqrt{3}/2)\,i; (1/2)-(\sqrt{3}/2)\,i; (-1/2)+(\sqrt{3}/2)\,i; (-1/2)-(\sqrt{3}/2)\,i\}$. **73.** (1) $\{1; (-1/2)+(\sqrt{3}/2)\,i; (-1/2)-(\sqrt{3}/2)\,i; 3; (-3/2)+(3\sqrt{3}/2)\,i; -(3/2)-(3\sqrt{3}/2)\,i\}$; (2) $\{-1/2; -2+(\sqrt{3}/2)\,i, -2-(\sqrt{3}/2)\,i; -1; (-7/4)+(\sqrt{3}/4)\,i; (-7/4)-(\sqrt{3}/4)\,i\}$. **Quiz No. 1. Form I.** (1) $\sqrt{2}$, $\varphi=\pi/4+2\pi k$, $k\in Z$; (2) $2i$, (3) 512; (4) $\sqrt{3}+i$; $-\sqrt{3}-i$; (5) $\{\sqrt{3}+i; \sqrt{3}-i; -\sqrt{3}+i; -\sqrt{3}-i\}$. **Form II.** (1) $\sqrt{2}$, $-\pi/4+2\pi k$, $k\in Z$; (2) $2i$; (3) -27; (4) 2, $-1+i\sqrt{3}$, $-1-i\sqrt{3}$; (5) $\sqrt{6}/2+(\sqrt{2}/2)\,i$, $(\sqrt{6}/2)-(\sqrt{2}/2)\,i$, $(-\sqrt{6}/2)+(\sqrt{2}/2)\,i$, $(-\sqrt{6}/2)-(\sqrt{2}/2)\,i$.

Chapter 15

3. (1) $y^3=x^3/3+C$; (2) $\sqrt{x}-\sqrt{y}=C$; (3) $y^{3/2}=3x^{3/2}+C$; (4) $y+1=C(x-1)$. **4.** (1) $y=C\sqrt{1+x^2}$; (2) $x=Ce^{1/y}+2$. **5.** (1) $(x+y)/(xy)+\ln(y/x)+C=0$; (2) $y=Cx^2e^{-3/x}$. **6.** (1) $2\sqrt{x}-\arctan y=C$; (2) $\arcsin y=C-\sqrt{1-x^2}$. **7.** (1) $y^2=x^2+12$; (2) $y=3x$. **8.** $s=t^3-t^2$. **9.** $y^3=x^3+8$. **10.** $x^2-y^2+4y-2x=0$. **11.** $(1-x)(1+y)=12$. **12.** $y=\ln(xy)+x$. **13.** $y=e^x$. **14.** $1/\cos^2 x=5-\cos^2 y$. **20.** $s=-2t^3+2t^2+6$. **21.** $y^2=x-1$. **22.** 35.6°. **23.** In 1575 years. **24.** 82.5 s. **27.** (1) $y=-1.5+Ce^{2x}$; (2) $y=Ce^x-1$; (3) $y=x^2/4+C/x^2$. **28.** (1) $y=1+Ce^{-x^2/2}$; (2) $y=(x+1)^2\times(x+C)$; (3) $y=(x+C)\sin x$. **29.** $y=x^3e^x$. **30.** $y=1/x+2/x^2$. **31.** $y=\tan x-1+e^{-\tan x}$. **36.** $y=x+2$. **37.** $y=2x^2-x$. **38.** $s=t^3+10t$. **39.** $s=3t^3+t^2+5t+4$. **40.** $0=(1/12)\,\omega^4+12\omega$. **41.** $y=e^x+1$. **42.** $y=0.25x^2+5$. **43.** $s=(1/2)\,gt^2+v_0t+s_0$. **50.** (1) $y=C_1e^{-3x}+C_2e^{2x}$; (2) $y=C_1e^{3x}+C_2e^{5x}$; (3) $y=C_1e^{-3x}+C_2e^{-2x}$. **51.** (1) $y=C_1e^{9x}+C_2$; (2) $y=C_1e^{-3x}+C_2$; (3) $y=C_1e^x+C_2$. **52.** (1) $y=C_1e^{-3x}+C_2e^{3x}$; (2) $y=C_1e^{-4}+C_2e^{4x}$; (3) $y=C_1e^{-x}+C_2e^x$. **53.** (1) $y=e^{-x}+e^x$; (2) $y=6e^{-x}+2e^{3x}$; (3) $y=0.8e^{-5x}+e^{4x}$. **54.** (1) $y=e^{3x}(C_1+C_2x)$; (2) $y=e^{-3x}(C_1+C_2x)$, (3) $y=e^{-5x}(C_1+C_2x)$. **55.** (1) $y=2e^{5x}-2xe^{5x}$; (2) $y=e^{-3x}+5xe^{-3x}$. **56.** (1) $y=C_1\cos 3x+C_2\sin 3x$; (2) $y=e^x(C_1\cos 2x+C_2\sin 2x)$; (3) $y=e^{-2x}\times(C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x)$. **57.** (1) $y=2\sin 3x-\cos 3x$; (2) $y=e^{2x}(\cos x-3\sin x)$. **58.** (1) $s=2\sin t$. **59.** $y=x+2$. **60.** $y=1/\sqrt{1-x^2}$. **61.** $y=(1/x)\,e^{1-1/x}$. **62.** $y=e^{2x}-2$. **63.** $y=e^{-x}(x+5)$. **64.** $s=\ln t+2$. **65.** $y=2e^{-4x}+2e^{2x}$. **Quiz No. 1. Form I.** (1) $y=(x^2+1)^2$; (2) $y=e^{-4x}+0.5$; (3) $s=t^3-2t^2+2t+1$; (4) $y=2e^{-3x}+3e^{2x}$; (5) $y=e^{2x}(2\cos 3x-\sin 3x)$. **Form II.** (1) $y^2=x^2+1$; (2) $y=e^{4x}+0.5$; (3) $s=t^3+4t^2-t+2$; (4) $y=e^{2x}+2e^{-x}$; (5) $y=e^{3x}(3\cos 2x+\sin 2x)$.

Chapter 16

7. $P_4=24$. **8.** (1) 40.320; (2) 35; (3) 3.624.480. **10.** (1) $n(n-1)$; (2) $\dfrac{1}{n(n-1)(n-2)}$; (3) $\dfrac{1}{(2m-2)!}$. **11.** (1) $\dfrac{n+2}{(n+1)!}$; (2) $-\dfrac{n}{(n+1)!}$; (3) $\dfrac{n(n-1)(n-2)}{(n-5)!}$. **12.** (1) $n\leqslant 4$; (2) $n\geqslant 6$. **13.** 3628800. **14.** 12! **15.** (1) 2730;

(2) $(m-1)(m-2)\ldots 6\cdot 5$. 16. (1) 390; (2) 9; (3) 1440. 17. 870. 18. (1) {6}; (2) {5}; (3) {6; 10}. 19. (1) {5}; (2) {5}; (3) {6}; (4) {6}; (5) {4}. 20. (1) 66; (2) 4950; (3) 56; (4) 101. 23. 14 or 3. 24. (1) {(18; 8)}; (2) {(5; 2)} 29. 0.13775. 30. 0.512. 31. (1) 0.417; (2) 0.151. 34. $p=1-C_8^3/C_{10}^3=5/6$. 35. 5/7. 38. 0.56. 39. 0.2. 42. 0.625. 43. 0.72. 44. 24/33. 45. 77/108. 47. 0.41. 48. 0.737. 49. 0.202. 50. (1) {9}; (2) {6}; (3) {3}; (4) {6}; (5) {4; 8}. 51. (1) {3; 4; 5}; (2) $n>7$, $n\in$ N. 52. {7; 8} 53. 3/8. 54. 1/9. 55. 0.008. 56. 0.995. 57. 3/8. 58. 0.478. 59. 0.00051. 60. 16/33. Quiz No. 1. Form I. (2) {8}; (3) {4}; (4) 0.1; (5) 67/120. Form II. (2) {3}; (3) {4}; (4) 1/4; (5) 14/23.

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2. (11; 1). 3. Into a triangle with vertices (−5; 1), (1; 3), (2; −2). 5. (1) 7; (2) 9. 6. Pairs (1) and (2). 13. $a+b+c=0$. 14. $R=12$. 6—N, $\alpha=11.5°$; $\beta=18.5°$. 15. (1) $m=\pm 1$; (2) $|m|>1$; (3) $|m|<1$. 16. (1) 1/2; (2) −1; (3) 2. 17. $\overrightarrow{CM}=-0.5AB-\overrightarrow{BC}$. 19. $\overrightarrow{OC}=b-a$, $\overrightarrow{OD}=-a$, $\overrightarrow{OE}=-b$, $\overrightarrow{OF}=a-b$. 26. (1) 4; (2) 0. 29. (1) $\mathrm{pr}_n a=\mathrm{pr}_n b$; (2) $\mathrm{pr}_n a=-\mathrm{pr}_n b$. 30. $l\perp a-b$. 31. (1) $0.5\sqrt{2}\,|a|$; (2) $-0.5\,|a|$; (3) $-0.5\sqrt{3}\,|a|$. 33. (1) (6; 1); (2) (3; −2). 34. (1) (3; −3); (2) (−5; 2); (3) (0; 2); (4) (−4; −6); (5) (−9; −4); (6) (−6; −13). 35. (1) $-2i-4j$; (2) $-i-8j$. 36. $\overrightarrow{AB}=4i+7j$; $\overrightarrow{BC}=3i-3j$; $\overrightarrow{CA}=-7i-4j$. 37. (1) $\overrightarrow{AB}\uparrow\downarrow\overrightarrow{CD}$; (2) $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are noncollinear; (3) $\overrightarrow{AB}\uparrow\uparrow\overrightarrow{CD}$. 41. (1) $|\overrightarrow{AB}|=5$; (2) $|\overrightarrow{BC}|=5\sqrt{5}$; (3) $|\overrightarrow{CA}|=10$. 42. (1) $15+5\sqrt{5}$; (2) $18+2\sqrt{17}$. 43. (1) $\cos\alpha=5/13$, $\cos\beta=12/13$; (2) $\cos\alpha=-0.8$; $\cos\beta=0.6$. 44. (1) (2; −1); (2) (4; −1). 45. (4; −3); $r=13$. 46. M_1 (−2; 0), M_2 (22; 0). 47. B_1 (0; −5), B_2 (0; 3). 48. (1) (0; −3); (2) (0; 5). 49. (1) (1; 0); (2) (−4; 0). 50. (1) M_1 (−13; −13) and M_2 (−5; −5); (2) M_1 (2; 2) and M_2 (10; 10). 51. M_1 (5; 2) and M_2 (5; 10). 52. M_1 (−5; 10) and M_2 (7; 10). 56. (1) C (2; −1); 51. M_1 (5; 2) and M_2 (5; 10). 52. M_1 (−5; 10) and M_2 (7; 10). 56. (1) C (2; −1); (2) C (2; −5). 57. C (5; 6). 58. C_1 (5; −4) and C_2 (8; −7). 59. (−2; −0.5); (2; 1.5). 60. B (9; 1), 61. (1) (−2; 1); (2) (−2/3; 2). 62. C (−2; −2), D (4; 1). (63. A (6; 8). 64. B (2; −1). 65. C (4; −9). 66. C (9; 5). 72. 409. 73. $16-12\sqrt{3}$. 74. (1) 12; (2) 2/3. 75. −85. 76. (1) −26; (2) −25. 77. 16.3°. 78. $11\sqrt{5}/25$. 79. (1) Yes; (2) yes; (3) no. 80. $|a|=|b|$. 81. 11. (units of work). 87. (1) (−5; 9); (2) (2; 2); (3) (−3; 1); (4) (0; 5). 88. (1) (−5; −5); (2) (1; −5); (3) (−5; −1); (4) (1; −1). 89. O (4; 4), O_1 (−4; −4). 90. (3; −5). 91. The abscissa and ordinate will interchange. 92. (1) $(\sqrt{2}; -3\sqrt{2})$; (2) $(2\sqrt{3}-1; -2, -\sqrt{3})$. 93. (1) (2; $2\sqrt{3}$); (2) (0; 4). 99. (1) (2; $-3\pi/4$), (4; $-\pi/3$), (1; $5\pi/6$); (2) (2; $-\pi/4$), (4; $-2\pi/3$), (1; $\pi/6$). 100. (1) (0; 2); (2) $(\sqrt{3}; -3)$; (3) (1; −1); (4) $(-\sqrt{2}; \sqrt{2})$. 101. (1) $(5; \pi/2)$; (2) $(\sqrt{3}; -\pi/2)$; (3) (2; $-5\pi/6$); (4) $(\sqrt{2}; -\pi/4)$. 102. Yes, if the point lies on the positive semiaxis Ox. 103. $\sqrt{31}\approx 5.6$. 114. (1) (2; 3); (2) (−2; 3); (3) (2; −3); (4) (−3; −2). 115. (1) (3; −1) and (1; −7); (2) (−3; −1) and (−1; −7); (3) (3; 1) and (1; 7), (4) (−1; 3) and (−7; 1). 116. (1) (−3; −2), (−7; −4) and (−1; −6); (2) (3; −2), (7; −4) and (1; −6); (3) (−3; 2), (−7; 4) and (−1; 6); (4) (2; 3), (4; 7) and (6; 1). 117. 13. 118. (6; −2), (−2; 4), (−6; −6). 119. (−5; −3). 120. (−7; 1). 121. (0; 9.6). 122. $\overrightarrow{AM_1}=a+0.5b$, $\overrightarrow{BM_2}=0.5b-0.5a$; $\overrightarrow{CM_3}=-0.5a-b$; $|AM_1|=10$. 123. 10. 124. A (−8; −3). 125. (−3; −3). 126. D (6; −3). 127. C (5; −5), D (1; −7). Quiz No. 1. Form I. (1) $\overrightarrow{AB}=(3; 7)$; (2) (4; 4); (3) M_1 (2; −2), M_2 (10; −10); (4) 90°. Form II. (1) $\overrightarrow{AB}=5i+9j$; (2) C (4; −3); (3) C (8; −2); (4) 63/65.

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11. 6 sq. un. 12. (1) $y+4=0$; (2) $x+6=0$. 13. (1) $y=0$, $x=-4$, $y=-3$, $x=0$; (2) $x=2$, $y=0$; $x=5$, $y=3$. 14. (1) $4x-5y-7=0$; (2) $3x-4y-7=0$. 15. (1) $2x+y+7=0$; (2) $5x-2y-6=0$; (3) $x-4y-10=0$. 16. (1) $x-4y=0$; (2) $4x+5y=0$. 17. 15. 18. (1) (2; −1); (2) (−4; −2). 23. (1) $x/3+y/3=1$; (2) $x/(-1/2)+y/(-1/3)=1$; (3) $x/3+y/2=1$; (4) $x/(-4)+y/3=1$. 24. (1) $x/(-2)+y/3=1$; (2) $x/3+y/(-4)=1$. 25. (1) 10; (2) 20; (3) 15. 37. (1) 30°; (2) 108.4°; (3) 81.9°; (4) 109.1°; (5) 149°; (6) 71°. 38. (1) $x+y=0$, (2) $4x-y=0$. 39. $y=(2/3)x-1/2$. 40. (1) $\sqrt{3}x-y=0$; (2) $\sqrt{3}x-3y=0$; (3) $x+y=0$; (4) $3x-y=0$; (5) $5x+y=0$. 41. (1) $y=x+3$; (2) $y=-\sqrt{3}x+3$; (3) $y=5x+3$. 42. (1) $y=(\sqrt{3}/3)x-2$; (2) $y=-x-2$; (3) $y=2x-2$. 43. (1) $2x+y=0$, (2) $5x-y=0$. 44. (1) P (4; 3); (2) (0; 0), (12; 0), (12; 16) and (0; 16). 45. $y=-2x-12$. 46. (1) $y=-2x+3$; (2) $y=-x-5$. 49. (1) $x-y=0$; (2) $2x+y-4=0$. 50. (1) $3x+y-7=0$; (2) $x-y+1=0$; (3) $x+y-5=0$. 55. (1) $x-y=0$; (2) $4x+3y-12=0$. 56. (1) $7x-4y+13=0$ (AB); $9x+7y-44=0$ (BC); $2x+11y+28=0$ (AC); (2) $2x-y-1=0$, $5x+y-20=0$ and $3x-5y-12=0$. 57. (1) $x+y-1=0$; (2) $x-2y-2=0$. 58. arctan 0.8 ≈ 38.7°. 59. (1) $a=5$, $b=-5$; (2) (0; 3). 60. (1) $x-y+3=0$; (2) $x+y-2=0$. 64. (1) (−1; −3); (2) (4; −2). 65. (1) (−2; −4), (5; 2) and (−5; 0); (2) (4; −1), (−3; −3); (1; 6). 66. (1) $3x-2y+7=0$; (2) $4x+5y-6=0$. 73. (1) arctan 2 ≈ 63.4°; (2) arctan (7/9) ≈ 37.9°; (3) arctan (14/23) ≈ 31.3°; (4) arccos (13/85) ≈ 81.2°. 74. arccos (63/65) ≈ 14.1°. 75. (1) ≈ 67.4°, ≈56.3°, 56.3°; (2) ≈ 76.3°, ≈43.8°, ≈59.9°; (3) ≈72.3°, ≈58.5°, 49.2°. 76. (1) ≈53.1°; (2) ≈46.8°. 77. arctan (25/21) ≈ 50°. 78. (1) $\hat{A}$ ≈ 25.8°, $\hat{B}$ ≈ 23.6°, $\hat{C}$ ≈ 130.6°; (2) $\hat{A}$ ≈ 67.4°, $\hat{B}=\hat{C}$ ≈ 56.3°. 79. arctan 0.5 ≈ 26.6°. 80. (1) ≈29.7°; (2) ≈48.4°. 81. arctan (27/47) ≈ 29.9°. 82. arctan (5/7) ≈ 35.5°. 83. 45°. 84. (1) $2x-y+9=0$ and $11x-2y+32=0$; (2) $y=0$ and $x=0$; (−1; 0) and (0; 1). 85. $y=2x$ and $y=7x$; $y=(1/7)x$ and $y=(1/2)x$. 86. (1) $y=-3x$ and $y=(-2/3)x$; (2) $y=(-3/2)x$ and $y=(-1/3)x$. 87. $y+2=0$. 89. The pairs of straight lines (1), (2) and (3) are parallel. 90. $a=10$. 91. (1) $5x-3y+21=0$; (2) $3x+4y+19=0$. 92. (1) $7x+5y+26=0$; (2) $x-6y-25=0$. 93. (1) $x-4y+6=0$; (2) $x-2y+2=0$. 97. The pairs of straight lines (1), (3) and (4) are perpendicular. 98. $k=-1/5$. 99. (1) $2x+5y+7=0$; (2) $5x+6y+14=0$; (3) $3x-2y=0$. 100. $5x-9y+26=0$. 101. $x+y+3=0$ and $x-y-3=0$. 102. $x+3y=0$. 103. $x-y-5=0$. 104. $2x-3y+10=0$. 105. (1) $4x+2y-9=0$; (2) $2x-3y+1=0$. 106. $7x+3y+20=0$. 107. (1) $10x+3y+2=0$, $5x+9y+2=0$ and $5x-6y=0$; (2) $x+y+10=0$, $15x+4y-18=0$ and $10x-y-68=0$; (3) $11x-9y-26=0$, $9x-2y-64=0$ and $2x-7y+38=0$. 108. (1) $2x-11y-29=0$; $5x-3y+10=0$ and $3x+8y+39=0$; (2) $x+2y-5=0$, $x-y=0$ and $2x+y-5=0$; (3) $4x-5y+4=0$, $3x+4y+14=0$ and $10x+3y+32=0$. 109. (1) 5; (2) 10. 110. (1) 10; (2) 13. 111. $k=2$. 112. (−2; −7). 113. $x-6y=0$, $3x-2y=0$. 114. $x-6y=0$, $2x-3y=0$. 115. $8x-11y+2=0$, $5x-2y-15=0$ and $2x+7y-32=0$. 116. $9x+11y+24=0$. 117. (6; 2). 118. $5x-12y=0$. 119. $8x-15y+11=0$. 120. $4x-y+36=0$, $x+4y-25=0$. 121. $4x+5y-16=0$, $4x+5y+25=0$. 122. (1) $2x-3y+4=0$; (2) $3x-11y-7=0$; (3) $3x+2y-7=0$; (4) 45° and 90°; (5) (7/3; 0). 123. $x-y=1=0$. 124. $42x+7y-68=0$. 125. $9x+2y-113=0$, $3x-10y-59=0$, $2x+3y-14=0$. 126. $x-3y-3=0$, $3x-2y-16=0$, $x+4y-10=0$, $5x-8y+6=0$. 127. $x-3y+8=0$, $5x-3y-32=0$, $5x-9y+4=0$, $y=1$. 128. $2x-y+3=0$, $2x-y-7=0$, $x+2y-1=0$, $x+2y-11=0$, $3x+y-8=0$ and $x-3y+4=0$. 129. $3x-y+3=0$, $x+3y-19=0$. 130. $3x-y+1=0$ and $x+3y-$

$13 = 0$. **Quiz No. 1.** Form I. (1) $4x - 3y + 37 = 0$; (2) $4x + 5y + 13 = 0$; (3) $3x - 4y - 10 = 0$; (4) arctan $(48/11) \approx 77.1°$; (5) $x + 1 = 0$. Form II. (1) $x - 2y + 18 = 0$; (2) $y - 4 = 0$; (3) $x - 3y + 2 = 0$; (4) arctan $(21/13) \approx 58.3°$; (5) $(-2/3;\ 4)$.

Chapter 19

64. $x - y - 4 = 0$. 5. $x^2 + y^2 - 3x = 0$. 6. $x^2 + y^2 = 16$. 7. $2x^2 + 2y^2 - 25 = 0$. 8. $x^2/36 + y^2/27 = 1$. 9. $x^2/4 + y^2/3 = 1$. 10. $x^2/16 - y^2/48 = 1$. 11. $x^2/9 - y^2/72 = 1$. 12. $x^2 + 4y + 4 = 0$. 13. $y^2 - 6x + 9 = 0$. 14. (1) $y^2 - 6y + 12x - 3 = 0$; (2) $x^2 + 6x - 12y + 21 = 0$. 15. $x^2 - 4x + 2y + 5 = 0$. 16. $x^2 - y^2 + 6x - 8y + 25 = 0$. 25. Point A belongs, points B and C do not; (2) The point K belongs, the point L does not. 26. (1) $x^2 + y^2 = 3$; (2) $(x + 2)^2 + (y + 5)^2 = 9$. 27. (1) $(x + 1)^2 + (y - 4)^2 = 17$; (2) $(x + 3)^2 + y^2 = 41$. 28. (1) $(x - 3)^2 + (y + 2)^2 = 34$; (2) $x^2 + (y - 4)^2 = 5$. 29. (1) $(x - 3)^2 + (y - 4)^2 = 25$; (2) $(x + 4)^2 + (y - 5)^2 = 41$. 30. (1) $(x + 2)^2 + (y - 3)^2 = 13$; (2) $(x - 3)^2 + (y + 5)^2 = 34$. 31. (2; 0), $(-6;\ 0)$, (0; 3) and (0; -4). 32. (1; 5) and (7; -3). 33. (1) $(x + 4)^2 + y^2 = 100$; (2) $(x - 1)^2 + (y + 2)^2 = 25$. 34. (1) $(x - 3)^2 + (y - 2)^2 = 25$; (2) $x^2 + (y + 3)^2 = 25$; (3) $(x - 2)^2 + (y + 3)^2 = 100$. 35. $(x - 5)^2 + (y - 4)^2 = 25$ and $(x + 5)^2 + (y - 4)^2 = 25$. 36. $(x - 3)^2 + (y - 5)^2 = 5^2$ and $(x - 27)^2 + (y - 29)^2 = 29^2$. 37. $(x - 29)^2 + (y - 29)^2 = 29^2$ and $(x - 5)^2 + (y - 5)^2 = 5^2$. 38. $x^2 + y^2 = 25$. 39. $(x - 4)^2 + y^2 = 41$. 40. $x^2 + (y - 2)^2 = 34$. 41. (1) $(x + 4)^2 + (y + 3)^2 = 52$; (2) $(x - 1)^2 + (y + 2)^2 = 20$. 42. $(x + 1)^2 + (y + 4)^2 = 25$. 43. $(x - 3)^2 + (y - 2)^2 = 13$. 44. $(x + 2)^2 + (y + 1.5)^2 = 6.25$. 45. (1) $O_1\ (-3;\ 5)$, $r = \sqrt{21}$; (2) $O_1\ (0;\ -6)$, $r = 7$; (3) $O_1\ (1/2;\ -5/2)$, $r = 7/2$; (4) $O_1\ (2;\ 5)$, $r = 0$ (there is no circle); (5) $O_1\ (-3,\ -7)$, $r = \sqrt{-23}$ (there is no circle). 46. (1) 10; (2) $\sqrt{157}$. 47. (1) $4x - 3y - 10 = 0$; (2) $4x - 7y + 29 = 0$. 48. $\sqrt{3}x - y + 3\sqrt{3} + 2 = 0$. 49. $x + y + 1 = 0$. 50. $5x + y - 21 = 0$. 51. $3x + 2y = 0$. 52. (1) $5x + 2y - 13 = 0$; (2) $x - y - 3 = 0$. 53. (1) $3x - 4y - 32 = 0$; (2) $2x + y = 0$. 54. $7x + y - 17 = 0$. 55. (1) $(x + 2)^2 + (y - 1)^2 = 100$; (2) $(x - 1)^2 + (y + 3)^2 = 97$. 65. (1) $x^2/16 + y^2/9 = 1$; (2) $x^2/4 + y^2/25 = 1$. 66. (1) $x^2/25 + y^2/16 = 1$; (2) $x^2/89 + y^2/64 = 1$; (3) $x^2/12 + y^2/16 = 1$. 67. (1) $x^2/36 + y^2/11 = 1$; (2) $x^2/20 + y^2/16 = 1$. 68. $x^2/25 + y^2/28 = 1$. 69. (1) $(-5;\ 0)$, (5; 0), (0; -3) and (0; 3); $2a = 10$ and $2b = 6$; (2) $(-4;\ 0)$, (4; 0), (0; -9) and (0; 9); $2a = 18$ and $2b = 8$; the foci are situated on the y-axis. 70. (1) $(-3;\ 0)$ and (3; 0); $2c = 6$; (2) (0; -4) and (0; 4); $2c = 8$. 71. (1) $e = 4/5$; (2) $e = 3/4$. 72. $x^2/27 + y^2/24 = 1$. 73. $x^2/100 + y^2/64 = 1$. 74. (1) $x^2/25 + y^2/16 = 1$; (2) $x^2/100 + y^2/64 = 1$. 75. (1) $x^2/25 + y^2/9 = 1$; (2) $x^2/169 + y^2/144 = 1$. 76. (1) $x^2/100 + y^2/25 = 1$; (2) $x^2/400 + y^2/225 = 1$. 77. (1) $x^2/100 + y^2/25 = 1$; (2) $x^2/10 + y^2/5 = 1$. 78. (1) (12; 3) and (9; 4); (2) (4; 1.8) and (3; 2.4). 79. (1) $\sqrt{17}$; (2) $\sqrt{245} \approx 15.6$. 89. $x^2/144 - y^2/400 = 1$. 90. $x^2/9 - y^3/36 = 1$. 91. $x^2/36 - y^2/64 = 1$. 92. $(-6;\ 0)$ and (6; 0), $(-5\sqrt{2};\ 0)$ and $(5\sqrt{2},\ 0)$. 93. (1) $2a = 10$, $2b = 14$; (2) $2a = 4\sqrt{3}$, $2b = 18$. 94. $(-6;\ 0)$ and (6; 0); $2c = 12$. 95. (1) $2c = 16$; (2) $2c = 12$. 96. (1) $e = 4/3$; (2) $e = 7/5$. 97. (1) $y = \pm(3/4)x$; (2) $y = \pm(2\sqrt{2}/3)x$. 98. (1) $x^2/2 - y^2/6 = 1$; (2) $x^2/18 - y^2/9 = 1$. 99. (1) $x^2/9 - y^2/16 = 1$; (2) $x^2/20 - y^2/16 = 1$. 100. (1) $x^2/16 - y^2/9 = 1$ and $x^2/9 - y^2/16 = 1$; (2) $x^2/256 - y^2/144 = 1$. 101. (1) $x^2/64 - y^2/16 = 1$; (2) $x^2/144 - y^2/36 = 1$. 102. (1) $x^2/8 - y^2/2 = 1$; (2) $x^2/32 - y^2/8 = 1$. 103. (1) $x^2/9 - y^2/16 = 1$; (2) $x^2/3 - y^2/6 = 1$; (3) $x^2/16 - y^2/48 = 1$. 104. (1) $x^2/27 - y^2/9 = 1$; (2) $x^2/8 - y^2/4 = 1$; (3) $x^2/12 - y^2/9 = 1$. 105. (0; -4) and (0; 4); (0, -5) and (0; 5); $e = 5/4$; $y = \pm(4/3)x$. 106. (1) $x^2 - y^2 = 9$; (2) $x^2 - y^2 = 60$. 107. $y^2 - x^2 = 2$. 115. (1) $y^2 = 20x$; (2) $y^2 = -16x$; (3) $x^2 = 8y$; (4) $x^2 = -12y$. 116. (1) $y^2 = 8x$; (2) $y^2 = -12x$; (3) $x^2 = 16y$; (4) $x^2 = -4y$. 117. (1) $y^2 = 1.8x$; (2) $y^2 = -x$; (3) $y^2 = -2x$. 118. (1) $x^2 = -(4/3)y$;

(2) $x^2 = 9y$. 119. (1) $x = -2$; (2) $x = 9/4$; (3) $y = -1$; (4) $y = 5/2$. 120. (1) (3/2; 0); (2) (−1; 0); (3) (0; 7/2 (4) (0; −5/4). 121. (1) (−2; 0), (2) (5; 0); (3) (0; −4); (4) (0; 6). 122. 20. 123 (1) (1; 4); (2) (1; 2) and (4; 4). 124. (1) (0; 0) and (1; 1); (2) (0; 0) and (9; 9). 133 (1) $(y + 2)^2 = 5(x + 4)$; (2) $(y + 4)^2 = 8(x + 2)$; (3) $(y + 1)^2 = -(2/3)(x - 3)$. 134. (1) $(x - 2)^2 = 16(y - 4)$; (2) $(x - 5)^2 = 5(y + 5)$; (3) $(x - 3)^2 = -1.8(y - 5)$. 135. (1) $(y - 6)^2 = -24(x - 4)$; (2) $(x - 3)^2 = 8(y + 2)$; (3) $(x + 1)^2 = -20(y - 1)$. 136. (1) $(y+3)^2 = -16(x - 1)$; (2) $(x + 2)^2 = 24(y - 4)$; (3) $(x + 3)^2 = -8(y - 5)$. 137. (1) $y^2 = 12(x - 3)$; (2) $y^2 = -24(x + 4)$. 138. (1) $x^2 = 8(y - 2)$; (2) $x^2 = 12(y + 2)$; (3) $x^2 = -12(y + 3)$. 139. (1) $(y + 1)^2 = -16(x + 2)$; (2) $y^2 = 8(x + 2)$; (3) $(x - 2)^2 = 12(y + 1)$; (4) $x^2 = -8(y - 2)$. 140. (1) (3; −5); (2) (−4; −1); (3) (−2; −3); (4) (2; 3). 141. (1) (−1; 4); (2) (0; 0); (3) (−5; −4); (4) (0; 0). 142. (1) $y = 5$; (2) $x = -8$. 143. (1) $x = -1.5$; (2) $x = 0$, (3) $y = 1.5$. 155. (1) 5; (2) 3. 156. (1) 63.4°; (2) 85.2°; (3) 153.4°, (4) arctan $4 \approx 76°$, arctan $2 \approx 63.4°$. 157. 99.5° and 80.5°. 158. (1) 45°; (2) 135°. 159. (1) 23.5°; (2) 60°. 160. (1) $x - y - 6 = 0$, $x + y - 2 = 0$; (2) $6x - y + 4 = 0$, $x + 6y + 13 = 0$. 161. (1) $3x - 4y + 25 = 0$, $4x + 3y = 0$; (2) $2x - 3y + 25 = 0$, $3x + 2y + 18 = 0$; (3) $10x + 3y + 32 = 0$, $3x - 10y + 75 = 0$; (4) $x + y + 2 = 0$, $x - y - 6 = 0$. 162. (1) $3x + y - \pi = 0$, $3x - 9y - \pi = 0$; (2) $x - 6y + 3\sqrt{3} - \pi = 0$, $12x + 2y - \sqrt{3} - 12\pi = 0$; (3) $6x + 2y - \pi = 0$; $6x - 18y - \pi = 0$. 163. (−2, −12). 164. $(\pi/6; 1/2)$. 165. (4; 2). 166. (1; 3). 167. arctan $(11/13) \approx 40.2°$; arctan $(9/7) \approx 52.1°$. 168. (1) 36.9°, (2) 34.7°. 169. 2.5°. 170. arctan (3/4). 171. arctan (16/13). 172. arctan 8, arctan (24/29). 173. $3x - y - 16 = 0$, $3x - y - 4 = 0$, $x + 3y + 12 = 0$, $x + 3y - 28 = 0$. 174. (−2; −1), (−2; 1), (2; −1) and (2; 1). 175. $(-\sqrt{3}; -1)$ and $(\sqrt{3}; -1)$. 176. $x = 0$, $y = 0$, $5x + 3y - 15 = 0$, $5x - 3y + 15 = 0$, $5x + 3y + 15 = 0$, $5x - 3y - 15 = 0$. 177. 28.8 sq. un. 178. 24 sq. un. 179. $x^2/12 - y^2/4 = 1$. 180. $\sqrt{5}$. 181. $x^2/100 + y^2/64 = 1$. 182. (0; 0) and (3; 6). 183. $y - 2 = 0$. 184. $p = 8$. **Quiz No. 1.** Form I. (1) $2x + 5y + 1 = 0$; (2) $x^2/144 + y^2/44 = 1$; (3) $e = 4/3$; (4) $y = 1$; (5) $y = -4$. Form II. (1) $x - y + 3 = 0$; (2) $e = 3/5$; (3) $x^2/729 - y^2/1296 = 1$; (4) $x = -3$; (5) $x = 5$.

Chapter 20

7. $a^2\sqrt{11}/16$. 8. $a^2\sqrt{2}/4$. 11. Three pairs. 13. 48 cm. 14. 12 cm. 15. 4.5 cm. 17. (1), (2) An infinite set of solutions. 26. 10 and 15 cm. 27. 18, 27 and 36 cm. 30. $m\sqrt{3}/3$. 31. $\sqrt{l^2 - a^2/3}$. 32. $0.5\sqrt{a^2 + b^2 + 4h^2}$. 34. $a\sqrt{2}/2$. 35. 12 and 20 cm. 36. 24 and 26 cm. 37. 51 cm. 38. 1 cm. 39. $\sqrt{h^2 + a^2/12}$. 40. $\tan \varphi_1 = \tan \varphi_2 = \sqrt{5}/5$, $\tan \varphi_3 = \sqrt{2}$. 41. $a\sqrt{3}$. 45. $m\sqrt{3}/2$. 46. $2m\sqrt{3}/3$. 47. $2\sqrt{13}$. 48. $a\sqrt{\cos 2\alpha}$. 49. $\sqrt{a^2 + b^2 + c^2}$, $\sqrt{a^2 + c^2}$, $\sqrt{b^2 + c^2}$. 50. 3 cm^2. 51. 75 cm^2. 52. $42\sqrt{3}$ cm^2. 53. $20\sqrt{2}$ cm^2. 54. $2\pi\sqrt{3}$ m^2. 56. $a\sqrt{6}/3$. 57. arccos $(\cos \alpha/\cos \times (\alpha/2))$. 58. $9a^2/8$. 59. $(a + b + c)(m + n)/m$. 61. 3.75 cm. 62. $\sqrt{m^2 - a^2}$. 63. 17 cm. 64. 15 and 20 cm. 65. arccos $(\sqrt{2} \cos \alpha)$. 66. 2 arccot $(2 \cos \alpha)$. 67. $0.5a \sin \beta/\sin (\alpha/2)$. 68. arccos $(3/4) = 41.4°$; 2 arccos $(3/4) = 82.8°$. 69. 2 arcsin $(\sin (\alpha/2)/\sqrt{2})$. 70. 45°. 71. 13 cm. 72. arcsin $(\sin \alpha \sin \beta)$. 73. $2a(\sqrt{3} + 1)$. 74. 90°. 75. arccos $(\tan \alpha/2) \cot \alpha)$. **Quiz No. 1.** Form (1) 32 m; (2) $12\sqrt{2}$ cm^2. Form II (1) 8 cm; (2) 54 cm^2.

Chapter 21

4. (1) $\overrightarrow{BA}$; (2) $\overrightarrow{AB}$; (3) 0. 5. (1) $\overrightarrow{BB_1}$; (2) $\overrightarrow{CC_1}$, (3) $\overrightarrow{AD}$; (4) $\overrightarrow{D_1B}$. 6. (1) $\overrightarrow{AC}$; (2) $\overrightarrow{AA_1}$; (3) $\overrightarrow{AA_1}$. 13. The triples of vectors (2), (6), (7) are coplanar the triples of

vectors (1), (3), (4), (5) are noncoplanar. 15. (1) $\overrightarrow{AD_1} = 0\cdot\mathbf{p} + 1\cdot\mathbf{q} + 1\cdot\mathbf{r}$; (2) $\overrightarrow{AC_1} = 1\cdot\mathbf{p} + 1\cdot\mathbf{q} + 1\cdot\mathbf{r}$; (3) $\overrightarrow{AM} = 1\cdot\mathbf{p} + 0\cdot\mathbf{q} + \frac{1}{2}\cdot\mathbf{r}$; (4) $\overrightarrow{AN} = 1\cdot\mathbf{p} + \frac{1}{2}\cdot\mathbf{q} + 1\cdot\mathbf{r}$; (5) $\overrightarrow{AP} = (3/7)\cdot\mathbf{p} + 1\cdot\mathbf{q} + 1\cdot\mathbf{r}$. 16. $\overrightarrow{DA} = -\overrightarrow{DB} - \overrightarrow{DC} + 3\overrightarrow{DM}$. Hint. Take advantage of the formula $\overrightarrow{OM} = (1/3)(\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OB})$ (see Problem 12). 17. (1) $\overrightarrow{MO} = 0.5\cdot\mathbf{a} + 0\cdot\mathbf{b} + 0.5\cdot\mathbf{c}$; (2) $\overrightarrow{MD} = 1\cdot\mathbf{a} - 1\cdot\mathbf{b} + 1\cdot\mathbf{c}$; (3) $\overrightarrow{MN} = 1\cdot\mathbf{a} - 0.5\cdot\mathbf{b} + 0.5\cdot\mathbf{c}$ .19. (1) (3; 2; −5); (2) (2; 0; −1); (3) (0.5; $\sqrt{2}$; 0); (4) (0; 0; 3); (5) (−4; 0; 0); (6) (0; 0; 0). 23. $\overrightarrow{AB} = (-6; 7; -5)$; $\overrightarrow{MN} = (-4; -5; -4)$. 24. (1) (1; 5; −3); (2) (5; 3; −2); (3) (−2; 5; −5); (4) (6; 9; −12); (5) (4; −3; 8); (6) (−5; 12; −9). 25. (1) Yes; (2) no. 26. $n = 6$, $p = 8/3$. 27. (1) 3; (2) $\sqrt{14}$; (3) $\sqrt{2}$; (4) 3. 28. (1) 5; (2) 0. 29. $6\sqrt{2}$. 30. 3. 31. 10. 32. C (5; −1; −2). 33. C (4; 0; −4); 34. (−2; 1; 3). 35. (1) $\cos\alpha = \cos\beta = \cos\gamma = \sqrt{3}/3$; (2) $\cos\alpha = 4/5$, $\cos\beta = 3/5$, $\cos\gamma = 0$; (3) $\cos\alpha = 0$, $\cos\beta = -\sqrt{10}/10$, $\cos\gamma = -3\sqrt{10}/10$; (4) $\alpha = 0$, $\beta = \gamma = 90°$. 39. (1) 23; (2) 1/3. 40. −8. 41. (1) 90°; (2) 90°; (3) 60°; (4) 60°; (5) 45°. 42. (1) arccos (63/65); (2) arccos (4/9); (3) arccos (1/11). 43. $\cos(\overrightarrow{CA}, \overrightarrow{CB}) = 2/3$. 44. (1) Yes; (2) no. 46. $y = -4$. 52. (1) 0; (2) 24e; (3) 12e, where e is the unit vector of the direction $\mathbf{a} \times \mathbf{b}$. 53. $5\mathbf{i} - 10\mathbf{j} - 5\mathbf{k}$. 54. $S = \sqrt{26} \approx 5.1$ (sq. un). 55. $S = 5\sqrt{2} = 7.07$ (sq. un). 56. $m_0(\vec{F}) = (3; 4; 9)$, $\alpha = 73°, 1$, $\beta = 67°, 1$, $\gamma = 29°, 1$. 58. (0; 2; 0). 60. 10. 61. (−1; 2; 5), (4; 3; −2). 62. $\mathbf{a} + \mathbf{b} = (1; 5; -1)$, $\mathbf{a} - \mathbf{b} = (5; -1; -9)$. 63. $\cos\alpha = 2/3$; $\cos\beta = -2/3$; $\cos\gamma = 1/3$. 64. $\cos(\widehat{a, b}) = 4/9$. 65. arccos ($\sqrt{5}/5$). **Quiz No. 1.** Form I. (1) $|\mathbf{a}| = 9$; $\cos\alpha = 4/9$; $\cos\beta = -7/9$; $\cos\gamma = 4/9$; (2) $\cos(\widehat{\mathbf{a}, \mathbf{b}}) = 3\sqrt{10}/10$; (3) $5\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$. Form II. (1) $\widehat{BAC} = 45°$; (2) 19/13; (3) $5\sqrt{3} = 8.66$ (sq. un.).

Chapter 22

11. The points A, B, and D. 12. $x + 3y - 2z - 5 = 0$. 13. $x + 3y + 2z + 9 = 0$. 14. $x - 3y + z - 13 = 0$. 15. $z + 1 = 0$. 16. (1) $z - 3 = 0$; (2) $y + 2 = 0$. 17. (1) $x - y = 0$; (2) $2x - z = 0$. 18. (1) $3x + 2z - 1 = 0$; (2) $2y - 3z + 11 = 0$. 19. (1) $8x + 7y + 18z + 35 = 0$; (2) $2x - 3y + z - 7 = 0$. 20. (1) $4x - 11y + 5z - 33 = 0$; (2) $x + y - 1 = 0$. 21. $2x + 3y + 4z - 3 = 0$. 22. ≈72°. 23. (1) 1; (2) 11. 24. $\sqrt{6}$. 32. (1) $\begin{cases} x - 2y - 3 = 0, \\ y - z - 2 = 0; \end{cases}$ (2) $\begin{cases} x - 2y - 1 = 0, \\ z + 2 = 0. \end{cases}$ 33. $\begin{cases} x - 2 = 0, \\ y + 1 = 0. \end{cases}$

34. (1) Parallel to the z-axis; (2) parallel to the y-axis; (3) parallel to the x-axis. 35. $(x - 2)/4 = (y + 3)/3 = (z + 1)/2$. 36. $x = 1 + t$, $y = 4 + 4t$, $z = -3 - 3t$. 37. $(x + 2)/2 = (y + 1)/3 = (z + 3)/4$. 38. $\cos\alpha = \pm 4/13$, $\cos\beta = \pm 3/13$, $\cos\gamma = \pm 12/13$. 40. 68.9°. 48. 21.1°. 49. $2x + 4y + 3z + 12 = 0$. 50. $\cos\alpha = \pm\sqrt{14}/7$; $\cos\beta = \pm 3\sqrt{14}/14$; $\cos\gamma = \pm\sqrt{14}/14$. 51. $(x + 1)/4 = (y - 1)/(-5) = (z + 2)/(-1)$. 52. (1; 7; −1). 53. $(x + 2)/1 = (y - 2)/3 = (z + 4)/2$; (1; −7; 2). 56. $2x + 4y - 3z - 11 = 0$. 57. $(x + 2)/2 = (y - 2)/(-4) = (z + 3)/5$; $x = -2 + 2t$, $y = 2 - 4t$, $z = -3 + 5t$. 58. $(x - 1)/(-1) = (y - 2)/1 = (z - 3)/\sqrt{2}$; $x = -t + 1$, $y = t + 2$, $z = \sqrt{2}t + 3$. 59. $\pi/3$. 60. $2x + y = 0$. 61. $2x - y + 2z - 9 = 0$. 62. (1; 4; −7). 63.

arcsin (6/91). 64. (0; 0; −2). **Quiz No.** 1. Form I. (1) $(x-2)/(-1) = (y+3)/4 = (z-4)/(-2)$; $x = 2-t$, $y = -3+4t$, $z = 4-2t$; (2) $7x - 2y + 3z - 6 = 0$; (3) $\sin\varphi = 4/9$. Form II. (1) $(x-1)/3 = (y-3)/(-4) = (z+5)/7$; $x = 1+3t$, $y = 3-4t$, $z = -5+7t$; (2) $x + 14y + 9z - 30 = 0$; (3) (10; 10; −3).

Chapter 23

1. Triangular prism: 5 faces, 6 vertices, 9 edges, 3 lateral edges; quadrangular prism: 6 faces, 8 vertices, 12 edges, 4 lateral edges; hexagonal prism: 8 faces, 12 vertices, 18 edges, 6 lateral edges. 2. Triangular prism; 18, 9, and 6; quadrangular: 24, 12, and 8; hexagonal: 36, 18, 12; n-gonal: $6n$, $3n$, and $2n$. 3. 0; 4; 18; $n(n-3)$. 4. (1) 0; 1; 3; $n-3$; (2) 2; 9; $n(n-3)/2$. 5. $16d$; $24d$; $40d$; $8d(n-1)$. 8. 180°; 360°; 720°; $180°(n-2)$. 11. $2m$ and $m\sqrt{5}$; $m^2\sqrt{3}$ and $2m^2$. 12. $4a^2\sqrt{3}/9$. 13. $\sqrt{n^2 + 0.5m^2}$. 15. 45°. 16. 180 cm². 17. 300 cm². 18. 4.5 cm. 19. (1) 29; (2) $2\sqrt{14} \approx 7.48$. 20. $a\sqrt{2}$ and $2a$. 22. $a = d\sqrt{3}/3$. 23. $d = d_1\sqrt{6}/2$. 24. $\arctan(\sqrt{2}/2) \approx 35.2°$. 25. $2\arctan(\sqrt{2}/2) \approx 70.5°$. 26. $120 + 40\sqrt{3} \approx$ 189.3 cm. 27. $a\sqrt{2}/2$. 28. $a\sqrt{6}/3$. 29. 21. 30. 50 and $25\sqrt{6}$ cm. 31. $h\sqrt{a^2+b^2}$. 32. a^2 and $a^2\sqrt{2}$. 33. (1), (2) (3) Yes; (4) No. 34. $a^2\sqrt{2}$. 35. $3a^2\sqrt{3}/4$. 36. $a^2\sqrt{3}/8$. 37. $3a\sqrt{2}$; $a^2\sqrt{3}/2$. 38. 16 cm. 40. $\arctan(H\sqrt{2}/a)$. 42. $\sqrt{6}S/6$. 43. (1) $2l^2$; (2) $3Q\sqrt{2}$. 44. $\sqrt{S} : \sqrt{Q}$. 45. $2\sqrt{Q+2Sh^2}$. 46. 432 cm². 48. 600 cm². 49. 10 and 24 cm. 50. 1056 cm². 51. 1152 cm². 52. $140\sqrt{2}$ cm². 53. (1) $3al + a^2\sqrt{3}/2$, (2) $4al + 2a^2$; (3) $6al + 3a^2\sqrt{3}$. 54. $75\sqrt{3}/8$ or $54\sqrt{3}$ cm². 55. $4l^2 \sin\alpha\sqrt{\cos 2\alpha}$ where $0 < \alpha < 90°$. 56. $48\sqrt{33}$ cm². 57. $3S$. 58. 940 cm². 59. 120 cm². 60. 1600 cm². 61. 2696 cm². 62. 1360 cm². 63. $ab(\sqrt{2}+1)$. 64. 480 cm². 65. $(a+b)c\sqrt{3}$. 66. (1) $\sqrt{b^2 - a^2/3}$; (2) $\sqrt{b^2 - a^2/2}$; (3) $\sqrt{b^2 - a^2}$. 67. (1) $(1/2)\sqrt{4h^2 + a^2/3}$; (2) $(1/2)\sqrt{4h^2 + a^2}$; (3) $(1/2)\sqrt{4h^2 + 3a^2}$. 68. (1) $\sqrt{h^2 + a^2/3}$; (2) $\sqrt{h^2 + a^2}/2$; (3) $\sqrt{h^2 + a^2}/2$. 70. $3a/2$; $a\sqrt{3}$. 71. $m\sqrt{3}\cos\alpha$. 72. $h\sqrt{1+\cos^2\alpha}$ $/\sin\alpha$. 73. $\alpha/(\sqrt{2}\sin\alpha)$. 74. $a/(\sqrt{3}\tan\alpha)$. 75. $a\sqrt{3m^2 - a^2}/4$. 76. $m^2(\sqrt{3}/4)\cos\alpha$. 77. $(a^2/2)\sin\alpha$. 78. $a^2/(4\cos\alpha)$. 79. $2\arctan(\sqrt{3}\cos\varphi)$. 80. $9\sqrt{33}$ cm². 84. 24 cm. 85. 6 cm. 87. 5 and 6 cm. 88. $2\sqrt{6}$ cm². 89. 60°, 90°, 120°, 90°. 90. 3 cm. 91. $S/4$. 92. (1) $\sqrt{2}/2$; (2) 1/2; (3) $\sqrt{mn}/n$. 93. 225, 100 and 25 cm². 94. 18 cm. 95. 36 cm². 96. (1) $\sqrt{c^2 - (a-b)^2/3}$; (2) $\sqrt{c^2 - (a-b)^2/2}$; (3) $\sqrt{c^2 - (a-b)^2}$. 97. $n(n-3)$. 98. $\sqrt{6}(a-b)/2$. 99. 6 dm. 100. 72 and 18 cm. 101. 9 cm. 103. $(a^2 - b^2)/4$. 104. (1) $a\sqrt{3}(\sqrt{12h^2 + a^2} + a)/4$; (2) $a(\sqrt{4h^2 + a^2} + a)$; (3) $3a(\sqrt{4h^2 + 3a^2} + a\sqrt{3})/2$. 105. 1458 cm². 106. 12 and 8 cm. 107. $a^2(\sqrt{7}+\sqrt{3})/4$. 108. $3a^2$. 109. $\sqrt{\sqrt{4h^4 + M^2} - 2h^2}$. 110. $81\sqrt{3}$ dm². 111. $3S$. 112. $3a^2/2$. 113. $432\sqrt{3}$ cm². 114. $3\sqrt{(b^2 + 3h^2)(b^2 - h^2)}/4$. 115. $2r(k + r)\sqrt{3}$. 116. 768 cm². 117. 18 cm². 118. 50 m². 119. $45\sqrt{3}$ cm². 120. (1) $(3/4)(a+b)\times$ $\sqrt{4h^2 + (a-b)^2/3}$; (2) $(a+b)\sqrt{4h^2 + (a-b)^2}$; (3) $(3/2)(a+b)\times$ $\sqrt{4h^2 + 3(a-b)^2}$; (4) $(1/4)n(a+b)\sqrt{4h^2 + (a-b)^2\cot^2(180°/n)}$. 121. 1728 cm². 122. 124 cm². 123. 432 cm². 124. 273 and 175 cm². 125. 1152 cm². 126. $a^2(2\sqrt{2}+\sqrt{3})$. 127. 72 or 112 cm². 128. 480 cm². 129. 928 cm². 130. 16, 30, and 34 cm. 131. $6r^2(6+\sqrt{3})$. 132. 72 cm². 133. $2a^2\sqrt{5}$. 138. $a^2(3+\sqrt{3})$. **Quiz No.** 1. Form I. (1) $144 + 15\sqrt{3} \approx 170$ cm²; (2) $16\sqrt{3}$ cm². Form II. (1) 906 cm²; (2) 30 cm².

Chapter 24

1. 10 cm; $\arcsin(4/5)$. 2. 25π cm^2. 3. 480 cm^2. 4. $30\sqrt{3}$ cm^2. 5. $3a^2(4+\sqrt{3})/8$. 6. 2 $\arctan(1/2)$. 7. 10 cm. 8. 0.5 $(\sqrt{1+\sin^2\alpha}-\sin\alpha)$. 9. $12r^2\sqrt{3}$. 10. 40 cm^2, 100 cm^2. 11. $0.5a^2\cot\alpha$. 12. 60 cm^2; 13 cm, $\arcsin(12/13)$. 13. $48\pi/\tan\alpha$. 14. $32\sqrt{2}$ cm^2. 15. $18\sqrt{3}$ cm^2. 16. $\pi R^2m^2/(m+n)^2$. 17. $R^2\sqrt{3}$. 18. $H^2/3$. 19. $0.5R\sqrt{2H^2+R^2}$. 20. $2H^2/3$. 21. $HR\sqrt{2}/(H+R\sqrt{2})$. 22. $HR\sqrt{3}/(H+R\sqrt{3})$. 24. $(a^2/12)\tan\alpha$. 25. $(R/\sqrt{R^2+H^2})\,360°$. 26. $\alpha=(R/L)\,360°$; in the case of an equilateral cone $\alpha=180°$. 27. 30°. 28. $L\alpha/360°$. 29. $\sqrt{H^2+(R-r)^2}$. 30. $(R-r)\tan\alpha$. 31. 34 cm. 32. $(R+r)\sqrt{l^2-(R-r)^2}$. 33. $(\sqrt{M_1}+\sqrt{M_2})^2/4$. 34. 18 and 8 cm, 64π and 81π cm^2. 38. $(\pm R;0;0)$; $(0;\pm R;0)$; $(0,0,\pm R)$. 39. (1) $(x-2)^2+(y-3)^2+(z-4)^2=25$; (2) $(x+3)^2+y^2+(z-4)^2=2$. 40. Points A, B and D. 41. (1) (0; 0; 0), $R=4$; (2) (1; −3; 5), $R=6$; (3) (3; −4; 2), $R=5$; (4) (−5; −1; −3); $R=6$. 42. $x^2+y^2+z^2=29$. 43. $(x-2)^2+(y+1)^2+(z-3)^2=14$. 44. $(x-2)^2+(y+3)^2+(z+1)^2=21$. 45. (10; −6; 8) and (−10; 6; −8). 46. (1) 1, (2) 5. 47. 4 cm. 48. (1) O (0; 0; 1), $r=2\sqrt{2}$, (2) O (0, 2, 0) $r=\sqrt{5}$. 49. (1) A point; (2) a circle. 50. $\pi(R^2-a^2)$. 51. $(R^2-m^2):R^2$. 52. $\pi R^2\cos^2\alpha$. 53. $a\sqrt{4R^2-a^2}/(2R)$. 54. $a/2$ and $a\sqrt{3}/2$. 55. 7 cm. 56. 11 cm. 57. $12R^2\sqrt{3}$. 58. $b^2/(2h)$. 59. $a\sqrt{6}/4$; $a\sqrt{6}/12$. 60. $b/(2\sin\alpha)$. 61. $2R\sin\alpha$. 62. $h\cot\alpha\tan(\alpha/2)$. 63. $3h/2$. 64. 13 cm. 65. 5 cm. 66. 2.5 cm. 67. 16π cm^2. 68. $l^2/(2h)$. 69. $2\pi r\,(l-r)/l$. 70. 12 cm. 71. (1) $l/(2\sin\alpha)$; (2) $l\cos\alpha\tan(\alpha/2)$. 72. $h\sqrt{2Rh-h^2}$, where $h<2R$. 73. $0.5\cos\alpha\times\sqrt{S\tan\alpha/\cos^2(\alpha/2)}$. 74. $4R^2/\sin\alpha$. 75. 5 cm. 76. $\sqrt{Rr}$. 77. $R/3$; $\pi R\sqrt{3}/3$. 78. $R\sin(\alpha/2)$: $(2\cos^2(45°-\alpha/4))$. 79. $0.5r\tan(\alpha/4)$. 80. 90 cm. 81. 69.2 cm^2. 82. (1) $a\sqrt{2}/2$; (2) $a\sqrt{6}/6$. 83. 8 cm. 84. $R^2\tan\alpha\sqrt{\cos(\varphi+\alpha)\cos(\varphi-\alpha)}/(\cos\alpha\cos^2\varphi)$. 85. $2\sqrt{2}d^2\sin(45°+\varphi)/(\sin\alpha\cos\varphi)$. 86. 24 cm^2. 87. 45°. 88. $\pi H^2/4$. 89. 5 cm. 90. 27π cm^2. 91. 150 cm^2. 92. 24π cm^2. 93. (1) $2\pi r$, (2) $r\tan(\alpha/4)$. **Quiz No. 1.** Form I. (1) $2h/(2+\cot(\alpha/2))$; (2) 144 cm^2. Form II. (1) $4d^2\tan(\alpha/2)$; (2) $a\tan(\alpha/2)/\cos\alpha$.

Chapter 25

1. (1) $l^3\sqrt{3}/9$; (2) $S\sqrt{6S}/36$. 2. $M\sqrt[4]{2M^2}/2$. 3. $\sqrt[3]{m^3+n^3}$. 4. 12 cm. 5. $\sqrt{S_1S_2S_3}$. 6. 9828 cm^3. 7. $48\sqrt{11}$ cm^3. 8. $l^3\sin\alpha\sin\beta\sqrt{\cos(\alpha+\beta)\cos(\alpha-\beta)}$. 9. $2048\sqrt{3}$ cm^2. 10. 5040 cm^3. 11. $3536\sqrt{3}$ cm^3. 12. 3840 cm^3. 13. $a^3\sin\alpha\cot\beta$. 14. $2a^3\sin\alpha\tan\beta\cos(\alpha/2)$. 15. (1) $a^2h\sqrt{3}/4$; (2) a^2h; (3) $3a^2h\sqrt{3}/2$. 16. $a^3/8$. 17. 1530 cm^3. 18. $\sqrt{0.5pqm\sin\alpha}$. 19. $24\sqrt{2}$ cm^3. 20. $0.5a^3\times\sqrt{\sin(\alpha+30°)\sin(\alpha-30°)}$. 22. $ah^2\cot\varphi\sin\beta$. 23. $abl\sin\alpha\sin\varphi$. 24. (1) $a^2\sqrt{3b^2-a^2}/12$; (2) $a^2\sqrt{4b^2-2a^2}/6$; (3) $a^2\sqrt{3(b^2-a^2)}/2$. 25. $h(k^2-h^2)\times\sqrt{3}$. 26. $\sqrt{Q(S^2-Q^2)}/6$. 27. $a^3\sqrt{2}/12$. 28. 1:27. 29. $8S\sin^2(\alpha/2)\sqrt{2S\cot\alpha/(3\sin\alpha)}$. 30. $a^3\sin\alpha/(24\cos(\alpha+30°)\cos(\alpha-30°))$, where $0<\alpha<60°$. 31. $S\cos\alpha\sqrt{S\sin\alpha}/6$. 32. $a^3\sqrt{3}/18$. 33. 500 cm^2. 34. 80 cm^3. 35. 1/3 m^3. 36. $(4a^3/3)\cos\alpha\cos\beta\sqrt{-\cos(\alpha+\beta)\cos(\alpha-\beta)}$. 37. $(a^3/6)\sin^2\alpha\tan\beta$. 38. $(1/12)(a^2+ab+b^2)\sqrt{3l^2-(a-b)^2}$, (2) $(1/3)(a^2+ab+b^2)\sqrt{l^2-(a-b)^2}/4$; (3) $(\sqrt{3}/2)(a^2+ab+b^2)\sqrt{l^2-(a-b)^2}$. 39. 784/3 cm^3. 40. $624\sqrt{3}$ cm^3. 41. $(1/12)(a^3-b^3)\tan\alpha$. 42. $(a^3-b^3)\sqrt{-\cos2\varphi}/(6\cos\varphi)$. 43. 1900 cm^3. 44. $ab\sqrt{a^2+b^2}\tan\alpha\,(n^3-1)/(12n^3)$. 46. 10 cm. 47. A cube with $\sqrt[3]{V}$ on a side. 48. The side of the base is equal to $\sqrt[3]{2V}$, the altitude equals $\sqrt[3]{2V}/2$. 49. The side of the base is equal to $\sqrt{3S}/3$, the altitude to $\sqrt{3S}/6$. 50. $\cos\alpha:\sin\alpha$. 51. $(\pi d^3/8)\sin2\alpha\cos\alpha$. 52. $\pi S\sqrt{S}/4$. 53. 21.5%. 54. $(\pi a^2/3)\sin\alpha\tan\alpha$.

55. $\alpha^2R^3\sqrt{4\pi^2-\alpha^2}/(24\pi^2)$. 56. $V_a : V_b = (a+2b) : (2a+b)$. 57. $2\pi a^3 \sin\alpha \cos^2\times(\alpha/2)$. 58. $(1/3)\,\pi r^3 \cot^3(\alpha/2)\tan\alpha$. 59. $\pi \tan\alpha\,(R^3 - r^3)/3$. 60. 11470 π cm^3. 61. $676\pi/3$ cm^3. 62. 793π cm^3. 63. $2\pi a^3 \sin\alpha \cos^2(\alpha/2)$. 64. $(7/6)\,\pi l^3 \sin 2\alpha \cos\alpha$. 65. 12 cm. 66. 47.6%. 67. $1 : 3\sqrt{3} \approx 0.19$. 68. 3 : 2 : 1. 69. 2149 cm^3. 70. 45π and 243π cm^3. 71. 5:16. 72. $512\,(16 - 9\sqrt{3})\,\pi/3$ cm^3. 73. $416\pi/3$ cm^3. 74. $\pi R^3/3$. 75. 112,500 cm^3. 76. $(4/3)\,\pi R^3 \sin^2(\alpha/4)$. 77. $\pi h^3/(6 \sin^4(\alpha/2))$. 78. $37.532\pi/3$ or $11968\pi/3$ cm^3. 79. 34182π cm^3. 82. The altitude of cylinder is $2R\sqrt{3}/3$, the radius of the base is equal to $R\sqrt{6}/3$. 83. $\alpha = \pi$. 84. The radius of the base is $l\sqrt{6}/3$, the altitude is $l\sqrt{3}/3$. 85. $2\pi\sqrt{6}/3$; the radius of the base equals $R\sqrt{6}/3$, the altitude is $R\sqrt{3}/3$. 93. (1) 1.5π cu. un.; (2) 12π cu. un; (3) 24π cu. un.; (4) 4π cu. un. 94. (1) $16/15\pi$ cu. un.; (2) 8.1π cu. un.; (3) $\pi/30$ cu.un. 95. $16\pi/3$ cu. un. 96. $4\pi r^3/3$. 97. $4\pi a^2 b/3$ cu. un. 98. (1) $32\pi/3$ cu. un.; (2) 16π cu. un. 99. (1) $\pi^2/2$ cu. un.; (2) $\pi^2/4$ cu. un. 100. 1.5π cu. un. 101. $152\pi/3$ cu. un. (2) 30π cu. un.; (3) 71.1π cu. un.; (4) $32\pi/3$ cu. un. 102. 3600 cm^3. 103. $3500\sqrt{3}$ cm^3. 104. $(m^3\sqrt{2}/128)\cos\varphi \sin 2\varphi \sin 2\alpha \sin^{-3}(45° + \varphi)$. 105. $a^3\sqrt{2}/24$. 106. $2a^2\sqrt{3}$, $a^3\sqrt{2}/3$. 107. $3a^3/4$. 108. $(2/3)\,h^3 \sin\alpha \sin\beta \sin(\alpha+\beta)\times\tan^2\varphi$. 109. $(2/3)\,H^3 \cot\beta \cot\gamma \sin\alpha$. 110. 96.75 cm^3. 111. $(\pi l^3/8)\cos\varphi \sin 2\varphi\times\sin^{-2}(\alpha/2)$. 112. $(\pi/12)\,a^3\sqrt{\cos\alpha}\sin^{-1}(\alpha/2)$. 113. $(2\pi/3)\,R^3 \sin^2\alpha \cos^2(\alpha/2)$. 114. $676\pi/3$ cm^3. 115. $\pi R^3\,(2 + 3\sqrt{2})/3$. 116. $3\pi R^3$. 117. $3\pi R^3/8$. 118. $3\pi R^3/4$. 119. $2\pi R^2\sqrt{R^2 - r^2}$. 120. $(10\pi l^3/81) \sin 2\alpha \cos\alpha$. 121. πa^3. **Quiz No. 1.** Form I. (1) $4\sqrt{3}$ cm^3; (2) $\pi a^3 \sin^2\alpha/(6 \cos\alpha)$; (3) $\pi/2$ cu. un. Form II. (1) 360 cm^2; (2) $2\pi a^3 \sin\alpha \cos^2(\alpha/2)$; (3) $16\pi/15$ cu. un.

Chapter 26

1. πS. 2. πl^2. 3. 20π cm^2. 4. $(\pi a/2)\,(a + 2H \sin(\varphi/2))\sin^{-2}(\varphi/2)$. 5. $32\pi\sqrt{3}/3$ 6. 9 cm. 7. $12\pi a^3$. 8. $\sqrt{M^2 + \pi^2 N^2}$. 9. $Q\pi\sqrt{3}$. 10. $0.5\pi a^2 \cos^2(\alpha/2)\sin^{-2}(\varphi/2)\times\cos^{-1}\alpha$. 11. $2\pi R^2 \sin(\beta + \alpha)/4 \cos(\beta - \alpha)/4 \sin^{-1}(\beta/2)$. 12. 40.8π cm^2. 13. $S\,(4\sqrt{3}+7)/6$. 14. $\pi r^2 \sin(180°/n)\sin^{-1}(\alpha/2)$. 15. 120π cm^2. 16. $0.5\pi a^2 \cos^2\times(\alpha/2)\cos^{-1}\alpha$. 17. $2\pi m^2 \cos^2(\alpha/2)\cos\alpha$. 18. $2\pi\,(R^2 - r^2)$. 19. $4\pi r^2/\sin^2\alpha$. 20. $2Rr/(R + r)$. 21. $2\pi l^2 \sin(15° + \alpha/2)\cos(15° - \alpha/2)$. 22. $\arccos((m - n)/p)$. 23. $\pi l^2 \cos\alpha\,(m + n)/(m - n)$. 24. 169π cm^2. 25. $4\pi r^2/\sin^2\alpha$. 26. (1) $\pi S/6$/(2) $\pi S\sqrt{3}/18$; (3) $\pi m^2/2$; (4) $4S/9$. 27. (1) 3:1; (2) 9:1; (3) 2:1; (4) 4:1. 28. $\sqrt[3]{36\pi V^2}$. 29. 2500π cm^2. 30. 275π cm^2. 31. $4\pi R^2 \sin(\beta + \alpha/2)\sin(\alpha/2)$. 32. $\pi R^2\times(\sqrt{3} + 2)/2$. 33. $2R \pm \sqrt{4R^2 - S/\pi}$. 34. $21\pi Q/(4\pi - 3\sqrt{3})$. 37. The radius of the base is equal to $0.5HR/(H - R)$, the altitude to $0.5H\,(H - 2R)/(H - R)$. 38. The radius of the base is equal to $\sqrt[3]{V/(2\pi)}$, the altitude to $2R$. 39. The radius of the base and altitude are equal to $\sqrt[3]{V/\pi}$. 40. The radius of the base is $R\sqrt{2}/2$, the altitude is $R\sqrt{2}$. 41. The radius of the base and the altitude are equal to $\sqrt{S/(3\pi)}$. 42. The radius of the base is $\sqrt{S/(6\pi)}$, the altitude is equal to $2\sqrt{S/(6\pi)}$. 43. The radius of the base is equal to $2\sqrt{2}R/3$, the altitude is equal to $4R/3$. 46. $2\pi rH$. 47. $2\pi rH$. 48. 8π sq. un. 49. 70π sq. un. 50. 40π sq. un. 51. $56\pi/3$ sq. un. 52. 49π sq. un. 53. 17,640π or 16,800π cm^2. 54. 90π sq. un. 55. $4\sqrt{2}$: 3; 4 : 3. 56. 2 : 3 (in both cases). 57. 30π cm^2. 58. 2.5 cm; 1.5 cm. 59. $\pi a^2 \cot^2(\alpha/2)\sin(15° + \alpha/4)\cos(15° - \alpha/4)$; (1/6) $\pi a^3 \cot^3(\alpha/2)\cos(\alpha/2)$. 60. $2\pi m\,(m + n)\sin\alpha$. 61. $\pi h^3/l$. 62. $6\pi a^2$. 63. $\pi h^2 \tan(\alpha/2)\sin^{-1}\beta$. 64. 50π and 2450π cm^2. 65. 10 cm. 67. $2S/\cos^2(\alpha/4)$. 68. $2S/3$. 69. $0.5 \cot^2(\alpha/2)\cos^2(\alpha/2)\times\cos^{-1}\alpha$. **Quiz No. 1.** Form I. (1) $4\sqrt{2}\pi S \cos(45° - \alpha/2)$; (2) $\pi l^2/\cos^2\alpha$. Form II. (1) $8\pi a^2 \cos^2(\alpha/2)$; (2) $4\pi l^2 \sin^2\alpha \tan^2(45° - \alpha/2)$.

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