

I. M. RÎJIC - I. S. GRADȘTEIN

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DIN PREFAȚA LA EDIȚIA ÎNTÂI

În manualele de referință existente în domeniul matematicii, atât sovietice cât și străine, numărul de formule privind integralele, sumele, seriile și produsele este, fără îndoială, insuficient, nu numai pentru cercetătorii care lucrează în știința matematică, dar și pentru inginerii care întreprind lucrări teoretice și de cercetare. Tabelele de față au fost întocmite cu scopul de a completa această lacună. Aici au fost strinse peste 5 000 de formule din diferite surse.

Cartea este destinată, în primul rând, cercetătorilor și inginerilor-cercetători din domeniul științelor fizico-matematice. De aceea, expunerile ocupă puțin loc în această carte. Ea este, în fond, o culegere de formule.

S-a acordat o deosebită atenție funcțiilor speciale, mai ales funcțiilor eliptice, cilindrice și sferice. Cartea cuprinde multe formule privind aceste funcții.

Profită de acest prilej pentru a exprima cele mai sincere sentimente de recunoștință profesorilor V. V. Stepanov, A. I. Markușevici și I. N. Bronștein, pentru sfaturile și indicațiile valoroase pe care le-am primit de la ei în cursul întocmirii acestei lucrări.

I. Rîjik

PREFAȚA LA EDIȚIA A TREIA

I. M. Rîjik, autorul primelor două ediții ale acestor tabele, și-a pierdut viața în timpul Marelui Război pentru Apărarea Patriei. La propunerea editurii¹, aceste tabele au fost prelucrate de mine. Această prelucrare se reduce, în fond, la următoarele. S-au scos secțiunile referitoare la teoria numerelor, la diferențele finite și la calculele aproximăte. În locul acestora, a fost mărit mult spațiul acordat funcțiilor speciale. În felul acesta, deși se păstrează vechiul titlu al cărții, conținutul ei poate fi definit pe scurt ca un manual de referință în domeniul funcțiilor (elementare și speciale) și al integralelor. În conformitate cu aceasta, s-a dat o oarecare extindere secțiunii funcțiilor elementare (în special, secțiunii funcțiilor trigonometrice inverse) și s-a modificat conținutul secțiunii funcțiilor speciale. În secțiunea funcțiilor speciale, se dau definițiile acestor funcții, reprezentările lor sub formă de integrale și dezvoltările în serie, formulele asimptotice, relațiile dintre diferite funcții, precum și teoremele care prezintă cele mai caracteristice proprietăți ale funcțiilor respective. Integralele funcțiilor speciale au fost separate: integralele nedefinite ale funcțiilor speciale au fost trecute la sfârșitul secțiunii 2, iar cele definite constituie secțiunea 4. Secțiunea integralelor nedefinite a fost, pe de o parte, redusă într-o oarecare măsură (de exemplu, „forma” $\sqrt{bx+cx^2}$ a fost contopită cu „forma” $\sqrt{a+bx+cx^2}$, cu care prilej integralele formei $\sqrt{bx+cx^2}$, acolo unde acest lucru nu putea duce la echivoc, au fost omise), iar pe de altă parte, lărgită.

În secțiunea integralelor definite, au fost introduse următoarele modificări. Toate „facultățile”, de tipul $2^{n-2/3}$ au fost înlocuite cu funcția Γ , iar

¹ Государственное издательство технико-теоретической литературы.

acolo unde a fost posibil, prin produse și factoriale obișnuite, deoarece am considerat că „facultățile” sînt puțin cunoscute cititorului de astăzi și nu fac decît să introducă dificultăți în plus. Acolo unde membrul al doilea al formulelor putea fi înlocuit prin vreo funcție specială sau prin vreun număr special, acest lucru a fost făcut. S-a adăugat o serie de integrale care duc la funcții speciale. Au fost omise integralele expresiilor care conțin mărimi complexe și alte cîteva integrale; în afară de aceasta, a fost modificată ordinea succesiunii formulelor. Am fost nevoiți, totuși, să menținem modul de notare a numerelor naturale, adoptat în primele două ediții: numerele naturale, în secțiunile integralelor definite (3, 4 și 5), au fost notate cu primele litere ale alfabetului ($a, b, \dots, h$), iar în aproape toate celelalte secțiuni, cu literele m și n .

A fost introdusă o nouă secțiune, care lipsea cu desăvîrșire în primele două ediții: transformările integrale.

A fost schimbat și modul numerotării formulelor. Toate formulele, definițiile și teoremele se împart în secțiuni numerotate. Principiul numerotării seamănă cu sistemul clasificării zecimale și poate fi înțeles ușor din tabla de materii. Tabla de materii indică numai secțiunile mai importante, ale căror numere de ordine conțin pînă la trei cifre. Secțiunile cele mai mici ale cărții au numere de ordine cu patru cifre. În aceste secțiuni, intră o formulă (o teoremă sau o definiție) sau mai multe, avînd numerele de ordine culese cu caractere drepte. Cifra „zero” a fost păstrată pentru secțiunile cu caracter general: introduceri, definiții etc. Prima secțiune a cărții, care conține o serie de teoreme cu caracter general și care îndeplinește rolul de introducere, a fost, de asemenea, numerotată cu zero.

Elementul nou în această ediție îl constituie trimerile de la sfîrșitul formulelor, care indică sursa de unde s-a scos formula¹. M-am străduit să dau trimiteri, în primul rînd, la ediții sovietice, îndeosebi originale, iar în al doilea rînd, la cărți străine și, în sfîrșit, la manuale de referință. Trimiteri la literatura periodică nu s-au făcut. Formulele luate din diverse cărți au fost uneori modificate. În acest caz, la sfîrșitul notei bibliografice, s-a pus litera m („modificat”). Între altele, litera m poate însemna și corectarea unei erori de tipar.

Pentru ca cititorul, atunci cînd găsește într-o formulă oarecare simbolul vreunei funcții, al vreunui număr special sau al vreunei operații, să poată afla despre ce funcție, număr sau operație este vorba, cartea are două indexuri terminologice: un index pentru notațiile folosite (pag. 455) și un index pe materii (pag. 448). Unele funcții și numere speciale sînt notate, iar uneori și tratate, în mod diferit de diverși autori (de exemplu, sînt normate diferit). La sfîrșitul cărții, sînt date diverse notații și interpretări ale funcțiilor și constantelor cuprinse în lucrarea de față. Tot astfel, sînt prezentate unele funcții înrudite cu funcțiile din tabele.

Folosesc acest prilej pentru a-mi exprima recunoștința față de tovarășii S. B. Stecikin, A. I. Dubovițki și I. N. Bronștein, pentru numeroasele sfaturi pe care mi le-au dat la prelucrarea cărții, tovarășilor B. S. Vilenskaia și I. V. Gheronimus, pentru revizuirea atentă a manuscrisului și a corecturilor, tovarășilor S. N. Ahlamov, G. N. Nelidova și L. O. Seceiko, pentru corectura minuțioasă și pentru efectuarea complicatei tehnoredactări a manuscrisului.

I. Gradștein

¹ Bibliografia pentru care există trimiteri se află la pp. 457—458.

După cifra care simbolizează cartea, referirile bibliografice conțin un număr. Numerele care nu sînt cuprinse între paranteze indică pagina; numerele între paranteze rotunde arată numărul formulei, iar cele între paranteze drepte, numărul tabelelor.

DESPRE SUCCESIUNEA FORMULELOR

Problema ordonării raționale a formulelor, mai cu seamă într-o secțiune ca aceea a integralelor definite, a fost extrem de dificilă. Ideea de a adopta o ordine analogă celei din dicționare se impunea în mod firesc. Dar stabilirea acestei ordini în formulele calculului integral este aproape imposibilă.

În adevăr, în orice formulă

$$\int_a^b f(x) dx = A$$

se poate face o serie întreagă de substituții de tipul $x = \varphi(t)$, obținându-se astfel o serie de „sinonime” pentru formula dată. Trebuie să spunem că atât tabela de integrale definite a lui Bierens de Haan, cit și primele ediții ale tabelelor de față suferă de un exces de asemenea „sinonime” și formule complicate. În ediția de față, am căutat să păstrăm numai cele mai simple dintre „formulele sinonime”. Simplitatea formulelor am apreciat-o îndeosebi după simplitatea argumentelor funcțiilor „exterioare”¹ care intră în expresia de sub semnul integrală. Acolo unde a fost cu putință, formulele complicate au fost înlocuite prin altele mai simple; uneori, mai multe formule complicate se reduc la o singură formulă mai simplă. În asemenea cazuri, am păstrat numai această formulă mai simplă. Uneori, în urma acestor substituții simplificatoare, am ajuns la o integrală care se putea calcula cu ajutorul formulelor din secțiunea 2 și al formulei lui Newton-Leibniz, sau la o integrală de forma

$$\int_{-a}^a f(x) dx,$$

unde $f(x)$ este o funcție impară. Atunci, această integrală este suprimată.

Să dăm un exemplu (nr. 26, p. 159, ediția a doua):

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg} x - 1)^{p-1}}{\sin^2 x} \ln \operatorname{tg} x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \quad (1)$$

Substituția firească este

$$\operatorname{ctg} x - 1 = u;$$

Cu ajutorul ei, obținem

$$\int_0^{\infty} u^{p-1} \ln(1+u) du = \frac{\pi}{p} \operatorname{cosec} p\pi. \quad (2)$$

¹ Dacă $h = f[g(x)]$, adică dacă h este o funcție de funcție de x , atunci f se numește funcția exterioară, iar g , funcția interioară. De exemplu, dacă $h = \sin(\cos x)$ atunci $f = \sin g$ este funcția exterioară, iar $g = \cos x$, funcția interioară. (Nota Editurii Tehnice).

Această integrală nu se găsea ca atare în tabele. Ea se putea deduce din alte formule mai complicate, cuprinse în tabele. Mai departe, nr. 59 și nr. 60 sînt forme particulare ale formulei nr. 26 de la p. 159. Toate aceste integrale au fost șterse în noua ediție. În locul lor, figurează formula (2) și formula care se obține din integrala (1), dacă punem $\operatorname{ctg} x = v$.

Alt exemplu (nr. 24, p. 172, ediția a doua):

$$I = \int_0^{\frac{\pi}{2}} \ln(\operatorname{tg}^p x + \operatorname{ctg}^p x) \ln \operatorname{tg} x = 0.$$

Substituția

$$\operatorname{tg} x = u$$

dă

$$I = \int_0^{\infty} \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} du.$$

Punem, mai departe,

$$v = \ln u.$$

Atunci

$$I = \int_{-\infty}^{\infty} \frac{ve^p}{1 + e^{2v}} \ln(e^{pv} + e^{-pv}) dv = \int_{-\infty}^{\infty} v \frac{\ln 2 \operatorname{ch} pv}{\operatorname{ch} v} dv.$$

Funcția de sub semnul integrală este impară și, deci, integrala este nulă.

Așadar, înainte de a căuta o integrală în tabele, expresia de sub semnul integrală trebuie simplificată, și anume în așa fel, încît argumentele funcției („funcțiile interioare”) care intră în expresia de sub semnul integrală să aibă forma cea mai simplă posibil.

Funcțiile sînt clasificate în modul următor:

Întîi funcțiile elementare:

1. Funcția $f(x) = x$.
2. Funcția exponențială.
3. Funcțiile hiperbolice.
4. Funcțiile trigonometrice.
5. Funcția logaritmică.
6. Funcțiile hiperbolice inverse. (În formule care conțin integrale definite, ele sînt înlocuite prin logaritmi respectivi).
7. Funcțiile trigonometrice inverse.

Urmează apoi funcțiile speciale:

8. Integralele eliptice.
9. Funcțiile eliptice.
10. Logaritmul integral, funcția exponențială integrală, sinusul integral și cosinusul integral.
11. Integralele de probabilitate și integralele lui Fresnel.
12. Funcția Γ și funcțiile înrudite.
13. Funcțiile cilindrice.
14. Funcțiile sferice.
15. Polinoamele $C_n^p(x)$.

16. Polinoamele ortogonale.

17. Funcția hipergeometrică și funcția hipergeometrică degenerată.

18. Funcția ζ a lui Riemann.

Dacă avem două funcții φ și ψ , vom spune că funcția φ este anterioară funcției ψ , dacă, în clasificarea de mai sus, funcția φ face parte dintr-o categorie cu număr de ordine mai mic decât funcția ψ . Vom aplica același criteriu și operatorilor.

În tabele, criteriul clasificării formulelor este, în primul rând, funcția exterioară. Dacă în mai multe expresii intră aceeași funcție exterioară, de exemplu, în expresiile $\sin e^x$, $\sin x$, $\sin \ln x$, avînd funcția exterioară — sinus — comună, atunci aceste expresii se ordonează după funcțiile interioare. De exemplu, cele trei funcții de mai sus se vor clasifica în următoarea ordine:

$$\sin x, \quad \sin e^x, \quad \sin \ln x.$$

Din lista dată de noi, lipsesc următoarele funcții: polinomul, funcția rațională, funcția algebrică și funcția putere. Funcția algebrică cuprinsă în tabelele de integrale definite se reduce, de obicei, la o combinație finită de radicali de indice rațional și, de aceea, putem considera convențional, în clasificarea noastră, că funcția putere este o generalizare a funcției algebrice și, deci, a funcției raționale¹. Toate aceste funcții le vom deosebi de cele enumerate mai sus, considerîndu-le drept operatori. Așadar, în expresia $\sin^2 e^x$, vom considera că funcției exterioare $\sin$ i se aplică operatorul de ridicare la pătrat.

În expresia $\frac{\sin x + \cos x}{\sin x - \cos x}$, vom considera că funcțiilor trigonometrice $\sin$ și $\cos$ li se aplică un operator rațional. Operatorii vor fi ordonați în modul următor:

1. Polinomul (ordonat, la rîndul său, după grad).

2. Operatorul rațional.

3. Operatorul algebric (în esență, $A^{\frac{p}{q}}$, unde $q > 0$ și p sînt numere raționale; cu cît q este mai mare, cu atît operatorul este așezat mai departe în tabele).

4. Operatorul putere.

Expresiile avînd aceleași funcții interioare și exterioare se ordonează după clasificarea corespunzătoare a operatorilor, de exemplu:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \sec x, \quad \frac{\sin x}{\cos x} = \operatorname{tg} x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Mai departe, dacă în expresia de sub semnul integrală intră două funcții exterioare $\varphi_1(x)$ și $\varphi_2(x)$ [$\varphi_2(x)$ fiind anterioară lui $\varphi_1(x)$], asupra cărora se efectuează una dintre operațiile de mai sus, integrala respectivă se pune după toate integralele care conțin numai funcția $\varphi_2(x)$, ținîndu-se seama de criteriul anteriorității pentru $\varphi_1(x)$. Astfel, după funcțiile trigonometrice, vin funcțiile trigonometrice și funcțiile putere [adică $\varphi_1(x) = x$], după care urmează

funcțiile trigonometrice și funcțiile exponențiale,

funcțiile trigonometrice, funcțiile exponențiale și funcțiile putere etc.,

funcțiile trigonometrice și funcțiile hiperbolice etc.

¹) Pentru n natural, funcția putere $(a+bx)^n$ a binomului $a+bx$ este un polinom; pentru n întreg și negativ $(a+bx)^n$ este o funcție rațională; pentru n irațional, funcția putere $(a+bx)^n$ nu mai este o funcție algebrică.

Integralele în care intră două funcții $\varphi_1(x)$ și $\varphi_2(x)$ se ordonează în secțiunea respectivă, numai după anterioritatea lui $\varphi_1(x)$. Dacă ordinea mai multor integrale numai după criteriul funcției anterioare coincide, aceste integrale se ordonează după funcția a doua.

La aceste reguli cu caracter general, se mai adaugă unele considerații speciale, care reies cu ușurință nemijlocit din tabele. De exemplu, potrivit celor de mai sus, funcția e^x este anterioară lui $e^{\frac{1}{x}}$, dar pentru $\ln x$ și $\ln \frac{1}{x}$, criteriul nu se mai aplică, deoarece $\ln \frac{1}{x} = -\ln x$: în secțiunea „funcții putere și funcții algebrice” (3.11—3.19), se formează din funcțiile putere de forma $(a+bx)^n$, $(a+\beta x)^n$ polinoame, funcții raționale și chiar funcții putere de funcții putere.

0. INTRODUCERE

0.1 SUME CU NUMĂR FINIT DE TERMENI

0.11 Progresii

0.111 Progresia aritmetică

$$\sum_{k=0}^{n-1} (a + kr) = \frac{n}{2} [2a + (n-1)r] = \frac{n}{2} (a + l) \quad [l \text{ fiind ultimul termen}].$$

0.112 Progresia geometrică

$$\sum_{k=0}^n a q^{k-1} = \frac{a (q^n - 1)}{q - 1}.$$

0.113 Progresia aritmetic-geometrică

$$\sum_{k=0}^{n-1} (a + kr) q^k = \frac{a - [a + (n-1)r] q^n}{1-q} + \frac{r q (1 - q^{n-1})}{(1-q)^2}. \quad \text{Jl (5)}$$

0.12 Sume de puteri ale numerelor naturale

0.121

$$\begin{aligned} \sum_{k=1}^n k^q &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2} \binom{q}{1} B_2 n^{q-1} + \frac{1}{4} \binom{q}{3} B_4 n^{q-3} + \frac{1}{6} \binom{q}{5} B_6 n^{q-5} + \dots \\ &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{q n^{q-1}}{12} - \frac{q(q-1)(q-2)}{720} n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30240} n^{q-5} - \dots \end{aligned}$$

[ultimul termen conține pe n sau pe n^2] (v. și 7.523 1.).

Ce 332

$$1. \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Ce 333

$$2. \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Ce 333

$$3. \quad \sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

Ce 333

$$4. \quad \sum_{k=1}^n k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).$$

Ce 333

$$5. \sum_{k=1}^n k^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^3}{12} = \frac{1}{12} n^3 (n+1)^2 (2n^2 + 2n - 1). \quad \text{Ce 333}$$

$$6. \sum_{k=1}^n k^6 = \frac{1}{42} n (n+1) (2n+1) (3n^4 + 6n^3 - 3n + 1). \quad \text{Ce 333}$$

$$7. \sum_{k=1}^n k^7 = \frac{1}{24} n^2 (n+1)^2 (3n^4 + 6n^3 - n^2 - 4n + 2). \quad \text{Ce 333}$$

$$0.122 \quad \sum_{k=1}^n (2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2} \binom{q}{1} 2^{q-1} B_2 n^{q-1} -$$

[ultimul termen conține pe n sau pe n^2]. $-\frac{1}{4} \binom{q}{3} 2^{q-3} (2^3 - 1) B_4 n^{q-3} - \dots$

$$1. \sum_{k=1}^n (2k-1) = n^2.$$

$$2. \sum_{k=1}^n (2k-1)^3 = \frac{1}{3} n (4n^2 - 1). \quad \text{Jl (32 a)}$$

$$3. \sum_{k=1}^n (2k-1)^5 = n^2 (2n^2 - 1). \quad \text{Jl (32 b)}$$

$$0.123 \quad \sum_{k=1}^n k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5).$$

$$0.124 \quad \sum_{k=1}^q k(n^2 - k^2) = \frac{1}{4} q(q+1)(2n^2 - q^2 - q).$$

$$0.125 \quad \sum_{k=1}^n k! \cdot k = (n+1)! - 1. \quad \text{A (188.1)}$$

$$0.126 \quad \sum_{k=1}^n \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}} K_{n+\frac{1}{2}} \left(\frac{1}{2} \right). \quad \text{W 94}$$

0.13 Sume ale inverselor numerelor naturale

$$0.131 \quad \sum_{k=1}^n \frac{1}{k} = C + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1) \dots (n+k-1)},$$

unde

$$A_k = \frac{1}{k} \int_0^1 x(1-x)(2-x)(3-x) \dots (k-1-x) dx.$$

$$A_2 = \frac{1}{12}, \quad A_3 = \frac{1}{12},$$

$$A_4 = \frac{19}{80}, \quad A_5 = \frac{9}{20},$$

Jl (59) A (1876)

$$0.132 \quad \sum_{k=1}^n \frac{1}{2k-1} = \frac{1}{2} (C + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{(2^3-1) B_4}{64n^4} + \dots \quad \text{Jl (71 a) } m$$

$$0.133 \quad \sum_{k=2}^n \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}. \quad \text{Jl (184 f)}$$

0.14 Sume de produse ale inverselor numerelor naturale

0.141

$$1. \quad \sum_{k=1}^{n+1} \frac{1}{[p+(k-1)q][p+kq]} = \frac{n}{p(p+nq)}. \quad \text{GC III (64) } m$$

$$2. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q][p+kq][p+(k+1)q]} = \frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]}. \quad \text{GC III (65) } m$$

$$3. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q][p+kq] \dots [p+(k+l)q]} = \frac{1}{(l+1)q} \left\{ \frac{1}{p(p+q) \dots (p+lq)} - \frac{1}{(p+nq)[p+(n+1)q] \dots [p+(n+l)q]} \right\}. \quad \text{A (1856) } m$$

$$4. \quad \sum_{k=1}^n \frac{1}{[1+(k-1)q][1+(k-1)q+p]} = \frac{1}{p} \left[\sum_{k=1}^n \frac{1}{1+(k-1)q} - \sum_{k=1}^n \frac{1}{1+(k-1)q+p} \right]. \quad \text{GC III (66) } m$$

$$0.142 \quad \sum_{k=1}^n \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!}. \quad \text{Jl (157)}$$

0.15 Sume de coeficienți binomiali (n natural)

0.151

$$1. \quad \sum_{k=0}^m \binom{n+k}{n} = \binom{n+m+1}{n+1}. \quad \text{Kr 64 (70.1)}$$

$$2. \quad 1 + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}.$$

$$3. \quad \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}.$$

$$4. \quad \sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}. \quad \text{Kr 64 (70.2)}$$

0.152

$$1. \quad \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right). \quad \text{Kr 62 (59.1)}$$

$$2. \quad \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-2)\pi}{3} \right). \quad \text{Kr 62 (59.2)}$$

$$3. \quad \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-4)\pi}{3} \right). \quad \text{Kr 62 (59.3)}$$

0.153

$$1. \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right). \quad \text{Kr 63 (60.1)}$$

$$2. \binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right). \quad \text{Kr 63 (60.2)}$$

$$3. \binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right). \quad \text{Kr 62 (60.3)}$$

$$4. \binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right). \quad \text{Kr 63 (60.4)}$$

0.154

$$1. \sum_{k=0}^n (k+1) \binom{n}{k} = 2^{n-1} (n+2). \quad \text{Kr 63 (66.1)}$$

$$2. \sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} = 0. \quad \text{Kr 63 (66.2)}$$

0.155

$$1. \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1}. \quad \text{Kr 63 (67)}$$

$$2. \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-1}{n+1}. \quad \text{Kr 63 (68.1)}$$

$$3. \sum_{k=0}^n \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1}-1}{n+1}. \quad \text{Kr 63 (68.2)}$$

$$4. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^n \frac{1}{m}. \quad \text{Kr 64 (69)}$$

0.156

$$1. \sum_{k=0}^p \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p} \quad [m \text{ natural}]. \quad \text{Kr 64 (71.1)}$$

$$2. \sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)! (n+p)!}. \quad \text{Kr 64 (71.2)}$$

0.157

$$1. \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}. \quad \text{Kr 64 (72.1)}$$

$$2. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}. \quad \text{Kr 64 (72.2)}$$

$$3. \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0. \quad \text{Kr 64 (72.3)}$$

$$4. \sum_{k=1}^n k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}. \quad \text{Kr 64 (72.4)}$$

0.2 SERII NUMERICE ȘI PRODUSE INFINITE

0.21 Convergența seriilor numerice

Seria

$$0.211 \quad \sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

se numește *absolut convergentă*, dacă este convergentă seria

$$0.212 \quad \sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \dots,$$

formată din valorile absolute (modulii) ale termenilor ei. Dacă însă seria 0.211 converge, iar seria 0.212 este divergentă, atunci seria 0.211 se numește *semi-convergentă*. Orice serie absolut convergentă este convergentă.

0.22 Criterii de convergență

0.221 Fie

$$\lim_{k \rightarrow \infty} |u_k|^{\frac{1}{k}} = q.$$

Dacă avem $q < 1$, atunci seria 0.211 este absolut convergentă; dacă $q > 1$, atunci seria 0.211 este divergentă. (Cauchy).

0.222 Fie

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = q.$$

Dacă avem $q < 1$, atunci seria 0.211 este absolut convergentă; dacă $q > 1$, atunci seria 0.211 este divergentă. Dacă $\left| \frac{u_{k+1}}{u_k} \right|$ tinde către 1, raportul rămânând supraunitar, atunci seria 0.211 este divergentă. (d'Alembert).

0.223 Fie

$$\lim_{k \rightarrow \infty} k \left\{ \left| \frac{u_k}{u_{k+1}} \right| - 1 \right\} = q.$$

Dacă $q > 1$, atunci seria 0.211 este absolut convergentă; dacă $q < 1$, atunci seria 0.211 este divergentă. (R a a b e).

0.224 Fie $f(x)$ o funcție pozitivă, descrescătoare și fie, pentru k natural,

$$\lim_{k \rightarrow \infty} \frac{e^k f(e^k)}{f(k)} = q.$$

Dacă, în aceste condiții, $q < 1$, atunci seria $\sum_{k=1}^{\infty} f(k)$ este absolut convergentă; dacă $q > 1$, atunci seria este divergentă. (E r m a k o v).

0.225 Fie

$$\left| \frac{u_k}{u_{k+1}} \right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

unde $p > 1$, iar $|v_k|$ mărginit, adică $|v_k|$ este mai mic decât un număr anumit M , independent de k . Dacă, în aceste condiții, $q > 1$, atunci seria 0.211 este absolut convergentă; dacă $q \leq 1$, atunci această serie este divergentă. (G a u s s).

0.226 Fie funcția $f(x)$ definită pentru $x \geq q$, continuă, pozitivă și monoton descrescătoare. Dacă aceste condiții sint satisfăcute, atunci seria

$$\sum_{k=1}^{\infty} f(k)$$

este convergentă sau divergentă, după cum este convergentă sau divergentă integrala

$$\int_q^{\infty} f(x) dx.$$

(Criteriul integral al lui Cauchy).

0.227 Fie toți termenii șirului $u_1, u_2, \dots, u_n$ pozitivi; în acest caz, seria

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \dots$$

se numește serie *alternată*.

Dacă termenii unei serii alternate descresc monoton în valoare absolută și tind către zero, adică dacă

$$2. \quad u_{k+1} < u_k \text{ și } \lim_{k \rightarrow \infty} u_k = 0,$$

atunci seria **0.227** 1. este convergentă. În acest caz, restul seriei va fi

$$3. \sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left| \sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^n (-1)^{k+1} u_k \right| < u_{n+1}.$$

(Leibniz)

0.228 Dacă seria

$$1. \sum_{k=1}^{\infty} v_k = v_1 + v_2 + \dots + v_k + \dots$$

este convergentă, iar numerele u_k formează un șir monoton și mărginit, adică există un număr M astfel încît, pentru orice k , $|u_k| < M$, atunci seria

$$2. \sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \dots + u_k v_k + \dots$$

este convergentă. (A bel).

F II 354

0.229 Dacă sumele parțiale ale seriei **0.228** 1. sint mărginite în totalitatea lor, iar numerele u_k formează un șir monoton care tinde către zero, adică dacă

$$\left| \sum_{k=1}^n v_k \right| < M \quad [n = 1, 2, 3, \dots] \text{ și } \lim_{k \rightarrow \infty} u_k = 0,$$

atunci seria **0.228** 2. este convergentă. (Dirichlet).

F II 355

0.23-0.24 Exemple de serii numerice

0.231 Progresii

$$1. \sum_{k=0}^{\infty} a q^k = \frac{a}{1-q} \quad [|q| < 1].$$

$$2. \sum_{k=0}^{\infty} (a + k r) q^k = \frac{a}{1-q} + \frac{r q}{(1-q)^2} \quad [|q| < 1] \quad (\text{cf. 0. 113}).$$

0.232

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2 \quad (\text{cf. 1.511}).$$

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4} \quad (\text{cf. 1.643}).$$

0.233

$$1. \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \zeta(p) \quad [\operatorname{Re} p > 1] \quad (\text{vezi și 7.422 1.})$$

WW II 44

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^p} = (1 - 2^{1-p}) \zeta(p) \quad [\operatorname{Re} p > 0] \quad (\text{vezi și 7.422 2.}).$$

WW II 46

$$3. \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}|. \quad \text{F II 721}$$

$$4. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2n}} = \frac{(2^{2n-1} - 1) \pi^{2n}}{(2n)!} |B_{2n}|. \quad \text{J I (165)}$$

$$5. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2 \cdot (2n)!} |B_{2n}|. \quad \text{J I (184 b)}$$

$$6. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2} (2n)!} |E_{2n}|. \quad \text{J I (184 d)}$$

0.234

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12}. \quad \text{E 158 m}$$

$$2. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}. \quad \text{E 163}$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G. \quad \text{F II 482}$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} = \frac{\pi^2}{32}. \quad \text{E 163}$$

$$5. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}. \quad \text{E 163}$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^6} = \frac{5\pi^6}{1536}. \quad \text{E 164}$$

$$7. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2.$$

$$0.235 \quad S_n = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^n},$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2-8}{16}, \quad S_3 = \frac{32-3\pi^2}{64}, \quad S_4 = \frac{\pi^4+30\pi^2-384}{768}. \quad \text{Jl (186)}$$

0.236

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)} = 2 \ln 2 - 1. \quad \text{Br}_{26} 51 m$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{k(9k^2-1)} = \frac{3}{2} (\ln 3 - 1). \quad \text{Br}_{26} 51 m$$

$$3. \quad \sum_{k=1}^{\infty} \frac{1}{k(36k^2-1)} = -3 + \frac{3}{2} \ln 3 + 2 \ln 2. \quad \text{Br}_{26} 52 A (6913.3)$$

$$4. \quad \sum_{k=1}^{\infty} \frac{k}{(4k^2-1)^2} = \frac{1}{8}. \quad \text{Br}_{26} 52$$

$$5. \quad \sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)^2} = \frac{3}{2} - 2 \ln 2. \quad \text{Br}_{26} 52$$

$$6. \quad \sum_{k=1}^{\infty} \frac{12k^2-1}{k(4k^2-1)^2} = 2 \ln 2. \quad A (6917.3) \text{ Br}_{26} 52$$

0.237

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}. \quad A (6917.2) \text{ Br}_{26} 52$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}.$$

$$3. \quad \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4} \quad (\text{cf. } 0.133).$$

$$4. \quad \sum'_{k=1, k \neq m} \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2} \quad [m \text{ intreg.}] \quad A (6916.1)$$

$$5. \quad \sum'_{k=1, k \neq m} \frac{(-1)^{k-1}}{(m-k)(m+k)} = \frac{3}{4m^2} \quad [m \text{ par.}] \quad A (6916.2)$$

0.238

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2}. \quad \text{GC III (93)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2} (1 - \ln 2). \quad \text{GC III (94) } m$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4} \ln 3 + \frac{\pi}{12\sqrt{3}}. \quad \text{GC III (95)}$$

0.239

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-2} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} + \ln 2 \right). \quad \text{GC III (85) Br}_{08} 161 (1)$$

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} - \ln 2 \right). \quad \text{Br}_{08} 161 (1)$$

$$3. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}} [\pi + 2 \ln (\sqrt{2} + 1)]. \quad \text{Br}_{08} 161 (1)$$

$$4. \sum_{k=1}^{\infty} (-1)^{E\left(\frac{k+3}{2}\right)} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2} \ln 2. \quad \text{GC III (87)}$$

$$5. \sum_{k=1}^{\infty} (-1)^{E\left(\frac{k+3}{2}\right)} \frac{1}{2k-1} = \frac{\pi}{2\sqrt{2}}.$$

$$6. \sum_{k=1}^{\infty} (-1)^{E\left(\frac{k+5}{3}\right)} \frac{1}{2k-1} = \frac{5\pi}{12}. \quad \text{GC III (88)}$$

$$7. \sum_{k=1}^{\infty} \frac{1}{(8k-1)(8k+1)} = \frac{1}{2} - \frac{\pi}{16} (\sqrt{2} + 1).$$

0.241

$$1. \sum_{k=1}^{\infty} \frac{1}{2^k k} = \ln 2. \quad \text{Jl (172g)}$$

$$2. \sum_{k=1}^{\infty} \frac{1}{2^k k^2} = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2. \quad \text{Jl (174)}$$

$$0.242 \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{2k}} = \frac{n^2}{n^2 + 1}.$$

0.243

$$1. \sum_{k=1}^{\infty} \frac{1}{[p+(k-1)q][p+kq] \dots [p+(k+l)q]} = \frac{1}{(l+1)q} \frac{1}{p(p+q) \dots (p+lq)} \quad (\text{vezi și } 0.141 \text{ 3.}).$$

$$2. \sum_{k=1}^{\infty} \frac{x^{k-1}}{[p+(k-1)q][p+(k-1)q+1][p+(k-1)q+2] \dots [p+(k-1)q+l-1]} = \\ = \frac{1}{l!} \int_0^1 \frac{t^{p-1} (1-t)^l}{1-xt^q} dt \quad [q > 0, x^2 \leq 1]. \quad \text{Br}_{08} 161 (2) m A (6.704)$$

0.244

$$1. \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \int_0^1 \frac{x^p - x^q}{1-x} dx \quad [p > -1, q > -1]. \quad \text{GC III (90)}$$

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{p+(k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q} dt \quad [p > 0, q > 0]. \quad \text{Br}_{08} 161 (1)$$

Sume ale inverselor factorialelor

0.245

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2,71828 \dots$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} = 0,36788 \dots$$

$$3. 2 \sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = 1 - \frac{1}{e} = 0,63212 \dots$$

$$4. \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2} \left(e + \frac{1}{e} \right) = 1,54308 \dots$$

$$6. \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2} \left(e - \frac{1}{e} \right) = 1,175201 \dots$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = \cos 57^{\circ}17'45'' = 0,54030 \dots$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = \sin 57^{\circ}17'45'' = 0,84147 \dots$$

0.246

$$1. \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2,27958530 \dots$$

$$2. \sum_{k=0}^{\infty} \frac{1}{k! (k+1)!} = I_1(2) = 1,590636855 \dots$$

$$3. \sum_{k=0}^{\infty} \frac{1}{k! (k+n)!} = I_n(2).$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0,22389078 \dots$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+1)!} = J_1(2) = 0,57672481 \dots$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} = J_n(2).$$

$$0.247 \quad \sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2) \cdot (n-1)!}.$$

$$0.248 \quad \sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$S_1 = e, \quad S_2 = 2e, \quad S_3 = 5e, \quad S_4 = 15e, \\ S_5 = 52e, \quad S_6 = 203e, \quad S_7 = 877e, \quad S_8 = 4140e.$$

Jl (185)

$$0.249 \quad \sum_{k=1}^{\infty} \frac{(k+1)^2}{k!} = 15e.$$

Jl (76)

0.25 Produse infinite

0.250 Fie șirul de numere $a_1, a_2, \dots, a_k, \dots$. Dacă limita $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1+a_k)$, finită sau infinită (dar de semn bine determinat), există, atunci această limită se numește valoarea *produsului infinit* $\prod_{k=1}^{\infty} (1+a_k)$ și scriem

$$1. \quad \lim_{n \rightarrow \infty} \prod_{k=1}^n (1+a_k) = \prod_{k=1}^{\infty} (1+a_k).$$

Dacă produsul infinit are o valoare finită diferită de zero, el se numește *convergent*. În caz contrar, produsul infinit se numește *divergent*. F II 400

0.251. Pentru ca produsul infinit 0.250 1. să fie convergent, este necesar ca $\lim_{k \rightarrow \infty} a_k = 0$. F II 403

0.252 Dacă, pentru toate valorile indicelui k (începînd de la o anumită valoare), orice $a_k > 0$ sau orice $a_k < 0$, atunci, pentru ca produsul 0.250 1. să fie convergent, este necesar și suficient ca seria $\sum_{k=1}^{\infty} a_k$ să fie convergentă.

0.253 Produsul $\prod_{k=1}^{\infty} (1+a_k)$ se numește *absolut convergent*, dacă produsul $\prod_{k=1}^{\infty} (1+|a_k|)$ este convergent. F II 466

0.254 Din convergența absolută a unui produs infinit, rezultă convergența lui.

0.255 Produsul $\prod_{k=1}^{\infty} (1+a_k)$ este absolut convergent, atunci și numai atunci cînd seria $\sum_{k=1}^{\infty} a_k$ este absolut convergentă. F II 406

0.26 Exemple de produse infinite

$$0.261 \quad \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}.$$

E 171

0.262

$$1. \quad \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}.$$

F II 401

$$2. \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi}. \quad \text{F II 401}$$

$$3. \sum_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4}. \quad \text{F II 401}$$

$$0.263 \quad \frac{2}{1} \cdot \left(\frac{4}{3}\right)^{\frac{1}{2}} \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{\frac{1}{4}} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{\frac{1}{8}} \dots = e.$$

$$0.264 \quad \prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}} = e^c. \quad \text{F II 402}$$

$$0.265 \quad \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2}} \dots = \frac{2}{\pi}. \quad \text{F II 402}$$

$$0.266 \quad \prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} \quad |x| < 1. \quad \text{F II 401}$$

0.3 SERII DE FUNCȚII

0.30 Definiții și teoreme

0.301 Seria

$$1. \quad \sum_{k=1}^{\infty} f_k(x),$$

formată din funcții se numește *serie de funcții*. Mulțimea valorilor variabilei independente x pentru care seria 0.301 1. este convergentă constituie *domeniul de convergență* al acestei serii.

0.302 Seria convergentă pentru toate valorile lui x din domeniul M se numește *uniform convergentă* în acest domeniu, dacă pentru orice $\epsilon > 0$ există un număr N , astfel încât, pentru $n > N$, inegalitatea

$$\left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \epsilon$$

să fie adevărată pentru orice valoare a lui x din M .

0.303 Dacă termenii seriei de funcții 0.301 1. satisfac, în domeniul M , inegalitățile

$$|f_k(x)| < u_k \quad (k=1, 2, 3, \dots),$$

unde u_k sînt termenii unei serii numerice convergente oarecare

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots,$$

atunci seria 0.301 1. este uniform convergentă în domeniul M . (Weierstrass). F II 449

0.304 Dacă funcțiile $g_k(x)$ formează (pentru orice x) un șir monoton și mărginit în totalitatea lui și dacă seria 0.301 1. este uniform convergentă în

M , adică dacă există un număr L , astfel încît, pentru orice n și x , sînt satisfăcute inegalitățile

$$1. \quad \left| \sum_{k=1}^n f_k(x) \right| \leq L;$$

atunci seria

$$2. \quad \sum_{k=1}^{\infty} f_k(x) g_k(x)$$

este uniform convergentă în domeniul M . (Abel).

F II 451

0.305 Fie sumele parțiale ale seriei **0.301** 1. mărginite în totalitatea lor, adică există un număr L , astfel încît, pentru orice n și x din domeniul M , sînt satisfăcute inegalitățile

$$\left| \sum_{k=1}^n f_k(x) \right| < L$$

și fie, totodată, funcțiile $g_n(x)$ care formează (pentru orice x) un șir monoton, care converge uniform către zero în domeniul M . Atunci seria **0.304** 2. converge uniform în domeniul M . (Dirichlet).

F II 451

0.306 Dacă funcțiile $f_k(x)$ ($k = 1, 2, 3, \dots$) sînt integrabile pe segmentul $[a, b]$, și seria **0.301** 1., formată din ele, este uniform convergentă pe acest segment, atunci ea poate fi integrată termen cu termen, adică

$$\int_a^x \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^x f_k(x) dx \quad [a \leq x \leq b]. \quad \text{F II 459}$$

0.307 Fie funcțiile $f_k(x)$ ($k = 1, 2, 3, \dots$) avînd derivatele $f'_k(x)$ continue pe segmentul $[a, b]$. Dacă, pe acest segment, seria **0.301** 1. este convergentă, iar seria $\sum_{k=1}^{\infty} f'_k(x)$, formată din derivate, este uniform convergentă, atunci seria **0.301** 1. poate fi derivată termen cu termen, adică

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f'_k(x). \quad \text{F II 460}$$

0.31 Serii de puteri

0.311 Seria de funcții de forma

$$1. \quad \sum_{k=0}^{\infty} a_k (x - \xi)^k = a_0 + a_1 (x - \xi) + a_2 (x - \xi)^2 + \dots$$

se numește *serie de puteri*. Pentru orice serie de puteri **0.311** 1., cu condiția ca ea să nu fie pretutindeni divergentă, domeniul de convergență este un cerc cu centrul în punctul ξ și de rază R ; în orice punct din interiorul acestui cerc, seria de puteri **0.311** 1. este absolut convergentă, iar în exteriorul lui, ea este divergentă. Acest cerc se numește *cerc de convergență*, iar raza lui, *rază de convergență*. Dacă seria este convergentă în toate punctele planului complex, se spune că *raza ei de convergență este egală cu infinit* ($R = +\infty$).

0.312 Seriiile de puteri pot fi integrate și derivate termen cu termen în interiorul cercului de convergență, adică

$$\int_{\xi}^x \left\{ \sum_{k=0}^{\infty} a_k (x-\xi)^k \right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-\xi)^{k+1},$$

$$\frac{d}{dx} \left\{ \sum_{k=0}^{\infty} a_k (x-\xi)^k \right\} = \sum_{k=1}^{\infty} k a_k (x-\xi)^{k-1}.$$

Raza de convergență a seriei care se obține în urma integrării sau a derivării termen cu termen coincide cu raza de convergență a seriei inițiale.

Operații asupra seriilor de puteri

0.313 Împărțirea seriilor de puteri

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

unde

$$c_n + \frac{1}{a_0} \sum_{k=1}^n c_{n-k} a_k - b_n = 0,$$

sau

$$c_n = \frac{(-1)^n}{a_0^n} \begin{vmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \dots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & \dots & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{vmatrix}. \quad A(6360)$$

0.314 Ridicarea la putere a seriilor de puteri

$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

unde

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0} \sum_{k=1}^m (kn - m + k) a_k c_{m-k} \text{ pentru } m \geq 1$$

(n natural) A(6361)

0.315 Substituirea unei serii într-o serie

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k, \quad y = \sum_{k=1}^{\infty} a_k x^k;$$

$$c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2a_1 a_2 b_2 + a_1^3 b_3,$$

$$c_4 = a_4 b_1 + a_2^2 b_2 + 2a_1 a_3 b_2 + 3a_1^2 a_2 b_3 + a_1^4 b_4, \dots$$

A(6362)

0.316 Înmulțirea seriilor de puteri

$$\sum_{k=1}^{\infty} a_k x^k \sum_{k=1}^{\infty} b_k x^k = \sum_{k=1}^{\infty} c_k x^k; \quad c_n = \sum_{k=1}^{n-1} a_k b_{n-k}.$$

F II 372

Seria Taylor

0.317 Dacă funcția $f(x)$ are, în vecinătatea punctului ξ , derivate de orice ordin, putem scrie seria

$$1. \quad f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots,$$

numită *seria Taylor* a funcției $f(x)$.

Seria Taylor converge către funcția $f(x)$, dacă *restul*

$$2. \quad R_n(x) = f(x) - f(\xi) - \sum_{k=1}^n \frac{(x-\xi)^k}{k!} f^{(k)}(\xi)$$

ține către zero, pentru $n \rightarrow \infty$.

Expresia restului în seria Taylor

$$3. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1]. \quad (\text{Lagrange})$$

$$4. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1]. \quad (\text{Cauchy})$$

$$5. \quad R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1].$$

(Schlömlich)

unde $\psi(x)$ este o funcție arbitrară care satisface următoarele două condiții:

1) este continuă împreună cu derivata ei $\psi'(x)$, în intervalul $(0, x-\xi)$;
2) derivata $\psi'(x)$ nu-și schimbă semnul în acest interval; punind $\psi(x) = x^{p+1}$, obținem următoarea formă pentru rest:

$$R_n(x) = \frac{(x-\xi)^{n+1} (1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < p \leq n; 0 < \theta < 1].$$

(Rocher)

$$6. \quad R_n(x) = \frac{1}{n!} \int_{\xi}^x f^{(n+1)}(t) (x-t)^n dt.$$

0.318 Seria Taylor se mai poate scrie

$$1. \quad f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

$$2. \quad f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

(Seria Mac-Laurin)

0.319 Seria Taylor pentru funcții de mai multe variabile

$$f(x, y) = f(\xi, \eta) + (x-\xi) \frac{\partial f(\xi, \eta)}{\partial x} + (y-\eta) \frac{\partial f(\xi, \eta)}{\partial y} + \\ + \frac{1}{2!} \left\{ (x-\xi)^2 \frac{\partial^2 f(\xi, \eta)}{\partial x^2} + 2(x-\xi)(y-\eta) \frac{\partial^2 f(\xi, \eta)}{\partial x \partial y} + (y-\eta)^2 \frac{\partial^2 f(\xi, \eta)}{\partial y^2} \right\} + \dots$$

0.32 Serii trigonometrice

0.320 Fie $f(x)$ o funcție periodică, avind perioada $2l$, absolut integrabilă (cel puțin în sens impropriu) în intervalul $(-l, l)$. Se numește *serie Fourier* a acestei funcții seria trigonometrică

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l},$$

al cărei coeficienți (*coeficienții Fourier*) se definesc prin formulele

$$2. \quad a_k = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{k\pi t}{l} dt = \frac{1}{l} \int_a^{a+2l} f(t) \cos \frac{k\pi t}{l} dt \quad (k=0, 1, 2, \dots),$$

$$3. \quad b_k = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{k\pi t}{l} dt = \frac{1}{l} \int_a^{a+2l} f(t) \sin \frac{k\pi t}{l} dt \quad (k=1, 2, \dots).$$

Criterii de convergență

0.321 Seria Fourier a funcției $f(x)$ converge, în punctul x_0 , către numărul

$$\frac{f(x_0+0) + f(x_0-0)}{2},$$

dacă, oricare ar fi $h>0$, există integrala

$$\int_0^h \frac{|f(x_0+t) + f(x_0-t) - f(x_0+0) - f(x_0-0)|}{t} dt$$

Se presupune că funcția $f(x)$ este, fie continuă în punctul x_0 , fie că are în x_0 discontinuități de speța I (salturi) și că ambele limite $f(x_0+0)$ și $f(x_0-0)$ există. (Dini). F III 524

0.322 Seria Fourier a funcției periodice $f(x)$ care satisface, pe segmentul $[a, b]$, condițiile lui Dirichlet converge în fiecare punct x_0 către valoarea $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$. (Dirichlet).

Despre funcția $f(x)$ se spune că satisface condițiile lui Dirichlet pe segmentul $[a, b]$, dacă este mărginită pe acest segment și dacă segmentul $[a, b]$ poate fi împărțit într-un număr finit de intervale, în interiorul cărora funcția $f(x)$ este continuă și monotonă.

0.323 Seria Fourier a funcției $f(x)$ converge, în punctul x_0 , către valoarea $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$, dacă într-un interval oarecare (x_0-h, x_0+h) , avînd centrul în acest punct, funcția $f(x)$ are o variație mărginită. (Jordan — Dirichlet). F III 528

Definiția funcției cu variație mărginită. Fie funcția $f(x)$ definită pe un segment oarecare $[a, b]$, unde $a < b$. Să împărțim acest segment în mod arbitrar, cu ajutorul unor puncte de diviziune:

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

și să formăm suma

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})|.$$

La diverse moduri de împărțire a segmentului $[a, b]$ (adică la alegeri diferite ale punctelor de diviziune x_i), corespund, în general, sume diferite. Dacă, în totalitatea lor, aceste sume sînt mărginite superior, se spune că funcția $f(x)$ are o *variație mărginită* pe segmentul $[a, b]$. Marginea superioară strictă a acestor sume se numește *variație totală* a funcției $f(x)$ pe segmentul $[a, b]$. F III 91

0.324 Fie funcția $f(x)$, continuă pe porțiuni pe segmentul $[a, b]$, și să admitem că, în fiecare porțiune de continuitate, ea are o derivată continuă pe porțiuni. Atunci, în fiecare punct x_0 al segmentului $[a, b]$, seria Fourier a funcției $f(x)$ converge către $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$.

0.325 Funcția $f(x)$, definită în intervalul $(0, l)$, poate fi dezvoltată în serie de cosinuși de forma

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

unde

$$2. \quad a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l} dt.$$

0.326. Funcția $f(x)$, definită în intervalul $(0, l)$, poate fi dezvoltată în serie de sinuși de forma

$$1. \quad \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

unde

$$2. \quad b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l} dt.$$

Criteriile de convergență pentru seriile 0.325 1. și 0.326 1. sînt analoge cu criteriile de convergență pentru seria 0.320 1. (v. 0.321—0.324).

0.327 Coeficienții Fourier a_k și b_k (definiți prin formulele 0.320 2. și 0.320 3.) ai unei funcții absolut integrabile tind către zero, pentru $k \rightarrow \infty$.

Pentru funcția $f(x)$, de pătrat integrabil în intervalul $(-l, l)$, este verificată *relația de completitudine* (de închidere)

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{l} \int_{-l}^l f^2(x) dx. \quad (\text{A. M. Liapunov}). \quad \text{F III 705}$$

0.328 Fie $f(x)$ și $\varphi(x)$ funcții de pătrat integrabil în intervalul $(-l, l)$, iar a_k, b_k și α_k, β_k coeficienții Fourier respectivi. Pentru aceste funcții, se verifică *relația de completitudine generalizată (egalitatea lui Parseval)*

$$\frac{a_0 a_0}{2} + \sum_{k=1}^{\infty} (a_k \alpha_k + b_k \beta_k) = \frac{1}{l} \int_{-l}^l f(x) \varphi(x) dx. \quad \text{F III 709}$$

Exemple de serii trigonometrice, v. 1.44, 1.45, 6.146, 6.180, 6.192, 6.343, 6.846, 7.421 2., precum și 6.117, 6.118, 6.461, 6.463.

0.33 Serii asimptotice

0.330 Printre seriile divergente, trebuie relevată în mod special importanta clasă de serii numite asimptotice. Deși aceste serii sînt divergente, valorile funcțiilor pe care le reprezintă pot fi calculate cu mare precizie, dacă luăm suma unui număr suficient de termeni din aceste serii. La seriile asimptotice alternate, se obține cea mai mare precizie dacă întrerupem seria la termenul care precede termenul cel mai mic în valoare absolută; în acest caz, eroarea (în valoare absolută) nu depășește valoarea absolută a primului dintre termenii eliminați (cf. 0.227 3.).

Seriile asimptotice au foarte multe proprietăți analoge cu proprietățile seriilor convergente și joacă, de aceea, un rol important în analiză.

Dezvoltarea asimptotică a unei funcții se notează

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}.$$

Definiția dezvoltării asimptotice. Seria divergentă $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ este dezvoltarea asimptotică a funcției $f(z)$ într-un domeniu dat de valori ale $\arg z$, dacă expresia $R_n(z) = z^n[(f(z) - S_n(z))]$, unde $S_n(z) = \sum_{k=0}^n \frac{A_k}{z^k}$, satisface condiția $\lim_{|z| \rightarrow \infty} R_n(z) = 0$ pentru n dat.

F II 820

Seria divergentă care reprezintă dezvoltarea asimptotică a unei funcții oarecare se numește *serie asimptotică*.

0.331 Proprietățile seriilor asimptotice:

1. Asupra seriilor asimptotice se pot efectua operațiile de adunare, scădere, înmulțire și ridicare la putere, ca și în cazul seriilor absolut convergente; seriile obținute ca rezultat al acestor operații vor fi, și ele, asimptotice.

2. Două serii asimptotice pot fi împărțite între ele, cu singura condiție ca termenul întâi A_0 al împărțitorului să nu fie egal cu zero. Seria obținută în urma împărțirii va fi, de asemenea, asimptotică.

F II 823-825

3. Seria asimptotică poate fi integrată termen cu termen, iar seria obținută va fi, de asemenea, asimptotică. În schimb, derivarea unei serii asimptotice nu este, în general, permisă.

F II 824

4. Una și aceeași dezvoltare asimptotică poate reprezenta funcții diferite. Dimpotrivă, o funcție dată poate fi dezvoltată numai într-un singur mod în serie asimptotică.

WW 1208

0.4 UNELE FORMULE DIN CALCULUL DIFERENȚIAL

0.41 Derivarea unei integrale definite în raport cu un parametru

$$0.410 \quad \frac{d}{da} \int_{\varphi(a)}^{\psi(a)} f(x, a) dx = f(\varphi(a), a) \frac{d\varphi(a)}{da} - f(\psi(a), a) \frac{d\psi(a)}{da} +$$

$$+ \int_{\varphi(a)}^{\psi(a)} \frac{d}{da} f(x, a) dx. \quad \text{F II 680}$$

0.411 În particular,

$$1. \quad \frac{d}{da} \int_a^a f(x) dx = f(a).$$

$$2. \quad \frac{d}{db} \int_b^a f(x) dx = -f(b).$$

0.42 Derivata de ordinul n a unui produs (Regula lui Leibniz)

Fie u și v funcții de x derivabile de n ori. Atunci

$$\frac{d^n(uv)}{dx^n} = u \frac{d^n v}{dx^n} + \frac{n}{1!} \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \left(\frac{n}{2}\right) \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} +$$

$$+ \left(\frac{n}{3}\right) \frac{d^3 u}{dx^3} \frac{d^{n-3} v}{dx^{n-3}} + \dots + v \frac{d^n u}{dx^n}$$

sau, simbolic,

$$\frac{d^n(uv)}{dx^n} = (u + v)^{(n)}.$$

FI 272

0.43 Derivata de ordinul n a unei funcții de funcție

0.430 Dacă $f(x) = F(y)$ și $y = \varphi(x)$, atunci

$$1. \quad \frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} F'(y) + \frac{U_2}{2!} F''(y) + \frac{U_3}{3!} F'''(y) + \dots + \frac{U_n}{n!} F^{(n)}(y),$$

unde

$$U_k = \frac{\partial^n}{\partial x^n} y^k - \frac{k}{1!} y \frac{\partial^n}{\partial x^n} y^{k-1} + \frac{k(k-1)}{2!} y^2 \frac{\partial^n}{\partial x^n} y^{k-2} - \dots \quad A(7361)$$

$$2. \quad \frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!j!h! \dots k!} \frac{d^m F}{dy^m} \left(\frac{y'}{1!}\right)^i \left(\frac{y''}{2!}\right)^j \left(\frac{y'''}{3!}\right)^h \dots \left(\frac{y^{(l)}}{l!}\right)^k,$$

semnul $\sum$ urmind a fi extins la toate soluțiile întregi și pozitive ale ecuației $i+2j+3h+\dots+lk=n$, iar $m=i+j+h+\dots+k$. Go I₁ (77)

0.431

$$1. \quad (-1)^n \frac{d^n}{dx^n} F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}} F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}} \frac{n}{1!} F^{(n-1)}\left(\frac{1}{x}\right) +$$

$$+ \frac{(n-1)(n-2)}{x^{2n-2}} \frac{n(n-1)}{2!} F^{(n-2)}\left(\frac{1}{x}\right) + \dots \quad A(7362.1)$$

$$2. \quad (-1)^n \frac{d^n}{dx^n} e^{\frac{a}{x}} = \frac{1}{x^n} e^{\frac{a}{x}} \left\{ \left(\frac{a}{x}\right)^n + (n-1) \binom{n}{1} \left(\frac{a}{x}\right)^{n-1} + \right.$$

$$\left. + (n-1)(n-2) \binom{n}{2} \left(\frac{a}{x}\right)^{n-2} + (n-1)(n-2)(n-3) \binom{n}{3} \left(\frac{a}{x}\right)^{n-3} + \dots \right\}.$$

A(7362.2)

0.432

$$1. \quad \frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!} (2x)^{n-2} F^{(n-1)}(x^2) +$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} F^{(n-2)}(x^2) +$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!} (2x)^{n-6} F^{(n-3)}(x^2) + \dots \quad A(7363.1)$$

$$2. \quad \frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2} \left\{ 1 + \frac{n(n-1)}{1! (4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2! (4ax^2)^2} + \right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3! (4ax^2)^3} + \dots \right\}. \quad A(7363.2)$$

$$3. \frac{d^n}{dx^n} (1 + ax^2)^p = \frac{p(p-1)(p-2) \dots (p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times \\ \times \left\{ 1 + \frac{n(n-1)}{1!(p-n+1)} \frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)} \left(\frac{1+ax^2}{4ax^2} \right)^2 + \dots \right\}. \quad A(7363.3)$$

$$4. \frac{d^{m-1}}{dx^{m-1}} (1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1} \frac{(2m-1)!!}{m} \sin(m \arccos x) \\ (\text{vezi și } 7.1102.). \quad A(7363.4)$$

0.433

$$1. \frac{d^n}{dx^n} F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!} \frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \\ + \frac{(n+1)n(n-1)(n-2)}{2!} \frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots \quad A(7364.1)$$

$$2. \frac{d^n}{dx^n} (1 + a\sqrt{x})^{2n-1} = \frac{(2n-1)!!}{2^n} \frac{a}{\sqrt{x}} \left(a^2 - \frac{1}{x} \right)^{n-1}. \quad A(7364.2)$$

$$0.434 \quad \frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{ - \binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n y^2}{dx^n} - \dots \right\}. \\ A(737.1)$$

$$0.435 \quad \frac{d^n}{dx^n} \ln y = \binom{n}{1} \left\{ \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n y^2}{dx^n} + \binom{n}{3} \frac{1}{3 \cdot y^3} \frac{d^n y^3}{dx^n} - \dots \right\}. \\ A(737.2)$$

1. FUNCȚII ELEMENTARE

1.1 PUTERI ALE BINOAMELOR

1.11 Serii de puteri

$$1.110 \quad (1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \dots + \frac{q(q-1)\dots(q-k+1)}{k!}x^k + \dots$$

Dacă q nu este nici număr natural, nici zero, atunci seria este absolut convergentă pentru $|x| < 1$ și divergentă pentru $|x| > 1$; dacă $x=1$, seria este convergentă pentru $q > -1$ și divergentă pentru $q \leq -1$; dacă $x=-1$, ea este absolut convergentă pentru $q > 0$; dacă $x=-1$, ea este absolut convergentă pentru $q > 0$ și divergentă pentru $q < 0$; dacă $q=n$ număr natural, seria 1.110 se transformă în suma cu număr finit de termeni 1.111. F II 425

1.111

$$(a+x)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k} \quad (\text{cf. 7.221 1.}).$$

1.112

$$1. \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

(vezi și 1.121 2., 7.221 24.).

$$2. \quad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{k=1}^{\infty} (-1)^{k-1} kx^{k-1}.$$

$$3. \quad (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots$$

$$4. \quad (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$$

$$1.113 \quad \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k. \quad [x^2 < 1].$$

1.114

$$1. \quad (1 + \sqrt{1+x})^q = 2^q \left\{ 1 + \frac{q}{1!} \left(\frac{x}{4} \right) + \frac{q(q-3)}{2!} \left(\frac{x}{4} \right)^2 + \right. \\ \left. + \frac{q(q-4)(q-5)}{3!} \left(\frac{x}{4} \right)^3 + \dots \right\} \quad [x^2 < 1, q \text{ real}].$$

A (6351.1)

$$2. (x + \sqrt{1+x^2})^q = 1 + \sum_{k=0}^{\infty} \frac{n^2 (n^2-2^2) (n^2-4^2) \dots [n^2-(2k)^2] x^{2k+2}}{(2k+2)!} +$$

$$+ nx + n \sum_{k=1}^{\infty} \frac{(n^2-1^2) (n^2-3^2) \dots [n^2-(2k-1)^2] x^{2k+1}}{(2k+1)!} x^{2k+1}$$

$$[x^2 < 1, q \text{ real}]. \quad A(6351.2)$$

1.12 Serii de fracții raționale

1.121

$$1. \frac{x}{1-x} = \sum_{k=1}^{\infty} \frac{2^{k-1} x^{2^{k-1}}}{1+x^{2^{k-1}}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1-x^{2^k}} \quad [x^2 < 1]. \quad A(6350.3)$$

$$2. \frac{1}{x-1} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{x^{2^{k-1}} + 1} \quad [x^2 > 1]. \quad A(6350.3)$$

1.2 FUNCȚIA EXPONENȚIALĂ

1.21 Dezvoltarea în serie

1.211

$$1. e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (\text{cf. 7.221 8}).$$

$$2. a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!}.$$

$$3. e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}.$$

$$1.212 \quad e^x (1+x) = \sum_{k=0}^{\infty} \frac{x^k (k+1)}{k!}.$$

$$1.213 \quad \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k)!} \quad [x < 2\pi]. \quad F \text{ II } 520$$

$$1.214 \quad e^{e^x} = e \left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots \right). \quad A(6460.3)$$

1.215

$$1. e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} + \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots \quad A(6460.4)$$

$$2. e^{\cos x} = e \left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots \right). \quad A(6460.5)$$

$$3. e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots \quad A(6460.6)$$

1.216

$$1. e^{\arcsin x} = 1 + x + \frac{x^3}{2!} + \frac{2x^3}{3!} + \frac{5x^5}{4!} + \dots \quad A(6480.7)$$

$$2. e^{\operatorname{arctg} x} = 1 + x + \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{7x^5}{4!} - \dots \quad A(6460.8)$$

1.217

$$1. \pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.421 3.}). \quad A(6707.1)$$

$$2. \frac{2\pi}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x \sum_{k=1}^{\infty} (-1)^k \frac{1}{x^2 + k^2} \quad (\text{cf. 1.422 3.}). \quad A(6707.2)$$

1.22 Relații funcționale

1.221

$$1. a^x = e^{x \ln a}.$$

$$2. a^{\ln_a x} = a^{\frac{1}{\ln_a a}} = x.$$

1.222

$$1. e^x = \operatorname{sh} x + \operatorname{ch} x.$$

$$2. e^{ix} = \sin x + i \cos x.$$

$$1.223 \quad e^{ax} - e^{bx} = (a-b)x \exp \left[\frac{1}{2}(a+b)x \right] \prod_{k=1}^{\infty} \left[1 + \frac{(a-b)^2 x^2}{4k^2 \pi^2} \right]. \quad \text{MO 216}$$

1.23 Serii de funcții exponențiale

$$1.231 \quad \sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x} \quad [a > 1 \text{ și } x < 0 \text{ sau } a < 1 \text{ și } x > 0].$$

1.232

$$1. \operatorname{th} x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx} \quad [x > 0].$$

$$2. \operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x} \quad [x > 0].$$

$$3. \operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x} \quad [x > 0].$$

1.3-1.4 FUNCȚII TRIGONOMETRICE ȘI FUNCȚII HIPERBOLICE

1.30 Introducere

Sinusul trigonometric și sinusul hiperbolic se leagă prin relațiile

$$\operatorname{sh} x = \frac{1}{i} \sin ix, \quad \sin x = \frac{1}{i} \operatorname{sh} ix.$$

Cosinusul trigonometric și cosinusul hiperbolic se leagă prin relațiile

$$\operatorname{ch} x = \cos ix, \quad \cos x = \operatorname{ch} ix.$$

Datorită acestei analogii, fiecărei relații în care intervin funcții trigonometrice i se poate asocia în mod formal o anumită relație în care intervin funcțiile hiperbolice corespunzătoare și, reciproc, fiecărei relații în care intervin funcții hiperbolice i se poate asocia în mod formal o anumită relație în care intervin funcții trigonometrice. În multe cazuri (dar nu în toate), ambele perechi de relații au în adevăr sens.

Dăm mai jos o listă care conține o serie de perechi de astfel de relații. Dar lista nu cuprinde toate formulele care au sens.

1.31 Relații funcționale fundamentale

1.311

1. $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$
 $= -i \operatorname{sh} ix$
2. $\operatorname{sh} x = \frac{1}{2}(e^x - e^{-x}).$
 $= -i \sin(ix)$
3. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}).$
 $= \operatorname{ch} ix$
4. $\operatorname{ch} x = \frac{1}{2}(e^x + e^{-x}).$
 $= \cos ix$
5. $\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{1}{i} \operatorname{th} ix.$
6. $\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{1}{i} \operatorname{tg} ix.$
7. $\operatorname{ctg} x = \frac{\cos x}{\sin x} = \frac{1}{\operatorname{tg} x} = i \operatorname{cth} ix.$
8. $\operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{1}{\operatorname{th} x} = i \operatorname{ctg} ix.$

1.312

1. $\cos^2 x + \sin^2 x = 1.$
2. $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1.$

1.313

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x.$
2. $\operatorname{sh}(x \pm y) = \operatorname{sh} x \operatorname{ch} y \pm \operatorname{sh} y \operatorname{ch} x.$
3. $\sin(x \pm iy) = \sin x \operatorname{ch} y \pm i \sin y \cos x.$
4. $\operatorname{sh}(x \pm iy) = \operatorname{sh} x \cos y \pm i \sin y \operatorname{ch} x.$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$
6. $\operatorname{ch}(x \pm y) = \operatorname{ch} x \operatorname{ch} y \pm \operatorname{sh} x \operatorname{sh} y.$
7. $\cos(x \pm iy) = \cos x \operatorname{ch} y \mp i \sin x \operatorname{sh} y.$
8. $\operatorname{ch}(x \pm iy) = \operatorname{ch} x \cos y \pm i \operatorname{sh} x \sin y.$
9. $\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}.$
10. $\operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x \operatorname{th} y}.$
11. $\operatorname{tg}(x \pm iy) = \frac{\operatorname{tg} x \pm i \operatorname{th} y}{1 \mp i \operatorname{tg} x \operatorname{th} y}.$
12. $\operatorname{th}(x \pm iy) = \frac{\operatorname{th} x \pm i \operatorname{tg} y}{1 \pm i \operatorname{th} x \operatorname{tg} y}.$

1.314

1. $\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y).$
2. $\operatorname{sh} x \pm \operatorname{sh} y = 2 \operatorname{sh} \frac{1}{2}(x \pm y) \operatorname{ch} \frac{1}{2}(x \mp y).$
3. $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y).$
4. $\operatorname{ch} x + \operatorname{ch} y = 2 \operatorname{ch} \frac{1}{2}(x + y) \operatorname{ch} \frac{1}{2}(x - y).$

$$5. \cos x - \cos y = 2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(y-x).$$

$$6. \operatorname{ch} x - \operatorname{ch} y = 2 \operatorname{sh} \frac{1}{2}(x+y) \operatorname{sh} \frac{1}{2}(x-y).$$

$$7. \operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}. \quad 8. \operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x \operatorname{ch} y}.$$

1.315

$$1. \sin^2 x - \sin^2 y = \sin(x+y) \sin(x-y) = \cos^2 y - \cos^2 x.$$

$$2. \operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{sh}(x+y) \operatorname{sh}(x-y) = \operatorname{ch}^2 x - \operatorname{ch}^2 y.$$

$$3. \cos^2 x - \sin^2 y = \cos(x+y) \cos(x-y) = \cos^2 y - \sin^2 x.$$

$$4. \operatorname{sh}^2 x + \operatorname{ch}^2 y = \operatorname{ch}(x+y) \operatorname{ch}(x-y) = \operatorname{ch}^2 x + \operatorname{sh}^2 y.$$

1.316

$$1. (\cos x + i \sin x)^n = \cos nx + i \sin nx. \quad 2. (\operatorname{ch} x + \operatorname{sh} x)^n = \operatorname{sh} nx + \operatorname{ch} ny$$

[n întreg]

1.317

$$1. \sin \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 - \cos x)}. \quad 2. \operatorname{sh} \frac{x}{2} = \pm \sqrt{\frac{1}{2}(\operatorname{ch} x - 1)}.$$

$$3. \cos \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 + \cos x)}. \quad 4. \operatorname{ch} \frac{x}{2} = \sqrt{\frac{1}{2}(\operatorname{ch} x + 1)}.$$

$$5. \operatorname{tg} \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}. \quad 6. \operatorname{th} \frac{x}{2} = \frac{\operatorname{ch} x - 1}{\operatorname{sh} x} = \frac{\operatorname{sh} x}{\operatorname{ch} x + 1}.$$

Semnul din fața radicalului în formulele 1.317 1., 1.317 2., 1.317 3. se alege în conformitate cu semnul membrului întâi; semnul membrului întâi, depinde de valoarea lui x .

1.32 Expriarea puterilor funcțiilor trigonometrice și hiperbolice prin funcții de argument (arc) multiplu

1.320

$$1. \sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}. \quad \text{Kr 56 (10,2)}$$

$$2. \operatorname{sh}^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \operatorname{ch} 2(n-k)x + \binom{2n}{n} \right\}.$$

$$3. \sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x.$$

Kr 56 (10,4)

$$4. \operatorname{sh}^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \operatorname{sh}(2n-2k-1)x.$$

$$5. \cos^{2n} x = \frac{1}{2^n} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}. \quad \text{Kr 56 (10,1)}$$

$$6. \operatorname{ch}^{2n} x = \frac{1}{2^n} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \operatorname{ch} 2(n-k)x + \binom{2n}{n} \right\}.$$

$$7. \cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos(2n-2k-1)x.$$

Kr 56 (10,3)

$$8. \operatorname{ch}^{2n-1} x = 2^{2-2n} \sum_{k=0}^{n-1} \binom{2n-1}{k} \operatorname{ch}(2n-2k-1)x.$$

Cazuri particulare

1.321

$$1. \sin^2 x = \frac{1}{2}(-\cos 2x + 1).$$

$$2. \sin^3 x = \frac{1}{4}(-\sin 3x + 3 \sin x).$$

$$3. \sin^4 x = \frac{1}{8}(\cos 4x - 4 \cos 2x + 3).$$

$$4. \sin^5 x = \frac{1}{16}(\sin 5x - 5 \sin 3x + 10 \sin x).$$

$$5. \sin^6 x = \frac{1}{32}(-\cos 6x + 6 \cos 4x - 15 \cos 2x + 10).$$

$$6. \sin^7 x = \frac{1}{64}(-\sin 7x + 7 \sin 5x - 21 \sin 3x + 35 \sin x).$$

1.322

$$1. \operatorname{sh}^2 x = \frac{1}{2}(\operatorname{ch} 2x - 1).$$

$$2. \operatorname{sh}^3 x = \frac{1}{4}(\operatorname{sh} 3x - 3 \operatorname{sh} x).$$

$$3. \operatorname{sh}^4 x = \frac{1}{8}(\operatorname{ch} 4x - 4 \operatorname{ch} 2x + 3).$$

$$4. \operatorname{sh}^5 x = \frac{1}{16}(\operatorname{sh} 5x - 5 \operatorname{sh} 3x + 10 \operatorname{sh} x).$$

$$5. \operatorname{sh}^6 x = \frac{1}{32}(\operatorname{ch} 6x - 6 \operatorname{ch} 4x + 15 \operatorname{ch} 2x - 10).$$

$$6. \operatorname{sh}^7 x = \frac{1}{64}(\operatorname{sh} 7x - 7 \operatorname{sh} 5x + 21 \operatorname{sh} 3x - 35 \operatorname{sh} x).$$

1.323

$$1. \cos^2 x = \frac{1}{2}(\cos 2x + 1).$$

$$2. \cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x).$$

$$3. \cos^4 x = \frac{1}{8}(\cos 4x + 4 \cos 2x + 3).$$

$$4. \cos^5 x = \frac{1}{16}(\cos 5x + 5 \cos 3x + 10 \cos x).$$

$$5. \cos^6 x = \frac{1}{32}(\cos 6x + 6 \cos 4x + 15 \cos 2x + 10).$$

$$6. \cos^7 x = \frac{1}{64}(\cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x).$$

1.324

$$1. \operatorname{ch}^2 x = \frac{1}{2}(\operatorname{ch} 2x + 1).$$

$$2. \operatorname{ch}^3 x = \frac{1}{4}(\operatorname{ch} 3x + 3 \operatorname{ch} x).$$

$$3. \operatorname{ch}^4 x = \frac{1}{8}(\operatorname{ch} 4x + 4 \operatorname{ch} 2x + 3).$$

$$4. \operatorname{ch}^5 x = \frac{1}{16} (\operatorname{ch} 5x + 5 \operatorname{ch} 3x + 10 \operatorname{ch} x).$$

$$5. \operatorname{ch}^6 x = \frac{1}{32} (\operatorname{ch} 6x + 6 \operatorname{ch} 4x + 15 \operatorname{ch} 2x + 10).$$

$$6. \operatorname{ch}^7 x = \frac{1}{64} (\operatorname{ch} 7x + 7 \operatorname{ch} 5x + 21 \operatorname{ch} 3x + 35 \operatorname{ch} x).$$

1.33 Exprimarea funcțiilor trigonometrice și hiperbolice de argument (arc) multiplu prin puteri ale acestor funcții

1.331

$$1. \sin nx = n \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x + \binom{n}{5} \cos^{n-5} x \sin^5 x - \dots;$$

$$= \sin x \left\{ 2^{n-1} \cos^{n-1} x - \binom{n-2}{1} 2^{n-3} \cos^{n-3} x + \right.$$

$$\left. + \binom{n-3}{2} 2^{n-5} \cos^{n-5} x - \binom{n-4}{3} 2^{n-7} \cos^{n-7} x + \dots \right\}. \quad A(3.175).$$

$$2. \operatorname{sh} nx = \operatorname{sh} x \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} \binom{n}{2k-1} \operatorname{sh}^{2k-1} x \operatorname{ch}^{n-2k+1} x;$$

$$= \operatorname{sh} x \sum_{k=0}^{E\left(\frac{n+1}{2}\right)} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1} \operatorname{ch}^{n-2k-1} x.$$

$$3. \cos nx = \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \dots;$$

$$= 2^{n-1} \cos^n x - \binom{n}{1} 2^{n-3} \cos^{n-2} x +$$

$$+ \frac{n}{2} \binom{n-3}{1} 2^{n-5} \cos^{n-4} x - \frac{n}{3} \binom{n-4}{2} 2^{n-7} \cos^{n-6} x + \dots \quad A(3.175)$$

$$4. \operatorname{ch} nx = \sum_{k=0}^{E\left(\frac{n}{2}\right)} \binom{n}{2k} \operatorname{sh}^{2k} x \operatorname{ch}^{n-2k} x =$$

$$= 2^{n-1} \operatorname{ch}^n x + n \sum_{k=1}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{1}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \operatorname{ch}^{n-2k} x.$$

1.332

$$1. \sin 2nx = 2n \cos x \left\{ \sin x - \frac{4n^2-2^2}{3!} \sin^3 x + \frac{(4n^2-2^2)(4n^2-4^2)}{5!} \sin^5 x - \dots \right\};$$

A(3.171)

$$= (-1)^{n-1} \cos x \left\{ 2^{2n-1} \sin^{2n-1} x - \frac{2n-2}{1!} 2^{2n-3} \sin^{2n-3} x + \right.$$

$$+ \frac{(2n-3)(2n-4)}{2!} 2^{2n-5} \sin^{2n-5} x -$$

$$\left. - \frac{(2n-4)(2n-5)(2n-6)}{3!} 2^{2n-7} \sin^{2n-7} x + \dots \right\}. \quad A(3.173)$$

$$\begin{aligned}
2. \quad \sin(2n-1)x &= (2n-1) \left\{ \sin x - \frac{(2n-1)^2-1^2}{3!} \sin^3 x + \right. \\
&\quad \left. + \frac{[(2n-1)^2-1^2][(2n-1)^2-3^2]}{5!} \sin^5 x - \dots \right\}; \quad A(3.172) \\
&= (-1)^{n-1} \left\{ 2^{2n-2} \sin^{2n-1} x - \frac{2n-1}{1!} 2^{2n-4} \sin^{2n-3} x + \right. \\
&\quad \left. + \frac{(2n-1)(2n-4)}{2!} 2^{2n-6} \sin^{2n-5} x - \right. \\
&\quad \left. - \frac{(2n-1)(2n-5)(2n-6)}{3!} 2^{2n-8} \sin^{2n-7} x + \dots \right\}. \quad A(3.174)m
\end{aligned}$$

$$\begin{aligned}
3. \quad \cos 2nx &= 1 - \frac{4n^2}{2!} \sin^2 x + \\
&\quad + \frac{4n^2(4n^2-2^2)}{4!} \sin^4 x - \frac{4n^2(4n^2-2^2)(4n^2-4^2)}{6!} \sin^6 x + \dots; \quad A(3.171) \\
&= (-1)^n \left\{ 2^{2n-1} \sin^{2n} x - \frac{2n}{1!} 2^{2n-3} \sin^{2n-2} x + \right. \\
&\quad \left. + \frac{2n(2n-3)}{2!} 2^{2n-5} \sin^{2n-4} x - \right. \\
&\quad \left. - \frac{2n(2n-4)(2n-5)}{3!} 2^{2n-7} \sin^{2n-6} x + \dots \right\}. \quad A(3.173)m
\end{aligned}$$

$$\begin{aligned}
4. \quad \cos(2n-1)x &= \cos x \left\{ 1 - \frac{(2n-1)^2-1^2}{2!} \sin^2 x + \right. \\
&\quad \left. + \frac{[(2n-1)^2-1^2][(2n-1)^2-3^2]}{4!} \sin^4 x - \dots \right\}; \quad A(3.172) \\
&= (-1)^{n-1} \cos x \left\{ 2^{2n-2} \sin^{2n-2} x - \frac{2n-3}{1!} 2^{2n-4} \sin^{2n-4} x + \right. \\
&\quad \left. + \frac{(2n-4)(2n-5)}{2!} 2^{2n-6} \sin^{2n-6} x - \right. \\
&\quad \left. - \frac{(2n-5)(2n-6)(2n-7)}{3!} 2^{2n-8} \sin^{2n-8} x + \dots \right\}. \quad A(3.174)
\end{aligned}$$

Folosind formulele și observația 1.30, putem scrie, pentru $\operatorname{sh} 2nx$, $\operatorname{sh}(2n-1)x$, $\operatorname{ch} 2nx$, $\operatorname{ch}(2n-1)x$, formulele analoge cu cele din 1.332, după cum am făcut în formulele 1.331.

Cazuri particulare

1.333

1. $\sin 2x = 2 \sin x \cos x$.
2. $\sin 3x = 3 \sin x - 4 \sin^3 x$.
3. $\sin 4x = \cos x (4 \sin x - 8 \sin^3 x)$.
4. $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$.
5. $\sin 6x = \cos x (6 \sin x - 32 \sin^3 x + 32 \sin^5 x)$.
6. $\sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$.

1.334

1. $\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$.
2. $\operatorname{sh} 3x = 3 \operatorname{sh} x + 4 \operatorname{sh}^3 x$.
3. $\operatorname{sh} 4x = \operatorname{ch} x (4 \operatorname{sh} x + 8 \operatorname{sh}^3 x)$.
4. $\operatorname{sh} 5x = 5 \operatorname{sh} x + 20 \operatorname{sh}^3 x + 16 \operatorname{sh}^5 x$.
5. $\operatorname{sh} 6x = \operatorname{ch} x (6 \operatorname{sh} x + 32 \operatorname{sh}^3 x + 32 \operatorname{sh}^5 x)$.
6. $\operatorname{sh} 7x = 7 \operatorname{sh} x + 56 \operatorname{sh}^3 x + 112 \operatorname{sh}^5 x + 64 \operatorname{sh}^7 x$.

1.335

1. $\cos 2x = 2 \cos^2 x - 1.$
2. $\cos 3x = 4 \cos^3 x - 3 \cos x.$
3. $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.$
4. $\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x.$
5. $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1.$
6. $\cos 7x = 64 \cos^7 x - 112 \cos^5 x + 56 \cos^3 x - 7 \cos x.$

1.336

1. $\operatorname{ch} 2x = 2 \operatorname{ch}^2 x - 1.$
2. $\operatorname{ch} 3x = 4 \operatorname{ch}^3 x - 3 \operatorname{ch} x.$
3. $\operatorname{ch} 4x = 8 \operatorname{ch}^4 x - 8 \operatorname{ch}^2 x + 1.$
4. $\operatorname{ch} 5x = 16 \operatorname{ch}^5 x - 20 \operatorname{ch}^3 x + 5 \operatorname{ch} x.$
5. $\operatorname{ch} 6x = 32 \operatorname{ch}^6 x - 48 \operatorname{ch}^4 x + 18 \operatorname{ch}^2 x - 1.$
6. $\operatorname{ch} 7x = 64 \operatorname{ch}^7 x - 112 \operatorname{ch}^5 x + 56 \operatorname{ch}^3 x - 7 \operatorname{ch} x.$

1.34 Unele sume de funcții trigonometrice și hiperbolice

1.341

1. $\sum_{k=0}^{n-1} \sin(x + ky) = \sin\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2}.$ A(361.8)
2. $\sum_{k=0}^{n-1} \operatorname{sh}(x + ky) = \operatorname{sh}\left(x + \frac{n-1}{2}y\right) \operatorname{sh} \frac{ny}{2} \frac{1}{\operatorname{sh} \frac{y}{2}}.$
3. $\sum_{k=0}^{n-1} \cos(x + ky) = \cos\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2}.$ A(361.9)
4. $\sum_{k=0}^{n-1} \operatorname{ch}(x + ky) = \operatorname{ch}\left(x + \frac{n-1}{2}y\right) \operatorname{sh} \frac{ny}{2} \frac{1}{\operatorname{sh} \frac{y}{2}}.$
5. $\sum_{k=0}^{2n-1} (-1)^k \cos(x + ky) = \sin\left(x + \frac{2n-1}{2}y\right) \sin ny \sec \frac{y}{2}.$ Jl(202)
6. $\sum_{k=0}^{n-1} (-1)^k \sin(x + ky) = \sin\left\{x + \frac{n-1}{2}(y + \pi)\right\} \sin \frac{n(y + \pi)}{2} \sec \frac{y}{2}.$ Jl(202a)

Cazuri particulare

1.342

1. $\sum_{k=1}^n \sin kx = \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}.$ A(361.1)
2. $\sum_{k=0}^n \cos kx = \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 = \cos \frac{nx}{2} \sin \frac{n+1}{2}x \operatorname{cosec} \frac{x}{2}.$ A(361.2)

$$3. \sum_{k=1}^n \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x. \quad A(361.7)$$

$$4. \sum_{k=1}^n \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x. \quad JI(207)$$

1.343

$$1. \sum_{k=1}^n (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos\left(\frac{2n+1}{2}x\right)}{2 \cos \frac{x}{2}}. \quad A(361.11)$$

$$2. \sum_{k=1}^n (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x}. \quad A(361.10)$$

$$3. \sum_{k=1}^n \cos(4k-3)x + \sum_{k=1}^n \sin(4k-1)x = \\ = \sin 2nx (\cos 2nx + \sin 2nx) (\cos x + \sin x) \operatorname{cosec} 2x. \quad JI(208)$$

1.344

$$1. \sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \operatorname{ctg} \frac{\pi}{2n}. \quad A(361.19)$$

$$2. \sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right). \quad A(361.18)$$

$$3. \sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right). \quad A(361.17)$$

1.35 Sume de puteri ale arcelor multiple

1.351

$$1. \sum_{k=1}^n \sin^2 kx = \frac{1}{4} [(2n+1) \sin x - \sin(2n+1)x] \operatorname{cosec} x; \\ = \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x}. \quad A(361.3)$$

$$2. \sum_{k=1}^n \cos^2 kx = \frac{n-1}{2} + \frac{1}{2} \cos nx \sin(n+1)x \operatorname{cosec} x; \\ = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x}. \quad A(361.4)m$$

$$3. \sum_{k=1}^n \sin^3 kx = \frac{3}{4} \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} - \\ - \frac{1}{4} \sin \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}. \quad JI(210)$$

$$4. \sum_{k=1}^n \cos^3 kx = \frac{3}{4} \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + \\ + \frac{1}{4} \cos \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}. \quad JI(211)m$$

$$5. \sum_{k=1}^n \sin^4 kx = \frac{1}{8} [3n - 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \\ + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x]. \quad \text{Jl (212)}$$

$$6. \sum_{k=1}^n \cos^4 kx = \frac{1}{8} [3n + 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \\ + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x]. \quad \text{Jl (213)}$$

1.352

$$1. \sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \frac{2n-1}{2} x}{2 \sin \frac{x}{2}}. \quad \text{A (361.5)}$$

$$2. \sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \frac{2n-1}{2} x}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}. \quad \text{A (361.6)}$$

1.353

$$1. \sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin(n-1)x}{1 - 2p \cos x + p^2}. \quad \text{A (361.12)m}$$

$$2. \sum_{k=1}^{n-1} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x - p^n \operatorname{sh} nx + p^{n+1} \operatorname{sh}(n-1)x}{1 - 2p \operatorname{ch} x + p^2}.$$

$$3. \sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos x + p^{n+1} \cos(n-1)x}{1 - 2p \cos x + p^2}. \quad \text{A (361.13)m}$$

$$4. \sum_{k=0}^{n-1} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x - p^n \operatorname{ch} nx + p^{n+1} \operatorname{ch}(n-1)x}{1 - 2p \operatorname{ch} x + p^2}. \quad \text{Jl (396)}$$

1.36 Sume de produse ale funcțiilor trigonometrice de arce multiple

1.361

$$1. \sum_{k=1}^n \sin kx \sin(k+1)x = \frac{1}{4} [(n+1) \sin 2x - \sin 2(n+1)x] \operatorname{cosec} x. \quad \text{Jl (214)}$$

$$2. \sum_{k=1}^n \sin kx \sin(k+2)x = \frac{n}{2} \cos 2x - \frac{1}{2} \cos(n+3)x \sin nx \operatorname{cosec} x. \quad \text{Jl (216)}$$

$$3. \sum_{k=1}^n \sin kx \cos(2k-1)y = \sin \left\{ ny + \frac{n+1}{2} x \right\} \sin \frac{n(x+2y)}{2} \operatorname{cosec} \frac{x+2y}{2} - \\ - \sin \left\{ ny - \frac{n+1}{2} x \right\} \sin \frac{n(2y-x)}{2} \operatorname{cosec} \frac{2y-x}{2}. \quad \text{Jl (217)}$$

1.362

$$1. \sum_{k=1}^n \left(2^k \sin^2 \frac{x}{2^k} \right)^2 = \left(2^n \sin^2 \frac{x}{2^n} \right)^2 - \sin^2 x. \quad A(361.15)$$

$$2. \sum_{k=1}^n \left(\frac{1}{2^k} \sec \frac{x}{2^k} \right)^2 = \operatorname{cosec}^2 x - \left(\frac{1}{2^n} \operatorname{cosec} \frac{x}{2^n} \right)^2. \quad A(361.14)$$

1.37 Sume de tangente ale arcelor multiple

1.371

$$1. \sum_{k=0}^n \frac{1}{2^k} \operatorname{tg} \frac{x}{2^k} = \frac{1}{2^n} \operatorname{ctg} \frac{x}{2^n} - 2 \operatorname{ctg} 2x. \quad A(361.16)$$

$$2. \sum_{k=0}^n \frac{1}{2^{2k}} \operatorname{tg}^2 \frac{x}{2^k} = \frac{2^{2n+2}-1}{3 \cdot 2^{2n-1}} + 4 \operatorname{ctg}^2 2x - \frac{1}{2^{2n}} \operatorname{ctg} \frac{x}{2^n}. \quad A(361.20)$$

Alte sume trigonometrice cu număr finit de termeni, v. 6.355 2., 6.395 1., 6.471, 6.811 4., 6.833—6.836.

1.38 Sume care duc la tangente hiperbolice și la cotangente hiperbolice

1.381

$$1. \sum_{k=0}^{n-1} \frac{\operatorname{th} x \frac{1}{n \sin^2 \frac{2k+1}{4n} \pi}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{2k+1}{4n} \pi}} = \operatorname{th} 2nx. \quad JI(402)m$$

$$2. \sum_{k=1}^{n-1} \frac{\operatorname{th} x \frac{1}{n \sin^2 \frac{k\pi}{2n}}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{k\pi}{2n}}} = \operatorname{cth} 2nx - \frac{1}{2n} (\operatorname{th} x + \operatorname{cth} x). \quad JI(403)$$

$$3. \sum_{k=0}^{n-1} \frac{\operatorname{th} x \frac{2}{(2n+1) \sin^2 \frac{2k+1}{2(2n+1)} \pi}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{2k+1}{2(2n+1)} \pi}} = \operatorname{th} (2n+1) x - \frac{\operatorname{th} x}{2n+1}. \quad JI(404)$$

$$4. \sum_{k=1}^n \frac{\operatorname{th} x \frac{2}{(2n+1) \sin^2 \frac{k\pi}{2n+1}}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{k\pi}{2n+1}}} = \operatorname{cth} (2n+1) x - \frac{\operatorname{cth} x}{2n+1}. \quad JI(405)$$

1.382

$$1. \sum_{k=0}^{n-1} \frac{1}{\sin^2 \frac{2k+1}{4n} \pi} = 2n \operatorname{th} nx. \quad JI(406)$$

$$\frac{1}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}$$

$$2. \sum_{k=1}^{n-1} \frac{1}{\frac{\sin^2 \frac{k\pi}{2n}}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = 2n \operatorname{cth} nx - 2 \operatorname{cth} x. \quad \text{Jl (407)}$$

$$\sum_{k=0}^{n-1} \frac{1}{\frac{\sin^2 \frac{2k+1}{2(2n+1)} \pi}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = (2n+1) \operatorname{th} \frac{(2n+1)x}{2} - \operatorname{th} \frac{x}{2}. \quad \text{Jl (408)}$$

$$4. \sum_{k=1}^n \frac{1}{\frac{\sin^2 \frac{k\pi}{2n+1}}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = (2n+1) \operatorname{cth} \frac{(2n+1)x}{2} - \operatorname{cth} \frac{x}{2}. \quad \text{Jl (409)}$$

1.39 Exprimarea cosinuşilor şi a sinuşilor de arce multiple sub formă de produse finite

1.391

$$1. \sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ par}]. \quad \text{Jl (568)}$$

$$2. \cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ par}]. \quad \text{Jl (569)}$$

$$3. \sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ impar}]. \quad \text{Jl (570)}$$

$$4. \cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ impar}]. \quad \text{Jl (571)m}$$

1.392

$$1. \sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(x + \frac{k\pi}{n} \right). \quad \text{Jl (548)}$$

$$2. \cos nx = 2^{n-1} \prod_{k=1}^n \sin \left(x + \frac{2k-1}{2n} \pi \right). \quad \text{Jl (549)}$$

1.393

$$1. \prod_{k=0}^{n-1} \cos \left(x + \frac{2k}{m} \pi \right) = \frac{1}{2^{n-1}} \cos nx \quad [n \text{ impar}].$$

$$= \frac{1}{2^{n-1}} [(-1)^{\frac{n}{2}} - \cos nx] \quad [n \text{ par}]. \quad \text{Jl (543)}$$

$$2. \prod_{k=0}^{n-1} \sin \left(x + \frac{2k}{n} \pi \right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \quad [n \text{ impar}].$$

$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \quad [n \text{ par}]. \quad \text{Jl (544)}$$

$$1.394 \quad \prod_{k=0}^{n-1} \left\{ x^2 - 2xy \cos \left(\alpha + \frac{2k\pi}{n} \right) + y^2 \right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}. \quad \text{Jl (573)}$$

1.395

$$1. \cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos x - \cos \left(y + \frac{2k\pi}{n} \right) \right\}. \quad \text{Jl (572)}$$

$$2. \operatorname{ch} nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \operatorname{ch} x - \cos \left(y + \frac{2k\pi}{n} \right) \right\}. \quad \text{Jl (538)}$$

1.396

$$1. \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{k\pi}{n} + 1 \right) = \frac{x^{2n} - 1}{x^2 - 1}. \quad \text{Kr 58 (28.1)}$$

$$2. \prod_{k=1}^n \left(x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x - 1}. \quad \text{Kr 58 (28.2)}$$

$$3. \prod_{k=1}^n \left(x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x + 1}. \quad \text{Kr 58 (28.3)}$$

$$4. \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{(2k+1)\pi}{2n} + 1 \right) = x^{2n} + 1. \quad \text{Kr 58 (28.4)}$$

1.41 Dezvoltarea funcțiilor trigonometrice și hiperbolice în serii de puteri

1.411

$$1. \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \quad 2. \operatorname{sh} x = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!}.$$

$$3. \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \quad 4. \operatorname{ch} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

$$5. \operatorname{tg} x = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} |B_{2k}| x^{2k-1} \left[x^2 < \frac{\pi^2}{4} \right]. \quad \text{F II 523}$$

$$6. \operatorname{th} x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315} x^7 + \dots = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} B_{2k} x^{2k-1} \left[x^2 < \frac{\pi^2}{4} \right].$$

$$7. \operatorname{ctg} x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2]. \quad \text{F II 523m}$$

$$8. \operatorname{cth} x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} [x^2 < \pi^2]. \text{ F II 522 } m$$

$$9. \sec x = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad \left[x^2 < \frac{\pi^2}{4} \right]. \quad \text{Ce 330 } m$$

$$10. \operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{61x^6}{720} + \dots = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!} x^{2k} \quad \left[x^2 < \frac{\pi^2}{4} \right].$$

Ce 330

$$11. \operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1) |B_{2k}| x^{2k-1}}{(2k)!} \quad [x^2 < \pi^2]. \quad \text{Ce 329 } m$$

$$12. \operatorname{cosech} x = \frac{1}{x} - \frac{1}{6} x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \dots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1) B_{2k}}{(2k)!} x^{2k-1}$$

$[x^2 < \pi^2]. \quad \text{Jl (418)}$

Cf. 7.221 9. + 7.221 23.

1.412

$$1. \sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}. \quad \text{Jl (452) } m$$

$$2. \cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}. \quad \text{Jl (443)}$$

$$3. \sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1} - 3}{(2k+1)!} x^{2k+1}. \quad \text{Jl (452a) } m$$

$$4. \cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(3^{2k+3} + 3) x^{2k}}{(2k)!}. \quad \text{Jl (443a)}$$

1 413

$$1. \operatorname{sh} x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k} x^{4k-2}}{(4k-2)!}. \quad \text{Jl (508)}$$

$$2. \operatorname{ch} x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{4k}}{(4k)!}. \quad \text{Jl (507)}$$

$$3. \operatorname{sh} x = x \sec x - \sec x \sum_{k=1}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!} = \sec x \sum_{k=1}^{\infty} (-1)^{E(\frac{k}{2})} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}.$$

Jl (510)

$$4. \operatorname{ch} x = x \operatorname{cosec} x + \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^{2k+1}}{(2k+1)!} =$$

$$= \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{E(\frac{k-1}{2})} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}. \quad \text{Jl (509)}$$

1.414

$$1. \cos [n \ln (x + \sqrt{1+x^2})] = \\ = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{(n^2+0^2)(n^2+2^2) \dots [n^2+(2k)^2]}{(2k+2)!} x^{2k+2} \quad [x^2 < 1]. \quad A (6456.1)$$

$$2. \sin [n \ln (x + \sqrt{1+x^2})] = \\ = nx - n^2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n^2+1^2)(n^2+3^2) \dots [n^2+(2k-1)^2]}{(2k+1)!} x^{2k+1} \quad [x^2 < 1]. \quad A (6456.2)$$

Seriile de puteri pentru $\ln \sin x$, $\ln \cos x$ și $\ln \operatorname{tg} x$, v. 1.518.

1.42 Descompunerea în fracții raționale

1.421

$$1. \operatorname{tg} \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2}. \quad \text{Br}_{08} (191) \quad A (6495.1)$$

$$2. \operatorname{th} \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}.$$

$$3. \operatorname{ctg} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2} = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k(x-k)}.$$

A (6495.2) JI (450a)

$$4. \operatorname{cth} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.217 1}).$$

$$5. \operatorname{tg}^2 \frac{\pi x}{2} = x^2 \sum_{k=1}^{\infty} \frac{2(2k-1)^2 - x^2}{(1^2 - x^2)^2 (3^2 - x^2)^2 \dots [(2k-1)^2 - x^2]^2}. \quad \text{JI (450)}$$

1.422

$$1. \sec \frac{\pi x}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(2k-1)^2 - x^2}. \quad A (6495.3)m$$

$$2. \sec^2 \frac{\pi x}{2} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-1-x)^2} + \frac{1}{(2k-1+x)^2} \right\}. \quad \text{JI (451) } m$$

$$3. \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad (\text{vezi și 1.217 2}). \quad A (6495.4)m$$

$$4. \operatorname{cosec}^3 \pi x = \frac{1}{\pi^3} \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^3} = \frac{1}{\pi^3 x^3} + \frac{2}{\pi^3} \sum_{k=1}^{\infty} \frac{x^2 + k^2}{(x^2 - k^2)^3}. \quad \text{JI (446)}$$

$$5. \frac{1+x \operatorname{cosec} x}{2x^3} = \frac{1}{x^3} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x^2 - k^2 \pi^2)}. \quad \text{JI (449)}$$

$$6. \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{1}{x-k} + \frac{1}{k} \right). \quad \text{JI (450 b)}$$

$$1.423 \quad \frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \operatorname{ctg} \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1-k^2 m^2)^2}. \quad \text{JI (477)}$$

1.43 Exprimarea sub formă de produs infinit

1.431

$$1. \sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right). \quad \text{E 149}$$

$$2. \operatorname{sh} x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2 \pi^2}\right). \quad \text{E 148}$$

$$3. \cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2 \pi^2}\right). \quad \text{E 149}$$

$$4. \operatorname{ch} x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2 \pi^2}\right). \quad \text{E 149}$$

1.432

$$1. \cos x - \cos y = 2 \left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(2k\pi + y)^2}\right) \left(1 - \frac{x^2}{(2k\pi - y)^2}\right). \quad \text{A (653.2)}$$

$$2. \operatorname{ch} x - \cos y = 2 \left(1 + \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(2k\pi + y)^2}\right) \left(1 + \frac{x^2}{(2k\pi - y)^2}\right). \quad \text{A (653.4)}$$

$$1.433 \quad \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4} = \prod_{k=1}^{\infty} \left[1 + \frac{(-1)^k x}{2k-1}\right]. \quad \text{Br}_{08} 189$$

$$1.434 \quad 1 + \sin x = \frac{1}{8} (\pi + 2x)^2 \prod_{k=1}^{\infty} \left[1 - \left(\frac{\pi + 2x}{2k\pi}\right)^2\right]^2. \quad \text{MO 216}$$

$$1.435 \quad \frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a}\right) \left(1 + \frac{x}{k+a}\right). \quad \text{MO 216}$$

$$1.436 \quad 1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{x}{k-a}\right)^2\right]. \quad \text{MO 216}$$

$$1.437 \quad \frac{\sin 3x}{\sin x} = - \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{2x}{x+k\pi}\right)^2\right]. \quad \text{MO 216}$$

$$1.438 \quad \frac{\operatorname{ch} x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty} \left[1 + \left(\frac{x}{2k\pi + a}\right)^2\right]. \quad \text{MO 216}$$

1.439

$$1. \sin x = x \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \quad [|x| < 1]. \quad \text{A (651)}$$

$$2. \frac{\sin x}{x} = \prod_{k=1}^{\infty} \left[1 - \frac{4}{3} \sin^2 \left(\frac{x}{3^k}\right)\right]. \quad \text{MO 216}$$

1.44-1.45 Serii trigonometrice

1.441

$$1. \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi-x}{2} \quad [0 < x < 2\pi]. \quad \text{F III 539}$$

$$2. \sum_{k=1}^{\infty} \frac{\cos kx}{k} = \frac{1}{2} \ln \frac{1}{2(1-\cos x)} \quad [0 < x < 2\pi]. \quad \text{A (6814)}$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \quad [-\pi < x < \pi]. \quad \text{F III 542}$$

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left(2 \cos \frac{x}{2} \right) \quad [-\pi < x < \pi]. \quad \text{F III 550}$$

1.442

$$1. \sum_{k=1}^{\infty} \frac{\sin (2k-1)x}{2k-1} = \frac{\pi}{4} \quad [0 < x < \pi]. \quad \text{F III 541}$$

$$2. \sum_{k=1}^{\infty} \frac{\cos (2k-1)x}{2k-1} = \frac{1}{4} \ln \operatorname{ctg}^2 \frac{x}{2} \quad [0 < x < \pi]. \quad \text{Br}_{08} 168 \text{ JI (266)}$$

$$3. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin (2k-1)x}{2k-1} = \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \\ \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right]. \quad \text{Br}_{08} 168 \text{ JI (268) } m$$

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos (2k-1)x}{2k-1} = \frac{\pi}{4} \quad [0 < x < \pi]. \quad \text{Br}_{08} 168 \text{ JI (269)}$$

1.443

$$1. \sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} B_{2n-k} \rho^k; \\ = (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \left(\frac{x}{2} \right) \\ \left[0 < x < 1, \rho = \frac{x}{2} - E \left(\frac{x}{2} \right) \right]. \quad \text{Ce 340}$$

$$2. \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^n 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} B_{2n-k+1} \rho^k; \\ = (-1)^n \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1} \left(\frac{x}{2} \right) \\ \left[0 < x < 1; \rho = \frac{x}{2} - E \left(\frac{x}{2} \right) \right]. \quad \text{Ce 340}$$

$$3. \sum_{k=1}^{\infty} \frac{\cos kx}{k^3} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad [0 < x < 2\pi]. \quad \text{F III 547}$$

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^3} = \frac{\pi^3}{12} - \frac{x^2}{4} \quad [-\pi < x < \pi]. \quad \text{F III 544}$$

$$\left. \begin{aligned} 5. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^3} &= \frac{\pi^2 x}{6} - \frac{\pi x^3}{4} + \frac{x^5}{12} \\ 6. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^4} &= \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^4}{12} - \frac{x^6}{48} \\ 7. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^5} &= \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^5}{48} - \frac{x^7}{240} \end{aligned} \right\} \quad \begin{aligned} & \\ & [0 < x < 2\pi]. \\ & \end{aligned} \quad \begin{aligned} & \text{A (6816)} \\ & \text{A (6817)} \\ & \text{A (6818)} \end{aligned}$$

1.444

$$\begin{aligned} 1. \quad \sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} &= \sin 2x - (\pi - 2x) \sin^2 x - \sin x \cos x \ln(4 \sin^2 x) \\ & \quad [0 \leq x \leq \pi]. \quad \text{Br}_{08} 168 \\ 2. \quad \sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} &= \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln(4 \sin^2 x) \\ & \quad [0 \leq x \leq \pi]. \quad \text{Br}_{08} 168 \\ 3. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} &= \sin x - \frac{x}{2} (1 + \cos x) - \sin x \ln \left| 2 \cos \frac{x}{2} \right|. \\ & \quad \text{MO (213)} \\ 4. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} &= \cos x - \frac{x}{2} \sin x - (1 + \cos x) \ln \left| 2 \cos \frac{x}{2} \right|. \\ & \quad \text{MO (213)} \\ 5. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} &= \frac{\pi}{4} x \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right]; \\ & \quad = \frac{\pi}{4} (\pi - x) \quad \left[\frac{\pi}{2} \leq x \leq \frac{3}{2} \pi \right]. \quad \text{MO (213)} \\ 6. \quad \sum_{k=1}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} &= \frac{\pi}{4} \left(\frac{\pi}{2} - |x| \right) \quad [-\pi \leq x \leq \pi]. \quad \text{F III 546} \\ 7. \quad \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} &= \frac{1}{2} - \frac{\pi}{4} \sin x \quad \left[0 \leq x \leq \frac{\pi}{2} \right]. \quad \text{Jl (591)} \end{aligned}$$

1.445

$$\begin{aligned} 1. \quad \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + a^2} &= \frac{\pi}{2} \frac{\operatorname{sh} a (\pi - x)}{\operatorname{sh} a \pi} \quad [0 < x < 2\pi]. \quad \text{Br}_{08} 257 \text{ Jl (411)} \\ 2. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + a^2} &= \frac{\pi}{2a} \frac{\operatorname{ch} a (\pi - x)}{\operatorname{sh} a \pi} - \frac{1}{2a^2} \quad [0 < x < 2\pi]. \quad \text{Br}_{08} 257 \text{ Jl (410)} \\ 3. \quad \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2 + a^2} &= \frac{\pi}{2a} \frac{\operatorname{ch} ax}{\operatorname{sh} a \pi} - \frac{1}{2a^2} \quad [-\pi \leq x \leq \pi]. \quad \text{F III 546} \end{aligned}$$

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2 + a^2} = \frac{\pi}{2} \frac{\operatorname{sh} ax}{\operatorname{sh} a\pi} \quad [-\pi < x < \pi]. \quad \text{F III 546}$$

$$5. \sum_{k=0}^{\infty} \frac{k \sin kx}{k^2 - a^2} = \pi \frac{\sin \{a[(2m+1)\pi - x]\}}{2 \sin a\pi} \\ [2m\pi \leq x \leq (2m+1)\pi, \quad a \text{ nu este egal cu un număr întreg}]. \quad \text{MO 213}$$

$$6. \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2} \frac{\cos \{a[(2m+1)\pi - x]\}}{a \sin a\pi} \\ [2m\pi \leq x \leq (2m+1)\pi, \quad a \text{ nu este egal cu un număr întreg}]. \quad \text{MO 213}$$

$$7. \sum_{k=0}^{\infty} (-1)^k \frac{k \sin kx}{k^2 - a^2} = \pi \frac{\sin [a(2m\pi - x)]}{2 \sin a\pi} \\ [(2m-1)\pi \leq x \leq (2m+1)\pi, \quad a \text{ nu este egal cu un număr întreg}]. \quad \text{F III 545}$$

$$8. \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2} \frac{\cos [a(2m\pi - x)]}{a \sin a\pi} \\ [(2m-1)\pi \leq x \leq (2m+1)\pi, \quad a \text{ nu este egal cu un număr întreg}]. \quad \text{F III 545}$$

1.446

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos (2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8} \cos^2 x - \frac{1}{8} \cos x \\ \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right]. \quad \text{Br}_{08} 256 \text{ GC III (189)}$$

1.447

$$\left. \begin{aligned} 1. \sum_{k=1}^{\infty} p^k \sin kx &= \frac{p \sin x}{1 - 2p \cos x + p^2} \\ 2. \sum_{k=0}^{\infty} p^k \cos kx &= \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \\ 3. 1 + 2 \sum_{k=1}^{\infty} p^k \cos kx &= \frac{1 - p^2}{1 - 2p \cos x + p^2} \end{aligned} \right\} [|p| < 1]. \quad \begin{array}{l} \text{F II 559} \\ \text{F II 559 m} \\ \text{MO 213} \end{array}$$

1.448

$$\left. \begin{aligned} 1. \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} &= \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \\ 2. \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} &= \ln \frac{1}{\sqrt{1 - 2p \cos x + p^2}} \\ 3. \sum_{k=1}^{\infty} \frac{p^{2k-1} \sin (2k-1)x}{2k-1} &= \frac{1}{2} \operatorname{arctg} \frac{2p \sin x}{1 - p^2} \\ 4. \sum_{k=1}^{\infty} \frac{p^{2k-1} \cos (2k-1)x}{2k-1} &= \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2} \end{aligned} \right\} \begin{array}{l} \text{F II 559} \\ \text{F II 559} \\ \text{Jl (594)} \\ \text{Jl (259)} \end{array} \quad \begin{array}{l} [0 < x < 2\pi, \\ p^2 \leq 1]. \end{array}$$

$$\left. \begin{aligned}
 5. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} &= \\
 &= \frac{1}{4} \ln \frac{1+2p \sin x + p^2}{1-2p \sin x + p^2} \quad [0 < x < \pi, p^2 \leq 1] \quad \text{Jl (261)} \\
 6. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} &= \\
 &= \frac{1}{2} \operatorname{arctg} \frac{2p \cos x}{1-p^2} \quad \text{Jl (597)}
 \end{aligned} \right\}$$

1.449

$$\left. \begin{aligned}
 1. \quad \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} &= e^{p \cos x} \sin(p \sin x) \\
 2. \quad \sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} &= e^{p \cos x} \cos(p \sin x)
 \end{aligned} \right\} [p^2 \leq 1].$$

Jl (486)
Jl (485)

Dezvoltarea funcțiilor
hiperbolice în serii trigonometrice

1.451

$$1. \quad \operatorname{sh} x = \cos x \sum_{k=0}^{\infty} \frac{(1^2 + 0^2)(1^2 + 2^2) \dots [1^2 + (2k)^2]}{(2k+1)!} \sin^{2k+1} x. \quad \text{Jl (504)}$$

$$2. \quad \operatorname{ch} x - \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2 + 1^2)(1^2 + 3^2) \dots [1^2 + (2k-1)^2]}{(2k)!} \sin^{2k} x. \quad \text{Jl (503)}$$

1.452

$$\left. \begin{aligned}
 1. \quad \operatorname{sh}(x \cos \theta) &= \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \cos(2k+1)\theta}{(2k+1)!} \\
 2. \quad \operatorname{ch}(x \cos \theta) &= \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k} \cos 2k\theta}{(2k)!} \\
 3. \quad \operatorname{sh}(x \cos \theta) &= \operatorname{cosec}(x \sin \theta) \sum_{k=1}^{\infty} \frac{x^{2k} \sin 2k\theta}{(2k)!} \\
 4. \quad \operatorname{ch}(x \cos \theta) &= \operatorname{cosec}(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \sin(2k+1)\theta}{(2k+1)!}
 \end{aligned} \right\} [x^2 < 1].$$

Jl (391)
Jl (390)
Jl (393)
Jl (392)

Vezi și 0.32, 6.113, 6.114, 6.146, 6.180,
6.192, 6.343 1., 6.46, 6.846, 7.421 2.

1.46 Serii de produse ale funcțiilor exponențiale și trigonometrice

1.461

$$\begin{aligned}
 1. \quad \sum_{k=0}^{\infty} e^{-kt} \sin kx &= \frac{1}{2} \frac{\sin x}{\operatorname{ch} t - \cos x} \quad [t > 0]. \quad \text{MO 213} \\
 2. \quad 1 + 2 \sum_{k=1}^{\infty} e^{-kt} \cos kx &= \frac{\operatorname{sh} t}{\operatorname{ch} t - \cos x} \quad [t > 0]. \quad \text{MO 213}
 \end{aligned}$$

$$1.462 \quad \sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[\frac{\sin^2 \frac{x+y}{2} + \operatorname{sh}^2 t}{\sin^2 \frac{x-y}{2} + \operatorname{sh}^2 t} \right]. \quad \text{MO 214}$$

1.463

$$1. \quad e^{x \cos \varphi} \cos(x \sin \varphi) = \sum_{k=0}^{\infty} \frac{x^k \cos k\varphi}{k!} \quad [x^2 < 1]. \quad \text{A (6476.1)}$$

$$2. \quad e^{x \cos \varphi} \sin(x \sin \varphi) = \sum_{k=1}^{\infty} \frac{x^k \sin k\varphi}{k!} \quad [x^2 < 1]. \quad \text{A (6476.2)}$$

1.47 Serii de funcții hiperbolice

1.471

$$1. \quad \sum_{k=1}^{\infty} \frac{\operatorname{sh} kx}{k!} = e^{\operatorname{ch} x} \operatorname{sh}(\operatorname{sh} x). \quad \text{Jl (395)}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{\operatorname{ch} kx}{k!} = e^{\operatorname{ch} x} \operatorname{ch}(\operatorname{sh} x). \quad \text{Jl (394)}$$

1.472

$$1. \quad \sum_{k=1}^{\infty} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x}{1 - 2p \operatorname{ch} x + p^2} \quad [p^2 < 1]. \quad \text{Jl (396)}$$

$$2. \quad \sum_{k=0}^{\infty} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x}{1 - 2p \operatorname{ch} x + p^2} \quad [p^2 < 1]. \quad \text{Jl (397) m}$$

1.48 „Unghiul de paralelism“ $\Pi(x)$ al lui Lobacevski

1.480 Definiție.

$$1. \quad \Pi(x) = 2 \operatorname{arctg} e^x = 2 \operatorname{arctg} e^{-x} \quad [x \geq 0]. \quad \text{Lo III 297 Lo I 120}$$

$$2. \quad \Pi(x) = \pi - \Pi(-x) \quad [x < 0]. \quad \text{Lo III 183 Lo I 93}$$

1.481 Relații funcționale

$$1. \quad \sin \Pi(x) = \frac{1}{\operatorname{ch} x}. \quad \text{Lo III 297}$$

$$2. \quad \cos \Pi(x) = \operatorname{th} x \quad \text{Lo III 297}$$

$$3. \quad \operatorname{tg} \Pi(x) = \frac{1}{\operatorname{sh} x}. \quad \text{Lo III 297}$$

$$4. \quad \operatorname{ctg} \Pi(x) = \operatorname{sh} x. \quad \text{Lo III 297}$$

$$5. \quad \sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}. \quad \text{Lo III 297}$$

$$6. \quad \cos \Pi(x+y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}. \quad \text{L III 183}$$

1.482 Legătura cu gudermanianul

$$\operatorname{gd}(-x) = \Pi(x) - \frac{\pi}{2}.$$

Integrala unghiului de paralelism (integrala definită), vezi 3.851.

1.49 Amplitudinea hiperbolică (gudermanianul) $\operatorname{gd} x$

1.490 Definiție

$$1. \quad \operatorname{gd} x = \int_0^x \frac{dt}{\operatorname{ch} t} = 2 \operatorname{arctg} e^x - \frac{\pi}{2}. \quad \text{JE 73}$$

$$2. \quad x = \int_0^{\operatorname{gd} x} \frac{dt}{\cos t} = \ln \operatorname{tg} \left(\frac{\operatorname{gd} x}{2} + \frac{\pi}{4} \right). \quad \text{JE 73}$$

1.491 Relații funcționale

$$1. \quad \operatorname{ch} x = \sec(\operatorname{gd} x). \quad \text{A (343.1)}$$

$$2. \quad \operatorname{sh} x = \operatorname{tg}(\operatorname{gd} x). \quad \text{A (343.2)}$$

$$3. \quad e^x = \sec(\operatorname{gd} x) + \operatorname{tg}(\operatorname{gd} x) = \operatorname{tg} \left(\frac{\pi}{4} + \frac{\operatorname{gd} x}{2} \right) = \frac{1 + \sin(\operatorname{gd} x)}{\cos(\operatorname{gd} x)}. \quad \text{A (343.3) A (344.5)}$$

$$4. \quad \operatorname{th} x = \sin(\operatorname{gd} x). \quad \text{A (344.3)}$$

$$5. \quad \operatorname{th} \frac{x}{2} = \operatorname{tg} \left(\frac{1}{2} \operatorname{gd} x \right). \quad \text{A (344.4)}$$

$$6. \quad \operatorname{arctg}(\operatorname{th} x) = \frac{1}{2} \operatorname{gd} 2x. \quad \text{A (344.6)m}$$

1.492 Dacă $\gamma = \operatorname{gd} x$, atunci $ix = \operatorname{gd} i\gamma$.

JE 74

1.493 Dezvoltarea în serie

$$1. \quad \frac{\operatorname{gd} x}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{tg}^{2k+1} \frac{x}{2}. \quad \text{JE 74}$$

$$2. \quad \frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \operatorname{tg}^{2k+1} \left(\frac{1}{2} \operatorname{gd} x \right). \quad \text{JE 74}$$

$$3. \quad \operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \dots \quad \text{JE 74}$$

$$4. \quad x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \dots \quad \left[\operatorname{gd} x < \frac{\pi}{2} \right]. \quad \text{JE 74}$$

1.5 FUNCȚIA LOGARITMICĂ

1.51 Dezvoltarea în serie

$$1.511 \quad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots =$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad [-1 < x \leq 1] \quad (\text{cf. } 7.221 \text{ 6.}).$$

1.512

$$1. \quad \ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots =$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} \quad [0 < x \leq 2].$$

$$2. \ln x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 \dots \right] =$$

$$= 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{x-1}{x+1} \right)^{2k-1} \quad [0 < x].$$

$$3. \ln x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots =$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x-1}{x} \right)^k \quad \left[x \geq \frac{1}{2} \right]. \quad A(644.6)$$

1.513

$$1. \ln \frac{1+x}{1-x} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1} \quad [x^2 < 1] \quad (\text{cf. 7.221 7.}) \quad F \parallel 421$$

$$2. \ln \frac{x+1}{x-1} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)x^{2k-1}} \quad [x^2 > 1]. \quad A(644.9)$$

$$3. \ln \frac{x}{1-x} = \sum_{k=1}^{\infty} \frac{1}{kx^k} \quad [x^2 > 1]. \quad J \parallel (88a)$$

$$4. \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad [x^2 < 1]. \quad J \parallel (88b)$$

$$5. \frac{1-x}{x} \ln \frac{1}{1-x} = 1 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} \quad [x^2 < 1]. \quad J \parallel (102)$$

$$6. \frac{1}{1-x} \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1]. \quad J \parallel (88e)$$

$$7. \frac{(1-x)^2}{2x^2} \ln \frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \quad [x^2 < 1]. \quad A(6445.1)$$

$$1.514 \quad \ln(1 - 2x \cos \varphi + x^2) = -2 \sum_{k=1}^{\infty} \frac{\cos k\varphi}{k} x^k. \quad MO98$$

$$\ln(x + \sqrt{1+x^2}) = \text{Arsh } x \quad \text{vezi } 1.631, 1.641, 1.642, 1.646.$$

1.515

$$1. \ln(1 + \sqrt{1+x^2}) = \ln 2 + \frac{1 \cdot 1}{2 \cdot 2} x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} x^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} x^6 + \dots;$$

$$= \ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k} (k!)^2} x^{2k} \quad [x^2 \leq 1]. \quad J \parallel (91)$$

$$2. \ln(1 + \sqrt{1+x^2}) = \ln x + \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots;$$

$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} \cdot k! (k-1)! (2k+1) x^{2k+1}}$$

$$[x^2 \geq 1]. \quad A(644.4)$$

$$3. \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) = x - \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1} (k-1)! k!}{(2k+1)!} x^{2k+1} \quad [x^2 < 1]. \quad \text{Jl (93)}$$

$$4. \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} (k!)^2}{(2k+1)!} x^{2k+1} \quad [x^2 < 1]. \quad \text{Jl (94)}$$

1.516

$$1. \frac{1}{2} \{\ln(1 \pm x)\}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1} x^{k+1}}{k+1} \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1]. \quad \text{Jl (86), Jl (85)}$$

$$2. \frac{1}{6} \{\ln(1+x)\}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+2}}{k+2} \sum_{n=1}^k \frac{1}{n+1} \sum_{m=1}^n \frac{1}{m} \quad [x^2 < 1]. \quad \text{A (644.14)}$$

$$3. -\ln(1+x) \cdot \ln(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n} \quad [x^2 < 1]. \quad \text{Jl (87)}$$

$$4. \frac{1}{4x} \left\{ \frac{1+x}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \ln(1-x) \right\} = \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(2k-1) 2k (2k+1)} \quad [0 < x < 1]. \quad \text{A (6445.2)}$$

1.517

$$1. \frac{1}{2x} \left\{ 1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \operatorname{arctg} x \right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1) 2k (2k+1)} \quad [0 < x \leq 1]. \quad \text{A (6445.3)}$$

$$2. \frac{1}{2} \operatorname{arctg} x \ln \frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{4k-2}}{2k-1} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{2n-1} \quad [x^2 < 1]. \quad \text{Br}_{08} 163$$

$$3. \frac{1}{2} \operatorname{arctg} x \ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2k+1} \sum_{n=1}^{2k} \frac{1}{n} \quad [x^2 < 1]. \quad \text{A (6455.3)}$$

1.518

$$1. \ln \sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots; \\ = \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k (2k)!} \quad [x^2 < \pi^2]. \quad \text{A (643.1) m}$$

$$2. \ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots; \\ = \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1} (2^{2k}-1) B_{2k}}{k (2k)!} x^{2k} = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} \quad \left[x^2 < \frac{\pi^2}{4} \right]. \quad \text{F II 524}$$

$$3. \ln \operatorname{tg} x = \ln x + \frac{x^2}{3} + \frac{7}{90} x^4 + \frac{62}{2835} x^6 + \frac{127}{18900} x^8 + \dots; \\ = \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1}-1) 2^{2k} B_{2k} x^{2k}}{k (2k)!} \quad \left[x^2 < \frac{\pi^2}{4} \right]. \quad \text{A (643.3) m}$$

Seriile de puteri pentru $\frac{\cos}{\sin} \left\{ n \ln(x + \sqrt{1+x^2}) \right\}$, vezi 1.414.

1.52 Serii de funcții logaritmice (Cf. 1.431)

1.521

$$1. \sum_{k=1}^{\infty} \ln \left(1 - \frac{4x^2}{(2k-1)^2\pi^2} \right) = \ln \cos x.$$

$$2. \sum_{k=1}^{\infty} \ln \left(1 - \frac{x^2}{k^2\pi^2} \right) = \ln \sin x - \ln x.$$

1.6 FUNCȚII TRIGONOMETRICE INVERSE ȘI FUNCȚII HIPERBOLICE INVERSE

1.61 Domeniul de definiție

Valorile principale ale funcțiilor trigonometrice inverse se definesc prin inegalitățile

$$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}; \quad 0 \leq \arccos x \leq \pi. \quad [-1 \leq x \leq 1]. \quad \text{F II 553}$$

$$-\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}; \quad 0 < \operatorname{arcctg} x < \pi \quad [-\infty < x < +\infty]. \quad \text{F II 552}$$

1.62-1.63 Relații funcționale

1.621 Legătura dintre funcțiile trigonometrice inverse și funcțiile trigonometrice directe corespunzătoare.

$$1. \arcsin(\sin x) = x - 2n\pi \quad \left[2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2} \right];$$

$$= -x + (2n+1)\pi \quad \left[(2n+1)\pi - \frac{\pi}{2} \leq x \leq (2n+1)\pi + \frac{\pi}{2} \right].$$

$$2. \arccos(\cos x) = x - 2n\pi \quad [2n\pi \leq x \leq (2n+1)\pi];$$

$$= -x + 2(n+1)\pi \quad [(2n+1)\pi \leq x \leq 2(n+1)\pi].$$

$$3. \operatorname{arctg}(\operatorname{tg} x) = x - n\pi \quad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2} \right].$$

$$4. \operatorname{arcctg}(\operatorname{ctg} x) = x - n\pi \quad [n\pi < x < (n+1)\pi].$$

1.622 Legătura dintre funcțiile trigonometrice inverse, funcțiile hiperbolice inverse și logaritmi.

$$1. \arcsin z = \frac{1}{i} \ln(iz + \sqrt{1-z^2}) = \frac{1}{i} \operatorname{Arsh}(iz).$$

$$2. \arccos z = \frac{1}{i} \ln(z + \sqrt{z^2-1}) = \frac{1}{i} \operatorname{Arch} z.$$

$$3. \operatorname{arctg} z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} = \frac{1}{i} \operatorname{Arth}(iz).$$

$$4. \operatorname{arcctg} z = \frac{1}{2i} \ln \frac{iz-1}{iz+1} = i \operatorname{Arcth}(iz).$$

$$5. \operatorname{Arsh} z = \ln(z + \sqrt{z^2+1}) = \frac{1}{i} \arcsin(iz).$$

$$6. \operatorname{Arch} z = \ln(z + \sqrt{z^2-1}) = i \arccos z.$$

$$7. \operatorname{Arth} z = \frac{1}{2} \ln \frac{1+z}{1-z} = \frac{1}{i} \operatorname{arctg} iz.$$

$$8. \operatorname{Arcth} z = \frac{1}{2} \ln \frac{z+1}{z-1} = \frac{1}{i} \operatorname{arcctg}(-iz).$$

Relații între diferitele funcții
trigonometrice inverse

1.623

$$1. \quad \arcsin x + \arccos x = \frac{\pi}{2}. \quad \text{No 43}$$

$$2. \quad \operatorname{arctg} x + \operatorname{arcctg} x = \frac{\pi}{2}. \quad \text{No 43}$$

1.624

$$1. \quad \begin{aligned} \arcsin x &= \arccos \sqrt{1-x^2} \quad [0 \leq x \leq 1]; \\ &= -\arccos \sqrt{1-x^2} \quad [-1 \leq x \leq 0]. \end{aligned} \quad \text{No 47 (5)}$$

$$2. \quad \arcsin x = \operatorname{arctg} \frac{x}{\sqrt{1-x^2}} \quad [x^2 < 1]. \quad \text{No 46 (2)}$$

$$3. \quad \begin{aligned} \arcsin x &= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1]; \\ &= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} - \pi \quad [-1 \leq x < 0]. \end{aligned} \quad \text{No 49 (10)}$$

$$4. \quad \begin{aligned} \arccos x &= \arcsin \sqrt{1-x^2} \quad [0 \leq x \leq 1]; \\ &= \pi - \arcsin \sqrt{1-x^2} \quad [-1 \leq x \leq 0]. \end{aligned} \quad \text{No 48 (6)}$$

$$5. \quad \begin{aligned} \arccos x &= \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1]; \\ &= \pi + \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [-1 \leq x < 0]. \end{aligned} \quad \text{No 48 (8)}$$

$$6. \quad \arccos x = \operatorname{arcctg} \frac{x}{\sqrt{1-x^2}} \quad [-1 \leq x < 1]. \quad \text{No 46 (4)}$$

$$7. \quad \operatorname{arctg} x = \arcsin \frac{x}{\sqrt{1+x^2}}. \quad \text{No 46 (3)}$$

$$8. \quad \begin{aligned} \operatorname{arctg} x &= \arccos \frac{1}{\sqrt{1+x^2}} \quad [x > 0]; \\ &= -\arccos \frac{1}{\sqrt{1+x^2}} \quad [x \leq 0]. \end{aligned} \quad \text{No 48 (7)}$$

$$9. \quad \begin{aligned} \operatorname{arctg} x &= \operatorname{arcctg} \frac{1}{x} \quad [x > 0]; \\ &= \operatorname{arcctg} \frac{1}{x} - \pi \quad [x < 0]. \end{aligned} \quad \text{No 49 (9)}$$

$$10. \quad \begin{aligned} \operatorname{arcctg} x &= \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x > 0]; \\ &= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x < 0]. \end{aligned} \quad \text{No 49 (11)}$$

$$11. \quad \operatorname{arcctg} x = \arccos \frac{x}{\sqrt{1+x^2}}. \quad \text{No 46 (4)}$$

$$12. \quad \begin{aligned} \operatorname{arcctg} x &= \operatorname{arctg} \frac{1}{x} \quad [x > 0]; \\ &= \pi + \operatorname{arctg} \frac{1}{x} \quad [x < 0]. \end{aligned} \quad \text{No 49 (12)}$$

1.625.

$$\begin{aligned}
 1. \quad \arcsin x + \arcsin y &= \arcsin (x \sqrt{1-y^2} + y \sqrt{1-x^2}) \\
 &\quad [xy \leq 0 \text{ sau } x^2 + y^2 \leq 1]; \\
 &= \pi - \arcsin (x \sqrt{1-y^2} + y \sqrt{1-x^2}) \\
 &\quad [x > 0, y > 0 \text{ și } x^2 + y^2 > 1]; \\
 &= -\pi - \arcsin (x \sqrt{1-y^2} + y \sqrt{1-x^2}) \\
 &\quad [x < 0, y < 0, \text{ și } x^2 + y^2 > 1]. \quad \text{No 54 (1)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \arcsin x + \arcsin y &= \arccos (\sqrt{1-x^2} \sqrt{1-y^2} - xy) \quad [x \geq 0, y \geq 0]; \\
 &= -\arccos (\sqrt{1-x^2} \sqrt{1-y^2} - xy) \quad [x < 0, y < 0]. \quad \text{No 55}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \arcsin x + \arcsin y &= \operatorname{arctg} \frac{x \sqrt{1-y^2} + y \sqrt{1-x^2}}{\sqrt{1-x^2} \sqrt{1-y^2} - xy} \\
 &\quad [xy \leq 0 \text{ sau } x^2 + y^2 < 1]; \\
 &= \operatorname{arctg} \frac{x \sqrt{1-y^2} + y \sqrt{1-x^2}}{\sqrt{1-x^2} \sqrt{1-y^2} - xy} + \pi \\
 &\quad [x > 0, y > 0 \text{ și } x^2 + y^2 > 1]; \\
 &= \operatorname{arctg} \frac{x \sqrt{1-y^2} + y \sqrt{1-x^2}}{\sqrt{1-x^2} \sqrt{1-y^2} - xy} - \pi \\
 &\quad [x < 0, y < 0 \text{ și } x^2 + y^2 > 1] \quad \text{No 56}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \arcsin x - \arcsin y &= \arcsin (x \sqrt{1-y^2} - y \sqrt{1-x^2}) \\
 &\quad [xy \geq 0 \text{ sau } x^2 + y^2 \leq 1]; \\
 &= \pi - \arcsin (x \sqrt{1-y^2} - y \sqrt{1-x^2}) \\
 &\quad [x > 0, y < 0 \text{ și } x^2 + y^2 > 1]; \\
 &= -\pi - \arcsin (x \sqrt{1-y^2} - y \sqrt{1-x^2}) \\
 &\quad [x < 0, y > 0 \text{ și } x^2 + y^2 > 1]. \quad \text{No 55 (2)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \arcsin x - \arcsin y &= \arccos (\sqrt{1-x^2} \sqrt{1-y^2} + xy) \quad [x > y]; \\
 &= -\arccos (\sqrt{1-x^2} \sqrt{1-y^2} + xy) \quad [x < y]. \quad \text{No 56}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \arccos x + \arccos y &= \arccos (xy - \sqrt{1-x^2} \sqrt{1-y^2}) \quad [x + y \geq 0]; \\
 &= 2\pi - \arccos (xy - \sqrt{1-x^2} \sqrt{1-y^2}) \quad [x + y < 0]. \quad \text{No 57 (3)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \arccos x - \arccos y &= -\arccos (xy + \sqrt{1-x^2} \sqrt{1-y^2}) \quad [x \geq y]; \\
 &= \arccos (xy + \sqrt{1-x^2} \sqrt{1-y^2}) \quad [x < y]. \quad \text{No 57 (4)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \operatorname{arctg} x + \operatorname{arctg} y &= \operatorname{arctg} \frac{x+y}{1-xy} \quad [xy < 1]; \\
 &= \pi + \operatorname{arctg} \frac{x+y}{1-xy} \quad [x > 0, xy > 1]; \\
 &= -\pi + \operatorname{arctg} \frac{x+y}{1-xy} \quad [x < 0, xy > 1]. \quad \text{No 59 (5)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \operatorname{arctg} x - \operatorname{arctg} y &= \operatorname{arctg} \frac{x-y}{1+xy} \quad [xy > -1]; \\
 &= \pi + \operatorname{arctg} \frac{x-y}{1+xy} \quad [x > 0, xy < -1]; \\
 &= -\pi + \operatorname{arctg} \frac{x-y}{1+xy} \quad [x < 0, xy < -1]. \quad \text{No 59 (6)}
 \end{aligned}$$

1.626

$$\begin{aligned}
 1. \quad 2 \arcsin x &= \arcsin (2x \sqrt{1-x^2}) \quad \left[|x| \leq \frac{1}{\sqrt{2}} \right]; \\
 &= \pi - \arcsin (2x \sqrt{1-x^2}) \quad \left[\frac{1}{\sqrt{2}} < x \leq 1 \right]; \\
 &= -\pi - \arcsin (2x \sqrt{1-x^2}) \quad \left[-1 \leq x < -\frac{1}{\sqrt{2}} \right].
 \end{aligned}$$

No 61 (7)

$$\begin{aligned}
 2. \quad 2 \arccos x &= \arccos (2x^2 - 1) \quad [0 \leq x \leq 1]; \\
 &= 2\pi - \arccos (2x^2 - 1) \quad [-1 \leq x < 0].
 \end{aligned}$$

No 61 (8)

$$\begin{aligned}
 3. \quad 2 \operatorname{arctg} x &= \operatorname{arctg} \frac{2x}{1-x^2} \quad [|x| < 1]; \\
 &= \operatorname{arctg} \frac{2x}{1-x^2} + \pi \quad [x > 1]; \\
 &= \operatorname{arctg} \frac{2x}{1-x^2} - \pi \quad [x < -1].
 \end{aligned}$$

No 61 (9)

1.627

$$\begin{aligned}
 1. \quad \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} &= \frac{\pi}{2} \quad [x > 0]; \\
 &= -\frac{\pi}{2} \quad [x < 0].
 \end{aligned}$$

GC I (878)

$$\begin{aligned}
 2. \quad \operatorname{arctg} x + \operatorname{arctg} \frac{1-x}{1+x} &= \frac{\pi}{4} \quad [x > -1]; \\
 &= -\frac{3}{4} \pi \quad [x < -1].
 \end{aligned}$$

No 62

1.628

$$\begin{aligned}
 1. \quad \arcsin \frac{2x}{1+x^2} &= -\pi - 2 \operatorname{arctg} x \quad [x < -1]; \\
 &= 2 \operatorname{arctg} x \quad [-1 \leq x \leq 1]; \\
 &= \pi - 2 \operatorname{arctg} x \quad [x > 1].
 \end{aligned}$$

No 65

$$\begin{aligned}
 2. \quad \arccos \frac{1-x^2}{1+x^2} &= 2 \operatorname{arctg} x \quad [x \geq 0]; \\
 &= -2 \operatorname{arctg} x \quad [x \leq 0].
 \end{aligned}$$

No 66

$$1.629 \quad \frac{2x-1}{2} - \frac{1}{\pi} \operatorname{arctg} \left(\operatorname{tg} \frac{2x-1}{2} \pi \right) = E(x).$$

GC I (886)

1.631 Relații între funcțiile hiperbolice inverse.

$$\begin{aligned}
 1. \quad \operatorname{Arsh} x &= \operatorname{Arch} \sqrt{x^2 + 1} = \operatorname{Arth} \frac{x}{\sqrt{x^2 + 1}}. & \text{JE 68} \\
 2. \quad \operatorname{Arch} x &= \operatorname{Arsh} \sqrt{x^2 - 1} = \operatorname{Arth} \frac{\sqrt{x^2 - 1}}{x}. & \text{JE 68} \\
 3. \quad \operatorname{Arth} x &= \operatorname{Arsh} \frac{x}{\sqrt{1-x^2}} = \operatorname{Arch} \frac{1}{\sqrt{1-x^2}} = \operatorname{Arcth} \frac{1}{x}. & \text{JE 68} \\
 4. \quad \operatorname{Arsh} x \pm \operatorname{Arsh} y &= \operatorname{Arsh} (x \sqrt{1+y^2} \pm y \sqrt{1+x^2}). & \text{JE 69} \\
 5. \quad \operatorname{Arch} x \pm \operatorname{Arch} y &= \operatorname{Arch} (xy \pm \sqrt{(x^2-1)(y^2-1)}). & \text{JE 69} \\
 6. \quad \operatorname{Arth} x \pm \operatorname{Arth} y &= \operatorname{Arth} \frac{x \pm y}{1 \pm xy}. & \text{JE 69}
 \end{aligned}$$

1.64 Dezvoltarea în serie

1.641

$$1. \quad \arcsin x = \frac{\pi}{2} - \arccos x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots;$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} = x F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) \quad [x^2 < 1]$$

F II 479

$$2. \quad \operatorname{Arsh} x = x - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots;$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1};$$

$$= x F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2\right) \quad [x^2 < 1].$$

F II 480

1.642

$$1. \quad \operatorname{Arsh} x = \ln 2x + \frac{1}{2} \frac{1}{2x^3} - \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4x^5} + \dots;$$

$$= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \quad [x^2 > 1].$$

A (6480.2) m

$$2. \quad \operatorname{Arch} x = \ln 2x - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 2k} x^{-2k} \quad [x^2 > 1].$$

A (6480.3) m

1.643

$$1. \quad \operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots;$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad [x^2 < 1] \quad (\text{cf. 7.221 27.}).$$

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$$2. \quad \operatorname{Arth} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad [x^2 < 1].$$

A (6480.4)

1.644

$$1. \quad \operatorname{arctg} x = \frac{x}{1+x^2} \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2}{(2k+1)!} \left(\frac{x^2}{1+x^2}\right)^k;$$

$$= \frac{x}{1+x^2} F\left(1, 1; \frac{3}{2}; \frac{x^2}{1+x^2}\right) \quad [x^2 < \infty].$$

A (641.3)

$$2. \quad \operatorname{arctg} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots;$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1) x^{2k+1}} \quad [x^2 \geq 1] \quad (\text{vezi și 1.643})$$

A (641.4)

1.645

$$1. \quad \operatorname{arcsec} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{2 \cdot 3x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} + \dots;$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)! x^{-(2k+1)}}{(k!)^2 2^{2k} (2k+1)};$$

$$= \frac{\pi}{2} - \frac{1}{x} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{x^2}\right) \quad [x > 1].$$

A (641.5)

$$2. (\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2 x^{2k+2}}{(2k+1)! (k+1)!} \quad [x^2 \leq 1]. \quad A (642.1)$$

$$3. (\arcsin x)^3 = x^3 + \frac{3!}{5!} 3^2 \left(1 + \frac{1}{3^2}\right) x^5 + \frac{3!}{7!} 3^2 \cdot 5^2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right) x^7 + \dots$$

[$x^2 \leq 1$]. Br₀₈ 188 A (642.2)

1.646

$$1. \operatorname{Arsh} \frac{1}{x} = \operatorname{Arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (k!)^2 (2k+1)!} x^{-2k-1} \quad [x^2 > 1].$$

A (6480.5)

$$2. \operatorname{Arch} \frac{1}{x} = \operatorname{Arsech} x = \ln \frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 2k} x^{2k} \quad [0 < x < 1].$$

A (6480.6)

$$3. \operatorname{Arsh} \frac{1}{x} = \operatorname{Arcosech} x = \ln \frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{2^{2k} (k!)^2 2k} x^{2k} \quad [0 < x < 1].$$

A (6480.7) m

$$4. \operatorname{Arth} \frac{1}{x} = \operatorname{Arcth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1} \quad [x^2 > 1]. \quad A (6480.8)$$

2. INTEGRALE NEDEFINITE

2.0 INTRODUCERE

2.00 Observații cu caracter general

În toate formulele din această secțiune, constanta de integrare s-a omis. De aceea, semnul de egalitate (=) înseamnă, în această secțiune, că funcțiile care stau de ambele părți ale acestui semn diferă printr-o constantă. De exemplu (vezi 201 15.), scriem

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arctg} x,$$

deși

$$\operatorname{arctg} x = -\operatorname{arctg} x + \frac{\pi}{2}.$$

La integrarea unor funcții, se obține logaritmul valorii absolute (de exemplu, $\int \frac{dx}{\sqrt{1+x^2}} = \ln |x + \sqrt{1+x^2}|$). În aceste formule, semnul de valoare absolută în argumentul logaritmului a fost omis, pentru simplitate.

În unele cazuri, este important să găsim o funcție primitivă pe deplin determinată. Asemenea funcții primitive, scrise sub formă de integrale definite, nu sînt clasificate în secțiunea 2, ci în alte secțiuni. Dăm lista formulelor care cuprind asemenea funcții primitive: 3.131, 3.133—3.141, 3.143—3.148, 3.213 1., 3.221, 3.222 1., 3.224 1., 3.225 1., 3.225 2., 3.273, 3.325 4., 3.326 6., 3.517 2., 3.522, 3.526 2., 3.611 1., 3.615 1., 3.615 6., 3.616 5., 3.617 1., 3.618 1., 3.618 4., 3.711, 3.762 4., 4.221 1., 4.231 1., 4.231 2., 4.241, 4.251, 4.321, 4.331, 4.401, 4.510, 4.511, 4.742, 6.525 7.

De aceste formule se leagă formulele în care limitele de integrare și funcția de sub semnul integrală depind de același parametru: 3.133, 3.137, 3.143—4.148, 3.155 1., 3.155 2., 3.171, 3.182, 3.185, 3.191 4., 3.276, 3.412, 3.441, 3.452, 3.517, 3.618 5., 3.618 6., 3.649, 3.676 1., 3.762 4., 3.769 3., 4.13 3., 4.321 1., 4.403, 4.405, 4.418, 4.451, 4.452 3., 4.465, 4.81.

Pentru anumite valori ale constantelor (parametrilor) sau pentru anumite relații între anumite constante, numeroase formule își pierd sensul (de exemplu, formula 2.02 8., pentru $n=-1$; formula 2.02 15., pentru $a=b$). În cea mai mare parte, aceste valori ale constantelor și aceste relații dintre ele se observă imediat din însăși structura membrului al doilea al formulei (cel care nu conține semnul integrală). De aceea, omitem, în această secțiune, să facem rezervele corespunzătoare. Totuși, dacă pentru valorile parametrilor pentru care o anumită formulă își pierde sensul, valoarea integralei este dată cu ajutorul unei alte formule, însoțim această a doua formulă cu explicațiile corespunzătoare.

Literele $x, y, t, \dots$ indică variabile independente; $f, g, \varphi, \dots$ înseamnă funcții de $x, y, t, \dots$; $f', g', \varphi', \dots, f'', g'', \varphi'', \dots$ derivatele lor de ordinul întâi, doi etc.; $a, b, m, p, \dots$ constante, prin care, în general, trebuie să înțelegem numere reale oarecare. Dacă o formulă oarecare este valabilă numai pentru anumite valori ale constantelor (de exemplu, pentru valori pozitive, sau numai pentru numere întregi), atunci se face rezerva necesară, în cazurile în care această limitare nu rezultă chiar din formulă (astfel, în formulele 2.148 4. și 2.442 6., nu s-au făcut nici un fel de rezerve, deoarece, din formulă însăși, reiese că n trebuie să fie un număr natural, adică întreg și pozitiv).

2.01 Integrale fundamentale

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1).$$

Pentru $n = -1$

$$2. \int \frac{dx}{x} = \ln x.$$

$$3. \int e^x dx = e^x.$$

$$4. \int a^x dx = \frac{a^x}{\ln a}.$$

$$5. \int \sin x dx = -\cos x.$$

$$6. \int \cos x dx = \sin x.$$

$$7. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$8. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$9. \int \frac{\sin x}{\cos^2 x} dx = \sec x.$$

$$10. \int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x.$$

$$11. \int \operatorname{tg} x dx = -\ln \cos x.$$

$$12. \int \operatorname{ctg} x dx = \ln \sin x.$$

$$13. \int \frac{dx}{\sin x} = \ln \operatorname{tg} \frac{x}{2}.$$

$$14. \int \frac{dx}{\cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln (\sec x + \operatorname{tg} x).$$

$$15. \int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arccotg} x.$$

$$16. \int \frac{dx}{1-x^2} = \operatorname{Arth} x = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

$$17. \int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arcsin} x = -\operatorname{arccos} x.$$

$$18. \int \frac{dx}{\sqrt{x^2+1}} = \operatorname{Arsh} x = \ln (x + \sqrt{x^2+1}).$$

$$19. \int \frac{dx}{\sqrt{x^2-1}} = \operatorname{Arch} x = \ln (x + \sqrt{x^2-1}).$$

$$20. \int \operatorname{sh} x dx = \operatorname{ch} x.$$

$$21. \int \operatorname{ch} x dx = \operatorname{sh} x$$

$$22. \int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$$

$$23. \int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$$

$$24. \int \operatorname{th} x dx = \ln \operatorname{ch} x.$$

$$25. \int \operatorname{cth} x dx = \ln \operatorname{sh} x.$$

$$26. \int \frac{dx}{\operatorname{sh} x} = \ln \operatorname{th} \frac{x}{2}.$$

2.02 Formule generale

1. $\int a f dx = a \int f dx.$
2. $\int [af \pm b\varphi \pm c\psi \pm \dots] dx = a \int f dx \pm b \int \varphi dx \pm c \int \psi dx \pm \dots$
3. $\frac{d}{dx} \int f dx = f.$
4. $\int f' dx = f.$
5. $\int f' \varphi dx = f\varphi - \int f\varphi' dx$ [integrare prin părți].
6. $\int f^{(n+1)} \varphi dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \dots + (-1)^n \varphi^{(n)} f +$
 $+ (-1)^{n+1} \int \varphi^{(n+1)} f dx.$
7. $\int f(x) dx = \int f[\varphi(y)] \varphi'(y) dy$ [$x = \varphi(y)$] [regula substituției].
8. $\int (f)^n f' dx = \frac{(f)^{n+1}}{n+1}.$

Pentru $n = -1,$

- $\int \frac{f' dx}{f} = \ln f.$
9. $\int (af + b)^n f' dx = \frac{(af + b)^{n+1}}{a(n+1)}.$
10. $\int \frac{f' dx}{\sqrt{af + b}} = \frac{2\sqrt{af + b}}{a}.$
11. $\int \frac{f'\varphi - \varphi'f}{\varphi^2} dx = \frac{f}{\varphi}.$
12. $\int \frac{f'\varphi - \varphi'f}{f\varphi} dx = \ln \frac{f}{\varphi}.$
13. $\int \frac{dx}{f(f \pm \varphi)} = \pm \int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi(f \pm \varphi)}.$
14. $\int \frac{f' dx}{\sqrt{f^2 + a}} = \ln(f + \sqrt{f^2 + a}).$
15. $\int \frac{f dx}{(f+a)(f+b)} = \frac{a}{a-b} \int \frac{dx}{(f+a)} - \frac{b}{a-b} \int \frac{dx}{(f+b)}.$

Pentru $a = b,$

- $\int \frac{f dx}{(f+a)^2} = \int \frac{dx}{f+a} - a \int \frac{dx}{(f+a)^2}.$
16. $\int \frac{f dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi dx}{(f+\varphi)^n}.$
17. $\int \frac{f' dx}{p^2 + q^2 f^2} = \frac{1}{pq} \operatorname{arctg} \frac{qf}{p}.$
18. $\int \frac{f' dx}{q^2 f^2 - p^2} = \frac{1}{2pq} \ln \frac{qf - p}{qf + p}.$
19. $\int \frac{f dx}{1-f} = -x + \int \frac{dx}{1-f}.$
20. $\int \frac{f^2 dx}{f^2 - a^2} = \frac{1}{2} \int \frac{f dx}{f-a} + \frac{1}{2} \int \frac{f dx}{f+a}.$
21. $\int \frac{f' dx}{\sqrt{a^2 - f^2}} = \arcsin \frac{f}{a}.$

$$22. \int \frac{f' dx}{af^2 + bf} = \frac{1}{b} \ln \frac{f}{af + b}.$$

$$23. \int \frac{f' dx}{f \sqrt{f^2 - a^2}} = \frac{1}{a} \operatorname{arccsc} \frac{f}{a}.$$

$$24. \int \frac{(f' \varphi - f \varphi') dx}{f^2 + \varphi^2} = \operatorname{arctg} \frac{f}{\varphi}.$$

$$25. \int \frac{(f' \varphi - f \varphi') dx}{f^2 - \varphi^2} = \frac{1}{2} \ln \frac{f - \varphi}{f + \varphi}.$$

2.1 FUNCȚII RAȚIONALE

2.10 Reguli generale de integrare

2.101 Pentru a integra orice funcție rațională $\frac{F(x)}{f(x)}$, unde $F(x)$ și $f(x)$ sînt polinoame care nu au factori comuni, trebuie întîi să separăm partea întreagă $E(x)$, dacă aceasta există, și să integrăm partea întreagă și restul.

$$\int \frac{F(x) dx}{f(x)} = \int E(x) dx + \int \frac{\varphi(x)}{f(x)} dx.$$

Integrarea restului, care este o funcție rațională ordinară (gradul numărătorului este mai mic decît gradul numitorului) se bazează pe descompunerea în fracții simple.

2.102 Dacă $a, b, c, \dots, m$ sînt rădăcinile ecuației $f(x)=0$, iar $\alpha, \beta, \gamma, \dots, \mu$, ordinul de multiplicitate respectiv, așa încît $f(x)=(x-a)^\alpha \cdot (x-b)^\beta \dots (x-m)^\mu$, atunci $\frac{\varphi(x)}{f(x)}$ poate fi descompusă în următoarele fracții simple:

$$\begin{aligned} \frac{\varphi(x)}{f(x)} = & \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \dots + \frac{A_1}{x-a} + \\ & + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \dots + \frac{B_1}{x-b} + \\ & + \dots + \dots + \dots + \\ & + \frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \dots + \frac{M_1}{x-m}, \end{aligned}$$

unde numărătorii diverselor fracții sînt definiți prin următoarele formule:

$$A_k = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \quad B_k = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \quad \dots, \quad M_k = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$\psi_1(x) = \frac{\varphi(x)(x-a)^\alpha}{f(x)}, \quad \psi_2(x) = \frac{\varphi(x)(x-b)^\beta}{f(x)}, \quad \dots,$$

$$\psi_m(x) = \frac{\varphi(x)(x-m)^\mu}{f(x)},$$

T 51 m

Dacă $a, b, \dots, m$ sînt rădăcini simple, adică $\alpha=\beta=\dots=\mu=1$, atunci

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{M}{x-m},$$

unde

$$A = \frac{\varphi(a)}{f'(a)}, \quad B = \frac{\varphi(b)}{f'(b)}, \quad \dots, \quad M = \frac{\varphi(m)}{f'(m)}.$$

Dacă unele rădăcini ale ecuației $f(x)=0$ sînt imaginare, atunci grupînd fracțiile simple corespunzătoare rădăcinilor conjugate, putem înlocui, după anumite transformări, perechile corespunzătoare de fracții prin fracții reale de forma

$$\frac{M_1x+N_1}{x^2+2Bx+C} + \frac{M_2x+N_2}{(x^2+2Bx+C)^2} + \dots + \frac{M_px+N_p}{(x^2+2Bx+C)^p}.$$

2.103 Așadar, integrarea unei fracții raționale ordinare $\frac{\varphi(x)}{f(x)}$ se reduce la integrale de forma $\int \frac{g dx}{(x-a)^\alpha}$ sau $\int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx$. Primele dau funcții raționale, pentru $\alpha > 1$, și logaritmi, pentru $\alpha=1$; celelalte dau funcții raționale și logaritmi sau arctangente:

1. $\int \frac{g dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}.$
2. $\int \frac{g dx}{(x-a)} = g \int \frac{d(x-a)}{(x-a)} = g \ln|x-a|.$
3. $\int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx = \frac{NB-MA+(NC-MB)x}{2(p-1)(AC-B^2)(A+2Bx+Cx^2)^{p-1}} +$
 $+ \frac{(2p-3)(NC-MB)}{2(p-1)(AC-B^2)} \int \frac{dx}{(A+2Bx+Cx^2)^{p-1}}.$
4. $\int \frac{dx}{A+2Bx+Cx^2} = \frac{1}{\sqrt{AC-B^2}} \operatorname{arctg} \frac{Cx+B}{\sqrt{AC-B^2}} \quad [AC > B^2];$
 $= \frac{1}{2\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| \quad [AC < B^2].$
5. $\int \frac{(Mx+N) dx}{A+2Bx+Cx^2} = \frac{M}{2C} \ln|A+2Bx+Cx^2| +$
 $+ \frac{NC-MB}{C\sqrt{AC-B^2}} \operatorname{arctg} \frac{Cx+B}{\sqrt{AC-B^2}}.$

Metoda lui Ostrogradski-Hermite

2.104 Cu ajutorul metodei lui Ostrogradski-Hermite, putem găsi partea rațională a integralei $\int \frac{\varphi(x)}{f(x)} dx$, fără a afla rădăcinile ecuației $f(x)=0$ și fără a descompune în fracții simple:

$$\int \frac{\varphi(x)}{f(x)} dx = \frac{M}{D} + \int \frac{N dx}{Q}. \quad \text{F II 49}$$

Aici M, N, D, Q sînt funcții de x raționale și întregi, iar D este cel mai mare comun divizor al funcției $f(x)$ și al derivatei ei $f'(x)$, $Q = \frac{f(x)}{D}$; M este un polinom de grad egal cel mult cu $m-1$, dacă m este gradul polinomului D , iar N , un polinom avînd gradul egal cu cel mult $n-1$, dacă n este gradul polinomului Q . Coeficienții polinoamelor M și N se determină prin egalarea coeficienților puterilor asemenea ale lui x din următoarea identitate:

$$\varphi(x) = M'Q - M(T-Q') + ND,$$

unde $T = \frac{f'(x)}{D}$, M' și Q' sînt derivatele polinoamelor M și Q .

2.11-2.13 Forme care conțin binoame $a+bx^k$ 2.110 Formule de reducere pentru $z_k = a + bx^k$

$$1. \int x^n z_k^m dx = \frac{x^{n+1} z_k^m}{km+n+1} + \frac{amk}{km+n+1} \int x^n z_k^{m-1} dx = \quad \text{La 126 (4)}$$

$$= x^{n+1} \sum_{s=0}^p \frac{(ak)^s m(m-1) \dots (m-s+1) z_k^{m-s}}{[mk+n+1][(m-1)k+n+1] \dots [(m-s)k+n+1]} +$$

$$+ \frac{(ak)^{p+1} m(m-1) \dots (m-s+1)(m-s)}{[mk+n+1][(m-1)k+n+1] \dots [(m-s)k+n+1]} \int x^n z_k^{m-p-1} dx.$$

$$2. \int x^n z_k^m dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} dx. \quad \text{La 126 (6)}$$

$$3. \int x^n z_k^m dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} dx. \quad \text{La 125 (1)}$$

$$4. \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} dx. \quad \text{La 125 (2)}$$

$$5. \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m dx. \quad \text{La 126 (3)}$$

$$6. \int x^n z_k^m dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m dx. \quad \text{La 126 (5)}$$

Forme care conțin binoame $z_1 = a + bx$

2.111

$$1. \int z_1^m dx = \frac{z_1^{m+1}}{b(m+1)}.$$

Pentru $m = -1$,

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1.$$

$$2. \int \frac{x^n dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} dx}{z_1^m}.$$

Pentru $n = m - 1$, putem folosi formula

$$3. \int \frac{x^n dx}{z_1^m} = -\frac{x^n}{z_1^{m-1}(m-1)b} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{z_1^{m-1}}.$$

Pentru $m = 1$,

$$\int \frac{x^n dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2 x^{n-2}}{(n-2)b^3} - \dots + (-1)^{n-1} \frac{a^{n-1} x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1.$$

$$4. \int \frac{x^n dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{ka^{k-1} x^{n-k}}{(n-k)b^{k+1}} +$$

$$+ (-1)^{n-1} \frac{a^n}{b^{n+1} z_1} + (-1)^{n+1} \frac{na^{n-1}}{b^{n+1}} \ln z_1.$$

2.112

$$1. \int \frac{xax}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1.$$

$$2. \int \frac{x^2 dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1.$$

2.113

1. $\int \frac{dx}{z_1^3} = -\frac{1}{bz_1}.$
2. $\int \frac{x dx}{z_1^3} = -\frac{x}{bz_1} + \frac{1}{b^2} \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2} \ln z_1.$
3. $\int \frac{x^2 dx}{z_1^3} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3} \ln z_1.$

2.114

1. $\int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}.$
2. $\int \frac{x dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2} \right] \frac{1}{z_1^2}.$
3. $\int \frac{x^2 dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3} \right] \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1.$
4. $\int \frac{x^3 dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2} x^2 - 2\frac{a^2}{b^3} x - \frac{5}{2} \frac{a^3}{b^4} \right] \frac{1}{z_1^2} - 3\frac{a}{b^4} \ln z_1.$

2.115

1. $\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}.$
2. $\int \frac{x dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2} \right] \frac{1}{z_1^3}.$
3. $\int \frac{x^2 dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3} \right] \frac{1}{z_1^3}.$
4. $\int \frac{x^3 dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2 x}{2b^3} + \frac{11a^3}{6b^4} \right] \frac{1}{z_1^3} + \frac{1}{b^4} \ln z_1.$

2.116

1. $\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}.$
2. $\int \frac{x dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2} \right] \frac{1}{z_1^4}.$
3. $\int \frac{x^2 dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3} \right] \frac{1}{z_1^4}.$
4. $\int \frac{x^3 dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2 x}{b^3} + \frac{a^3}{4b^4} \right] \frac{1}{z_1^4}.$

2.117

1. $\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z_1^m}.$
2. $\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}.$
3. $\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a} \int \frac{dx}{xz_1^{m-1}}.$
4. $\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n} \ln \frac{z_1}{x}.$

2.118

1. $\int \frac{dx}{xz_1} = -\frac{1}{a} \ln \frac{z_1}{x}.$
2. $\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{z_1}{x}.$
3. $\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2 x} - \frac{b^2}{a^3} \ln \frac{z_1}{x}.$

2.119

1. $\int \frac{dx}{x^2 z_1^2} = \frac{1}{ax_1} - \frac{1}{a^2} \ln \frac{x_1}{x}.$
2. $\int \frac{dx}{x^3 z_1^3} = - \left[\frac{1}{ax} + \frac{2b}{a^2} \right] \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{x_1}{x}.$
3. $\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2 x} + \frac{3b^2}{a^3} \right] \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{x_1}{x}.$

2.121

1. $\int \frac{dx}{x^2 z_1^2} = \left[\frac{3}{2a} + \frac{bx}{a^2} \right] \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{x_1}{x}.$
2. $\int \frac{dx}{x^3 z_1^3} = - \left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2 x}{a^3} \right] \frac{1}{z_1^3} + \frac{3b^4}{a^4} \ln \frac{x_1}{x}.$
3. $\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2 x} + \frac{9b^2}{a^3} + \frac{6b^3 x}{a^4} \right] \frac{1}{z_1^3} - \frac{6b^4}{a^5} \ln \frac{x_1}{x}.$

2.122

1. $\int \frac{dx}{x^2 z_1^2} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2 x^2}{a^3} \right] \frac{1}{z_1^2} - \frac{1}{a^4} \ln \frac{x_1}{x}.$
2. $\int \frac{dx}{x^3 z_1^3} = - \left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2 x}{a^3} + \frac{4b^3 x^2}{a^4} \right] \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{x_1}{x}.$
3. $\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2 x} + \frac{55b^2}{3a^3} + \frac{25b^3 x}{a^4} + \frac{10b^4 x^2}{a^5} \right] \frac{1}{z_1^3} - \frac{10b^4}{a^6} \ln \frac{x_1}{x}.$

2.123

1. $\int \frac{dx}{x^2 z_1^2} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2 x^2}{2a^3} + \frac{b^3 x^3}{a^4} \right] \frac{1}{z_1^2} - \frac{1}{a^5} \ln \frac{x_1}{x}.$
2. $\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2 x}{3a^3} - \frac{35b^3 x^2}{2a^4} - \frac{5b^4 x^3}{a^5} \right] \frac{1}{z_1^3} + \frac{5b}{a^6} \ln \frac{x_1}{x}.$
3. $\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2 x} + \frac{125b^2}{4a^3} + \frac{65b^3 x}{a^4} + \frac{105b^4 x^2}{2a^5} + \frac{15b^5 x^3}{a^6} \right] \frac{1}{z_1^3} - \frac{15b^5}{a^7} \ln \frac{x_1}{x}.$

2.124 Forme care conțin binoame $z_2 = a + bx^2$.

1. $\int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}} \operatorname{arctg} x \sqrt{\frac{b}{a}} \quad [ab > 0] \quad (\text{vezi și 2.141 2.});$
 $= \frac{1}{2i\sqrt{ab}} \ln \frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \quad [ab < 0] \quad (\text{vezi și 2.143 2. și 2.143 3.}).$
2. $\int \frac{x dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \quad (\text{vezi și 2.145 2., 2.145 6. și 2.18}).$

Forme care conțin binoame $z_3 = a + bx^3$ Notație: $\alpha = \sqrt[3]{\frac{a}{b}}$

2.125

1. $\int \frac{x^n dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-2}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3} dx}{z_3^m}.$
2. $\int \frac{x^n dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-2m}{3a(m-1)} \int \frac{x^n dx}{z_3^{m-1}}.$

La 133 (1)

2.126

1. $\int \frac{dx}{z_3} = \frac{a}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^3}{x^3 - ax + a^3} + \sqrt{3} \operatorname{arctg} \frac{x\sqrt{3}}{2a-x} \right\};$
 $= \frac{a}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^3}{x^3 - ax + a^3} + \sqrt{3} \operatorname{arctg} \frac{2x-\alpha}{\alpha\sqrt{3}} \right\}$
 (vezi și 2.141 3. și 2.143 4.).

$$2. \int \frac{x dx}{z_3} = -\frac{1}{3ba} \left\{ \frac{1}{2} \ln \frac{(x+a)^2}{x^2-ax+a^2} - \sqrt{3} \operatorname{arctg} \frac{2x-a}{a\sqrt{3}} \right\}$$

(vezi și 2.145 3. și 2.145 7.).

$$3. \int \frac{x^3 dx}{z_3} = -\frac{1}{3b} \ln(1+x^3\alpha^{-3}) = \frac{1}{3b} \ln z_3.$$

$$4. \int \frac{x^3 dx}{z_3} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{z_3} \quad (\text{vezi } 2.126 \text{ 1.}).$$

$$5. \int \frac{x^4 dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b} \int \frac{x dx}{z_3} \quad (\text{vezi } 2.126 \text{ 2.}).$$

2.127

$$1. \int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a} \int \frac{dx}{z_3} \quad (\text{vezi } 2.126 \text{ 1.}).$$

$$2. \int \frac{x dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a} \int \frac{x dx}{z_3} \quad (\text{vezi } 2.126 \text{ 2.}).$$

$$3. \int \frac{x^2 dx}{z_3^2} = -\frac{1}{3bz_3}.$$

$$4. \int \frac{x^3 dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b} \int \frac{dx}{z_3} \quad (\text{vezi } 2.126 \text{ 1.}).$$

2.128

$$1. \int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)} \int \frac{dx}{x^{n-3}z_3^m}.$$

$$2. \int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)} \int \frac{dx}{x^{n-3}z_3^{m-1}}.$$

La 133 (2)

2.129

$$1. \int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}.$$

$$2. \int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x dx}{z_3} \quad (\text{vezi } 2.126 \text{ 2.}).$$

$$3. \int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3} \quad (\text{vezi } 2.126 \text{ 1.}).$$

2.131

$$1. \int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}.$$

$$2. \int \frac{dx}{x^2 z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2} \right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x dx}{z_3} \quad (\text{vezi } 2.126 \text{ 2.}).$$

$$3. \int \frac{dx}{x^3 z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2} \right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3} \quad (\text{vezi } 2.126 \text{ 1.}).$$

Forme care conțin binoame $z_3 = a + bx^4$

Notății: $a = \sqrt[4]{\frac{a}{b}}, \quad a' = \sqrt[4]{\frac{-a}{b}}.$

2.132

$$1. \int \frac{dx}{z_4} = \frac{a}{4a\sqrt{2}} \left\{ \ln \frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + 2 \operatorname{arctg} \frac{ax\sqrt{2}}{a^2-x^2} \right\} \quad [ab > 0]$$

(vezi și 2.141 4.);

$$= \frac{a'}{4a} \left\{ \ln \frac{x+a'}{x-a'} + 2 \operatorname{arctg} \frac{x}{a'} \right\} \quad [ab < 0] \quad (\text{vezi și } 2.143 \text{ 5.}).$$

2. $\int \frac{x dx}{z_4} = \frac{1}{2\sqrt{ab}} \operatorname{arctg} x^2 \sqrt{\frac{b}{a}} \quad [ab > 0] \quad (\text{vezi și 2.145 4.});$
 $= \frac{1}{4i\sqrt{ab}} \ln \frac{a+x^2i\sqrt{ab}}{a-x^2i\sqrt{ab}} \quad [ab < 0] \quad (\text{vezi și 2.145 8.}).$
3. $\int \frac{x^3 dx}{z_4} = \frac{1}{4ba\sqrt{2}} \left\{ \ln \frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2} + 2 \operatorname{arctg} \frac{ax\sqrt{2}}{a^2-x^2} \right\} \quad [ab > 0];$
 $= -\frac{1}{4ba'} \left\{ \ln \frac{x+a'}{x-a'} - 2 \operatorname{arctg} \frac{x}{a'} \right\} \quad [ab < 0].$
4. $\int \frac{x^3 dx}{z_4} = \frac{1}{4b} \ln z_4.$

2.133

1. $\int \frac{x^n dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{z_4^{m-1}}. \quad \text{La 134 (1)}$
2. $\int \frac{x^n dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4} dx}{z_4^m}.$

2.134

1. $\int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a} \int \frac{dx}{z_4} \quad (\text{vezi 2.132 1.}).$
2. $\int \frac{x dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a} \int \frac{x dx}{z_4} \quad (\text{vezi 2.132 2.}).$
3. $\int \frac{x^3 dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a} \int \frac{x^3 dx}{z_4} \quad (\text{vezi 2.132 3.}).$
4. $\int \frac{x^5 dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}.$

$$2.135 \quad \int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a} \int \frac{dx}{x^{n-4}z_4^m}.$$

Pentru $n=1$,

$$\int \frac{dx}{x^n z_4^m} = \frac{1}{a} \int \frac{dx}{x^n z_4^{m-1}} - \frac{b}{a} \int \frac{dx}{x^{n-4}z_4^m}.$$

2.136

1. $\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a} \ln \frac{x^4}{z_4}.$
2. $\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x^3 dx}{z_4} \quad (\text{vezi 132 3.}).$

2.14 Forme care conțin binoame $1 \pm x^n$

2.141

1. $\int \frac{dx}{1+x} = \ln(1+x).$
2. $\int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arcctg} x \quad (\text{vezi și 2.124 1.}).$
3. $\int \frac{dx}{1+x^3} = \frac{1}{3} \ln \frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{2-x} \quad (\text{vezi și 2.126 1.}).$
4. $\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} \quad (\text{vezi și 2.132 1.}).$

$$2.142 \quad \int \frac{dx}{1+x^n} = -\frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} Q_k \sin \frac{2k+1}{n} \pi$$

[n par și pozitiv] T (43)m

$$= \frac{1}{n} \ln(1+x) - \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi$$

[n impar și pozitiv] T (45)

$$P_k = \frac{1}{2} \ln \left(x^2 - 2x \cos \frac{2k+1}{n} \pi + 1 \right),$$

$$Q_k = \operatorname{arctg} \frac{x \sin \frac{2k+1}{n} \pi}{1 - x \cos \frac{2k+1}{n} \pi} = \operatorname{arctg} \frac{x - \cos \frac{2k+1}{n} \pi}{\sin \frac{2k+1}{n} \pi}$$

2.143

$$1. \quad \int \frac{dx}{1-x} = -\ln(1-x).$$

$$2. \quad \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{Arth} x \quad [-1 < x < 1] \quad (\text{vezi și 2.141 1.}).$$

$$3. \quad \int \frac{dx}{x^2-1} = -\operatorname{Arcth} x \quad [x > 1, x < -1].$$

$$4. \quad \int \frac{dx}{1-x^3} = \frac{1}{3} \ln \frac{\sqrt{1+x+x^3}}{1-x} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{2+x} \quad (\text{vezi și 2.126 1.}).$$

$$5. \quad \int \frac{dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \operatorname{arctg} x = \frac{1}{2} (\operatorname{Arth} x + \operatorname{arctg} x)$$

(vezi și 2.132 1.).

2.144

$$1. \quad \int \frac{dx}{1-x^n} = \frac{1}{n} \ln \frac{1+x}{1-x} - \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} P_k \cos \frac{2k}{n} \pi + \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} Q_k \sin \frac{2k}{n} \pi$$

[n par și pozitiv]

$$P_k = \frac{1}{2} \ln \left(x^2 - 2x \cos \frac{2k}{n} \pi + 1 \right), \quad Q_k = \operatorname{arctg} \frac{x - \cos \frac{2k}{n} \pi}{\sin \frac{2k}{n} \pi} \quad T (47)$$

$$2. \quad \int \frac{dx}{1-x^n} = -\frac{1}{n} \ln(1-x) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi +$$

$$+ \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi \quad [n \text{ impar și pozitiv}] \quad T (49)$$

$$P_k = \frac{1}{2} \ln \left(x^2 + 2x \cos \frac{2k+1}{n} \pi + 1 \right), \quad Q_k = \operatorname{arctg} \frac{x + \cos \frac{2k+1}{n} \pi}{\sin \frac{2k+1}{n} \pi}$$

2.145

1. $\int \frac{x dx}{1+x} = x - \ln(1+x).$
2. $\int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2).$
3. $\int \frac{x dx}{1+x^3} = -\frac{1}{6} \ln \frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} \quad (\text{vezi și 2.126 2.}).$
4. $\int \frac{x dx}{1+x^4} = \frac{1}{2} \operatorname{arctg} x^2.$
5. $\int \frac{x dx}{1-x} = -\ln(1-x) - x.$
6. $\int \frac{x dx}{1-x^2} = -\frac{1}{2} \ln(1-x^2).$
7. $\int \frac{x dx}{1-x^3} = -\frac{1}{6} \ln \frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \quad (\text{vezi și 2.126 2.}).$
8. $\int \frac{x dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x^2}{1-x^2} \quad (\text{vezi și 2.132 2.}).$

2.146 Pentru m și n naturali,

1.
$$\int \frac{x^{m-1} dx}{1+x^{2n}} = -\frac{1}{2n} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n} \pi + x^2 \right\} +$$

$$+ \frac{1}{n} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n} \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n} \pi}{\sin \frac{2k-1}{2n} \pi} \quad [m < 2n]. \quad T(44)$$
2.
$$\int \frac{x^{m-1} dx}{1+x^{2n+1}} = (-1)^{m+1} \frac{\ln(1+x)}{2n+1} -$$

$$- \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right\} +$$

$$+ \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \quad [m \leq 2n]. \quad T(46)$$
3.
$$\int \frac{x^{m-1} dx}{1-x^{2n}} = \frac{1}{2n} \{ (-1)^{m+1} \ln(1+x) - \ln(1-x) \} -$$

$$- \frac{1}{2n} \sum_{k=1}^{n-1} \cos \frac{km\pi}{n} \ln \left(1 - 2x \cos \frac{k\pi}{n} + x^2 \right) +$$

$$+ \frac{1}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \operatorname{arctg} \frac{x - \cos \frac{k\pi}{n}}{\sin \frac{k\pi}{n}} \quad [m < 2n]. \quad T(48)$$
4.
$$\int \frac{x^{m-1} dx}{1-x^{2n+1}} = -\frac{1}{2n+1} \ln(1-x) +$$

$$+ (-1)^{m+1} \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left(1 + 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right) +$$

$$+ (-1)^{m+1} \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \operatorname{arctg} \frac{x + \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \quad [m \leq 2n]. \quad T(50)$$

2.147

1. $\int \frac{x^m dx}{1-x^{2n}} = \frac{1}{2} \int \frac{x^m dx}{1-x^n} + \frac{1}{2} \int \frac{x^m dx}{1+x^n}.$
2. $\int \frac{x^m dx}{(1+x^2)^n} = -\frac{1}{2n-m-1} \cdot \frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1+x^2)^n}.$ La 139 (28)
3. $\int \frac{x^m}{1+x^2} dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2} dx.$
4. $\int \frac{x^m dx}{(1-x^2)^n} = \frac{1}{2n-m-1} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1-x^2)^n};$
 $= \frac{1}{2n-2} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-2} \int \frac{x^{m-2} dx}{(1-x^2)^{n-1}}.$ La 139 (33)
5. $\int \frac{x^m dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2} dx}{1-x^2}.$

2.148

1. $\int \frac{dx}{x^m (1+x^2)^n} = -\frac{1}{m-1} \frac{1}{x^{m-1} (1+x^2)^{n-1}} - \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1+x^2)^n}.$ La 139 (29)

Pentru $m=1$,

$$\int \frac{dx}{x(1+x^2)^n} = \frac{1}{2n-2} \frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x(1+x^2)^{n-1}}.$$
 La 139 (31)

Pentru $m=1$ și $n=1$,

$$\int \frac{dx}{x(1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}}.$$

2. $\int \frac{dx}{x^m (1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2} (1+x^2)}.$
3. $\int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}}.$ La 139 (30)
4. $\int \frac{dx}{(1+x^2)^n} = x \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k) (1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1} (n-1)!} \arctg x.$ T(91)

2.149

1. $\int \frac{dx}{x^m (1-x^2)^n} = -\frac{1}{(m-1)x^{m-1} (1-x^2)^{n-1}} + \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1-x^2)^n}.$ La 139 (34)

Pentru $m=1$,

$$\int \frac{dx}{x(1-x^2)^n} = \frac{1}{2(n-1)(1-x^2)^{n-1}} + \int \frac{dx}{x(1-x^2)^{n-1}}.$$
 La 139 (36)

Pentru $m=1$ și $n=1$,

$$\int \frac{dx}{x(1-x^2)} = \ln(x/\sqrt{1-x^2}).$$

2. $\int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2} \frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1-x^2)^{n-1}}.$ La 139 (35)
3. $\int \frac{dx}{(1-x^2)^n} = x \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k) (1-x^2)^{n-k}} +$
 $+ \frac{(2n-3)!!}{2^n \cdot (n-1)!} \ln \frac{1+x}{1-x}.$ T(91)

2.15 Forme care conțin perechi de binoame $a+bx$ și $\alpha+\beta x$

Notații: $z=a+bx$; $t=\alpha+\beta x$; $\Delta=a\beta-b^2$.

$$2.151 \quad \int z^n t^m dx = \frac{z^{n+1} t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b} \int z^n t^{m-1} dx.$$

2.152

$$1. \quad \int \frac{z}{t} dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2} \ln t.$$

$$2. \quad \int \frac{t}{z} dx = \frac{\beta x}{b} - \frac{\Delta}{b^2} \ln z.$$

$$2.153 \quad \begin{aligned} \int \frac{t^m dx}{z^n} &= \frac{1}{(m-n+1)b} \frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b} \int \frac{t^{m-1} dx}{z^n}; \\ &= \frac{1}{(n-1)\Delta} \frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta} \int \frac{t^m dx}{z^{n-1}}; \\ &= -\frac{1}{(n-1)b} \frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b} \int \frac{t^{m-1} dx}{z^{n-1}}. \end{aligned}$$

$$2.154 \quad \int \frac{dx}{zt} = \frac{1}{\Delta} \ln \frac{t}{z}.$$

$$2.155 \quad \begin{aligned} \int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta} \int \frac{dx}{t^{m-1} z^n}; \\ &= \frac{1}{(n-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta} \int \frac{dx}{t^m z^{n-1}}. \end{aligned}$$

$$2.156 \quad \int \frac{x dx}{zt} = \frac{1}{\Delta} \left(\frac{a}{b} \ln z - \frac{\alpha}{\beta} \ln t \right).$$

2. 2.16 Forme care conțin trinoame $a+bx^k+cx^{2k}$

2.160 Formule de reducere pentru $R_k=a+bx^k+cx^{2k}$

$$1. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma} \int x^{m+k-1} R_k^n dx - \frac{(m+2k+2kn)c}{ma} \int x^{m+2k-1} R_k^n dx.$$

$$2. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m} \int x^{m+k-1} R_k^{n-1} dx - \frac{2ekn}{m} \int x^{m+2k-1} R_k^{n-1} dx.$$

$$3. \quad \begin{aligned} \int x^{m-1} R_k^n dx &= \frac{x^{m-2k} R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c} \int x^{m-2k-1} R_k^n dx - \\ &\quad - \frac{(m-k+kn)b}{(m+2kn)c} \int x^{m-k-1} R_k^n dx; \\ &= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn} \int x^{m-1} R_k^{n-1} dx + \frac{bkn}{m+2kn} \int x^{m+k-1} R_k^{n-1} dx. \end{aligned}$$

2.161 Forme care conțin trinomialul $R_2=a+bx^2+cx^4$

Notații: $f=\frac{b}{2}-\frac{1}{2}\sqrt{b^2-4ac}$, $g=\frac{b}{2}+\frac{1}{2}\sqrt{b^2-4ac}$, $h=\sqrt{b^2-4ac}$,

$$q=\sqrt[4]{\frac{a}{c}}, \quad l=2a(n-1)(b^2-4ac), \quad \cos \alpha = -\frac{b}{2\sqrt{ac}}.$$

$$1. \quad \int \frac{dx}{R_2} = \frac{c}{h} \left\{ \int \frac{dx}{cx^2+f} - \int \frac{dx}{cx^2+g} \right\} \quad [h^2 > 0]; \quad \text{La 146 (5)}$$

$$= \frac{1}{4cq^2 \sin \alpha} \left\{ \sin \frac{\alpha}{2} \ln \frac{x^2 + 2qx \cos \frac{\alpha}{2} + q^2}{x^2 - 2qx \cos \frac{\alpha}{2} + q^2} + 2 \cos \frac{\alpha}{2} \operatorname{arctg} \frac{x^2 - q^2}{2qx \sin \frac{\alpha}{2}} \right\}$$

$[h^2 < 0]$. La 146 (8) m

$$2. \quad \int \frac{x dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2 + f}{cx^2 + g} \quad [h^2 > 0]; \quad \text{La 146 (6)}$$

$$= \frac{1}{2cq^2 \sin \alpha} \operatorname{arctg} \frac{x^2 - q^2 \cos \alpha}{q^2 \sin \alpha} \quad [h^2 < 0]. \quad \text{La 146 (9) m}$$

$$3. \quad \int \frac{x^3 dx}{R_2} = \frac{g}{h} \int \frac{dx}{cx^2 + g} - \frac{f}{h} \int \frac{dx}{cx^2 + f} \quad [h^2 > 0]. \quad \text{La 146 (7)}$$

$$4. \quad \int \frac{dx}{R_2^2} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2} + \frac{b^2 - 6ac}{l} \int \frac{dx}{R_2} + \frac{bc}{l} \int \frac{x^2 dx}{R_2}.$$

$$5. \quad \int \frac{dx}{R_2^n} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2^{n-1}} + \frac{(4n-7)bc}{l} \int \frac{x^2 dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l} \int \frac{dx}{R_2^{n-1}} \quad [n > 1]. \quad \text{La 146 (10)}$$

$$6. \quad \int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a} \int \frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)b}{(m-1)a} \int \frac{dx}{x^{m-4}R_2^n}. \quad \text{La 147 (12) m}$$

2.17 Forme care conțin trinomiul de gradul doi $a+bx+cx^2$ și puteri ale lui x

Notății: $R=a+bx+cx^2$; $\Delta=4ac-b^2$.

2.171

$$1. \quad \int x^{m+1} R^n dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)} \int x^{m-1} R^n dx - \frac{b(m+n+1)}{c(m+2n+2)} \int x^m R^n dx. \quad \text{T (97)}$$

$$2. \quad \int \frac{R^n dx}{x^{m+1}} = -\frac{R^{n+1}}{amx^m} + \frac{b(n-m+1)}{am} \int \frac{R^n dx}{x^m} + \frac{c(2n-m+2)}{am} \int \frac{R^n dx}{x^{m-1}}. \quad \text{La 142 (3)}$$

$$3. \quad \int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta} \int \frac{dx}{R^n}. \quad \text{T (94)}$$

$$4. \quad \int \frac{dx}{R^{n+1}} = (2cx+b) \sum_{k=0}^{n-1} \frac{2^k (2n-1)(2n-3) \dots (2n-2k+1) c^k}{n(n-1) \dots (n-k) \Delta^{k+1} R^{n-k}} + 2^n \frac{(2n-1)!! c^n}{n! \Delta^n} \int \frac{dx}{R}. \quad \text{T (96)}$$

$$2.172 \quad \int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}} \ln \frac{b+2cx-\sqrt{-\Delta}}{b+2cx+\sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} \operatorname{Arth} \frac{b+2cx}{\sqrt{-\Delta}} \quad [\Delta < 0];$$

$$= \frac{-2}{b+2cx} \quad [\Delta = 0];$$

$$= \frac{2}{\sqrt{\Delta}} \operatorname{arctg} \frac{b+2cx}{\sqrt{\Delta}} \quad [\Delta > 0].$$

2.173

$$1. \quad \int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$2. \quad \int \frac{dx}{R^3} = \frac{b+2cx}{\Delta} \left\{ \frac{1}{2R^2} + \frac{3c}{\Delta R} \right\} + \frac{6c^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

2.174

$$1. \int \frac{x^m dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c} \int \frac{x^{m-1} dx}{R^n} + \\ + \frac{(m-1)a}{(2n-m-1)c} \int \frac{x^{m-2} dx}{R^n}.$$

Pentru $m=2n-1$, această formulă este inaplicabilă; în locul ei se poate folosi

$$2. \int \frac{x^{2n-1} dx}{R^n} = \frac{1}{c} \int \frac{x^{2n-3} dx}{R^{n-1}} - \frac{a}{c} \int \frac{x^{2n-3} dx}{R^n} - \frac{b}{c} \int \frac{x^{2n-2} dx}{R^n}.$$

2.175

$$1. \int \frac{x dx}{R} = \frac{1}{2c} \ln R - \frac{b}{2c} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$2. \int \frac{x dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$3. \int \frac{x dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$4. \int \frac{x^2 dx}{R} = \frac{x}{c} - \frac{b}{2c^2} \ln R + \frac{b^2-2ac}{2c^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$5. \int \frac{x^2 dx}{R^2} = \frac{ab+(b^2-2ac)x}{c\Delta R} + \frac{2a}{\Delta} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$6. \int \frac{x^2 dx}{R^3} = \frac{ab+(b^2-2ac)x}{2c\Delta R^2} + \frac{(2ac+b^2)(b+2cx)}{2c\Delta^2 R} + \frac{2ac+b^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$7. \int \frac{x^3 dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^2} \ln R - \frac{b(b^2-3ac)}{2c^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$8. \int \frac{x^3 dx}{R^2} = \frac{1}{2c^2} \ln R + \frac{a(2ac-b^2)+b(3ac-b^2)x}{c^2\Delta R} - \frac{b(6ac-b^2)}{2c^2\Delta} \int \frac{dx}{R} \\ (\text{vezi 2.172}).$$

$$9. \int \frac{x^3 dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right) \frac{1}{2R^2} - \frac{3ab}{2c\Delta} \int \frac{dx}{R^2} \quad (\text{vezi 2.173 1.}).$$

$$2.176. \int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)} \int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)} \int \frac{dx}{x^{m-2}R^n},$$

2.177

$$1. \int \frac{dx}{xR} = \frac{1}{2a} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$2. \int \frac{dx}{xR^2} = \frac{1}{2a^2} \ln \frac{x^2}{R} + \frac{1}{2aR} \left\{ 1 - \frac{b(b+2cx)}{\Delta} \right\} - \frac{b}{2a^2} \left(1 + \frac{2ac}{\Delta} \right) \int \frac{dx}{R} \\ (\text{vezi 2.172}).$$

$$3. \int \frac{dx}{xR^3} = \frac{1}{4aR^3} + \frac{1}{2a^2R} + \frac{1}{2a^3} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R^3} - \frac{b}{2a^2} \int \frac{dx}{R^2} - \frac{b}{2a^2} \int \frac{dx}{R} \\ (\text{vezi 2.172, 2.173}).$$

$$4. \int \frac{dx}{x^2 R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2-2ac}{2a^2} \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

$$5. \int \frac{dx}{x^2 R^2} = -\frac{b}{a^2} \ln \frac{x^2}{R} - \frac{a+bx}{a^2 xR} + \frac{(b^2-3ac)(b+2cx)}{a^2 \Delta R} - \\ - \frac{1}{\Delta} \left(\frac{b^4}{a^3} - \frac{6b^2c}{a^3} + \frac{6c^2}{a} \right) \int \frac{dx}{R} \quad (\text{vezi 2.172}).$$

6. $\int \frac{dx}{xR^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3}$ (vezi 2.173 și 2.177 3.).
7. $\int \frac{dx}{x^3R} = -\frac{ac-b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b(3ac-b^2)}{2a^3} \int \frac{dx}{R}$ (vezi 2.172).
8. $\int \frac{dx}{x^3R^3} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x}\right) \frac{1}{R} + \left(\frac{3b^2}{a^3} - \frac{2c}{a}\right) \int \frac{dx}{xR^3} + \frac{9bc}{2a^3} \int \frac{dx}{R^3}$
(vezi 2.173 1. și 2.177 2.).
9. $\int \frac{dx}{x^3R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2x}\right) \frac{1}{R^2} + \left(\frac{6b^2}{a^3} - \frac{3c}{a}\right) \int \frac{dx}{xR^3} + \frac{10bc}{a^3} \int \frac{dx}{R^3}$
(vezi 2.173 2., 2.177 3.).

2.18 Forme care conțin trinomialul de gradul doi $a+bx+cx^2$ și binomialul $\alpha+\beta x$

Notății: $R = a + bx + cx^2$; $z = \alpha + \beta x$; $A = a\beta^2 - ab\beta + ca^2$; $B = b\beta - 2ca$; $\Delta = 4ac - b^2$.

1. $\int z^m R^n dx = \frac{\beta z^{m-1} R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c} \int z^{m-1} R^n dx.$
2. $\int \frac{R^n dx}{z^m} = \frac{1}{(m-2n-1)\beta} \frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^m} -$
 $-\frac{nB}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}};$ La 148 (4) m
 $= \frac{-\beta}{(m-1)A} \frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A} \int \frac{R^n dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A} \int \frac{R^n dx}{z^{m-2}};$ La 148 (5)
 $= -\frac{1}{(m-1)\beta} \frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-2}}$ La 148 (6)
3. $\int \frac{z^m dx}{R^n} = \frac{\beta}{(m-2n+1)c} \frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c} \int \frac{z^{m-1} dx}{R^n} -$
 $-\frac{(m-1)A}{(m-2n+1)c} \int \frac{z^{m-2} dx}{R^n};$ La 147 (1)
 $= \frac{B+2cx}{(n-1)\Delta} \frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta} \int \frac{z^m dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta} \int \frac{z^{m-1} dx}{R^{n-1}}$ La 148 (3)
4. $\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{(m+n-2)B}{(m-1)A} \int \frac{dx}{z^{m-1} R^n} -$
 $-\frac{(m+2n-3)c}{(m-1)A} \int \frac{dx}{z^{m-2} R^n};$ La 148 (7)
 $= \frac{\beta}{2(n-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{B}{2A} \int \frac{dx}{z^{m-1} R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A} \int \frac{dx}{z^m R^{n-1}}.$ La 148 (8)

Pentru $m=1$ și $n=1$,

$$\int \frac{dx}{zR} = \frac{\beta}{2A} \ln \frac{z^2}{R} - \frac{B}{2A} \int \frac{dx}{R}.$$

Pentru $A=0$,

$$\int \frac{dx}{z^m R^n} = \frac{\beta}{(m+n-1)B} \frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B} \int \frac{dx}{z^{m-1} R^n}. \quad \text{La 148 (9)}$$

2.2 FUNCȚII ALGEBRICE

2.20 Introducere

2.201 Integralele $\int R\left(x, \left(\frac{ax+\beta}{\gamma x+\delta}\right)^r, \left(\frac{ax+\beta}{\gamma x+\delta}\right)^s, \dots\right) dx$, unde $r, s, \dots$ sînt numere raționale, se reduc la integrale de funcții raționale prin substituția

$$\frac{ax+\beta}{\gamma x+\delta} = t^m, \quad \text{F II 57}$$

unde m este numitorul comun al fracțiilor $r, s, \dots$

2.202 Integralele de forma $\int x^m (a+bx^n)^p dx$ (integralele binoame), unde m, n și p sînt numere raționale, se exprimă prin funcții elementare numai în următoarele cazuri:

a) cînd p este un număr întreg; atunci această integrală are forma unei sume de integrale de tipul celei din 2.201;

b) cînd $\frac{m+1}{n}$ este un număr întreg; prin substituția $x^n = z$, această integrală se aduce la forma $\frac{1}{n} \int (a+bz)^p z^{\frac{m+1}{n}-1} dz$, tratată în 2.201;

c) cînd $\frac{m+1}{n} + p$ este un număr întreg; cu ajutorul aceleiași substituții $x^n = z$, integrala dată se reduce la o integrală de forma $\frac{1}{n} \int \left(\frac{a+bz}{z}\right)^p z^{\frac{m+1}{n}+p-1} dz$, tratată în 2.201.

Formule de reducere pentru integralele binoame, vezi 2.110

2.21 Forme care conțin binomul $a+bx^k$ și $\sqrt{x}$

Notație: $z_1 = a+bx$

$$\begin{aligned} 2.211 \quad \int \frac{dx}{z_1 \sqrt{x}} &= \frac{2}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{bx}{a}} \quad [ab > 0]; \\ &= \frac{1}{i \sqrt{ab}} \ln \frac{a-bx+2i\sqrt{xab}}{z_1} \quad [ab < 0]. \end{aligned}$$

$$2.212 \quad \int \frac{x^m \sqrt{x}}{z_1} dx = 2 \sqrt{x} \sum_{k=0}^m \frac{(-1)^k a^k x^{m-k}}{(m-2k+1) b^{k+1}} + (-1)^{m+1} \frac{a^{m+1}}{b^{m+1}} \int \frac{dx}{z_1 \sqrt{x}}$$

(vezi 2.211).

2.213

1. $\int \frac{\sqrt{x} dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi } 2.211).$
2. $\int \frac{x \sqrt{x} dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right) 2\sqrt{x} + \frac{a^2}{b^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi } 2.211).$
3. $\int \frac{x^2 \sqrt{x} dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^2}\right) 2\sqrt{x} - \frac{a^3}{b^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi } 2.211).$
4. $\int \frac{dx}{z_1^2 \sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi } 2.211).$

$$5. \int \frac{\sqrt{x} dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi 2.211}).$$

$$6. \int \frac{x \sqrt{x} dx}{z_1^2} = \frac{2x \sqrt{x}}{bz_1} - \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{vezi 2.213 5}).$$

$$7. \int \frac{x^2 \sqrt{x} dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2} \right) \frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{vezi 2.213 5}).$$

$$8. \int \frac{dx}{z_1^2 \sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2 z_1} \right) \sqrt{x} + \frac{3}{8a^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi 2.211}).$$

$$9. \int \frac{\sqrt{x} dx}{z_1^2} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1} \right) \sqrt{x} + \frac{1}{8ab} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{vezi 2.211}).$$

$$10. \int \frac{x \sqrt{x} dx}{z_1^2} = -\frac{2x \sqrt{x}}{bz_1^2} + \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{vezi 2.213 9}).$$

$$11. \int \frac{x^2 \sqrt{x} dx}{z_1^2} = \left(\frac{x^2}{b} + \frac{5ax}{b^2} \right) \frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{vezi 2.213 9}).$$

$$\text{Notății: } z_2 = a + bx^2, \quad \alpha = \sqrt[4]{\frac{a}{b}} \quad \alpha' = \sqrt[4]{-\frac{a}{b}}.$$

$$2.214 \quad \int \frac{dx}{z_2 \sqrt{x}} = \frac{1}{ba^2 \sqrt{2}} \left[\ln \frac{x + \alpha \sqrt{2x + \alpha^2}}{\sqrt{x_2}} + \operatorname{arctg} \frac{\alpha \sqrt{2x}}{\alpha^2 - x} \right] \quad \left[\frac{a}{b} > 0 \right];$$

$$= \frac{1}{2ba'^2} \left(\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2 \operatorname{arctg} \frac{\sqrt{x}}{\alpha'} \right) \quad \left[\frac{a}{b} < 0 \right].$$

$$2.215 \quad \int \frac{\sqrt{x} dx}{z_2} = \frac{1}{ba \sqrt{2}} \left[-\ln \frac{x + \alpha \sqrt{2x + \alpha^2}}{\sqrt{x_2}} + \operatorname{arctg} \frac{\alpha \sqrt{2x}}{\alpha^2 - x} \right] \quad \left[\frac{a}{b} > 0 \right];$$

$$= \frac{1}{2ba'} \left[\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} + 2 \operatorname{arctg} \frac{\sqrt{x}}{\alpha'} \right] \quad \left[\frac{a}{b} < 0 \right].$$

2.216

$$1. \int \frac{x \sqrt{x} dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_2 \sqrt{x}} \quad (\text{vezi 2.214}).$$

$$2. \int \frac{x^2 \sqrt{x} dx}{z_2} = \frac{2x \sqrt{x}}{3b} - \frac{a}{b} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{vezi 2.215}).$$

$$3. \int \frac{dx}{z_2^2 \sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a} \int \frac{dx}{z_2 \sqrt{x}} \quad (\text{vezi 2.214}).$$

$$4. \int \frac{\sqrt{x} dx}{z_2^2} = \frac{x \sqrt{x}}{2az_2} + \frac{1}{4a} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{vezi 2.215}).$$

$$5. \int \frac{x \sqrt{x} dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b} \int \frac{dx}{z_2 \sqrt{x}} \quad (\text{vezi 2.214}).$$

$$6. \int \frac{x^2 \sqrt{x} dx}{z_2^2} = -\frac{x \sqrt{x}}{2bz_2} + \frac{3}{4b} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{vezi 2.215}).$$

$$7. \int \frac{dx}{z_2^2 \sqrt{x}} = \left(\frac{1}{4az_2^2} + \frac{7}{16a^2 z_2} \right) \sqrt{x} + \frac{21}{32a^2} \int \frac{dx}{z_2 \sqrt{x}} \quad (\text{vezi 2.214}).$$

$$8. \int \frac{\sqrt{x} dx}{z_2^2} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2 z_2} \right) x \sqrt{x} + \frac{5}{32a^2} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{vezi 2.215}).$$

$$9. \int \frac{x \sqrt{x} dx}{z_2^2} = \frac{(bx^2 - 3a) \sqrt{x}}{16abz_2^2} + \frac{3}{32ab} \int \frac{dx}{z_2 \sqrt{x}} \quad (\text{vezi 2.214}).$$

$$10. \int \frac{x^2 \sqrt{x} dx}{z_2^2} = -\frac{2x \sqrt{x}}{5bz_2^2} + \frac{3a}{5b} \int \frac{\sqrt{x} dx}{z_2^2} \quad (\text{vezi 2.216 8}).$$

2.22-2.23 Forme care conțin $\sqrt[n]{(a+bx)^k}$ Notație: $z=a+bx$

$$2.220 \quad \int x^n \sqrt[l]{z^{lm+f}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{ln-lk+l(m+1)+f} \right\} \frac{l \sqrt[l]{z^{l(m+1)+f}}}{b^{k+1}}.$$

Rădăcina pătrată

$$2.221 \quad \int x^n \sqrt{z^{2m+1}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{2n-2k+2m+1} \right\} \frac{2 \sqrt{z^{2m+1}}}{b^{n+1}}.$$

2.222

$$1. \quad \int \frac{dx}{\sqrt{z}} = \frac{2}{b} \sqrt{z}.$$

$$2. \quad \int \frac{x dx}{\sqrt{z}} = \left(\frac{1}{3} z - a \right) \frac{2 \sqrt{z}}{b^3}.$$

$$3. \quad \int \frac{x^2 dx}{\sqrt{z}} = \left(\frac{1}{5} z^2 - \frac{2}{3} az + a^2 \right) \frac{2 \sqrt{z}}{b^5}.$$

2.223

$$1 \quad \int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b \sqrt{z}}.$$

$$2 \quad \int \frac{x dx}{\sqrt{z^3}} = (z+a) \frac{2}{b^3 \sqrt{z}}.$$

$$3. \quad \int \frac{x^2 dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2 \right) \frac{2}{b^5 \sqrt{z}}.$$

2.224

$$1. \quad \int \frac{z^m dx}{x^n \sqrt{z}} = -\frac{z^m \sqrt{z}}{(n-1) a x^{n-1}} + \frac{2m-2n+3}{2(n-1)} \frac{b}{a} \int \frac{z^m dx}{x^{n-1} \sqrt{z}}.$$

$$2. \quad \int \frac{z^m dx}{x^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1) a x^{n-1}} + \sum_{k=1}^{n-2} \frac{(2m-2n+3)(2m-2n+5) \dots (2m-2n+2k+1)}{2^k (n-1)(n-2) \dots (n-k-1) x^{n-k-1}} \frac{b^k}{a^{k+1}} + \frac{(2m-2n+3)(2m-2n+5) \dots (2m-3)(2m-1)}{2^{n-1} (n-1)! x} \frac{b^{n-1}}{a^{n-1}} \int \frac{z^m dx}{x \sqrt{z}} \right\}.$$

Pentru $n=1$,

$$\int \frac{z^m}{x \sqrt{z}} dx = \frac{2z^m}{(2m-1) \sqrt{z}} + a \int \frac{z^{m-1}}{x \sqrt{z}} dx.$$

$$3. \quad \int \frac{z^m}{x \sqrt{z}} dx = \sum_{k=1}^m \frac{2a^{m-k} z^k}{(2k-1) \sqrt{z}} + a^m \int \frac{dx}{x \sqrt{z}}.$$

$$4. \quad \int \frac{dx}{x \sqrt{z}} = \frac{1}{\sqrt{a}} \ln \frac{\sqrt{z}-\sqrt{a}}{\sqrt{z}+\sqrt{a}} \quad [a > 0];$$

$$= \frac{2}{\sqrt{-a}} \operatorname{arctg} \frac{\sqrt{z}}{\sqrt{-a}} \quad [a < 0].$$

2.225

$$1. \quad \int \frac{\sqrt{z} dx}{x} = 2 \sqrt{z} + a \int \frac{dx}{x \sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$2. \int \frac{\sqrt{z} dx}{x^3} = -\frac{\sqrt{z}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$3. \int \frac{\sqrt{z} dx}{x^5} = -\frac{\sqrt{z^3}}{2ax^3} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

2.226

$$1. \int \frac{\sqrt{z^3} dx}{x} = \left(\frac{z}{3} + a\right) 2\sqrt{z} + a^3 \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$2. \int \frac{\sqrt{z^3} dx}{x^3} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{vezi 2.226 1.}).$$

$$3. \int \frac{\sqrt{z^3} dx}{x^5} = -\left(\frac{1}{2ax^3} + \frac{b}{4a^2x}\right) \sqrt{z^5} + \frac{3b^2}{8a^2} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{vezi 2.226 1.}).$$

$$2.227 \quad \int \frac{dx}{xz^m \sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1) a^{m-k} z^k \sqrt{z}} + \frac{1}{a^m} \int \frac{dx}{x\sqrt{z}}. \quad (\text{vezi 2.224 4.}).$$

2.228

$$1. \int \frac{dx}{x^3 \sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$2. \int \frac{dx}{x^5 \sqrt{z}} = \left(-\frac{1}{2ax^3} + \frac{3b}{4a^2x}\right) \sqrt{z} + \frac{3b^2}{8a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

2.229

$$1. \int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}} + \frac{1}{a} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$2. \int \frac{dx}{x^3 \sqrt{z^3}} = \left(-\frac{1}{ax} - \frac{3b}{a^2}\right) \frac{1}{\sqrt{z}} - \frac{3b}{2a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

$$3. \int \frac{dx}{x^5 \sqrt{z^3}} = \left(-\frac{1}{2ax^3} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3}\right) \frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3} \int \frac{dx}{x\sqrt{z}} \quad (\text{vezi 2.224 4.}).$$

Rădăcina cubică

2.231

$$1. \int \sqrt[3]{z^{3m+1}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n-3k+3(m+1)+1} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}.$$

$$2. \int \frac{x^n dx}{\sqrt[3]{z^{3m+2}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n-3k-3(m-1)-2} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+2}}}.$$

$$3. \int \sqrt[3]{z^{3m+2}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n-3k+3(m+1)+2} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}.$$

$$4. \int \frac{x^n dx}{\sqrt[3]{z^{3m+1}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n-3k-3(m-1)-1} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+1}}}.$$

$$5. \int \frac{z^n dx}{x^m \sqrt[3]{z^3}} = -\frac{z^{\frac{n+1}{3}}}{(m-1) a x^{m-1}} + \frac{3n-3m+4}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z^3}}.$$

Pentru $m = 1$,

$$\int \frac{z^n dx}{x \sqrt[3]{z^3}} = \frac{3z^n}{(3n-2) \sqrt[3]{z^3}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z^3}}.$$

$$6. \int \frac{dx}{x^{2n} \sqrt[n]{z}} = \frac{3 \sqrt[n]{z}}{(3n-1) a z^n} + \frac{1}{a} \int \frac{\sqrt[n]{z} dx}{x z^n}.$$

$$2.232 \int \frac{dx}{x \sqrt[n]{z}} = \frac{1}{\sqrt[n]{a^3}} \left\{ \frac{3}{2} \ln \frac{\sqrt[n]{z} - \sqrt[n]{a}}{\sqrt[n]{z}} - \sqrt{3} \operatorname{arctg} \frac{\sqrt{3} \sqrt[n]{z}}{\sqrt[n]{z} + 2 \sqrt[n]{a}} \right\}.$$

2.233

$$1. \int \frac{\sqrt[n]{z} dx}{x} = 3 \sqrt[n]{z} + a \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.232).$$

$$2. \int \frac{\sqrt[n]{z} dx}{x^2} = -\frac{x \sqrt[n]{z}}{a x} + \frac{b}{a} \sqrt[n]{z} + \frac{b}{3} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.232).$$

$$3. \int \frac{\sqrt[n]{z} dx}{x^3} = \left(-\frac{1}{2ax^3} + \frac{b}{3a^2x} \right) z \sqrt[n]{z} - \frac{b^3}{3a^3} \sqrt[n]{z} - \frac{b^3}{9a} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.232).$$

$$4. \int \frac{dx}{x^3 \sqrt[n]{z^3}} = -\frac{\sqrt[n]{z}}{a x} - \frac{2b}{3a} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.232).$$

$$5. \int \frac{dx}{x^3 \sqrt[n]{z^3}} = \left[-\frac{1}{2ax^3} + \frac{5b}{6a^2x} \right] \sqrt[n]{z} + \frac{5b^3}{9a^3} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.232).$$

2.234

$$1. \int \frac{z^n dx}{x^m \sqrt[n]{z}} = -\frac{z^n \sqrt[n]{z^3}}{(m-1) a x^{m-1}} + \frac{3n-3m+5}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[n]{z}}.$$

Pentru $m=1$,

$$\int \frac{z^n dx}{x \sqrt[n]{z}} = \frac{3z^n}{(3n-1) \sqrt[n]{z}} + a \int \frac{z^{n-1} dx}{x \sqrt[n]{z}}.$$

$$2. \int \frac{dx}{x^{2n} \sqrt[n]{z}} = \frac{3 \sqrt[n]{z^3}}{(3n-2) a z^n} + \frac{1}{a} \int \frac{\sqrt[n]{z^3} dx}{x z^n}.$$

$$2.235 \int \frac{dx}{x \sqrt[n]{z}} = \frac{1}{\sqrt[n]{a^3}} \left\{ \frac{3}{2} \ln \frac{\sqrt[n]{z} - \sqrt[n]{a}}{\sqrt[n]{z}} + \sqrt{3} \operatorname{arctg} \frac{\sqrt{3} \sqrt[n]{z}}{\sqrt[n]{z} + 2 \sqrt[n]{a}} \right\}.$$

2.236

$$1. \int \frac{\sqrt[n]{z^3} dx}{x} = \frac{3}{2} \sqrt[n]{z^3} + a \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.235).$$

$$2. \int \frac{\sqrt[n]{z^3} dx}{x^2} = -\frac{\sqrt[n]{z^3}}{a x} + \frac{b}{a} \sqrt[n]{z^3} + \frac{2b}{3} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.235).$$

$$3. \int \frac{\sqrt[n]{z^3} dx}{x^3} = \left[-\frac{1}{2ax^3} + \frac{b}{6a^2x} \right] z^{\frac{3}{2}} - \frac{b^3}{6a^3} \sqrt[n]{z^3} - \frac{b^3}{9a} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.235).$$

$$4. \int \frac{dx}{x^3 \sqrt[n]{z^3}} = -\frac{\sqrt[n]{z^3}}{a x} - \frac{b}{3a} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.235).$$

$$5. \int \frac{dx}{x^3 \sqrt[n]{z^3}} = \left[-\frac{1}{2ax^3} + \frac{2b}{3a^2x} \right] \sqrt[n]{z^3} + \frac{2b^3}{9a^3} \int \frac{dx}{x \sqrt[n]{z^3}} \quad (\text{vezi } 2.235).$$

2.24 Forme care conțin $\sqrt{a+bx}$ și binomul $\alpha+\beta x$

Notații: $z = a + bx$, $t = \alpha + \beta x$, $\Delta = a\beta - b\alpha$.

2.241

$$1. \int \frac{z^m t^n dx}{\sqrt{z}} = \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1} \sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta} \int \frac{z^{m-1} t^n dx}{\sqrt{z}}.$$

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$$2. \int \frac{t^n z^m dx}{\sqrt{z}} = 2 \sqrt{z^{2m+1}} \sum_{k=0}^n \binom{n}{k} \frac{\alpha^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} \alpha^p}{2k-2p+2m+1}.$$

2.242

1. $\int \frac{t dx}{\sqrt{z}} = \frac{2a\sqrt{z}}{b} + \beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2}.$
2. $\int \frac{t^2 dx}{\sqrt{z}} = \frac{2a^2\sqrt{z}}{b} + 2a\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + \beta^2 \left(\frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3}.$
3. $\int \frac{t^3 dx}{\sqrt{z}} = \frac{2a^3\sqrt{z}}{b} + 3a^2\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + 3a\beta^2 \left(\frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3} +$
 $+ \beta^3 \left(\frac{z^3}{7} - \frac{3z^2a}{5} + za^2 - a^3 \right) \frac{2\sqrt{z}}{b^4}.$
4. $\int \frac{tz dx}{\sqrt{z}} = \frac{2a\sqrt{z^3}}{3b} + \beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2}.$
5. $\int \frac{t^2z dx}{\sqrt{z}} = \frac{2a^2\sqrt{z^3}}{3b} + 2a\beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} + \beta^2 \left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3}.$
6. $\int \frac{t^3z dx}{\sqrt{z}} = \frac{2a^3\sqrt{z^3}}{3b} + 3a^2\beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} +$
 $+ 3a\beta^2 \left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3} + \beta^3 \left(\frac{z^3}{9} - \frac{3z^2a}{7} + \frac{3za^2}{5} - \frac{a^3}{3} \right) \frac{2\sqrt{z^3}}{b^4}.$
7. $\int \frac{tz^2 dx}{\sqrt{z}} = \frac{2a\sqrt{z^5}}{5b} + \beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2}.$
8. $\int \frac{t^2z^2 dx}{\sqrt{z}} = \frac{2a^2\sqrt{z^5}}{5b} + 2a\beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + \beta^2 \left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3}.$
9. $\int \frac{t^3z^2 dx}{\sqrt{z}} = \frac{2a^3\sqrt{z^5}}{5b} + 3a^2\beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + 3a\beta^2 \left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3} +$
 $+ \beta^3 \left(\frac{z^3}{11} - \frac{3z^2a}{9} + \frac{3za^2}{7} - \frac{a^3}{5} \right) \frac{2\sqrt{z^5}}{b^4}.$
10. $\int \frac{tz^3 dx}{\sqrt{z}} = \frac{2a\sqrt{z^7}}{7b} + \beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2}.$
11. $\int \frac{t^2z^3 dx}{\sqrt{z}} = \frac{2a^2\sqrt{z^7}}{7b} + 2a\beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + \beta^2 \left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3}.$
12. $\int \frac{t^3z^3 dx}{\sqrt{z}} = \frac{2a^3\sqrt{z^7}}{7b} + 3a^2\beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} +$
 $+ 3a\beta^2 \left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3} + \beta^3 \left(\frac{z^3}{13} - \frac{3z^2a}{11} + \frac{3za^2}{9} - \frac{a^3}{7} \right) \frac{2\sqrt{z^7}}{b^4}.$

2.243

1. $\int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{t^{n+1}}{z^m} \sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta} \int \frac{t^n dx}{z^{m-1} \sqrt{z}};$
 $= -\frac{2}{(2m-1)b} \frac{t^n}{z^m} \sqrt{z} + \frac{2n\beta}{(2m-1)b} \int \frac{t^{n-1} dx}{z^{m-1} \sqrt{z}}. \quad \text{La 176 (2)}$
2. $\int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}} \sum_{k=0}^n \binom{n}{k} \frac{a^{n-k}\beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p}a^p}{2k-2p-2m+1}.$

2.244

1. $\int \frac{t dx}{z \sqrt{z}} = -\frac{2a}{b \sqrt{z}} + \frac{2\beta(z+a)}{b^2 \sqrt{z}}.$
2. $\int \frac{t^2 dx}{z \sqrt{z}} = -\frac{2a^2}{b \sqrt{z}} + \frac{4a\beta(z+a)}{b^2 \sqrt{z}} + \frac{2\beta^2\left(\frac{z^2}{3} - 2za - a^2\right)}{b^3 \sqrt{z}}.$
3. $\int \frac{t^3 dx}{z \sqrt{z}} = -\frac{2a^3}{b \sqrt{z}} + \frac{6a^2\beta(z+a)}{b^2 \sqrt{z}} + \frac{6a\beta^2\left(\frac{z^2}{3} - 2za - a^2\right)}{b^3 \sqrt{z}} +$
 $+\frac{2\beta^3\left(\frac{z^3}{5} - \frac{3z^2a}{3} + 3za^2 + a^3\right)}{b^4 \sqrt{z}}.$
4. $\int \frac{t dx}{z^2 \sqrt{z}} = -\frac{2a}{3b \sqrt{z^3}} - \frac{2\beta\left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}}.$
5. $\int \frac{t^2 dx}{z^2 \sqrt{z}} = -\frac{2a^2}{3b \sqrt{z^3}} - \frac{4a\beta\left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}} + \frac{2\beta^2\left(z^2 + 2az - \frac{a^2}{3}\right)}{b^3 \sqrt{z^3}}.$
6. $\int \frac{t^3 dx}{z^2 \sqrt{z}} = -\frac{2a^3}{3b \sqrt{z^3}} - \frac{6a^2\beta\left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}} + \frac{6a\beta^2\left(z^2 + 2za - \frac{a^2}{3}\right)}{b^3 \sqrt{z^3}} +$
 $+\frac{2\beta^3\left(\frac{z^3}{3} - 3z^2a - 3za^2 + \frac{a^3}{9}\right)}{b^4 \sqrt{z^3}}.$
7. $\int \frac{t dx}{z^3 \sqrt{z}} = -\frac{2a}{5b \sqrt{z^5}} - \frac{2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}}.$
8. $\int \frac{t^2 dx}{z^3 \sqrt{z}} = -\frac{2a^2}{5b \sqrt{z^5}} - \frac{4a\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}} - \frac{2\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3 \sqrt{z^5}}.$
9. $\int \frac{t^3 dx}{z^3 \sqrt{z}} = -\frac{2a^3}{5b \sqrt{z^5}} - \frac{6a^2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}} - \frac{6a\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3 \sqrt{z^5}} +$
 $+\frac{2\beta^3\left(z^3 + 3z^2a - za^2 + \frac{a^3}{5}\right)}{b^4 \sqrt{z^5}}.$

2.245

1. $\int \frac{z^m dx}{t^n \sqrt{z}} = -\frac{2}{(2n-2m-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta} \int \frac{z^{m-1} dx}{t^n \sqrt{z}}; \text{ La 176 (3)}$
 $= -\frac{1}{(n-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta} \int \frac{z^{m-1}}{t^{n-1} \sqrt{z}} dx;$
 $= -\frac{1}{(n-1)\Delta} \frac{z^m}{t^{n-1}} \sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta} \int \frac{z^m dx}{t^{n-1} \sqrt{z}}.$
2. $\int \frac{z^m dx}{t^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \right.$
 $+\sum_{k=2}^{n-1} \frac{(2n-2m-3)(2n-2m-5)\dots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k} \frac{1}{t^{n-k}} \left. \right\} -$
 $-\frac{(2n-2m-3)(2n-2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}\cdot(n-1)!\Delta^n} \int \frac{z^m dx}{t \sqrt{z}}.$

Pentru $n=1$,

$$\int \frac{z^m dx}{t \sqrt{z}} = \frac{2}{(2m-1)\beta} \frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta} \int \frac{z^{m-1} dx}{t \sqrt{z}}.$$

$$3. \int \frac{z^m dx}{t \sqrt{z}} = 2 \sum_{k=0}^{m-1} \frac{\Delta^k}{(2m-2k-1)\beta^{k+1}} \frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m} \int \frac{dx}{t \sqrt{z}}.$$

$$\begin{aligned} 2.246 \quad \int \frac{dx}{t \sqrt{z}} &= \frac{1}{\sqrt{\beta\Delta}} \ln \frac{\beta \sqrt{z} - \sqrt{\beta\Delta}}{\beta \sqrt{z} + \sqrt{\beta\Delta}} \quad [\beta\Delta > 0]; \\ &= \frac{2}{\sqrt{-\beta\Delta}} \arctg \frac{\beta \sqrt{z}}{\sqrt{-\beta\Delta}} \quad [\beta\Delta < 0]; \\ &= -\frac{2\sqrt{z}}{bt} \quad [\Delta = 0]. \end{aligned}$$

$$2.247. \int \frac{dx}{t z^m \sqrt{z}} = \frac{2}{z^{m-1} \sqrt{z}} \sum_{k=1}^m + \frac{\beta^{k-1} z^k}{\Delta^k (2m-2k+1)} + \frac{\beta^m}{\Delta^m} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

2.248

$$1. \int \frac{dx}{t z \sqrt{z}} = \frac{2}{\Delta \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$2. \int \frac{dx}{t z^2 \sqrt{z}} = \frac{2}{3\Delta z \sqrt{z}} + \frac{2\beta}{\Delta^2 \sqrt{z}} + \frac{\beta^2}{\Delta^2} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$3. \int \frac{dx}{t z^3 \sqrt{z}} = \frac{2}{5\Delta z^2 \sqrt{z}} + \frac{2\beta}{3\Delta^2 z \sqrt{z}} + \frac{2\beta^2}{\Delta^3 \sqrt{z}} + \frac{\beta^3}{\Delta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$4. \int \frac{dx}{t^2 \sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$5. \int \frac{dx}{t^2 z \sqrt{z}} = -\frac{1}{\Delta t \sqrt{z}} - \frac{3b}{\Delta^2 \sqrt{z}} - \frac{3b\beta}{2\Delta^2} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$6. \int \frac{dx}{t^2 z^2 \sqrt{z}} = -\frac{1}{\Delta t z \sqrt{z}} - \frac{5b}{3\Delta^2 z \sqrt{z}} - \frac{5b\beta}{\Delta^3 \sqrt{z}} - \frac{5b\beta^2}{2\Delta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$\begin{aligned} 7. \int \frac{dx}{t^2 z^3 \sqrt{z}} &= -\frac{1}{\Delta t z^2 \sqrt{z}} - \frac{7b}{5\Delta^2 z^2 \sqrt{z}} - \frac{7b\beta}{3\Delta^3 z \sqrt{z}} - \frac{7b\beta^2}{\Delta^4 \sqrt{z}} - \\ &\quad - \frac{7b\beta^3}{2\Delta^4} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}). \end{aligned}$$

$$8. \int \frac{dx}{t^3 \sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b \sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$9. \int \frac{dx}{t^3 z \sqrt{z}} = -\frac{1}{2\Delta t^2 \sqrt{z}} + \frac{5b}{4\Delta^2 t \sqrt{z}} + \frac{15b^2}{4\Delta^3 \sqrt{z}} + \frac{15b^2\beta}{8\Delta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$\begin{aligned} 10. \int \frac{dx}{t^3 z^2 \sqrt{z}} &= -\frac{1}{2\Delta t^2 z \sqrt{z}} + \frac{7b \sqrt{z}}{4\Delta^2 t z \sqrt{z}} + \frac{35b^2}{12\Delta^3 z \sqrt{z}} + \\ &\quad + \frac{35b^2\beta}{4\Delta^4 \sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}). \end{aligned}$$

$$\begin{aligned} 11. \int \frac{dx}{t^3 z^3 \sqrt{z}} &= -\frac{1}{2\Delta t^2 z^2 \sqrt{z}} + \frac{9b}{4\Delta^2 t z^2 \sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2 \sqrt{z}} + \\ &\quad + \frac{21b^2\beta}{4\Delta^4 z \sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5 \sqrt{z}} + \frac{63b^2\beta^3}{8\Delta^5} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}). \end{aligned}$$

$$12. \int \frac{z dx}{t \sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$13. \int \frac{z^2 dx}{t \sqrt{z}} = \frac{2z \sqrt{z}}{3\beta} + \frac{2\Delta \sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{vezi 2.246}).$$

$$14. \int \frac{z^3 dx}{t \sqrt{z}} = \frac{2z^2 \sqrt{z}}{5\beta} + \frac{2\Delta z \sqrt{z}}{3\beta^2} + \frac{2\Delta^2 \sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$15. \int \frac{z dx}{t^3 \sqrt{z}} = -\frac{z \sqrt{z}}{\Delta t} + \frac{b \sqrt{z}}{\beta \Delta} + \frac{b}{2\beta} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$16. \int \frac{z^2 dx}{t^3 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{\Delta t} + \frac{bz \sqrt{z}}{\beta \Delta} + \frac{3b \sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$17. \int \frac{z^3 dx}{t^3 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{\Delta t} + \frac{bz^2 \sqrt{z}}{\beta \Delta} + \frac{5bz \sqrt{z}}{3\beta^2} + \frac{5b\Delta \sqrt{z}}{\beta^3} + \\ + \frac{5\Delta^2 b}{2\beta^3} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$18. \int \frac{z dx}{t^3 \sqrt{z}} = -\frac{z \sqrt{z}}{2\Delta t^2} - \frac{bz \sqrt{z}}{4\Delta^2 t} + \frac{b^2 \sqrt{z}}{4\beta \Delta^2} + \frac{b^2}{8\beta \Delta} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$19. \int \frac{z^2 dx}{t^3 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{2\Delta t^2} + \frac{bz^2 \sqrt{z}}{4\Delta^2 t} + \frac{b^2 z \sqrt{z}}{4\beta \Delta^2} + \\ + \frac{3b^2 \sqrt{z}}{4\beta^2 \Delta} + \frac{3b^2}{8\beta^2} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

$$20. \int \frac{z^3 dx}{t^3 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{2\Delta t^2} + \frac{3bz^2 \sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2 \sqrt{z}}{4\beta \Delta^2} + \frac{5b^2 z \sqrt{z}}{4\beta^2 \Delta} + \\ + \frac{15b^2 \sqrt{z}}{4\beta^2} + \frac{15b^2 \Delta}{8\beta^2} \int \frac{dx}{t \sqrt{z}} \text{ (vezi 2.246).}$$

2.249

$$1. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{\sqrt{z}}{t^{n-1} z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta} \int \frac{dx}{t^{n-2} z^{m-1} \sqrt{z}}; \text{ La 177 (4.)} \\ = -\frac{1}{(n-1)\Delta} \frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta} \int \frac{dx}{t^{n-1} z^m \sqrt{z}}.$$

$$2. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{\sqrt{z}}{z^m} \left\{ \frac{-1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \right. \\ \left. + \sum_{k=2}^{n-1} (-1)^k \frac{(2n+2m-3)(2n+2m-5) \dots (2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2) \dots (n-k)\Delta^k} \cdot \frac{1}{t^{n-k}} \right\} + \\ + (-1)^{n-1} \frac{(2n+2m-3)(2n+2m-5) \dots (-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}} \int \frac{dx}{t z^n \sqrt{z}}$$

Pentru $n=1$

$$\int \frac{dx}{z^m t \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{1}{z^{m-1} \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t z^{m-1} \sqrt{z}}.$$

2.25 Forme care conțin $\sqrt{a+bx+cx^2}$

Metode de integrare

2.251 Raționalizarea expresiei de sub semnul integrală în integralele de forma $\int R(x, \sqrt{a+bx+cx^2}) dx$ se face cu ajutorul a cel puțin una dintre următoarele trei substituții, numite *substituțiile lui Euler*:

$$1. \sqrt{a+bx+cx^2} = xt \pm \sqrt{a} \text{ pentru } a > 0;$$

$$2. \sqrt{a+bx+cx^2} = t \pm x\sqrt{c} \text{ pentru } c > 0;$$

3. $\sqrt{c(x-x_1)(x-x_2)} = t(x-x_1)$, cu condiția ca rădăcinile x_1 și x_2 ale ecuației $a+bx+cx^2=0$ să fie reale.

2.252 În afară de substituțiile lui Euler, mai există următoarea metodă pentru calcularea integralelor de forma $\int R(x, \sqrt{a+bx+cx^2}) dx$. Cu ajutorul raționalizării numitorului și al unor operații algebrice simple, expresia de sub semnul integrală poate fi redusă la suma unei anumite funcții raționale de x și la o expresie de forma $\frac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, unde $P_1(x)$ și $P_2(x)$ sînt două polinoame.

Cu ajutorul separării părții întregi din funcția rațională $\frac{P_1(x)}{P_2(x)}$ și al descompunerii restului în fracții simple, integrala acestei din urmă expresii se reduce la o sumă de integrale, avînd fiecare una dintre următoarele trei forme:

- I. $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$, unde $P(x)$ este un polinom de grad oarecare r ;
- II. $\int \frac{dx}{(x+p)^k \sqrt{a+bx+cx^2}}$;
- III. $\int \frac{(Mx+N) dx}{(a+\beta x+x^2)^m \sqrt{c(a+bx+x^2)}}$.

$$I. \int \frac{P(x) dx}{\sqrt{a+bx+cx^2}} = Q(x) \sqrt{a+bx+cx^2} + \lambda \int \frac{dx}{\sqrt{a+bx+cx^2}},$$

unde $Q(x)$ este un polinom de gradul $r-1$. Coeficienții lui, precum și numărul λ se calculează după metoda coeficienților nedeterminați din identitatea

$$P(x) = Q'(x)(a+bx+cx^2) + \frac{1}{2} Q(x)(b+2c) + \lambda. \quad \text{F II 77}$$

Integralele de forma $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$ (pentru $r \leq 3$) se mai pot calcula, folosind formulele 2.26.

II. Integralele de forma $\int \frac{P(x) dx}{(x+p)^k \sqrt{a+bx+cx^2}}$, cu condiția ca gradul n al polinomului $P(x)$ să fie mai mic decît k , se reduc, cu ajutorul substituției $t = \frac{1}{x+p}$, la o integrală de forma $\int \frac{P(t) dt}{\sqrt{a+\beta t+\gamma t^2}}$ (vezi și 2.281).

III. Integralele de forma $\int \frac{(Mx+N) dx}{(a+\beta x+x^2)^m \sqrt{c(a+bx+x^2)}}$ se calculează în modul următor.

Dacă $b \neq \beta$, atunci, cu ajutorul substituției

$$x = \frac{a-\alpha}{\beta-\beta} + \frac{t-1}{t+1} \frac{\sqrt{(a-\alpha)^2 - (a\beta - \alpha\beta)(\beta-\beta)}}{\beta-\beta}$$

această integrală se reduce la o integrală de forma $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{c(t^2+q)}}$, unde $P(t)$ este un polinom de grad mai mic sau egal cu $2m-1$. Integrala $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{t^2+q}}$ se reduce la o sumă de integrale de forma $\int \frac{t dt}{(t^2+p)^k \sqrt{t^2+q}}$ și $\int \frac{dt}{(t^2+p)^k \sqrt{t^2+q}}$.

În cazul cînd $b=\beta$, atunci se aplică integralelor de forma $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{c(t^2+q)}}$ substituția $t = x + \frac{b}{2}$.

Integrala $\int \frac{tdt}{(t^2+p)^k \sqrt{c(t^2+q)}}$ se rezolvă cu ajutorul substituției $t^2+q=u^2$.
 Integrala $\int \frac{dt}{(t^2+p)^k \sqrt{c(t^2+q)}}$ se rezolvă cu ajutorul substituției $\frac{t}{\sqrt{t^2+q}} = v$
 (vezi și 2.283). F II 78—82

2.26 Forme care conțin $\sqrt{a+bx+cx^2}$ și puteri întregi ale lui x

Notății: $R=a+bx+cx^2$, $\Delta=4ac-b^2$.

Formule simplificate pentru cazul $b=0$, vezi 2.27.

2.260

$$1. \int x^m \sqrt{R^{2n+1}} dx = \frac{x^{m-1} \sqrt{R^{2n+3}}}{(m+2n+2)c} - \frac{(2m+2n+1)b}{2(m+2n+2)c} \int x^{m-1} \sqrt{R^{2n+1}} dx - \frac{(m-1)a}{(m+2n+2)c} \int x^{m-2} \sqrt{R^{2n+1}} dx. \quad T(192)m$$

$$2. \int \sqrt{R^{2n+1}} dx = \frac{2cx+b}{4(n+1)c} \sqrt{R^{2n+1}} + \frac{2n+1}{8(n+1)} \frac{\Delta}{c} \int \sqrt{R^{2n-1}} dx. \quad T(188)$$

$$3. \int \sqrt{R^{2n+1}} dx = \frac{(2cx+b)\sqrt{R}}{4(n+1)c} \left\{ R^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)} \left(\frac{\Delta}{c}\right)^{k+1} R^{n-k-1} + \frac{(2n+1)!!}{8^{n+1} \cdot n!} \left(\frac{\Delta}{c}\right)^{n+1} \int \frac{dx}{\sqrt{R}} \right\}. \quad T(190)$$

2.261 Pentru $n=-1$,

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln(2\sqrt{cR} + 2cx + b) \quad [c > 0]; \quad T(127)$$

$$= \frac{1}{\sqrt{c}} \operatorname{Arsh} \frac{2cx+b}{\sqrt{\Delta}} \quad [c > 0, \Delta > 0]; \quad D(380\ 004)$$

$$= \frac{-1}{\sqrt{-c}} \arcsin \frac{2cx+b}{\sqrt{-\Delta}} \quad [c < 0, \Delta < 0]; \quad T(128)$$

$$= \frac{1}{\sqrt{c}} \ln(2cx+b) \quad [c > 0, \Delta = 0]. \quad D(380\ 004)$$

2.262

$$1. \int \sqrt{R} dx = \frac{(2cx+b)\sqrt{R}}{4c} + \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$2. \int x \sqrt{R} dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx+b)b}{8c^2} \sqrt{R} - \frac{b\Delta}{16c^2} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$3. \int x^2 \sqrt{R} dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right) \sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{(2cx+b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$4. \int x^3 \sqrt{R} dx = \left(\frac{x^3}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right) \sqrt{R^3} - \left(\frac{7b^2}{32c^2} - \frac{3ab}{8c^2}\right) \frac{(2cx+b)\sqrt{R}}{4c} - \left(\frac{7b^2}{32c^2} - \frac{3ab}{8c^2}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$5. \int \sqrt{R^3} dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right) (2cx+b) \sqrt{R} + \frac{3\Delta^2}{128c^2} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261})$$

$$6. \int x \sqrt{R^3} dx = \frac{\sqrt{R}}{5c} - (2cx + b) \left(\frac{b}{16c^3} \sqrt{R^3} + \frac{3\Delta b}{128c^3} \sqrt{R} \right) - \frac{3\Delta^3 b}{256c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$7. \int x^3 \sqrt{R^3} dx = \left(\frac{x}{6c} - \frac{7b}{60c^3} \right) \sqrt{R^3} + \left(\frac{7b^2}{24c^3} - \frac{a}{6c} \right) \left(2x + \frac{b}{c} \right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R} \right) + \left(\frac{7b^2}{4c} - a \right) \frac{\Delta^3}{256c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$8. \int x^3 \sqrt{R^3} dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^3} + \frac{3b^2}{40c^3} - \frac{2a}{35c^3} \right) \sqrt{R^3} - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^3} \right) \left(2x + \frac{b}{c} \right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R} \right) - \left(\frac{3b^3}{4c} - a \right) \frac{3\Delta^3 b}{512c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

2.263

$$1. \int \frac{x^m dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c \sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c} \int \frac{x^{m-1} dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c} \int \frac{x^{m-2} dx}{\sqrt{R^{2n+1}}}. \quad T(193)$$

Pentru $m = 2n$,

$$2. \int \frac{x^{2n} dx}{\sqrt{R^{2n+1}}} = - \frac{x^{2n-1}}{(2n-1)c \sqrt{R^{2n-1}}} - \frac{b}{2c} \int \frac{x^{2n-1}}{\sqrt{R^{2n+1}}} dx + \frac{1}{c} \int \frac{x^{2n-2}}{\sqrt{R^{2n-1}}} dx. \quad T(194)m$$

$$3. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta \sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta} \int \frac{dx}{\sqrt{R^{2n-1}}}. \quad T(189)$$

$$4. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta \sqrt{R^{2n-1}}} \times \times \left\{ 1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \frac{c^k}{\Delta^k} R^k \right\} \quad [n \geq 1]. \quad T(191)$$

2.264

$$1. \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$2. \int \frac{x dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$3. \int \frac{x^3 dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^3} \right) \sqrt{R} + \left(\frac{3b^3}{8c^3} - \frac{a}{2c} \right) \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$4. \int \frac{x^3 dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^3} + \frac{5b^2}{8c^3} - \frac{2a}{3c^3} \right) \sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^3} \right) \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$5. \int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta \sqrt{R}}.$$

$$6. \int \frac{x dx}{\sqrt{R^3}} = - \frac{2(2a+bx)}{\Delta \sqrt{R}}.$$

$$7. \int \frac{x^3 dx}{\sqrt{R^3}} = - \frac{(\Delta - b^2)x - 2ab}{c\Delta \sqrt{R}} + \frac{1}{c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$8. \int \frac{x^3 dx}{\sqrt{R^3}} = \frac{c\Delta x^3 + b(16ac - 6b^2)x + 2a(4ac - 3b^2)}{c^3 \Delta \sqrt{R}} - \frac{3b}{2c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261}).$$

$$2.265 \quad \int \frac{\sqrt{R^{2n+1}}}{x^m} dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-1}} dx + \\ + \frac{(2n-m+4)c}{(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-2}} dx. \quad T(195)$$

Pentru $m=1$,

$$\int \frac{\sqrt{R^{2n+1}}}{x} dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2} \int \sqrt{R^{2n-1}} dx + a \int \frac{\sqrt{R^{2n-1}}}{x} dx. \quad T(198)$$

Pentru $a=0$,

$$\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m} dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \\ + \frac{2(m-2n-3)c}{(2n-2m+3)b} \int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}} dx. \quad \text{La 169 (3)}$$

Pentru $m=0$ vezi 2.260 2. și 2.260 3.

Pentru $n=-1$ și $m=1$,

$$2.266 \quad \int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \quad [a > 0]; \quad T(137)$$

$$= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \quad [a < 0, \Delta < 0]; \quad T(138)$$

$$= \frac{1}{\sqrt{-a}} \operatorname{arctg} \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \quad [a < 0]; \quad \text{La 178 (6)m}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{Arsh} \frac{2a+bx}{x\sqrt{\Delta}} \quad [a > 0, \Delta > 0]; \quad D(380111)$$

$$= -\frac{1}{\sqrt{a}} \operatorname{Arth} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \quad [a > 0];$$

$$= \frac{1}{\sqrt{a}} \ln \frac{x}{a+bx} \quad [a > 0, \Delta = 0];$$

$$= -\frac{2\sqrt{bx+cx^2}}{bx} \quad [a=0, b \neq 0]. \quad \text{La 170 (16)}$$

2.267

$$1. \quad \int \frac{\sqrt{R} dx}{x} = \sqrt{R} + a \int \frac{dx}{x\sqrt{R}} + \frac{b}{2} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261 și 2.266}).$$

$$2. \quad \int \frac{\sqrt{R} dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{R}} + c \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261 și 2.266}).$$

Pentru $a=0$,

$$\int \frac{\sqrt{bx+cx^2}}{x^2} dx = -\frac{2\sqrt{bx+cx^2}}{x} + c \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{vezi 2.261}).$$

$$3. \quad \int \frac{\sqrt{R} dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right) \sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right) \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{\sqrt{bx+cx^2}}{x^3} dx = -\frac{2\sqrt{(bx+cx^2)^3}}{3bx^3}$$

$$4. \quad \int \frac{\sqrt{R^3}}{x} dx = \frac{\sqrt{R^3}}{3} + \frac{2bcx+b^2+8ac}{8c} \sqrt{R} + \\ + a^2 \int \frac{dx}{x\sqrt{R}} + \frac{b(12ac-b^2)}{16c} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261 și 2.266}).$$

$$5. \int \frac{\sqrt{R^3}}{x^3} dx = -\frac{\sqrt{R^3}}{ax} + \frac{cx+b}{a} \sqrt{R^3} + \frac{3}{4}(2cx+3b) \sqrt{R} + \\ + \frac{3}{2}ab \int \frac{dx}{x\sqrt{R}} + \frac{3(4ac+b^2)}{8} \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261 și 2.266}).$$

Pentru $a=0$,

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^3} = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b}{4} \sqrt{bx+cx^2} + \frac{3b^2}{8} \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{vezi 2.261}).$$

$$6. \int \frac{\sqrt{R^3}}{x^3} dx = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right) \sqrt{R^3} + \frac{bcx+2ac+b^2}{4a^2} \sqrt{R^3} + \\ + \frac{3(bcx+2ac+b^2)}{4a} \sqrt{R} + \frac{3}{8}(4ac+b^2) \int \frac{dx}{x\sqrt{R}} + \frac{3}{2}bc \int \frac{dx}{\sqrt{R}} \quad (\text{vezi 2.261 și 2.266}).$$

Pentru $a=0$,

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^3} dx = \left(c - \frac{2b}{x}\right) \sqrt{bx+cx^2} + \frac{3bc}{2} \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{vezi 2.261}).$$

$$2.268 \quad \int \frac{dx}{x^m \sqrt{R^{2n+1}}} = -\frac{1}{(m-1)ax^{m-1} \sqrt{R^{2n+1}}} - \\ - \frac{(2n+2m-3)b}{2(m-1)a} \int \frac{dx}{x^{m-1} \sqrt{R^{2n+1}}} - \frac{(2n+2m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2} \sqrt{R^{2n+1}}} \quad T(196)$$

Pentru $m=1$,

$$\int \frac{dx}{x \sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a \sqrt{R^{2n-1}}} - \frac{b}{2a} \int \frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a} \int \frac{dx}{x \sqrt{R^{2n-1}}} \quad T(199)$$

Pentru $a=0$,

$$\int \frac{dx}{x^m \sqrt{(bx+cx^2)^{2n+1}}} = -\frac{2}{(2n+2m-1)bx^{m-1} \sqrt{(bx+cx^2)^{2n-1}}} - \\ - \frac{(4n+2m-2)c}{(2n+2m-1)b} \int \frac{dx}{x^{m-1} \sqrt{(bx+cx^2)^{2n+1}}} \quad (\text{cf. 2.265}).$$

2.269

$$1. \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

$$2. \int \frac{dx}{x^2 \sqrt{R}} = -\frac{\sqrt{R}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{dx}{x^3 \sqrt{bx+cx^2}} = \frac{2}{3} \left(-\frac{1}{bx^3} + \frac{2c}{b^2x}\right) \sqrt{bx+cx^2}.$$

$$3. \int \frac{dx}{x^3 \sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right) \sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right) \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{dx}{x^3 \sqrt{bx+cx^2}} = \frac{2}{5} \left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x}\right) \sqrt{bx+cx^2}.$$

$$4. \int \frac{dx}{x\sqrt{R^3}} = -\frac{2(bcx-2ac+b^2)}{a\Delta\sqrt{R}} + \frac{1}{a} \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{dx}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3} \left(-\frac{1}{bx} + \frac{4c}{b^3} - \frac{8c^2x}{b^3} \right) \frac{1}{\sqrt{bx+cx^2}}.$$

$$5. \int \frac{dx}{x^3\sqrt{R^3}} = \left(-\frac{1}{ax} + \frac{2bc}{a\Delta} + \frac{c(3b^2-8ac)x}{a^2\Delta} \right) \frac{1}{\sqrt{R}} - \frac{3b}{2a^3} \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^2x}{b^4} \right) \frac{1}{\sqrt{bx+cx^2}}.$$

$$6. \int \frac{dx}{x^3\sqrt{R^3}} = \left(-\frac{1}{ax^3} + \frac{5b}{2a^2x} - \frac{15b^2-62acb^2+24a^2c^2}{2a^3\Delta} - \frac{bc(15b^2-52ac)x}{2a^3\Delta} \right) \frac{1}{2\sqrt{R}} + \frac{15b^3-12ac}{8a^3} \int \frac{dx}{x\sqrt{R}} \quad (\text{vezi 2.266}).$$

Pentru $a=0$,

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{7} \left(-\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5} \right) \frac{1}{\sqrt{bx+cx^2}}.$$

2.27 Forme care conțin $\sqrt{a+cx^2}$ și puteri întregi ale lui x

Notăție: $u = \sqrt{a+cx^2}$.

$$I_1 = \frac{1}{\sqrt{c}} \ln(x\sqrt{c}+u) \quad [c > 0];$$

$$= \frac{1}{\sqrt{-c}} \arcsin x \sqrt{-\frac{c}{a}} \quad [c < 0 \text{ și } a > 0].$$

$$I_2 = \frac{1}{2\sqrt{a}} \ln \frac{u-\sqrt{a}}{u+\sqrt{a}} \quad [a > 0 \text{ și } c > 0];$$

$$= \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}-u}{\sqrt{a}+u} \quad [a > 0 \text{ și } c < 0];$$

$$= \frac{1}{\sqrt{-a}} \operatorname{arcsec} x \sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}} \arccos \frac{1}{x} \sqrt{-\frac{a}{c}} \quad [a < 0 \text{ și } c > 0].$$

2.271

$$1. \int u^5 dx = \frac{1}{6} xu^5 + \frac{5}{24} axu^3 + \frac{5}{16} a^2 xu + \frac{5}{16} a^3 I_1. \quad D(230.05)m$$

$$2. \int u^3 dx = \frac{1}{4} xu^3 + \frac{3}{8} axu + \frac{3}{8} a^2 I_1. \quad D(230.03)m$$

$$3. \int u dx = \frac{1}{2} xu + \frac{1}{2} a I_1. \quad D(230.01)m$$

$$4. \int \frac{dx}{u} = I_1. \quad D(200.01)m$$

$$5. \int \frac{dx}{u^3} = \frac{1}{a} \frac{x}{u} \quad D(200.03)m$$

$$6. \int \frac{dx}{u^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{u^{2k+1}}$$

$$7. \int \frac{x dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}}. \quad D(201.9)m$$

2.272

1. $\int x^2 u^3 dx = \frac{1}{6} \frac{xu^5}{c} - \frac{1}{24} \frac{axu^3}{c} - \frac{1}{16} \frac{a^2xu}{c} - \frac{1}{16} \frac{a^3}{c} I_1.$ D (232.03)m
2. $\int x^2 u dx = \frac{1}{4} \frac{xu^3}{c} - \frac{1}{8} \frac{axu}{c} - \frac{1}{8} \frac{a^2}{c} I_1.$ D (232.01)m
3. $\int \frac{x^2}{u} dx = \frac{1}{2} \frac{xu}{c} - \frac{1}{2} \frac{a}{c} I_1.$ D (202.01)m
4. $\int \frac{x^2}{u^3} dx = -\frac{x}{cu} + \frac{1}{c} I_1.$ D (202.03)m
5. $\int \frac{x^2}{u^5} dx = \frac{1}{3} \frac{x^3}{au^3}.$ D (202.05)m
6. $\int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{u^{2k+3}}.$
7. $\int \frac{x^3 dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^3 u^{2n-3}} + \frac{a}{(2n-1)c^3 u^{2n-1}}.$ D (203.9)m

2.273

1. $\int x^4 u^3 dx = \frac{1}{8} \frac{x^5 u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2xu^3}{64c^3} + \frac{3a^3xu}{128c^4} + \frac{3a^4}{128c^4} I_1.$ D (234.03)m
2. $\int x^4 u dx = \frac{1}{6} \frac{x^5 u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2xu}{16c^3} + \frac{a^3}{16c^3} I_1.$ D (234.01)m
3. $\int \frac{x^4}{u} dx = \frac{1}{4} \frac{x^5 u}{c} - \frac{3}{8} \frac{axu}{c^2} + \frac{3}{8} \frac{a^2}{c^2} I_1.$ D (204.01)m
4. $\int \frac{x^4}{u^3} dx = \frac{1}{2} \frac{xu}{c^3} + \frac{ax}{c^3 u} - \frac{3}{2} \frac{a}{c^3} I_1.$ D (204.03)m
5. $\int \frac{x^4}{u^5} dx = -\frac{x}{c^5 u} - \frac{1}{3} \frac{x^3}{cu^3} + \frac{1}{c^3} I_1.$ D (204.05)m
6. $\int \frac{x^4}{u^7} dx = \frac{1}{5} \frac{x^5}{au^5}.$ D (204.07)m
7. $\int \frac{x^4 dx}{u^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{u^{2k+5}}.$
8. $\int \frac{x^5 dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^5 u^{2n-5}} + \frac{2a}{(2n-3)c^5 u^{2n-3}} - \frac{a^2}{(2n-1)c^5 u^{2n-1}}.$ D (205.9)m

2.274

1. $\int x^6 u^3 dx = \frac{1}{10} \frac{x^5 u^5}{c} - \frac{ax^3 u^5}{16c^2} + \frac{a^2 x u^5}{32c^3} - \frac{a^3 x u^3}{128c^4} - \frac{3a^4 x u}{256c^5} - \frac{3}{256} \frac{a^5}{c^5} I_1.$
2. $\int x^6 u dx = \frac{1}{8} \frac{x^5 u^3}{c} - \frac{5}{48} \frac{ax^3 u^3}{c^2} + \frac{5a^2 x u^3}{64c^3} - \frac{5a^3 x u}{128c^4} - \frac{5}{128} \frac{a^4}{c^4} I_1.$
3. $\int \frac{x^6}{u} dx = \frac{1}{6} \frac{x^5 u}{c} - \frac{5}{24} \frac{ax^3 u}{c^2} + \frac{5}{16} \frac{a^2 x u}{c^3} - \frac{5}{16} \frac{a^3}{c^3} I_1.$ D (206.01)m
4. $\int \frac{x^6}{u^3} dx = \frac{1}{4} \frac{x^5}{cu} - \frac{5}{8} \frac{ax^3}{c^2 u} - \frac{15}{8} \frac{a^2 x}{c^3 u} + \frac{15}{8} \frac{a^3}{c^3} I_1.$ D (206.03)m
5. $\int \frac{x^6}{u^5} dx = \frac{1}{2} \frac{x^5}{cu^3} + \frac{10}{3} \frac{ax^3}{c^2 u^3} + \frac{5}{2} \frac{a^2 x}{c^3 u^3} - \frac{5}{2} \frac{a}{c^3} I_1.$ D (206.05)m
6. $\int \frac{x^6}{u^7} dx = -\frac{23}{15} \frac{x^5}{cu^5} - \frac{7}{3} \frac{ax^3}{c^2 u^5} - \frac{a^2 x}{c^3 u^5} + \frac{1}{c^3} I_1.$ D (206.07)m
7. $\int \frac{x^6}{u^9} dx = \frac{1}{7} \frac{x^7}{au^7}.$ D (206.09)m

$$8. \int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-3}} \sum_{k=0}^{n-4} \frac{(-1)^k}{2k+7} \binom{n-4}{k} \frac{c^k x^{2k+7}}{u^{2k+7}}.$$

$$9. \int \frac{x^7 dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4 u^{2n-7}} + \frac{3a}{(2n-5)c^4 u^{2n-5}} - \frac{3a^2}{(2n-3)c^4 u^{2n-3}} + \frac{a^3}{(2n-1)c^4 u^{2n-1}}. \quad D (207.9)m$$

2.275

$$1. \int \frac{u^5}{x} dx = \frac{u^5}{5} + \frac{1}{3} au^3 + a^2 u + a^3 I_2. \quad D (241.05)m$$

$$2. \int \frac{u^3}{x} dx = \frac{u^3}{3} + au + a^3 I_2. \quad D (241.03)m$$

$$3. \int \frac{u}{x} dx = u + a I_2. \quad D (241.01)m$$

$$4. \int \frac{dx}{xu} = I_2. \quad D (221.01)m$$

$$5. \int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1) a^{n-k} u^{2k+1}}.$$

$$6. \int \frac{u^5}{x^2} dx = -\frac{u^5}{x} + \frac{5}{4} cxu^3 + \frac{15}{8} acxu + \frac{15}{8} a^2 I_1. \quad D (242.05)m$$

$$7. \int \frac{u^3}{x^2} dx = -\frac{u^3}{x} + \frac{3}{2} cxu + \frac{3}{2} a I_1. \quad D (242.03)m$$

$$8. \int \frac{u}{x^2} dx = -\frac{u}{x} + I_1. \quad D (242.01)m$$

$$9. \int \frac{dx}{x^2 u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u} \right)^{2k-1} \right\}.$$

2.276

$$1. \int \frac{u^5}{x^3} dx = -\frac{u^5}{2x^2} + \frac{5}{6} cu^3 + \frac{5}{2} acu + \frac{5}{2} a^2 c I_2 \quad D (243.05)m$$

$$2. \int \frac{u^3}{x^3} dx = -\frac{u^3}{2x^2} + \frac{3}{2} cu + \frac{3}{2} ac I_2. \quad D (243.03)m$$

$$3. \int \frac{u}{x^3} dx = -\frac{u}{2x^2} + \frac{c}{2} I_2. \quad D (243.01)m$$

$$4. \int \frac{dx}{x^2 u} = -\frac{u}{2ax^2} - \frac{c}{2a} I_2. \quad D (223.01)m$$

$$5. \int \frac{dx}{x^2 u^3} = -\frac{1}{2ax^2 u} - \frac{3c}{2a^2 u} - \frac{3c}{2a^2} I_2. \quad D (223.03)m$$

$$6. \int \frac{dx}{x^2 u^5} = -\frac{1}{2ax^2 u^3} - \frac{5}{6} \frac{c}{a^2 u^3} - \frac{5}{2} \frac{c}{a^2 u} - \frac{5}{2} \frac{c}{a^2} I_2. \quad D (223.05)m$$

$$7. \int \frac{u^5}{x^4} dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2 xu}{2} + \frac{5}{2} ac I_1. \quad D (244.05)m$$

$$8. \int \frac{u^3}{x^4} dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + c I_1. \quad D (244.03)m$$

$$9. \int \frac{u}{x^4} dx = -\frac{u^3}{3ax^3}. \quad D (244.01)m$$

$$10. \int \frac{dx}{x^4 u^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{u^3}{3x^3} + (n+1) \frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left(\frac{x}{u} \right)^{2k-3} \right\}.$$

2.277

$$1. \int \frac{u^3}{x^5} dx = -\frac{u^3}{4x^4} - \frac{3}{8} \frac{cu^3}{ax^3} + \frac{3}{8} \frac{c^2u}{a} + \frac{3}{8} c^2 I_2. \quad D (245.03)m$$

$$2. \int \frac{u}{x^5} dx = -\frac{u}{4x^4} - \frac{1}{8} \frac{cu}{ax^3} - \frac{1}{8} \frac{c^2}{a} I_2. \quad D (245.01)m$$

$$3. \int \frac{dx}{x^5 u} = -\frac{u}{4ax^4} + \frac{3}{8} \frac{cu}{a^2 x^3} + \frac{3}{8} \frac{c^2}{a^2} I_2. \quad D (225.01)m$$

$$4. \int \frac{dx}{x^5 u^3} = -\frac{1}{4ax^4 u} + \frac{5}{8} \frac{c}{a^2 x^3 u} + \frac{15}{8} \frac{c^2}{a^3 u} + \frac{15}{8} \frac{c^2}{a^3} I_2. \quad D (225.03)m$$

2.278

$$1. \int \frac{u^3}{x^5} dx = -\frac{u^3}{5ax^4}. \quad D (246.03)m$$

$$2. \int \frac{u}{x^5} dx = -\frac{u^3}{5ax^4} + \frac{2}{15} \frac{cu^3}{a^2 x^3}. \quad D (246.01)m$$

$$3. \int \frac{dx}{x^5 u} = \frac{1}{a^3} \left(-\frac{u^3}{5x^4} + \frac{2}{3} \frac{cu^3}{x^3} - \frac{c^2 u}{x} \right). \quad D (226.01)m$$

$$4. \int \frac{dx}{x^5 u^{2n+1}} = \frac{1}{a^{n+3}} \left\{ -\frac{u^3}{5x^4} + \frac{1}{3} \binom{n+2}{1} \frac{cu^3}{x^3} - \binom{n+2}{2} \frac{c^2 u}{x} + \sum_{k=3}^{n+2} \frac{(-1)^k}{2k-5} \binom{n+2}{k} c^k \left(\frac{x}{u} \right)^{2k-5} \right\}$$

2.28 Forme care conțin $\sqrt{a+bx+cx^2}$ și polinoame de gradul întâi și doi

Notăție: $R=a+bx+cx^2$

Vezi și 2.252.

$$2.281 \quad \int \frac{dx}{(x+p)^n \sqrt{R}} = -\int \frac{t^{n-1} dt}{\sqrt{c+(b-2pc)t+(a-bp+cp^2)t^2}} \left[t = \frac{1}{x+p} \right].$$

2.282

$$1. \int \frac{\sqrt{R} dx}{x+p} = c \int \frac{x dx}{\sqrt{R}} + (b-cp) \int \frac{dx}{\sqrt{R}} + (a-bp+cp^2) \int \frac{dx}{(x+p)\sqrt{R}}.$$

$$2. \int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}.$$

$$3. \int \frac{\sqrt{R} dx}{(x+p)(x+q)} = \frac{1}{q-p} \int \frac{\sqrt{R} dx}{x+p} + \frac{1}{p-q} \int \frac{\sqrt{R} dx}{x+q}.$$

$$4. \int \frac{(x+p)\sqrt{R} dx}{x+q} = \int \sqrt{R} dx + (p-q) \int \frac{\sqrt{R} dx}{x+q}.$$

$$5. \int \frac{(rx+s) dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}.$$

$$2.283 \quad \int \frac{(Ax+B) dx}{(p+R)^n \sqrt{R}} = \frac{A}{c} \int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c} \int \frac{(1-cv^2)^{n-1} dv}{\left[p+a-\frac{b^2}{4c}-cpv^2 \right]^n},$$

unde $u = \sqrt{R}$ și $v = \frac{b+2cx}{2c\sqrt{R}}$.

$$2.284 \int \frac{Ax+B}{(p+R)\sqrt{R}} dx = \frac{A}{c} I_1 + \frac{2Bc-Ab}{\sqrt{c^2p[b^2-4(a+p)c]}} I_2,$$

unde

$$I_1 = \frac{1}{\sqrt{p}} \operatorname{arctg} \sqrt{\frac{R}{p}} \quad [p > 0];$$

$$= \frac{1}{2\sqrt{-p}} \ln \frac{\sqrt{-p}-\sqrt{R}}{\sqrt{-p}+\sqrt{R}} \quad [p < 0].$$

$$I_2 = \operatorname{arctg} \sqrt{\frac{p}{b^2-4(a+p)c}} \frac{b+2cx}{\sqrt{R}} \quad [p\{b^2-4(a+p)c\} > 0, p < 0];$$

$$= -\operatorname{arctg} \sqrt{\frac{p}{b^2-4(a+p)c}} \frac{b+2cx}{\sqrt{R}} \quad [p\{b^2-4(a+p)c\} > 0, p > 0];$$

$$= \frac{1}{2i} \ln \frac{\sqrt{4(a+p)c-b^2}\sqrt{R}+\sqrt{p}(b+2cx)}{\sqrt{4(a+p)c-b^2}\sqrt{R}-\sqrt{p}(b+2cx)} \quad [p\{b^2-4(a+p)c\} < 0, p > 0];$$

$$= \frac{1}{2i} \ln \frac{\sqrt{b^2-4(a+p)c}\sqrt{R}-\sqrt{-p}(b+2cx)}{\sqrt{b^2-4(a+p)c}\sqrt{R}+\sqrt{-p}(b+2cx)} \quad [p\{b^2-4(a+p)c\} < 0, p < 0];$$

2.29 Integrale reductibile la integrale eliptice și pseudoeliptice

2.290 Integralele $\int R(x, \sqrt{P(x)}) dx$, unde $P(x)$ este un polinom de gradul trei sau patru, se reduc, prin transformări algebrice, la o sumă de integrale care se exprimă prin funcții elementare și la integrale eliptice (vezi 6.11). Deoarece substituțiile care transformă integrala dată într-o integrală eliptică de formă normală Legendre sînt diferite pentru diferite intervale de integrare, formulele respective sînt date în secțiunea integralelor definite (vezi 3.13, 3.14).

2.291 La integralele de forma $\int R(x, \sqrt{P(x)}) dx$ se reduc unele integrale de forma $\int R(x, \sqrt[k]{P_n(x)}) dx$, unde $k \geq 2$, iar $P_n(x)$ este un polinom de grad mai mare decît patru. Dăm mai jos exemple de astfel de reduceri.

$$1. \int \frac{dx}{\sqrt{1-x^6}} = - \int \frac{dz}{\sqrt{3+3z^2+z^4}} \quad [x^2 = \frac{1}{1+z^2}].$$

$$2. \int \frac{dx}{\sqrt{a+bx^3+cx^4+dx^5}} = \frac{1}{2} \int \frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}} \quad [x^2=z].$$

$$3. \int (a+2bx+cx^2+gx^3)^{\pm \frac{1}{3}} dx = \frac{3}{2} \int \frac{z^2 A^{\pm \frac{1}{3}} dz}{B} \quad [a+2bx+cx^2+gx^3=z^3, \\ A = g \left(\frac{-b+\sqrt{b^2+(z^3-a)c}}{c} \right)^3 + z^3, B = \sqrt{b^2+(z^3-a)c}].$$

$$4. \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}} = \\ = -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x=z+\sqrt{z^2-1}]; \\ = -\frac{1}{\sqrt{2}} \int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x=z-\sqrt{z^2-1}],$$

unde $p=2a(4z^3-3z)+2b(2z^2-1)+2cz+d$.

$$\begin{aligned}
 5. \quad \int \frac{dx}{\sqrt{a+bx^3+cx^4+bx^4+ax^5}} &= \frac{1}{2} \int \frac{dy}{\sqrt{y} \sqrt{a+by+cy^2+by^3+ay^4}} \quad [x=\sqrt{y}]; \\
 &= -\frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [y=z+\sqrt{z^2-1}]; \\
 &= -\frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [y=z-\sqrt{z^2-1}].
 \end{aligned}$$

unde $p = 2a(2z^2-1) + 2bz + c$.

$$\begin{aligned}
 6. \quad \int \frac{dx}{\sqrt{a+bx^4+cx^5}} &= \frac{1}{2} \sqrt[8]{\frac{a}{c}} \int \frac{dt}{\sqrt{t} \sqrt{a+b \frac{t^2}{c} + at^4}} \quad \left[x = \sqrt[8]{\frac{a}{c}} \sqrt[4]{t} \right]; \\
 &= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}} \right\} \quad [t=z+\sqrt{z^2-1}]; \\
 &= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}} \right\} \quad [t=z-\sqrt{z^2-1}],
 \end{aligned}$$

unde $p = 2a(2z^2-1) + b_1$; $b_1 = b \sqrt[8]{\frac{a}{c}}$.

$$\begin{aligned}
 7. \quad \int \frac{x dx}{\sqrt{a+bx^3+cx^4}} &= 2 \int \frac{z^2 dz}{\sqrt{A+Bz^4}} \\
 [a+bx^3+cx^4 &= z^4, \quad A=b^2-4ac, \quad B=4c].
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int \frac{dx}{\sqrt{a+2bx^3+cx^4}} &= \int \frac{\sqrt{b^2-a(c-z^4)}+b}{(c-z^4)\sqrt{b^2-a(c-z^4)}} z^2 dz = \\
 &= \int R_1(z^4) z^2 dz + \int \frac{R_2(z^4) z^2 dz}{\sqrt{b^2-a(c-z^4)}},
 \end{aligned}$$

unde $R_1(z^4)$ și $R_2(z^4)$ sînt funcții raționale de z^4 ; $a+2bx^3+cx^4-x^4z^4$.

2.292 În unele cazuri, integralele $\int R(x, \sqrt{P(x)}) dx$, unde $P(x)$ este un polinom de gradul trei sau patru, pot fi exprimate cu ajutorul unor funcții elementare. Aceste integrale se numesc *pseudoeliptice*.

Astfel, dacă au loc relațiile

$$\begin{aligned}
 f_1(x) &= -f_1\left(\frac{1}{k^2x}\right), \quad f_2(x) = -f_2\left(\frac{1-k^2x}{k^2(1-x)}\right), \\
 f_3(x) &= -f_3\left(\frac{1-x}{1-k^2x}\right),
 \end{aligned}$$

avem

$$\begin{aligned}
 1. \quad \int \frac{f_1(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_1(z) dz \quad [zx = \sqrt{x(1-x)(1-k^2x)}]; \\
 2. \quad \int \frac{f_2(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_2(z) dz \quad \left[z = \frac{\sqrt{x(1-k^2x)}}{\sqrt{1-x}} \right]; \\
 3. \quad \int \frac{f_3(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_3(z) dz \quad \left[z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}} \right].
 \end{aligned}$$

unde $R_1(z)$, $R_2(z)$, $R_3(z)$ sînt funcții raționale de z .

2.3 FUNCȚIA EXPONENȚIALĂ. FUNCȚII HIPERBOLICE

2.31 Forme care conțin e^{ax}

$$2.311 \quad \int e^{ax} dx = \frac{e^{ax}}{a}.$$

2.312 a^x în funcțiile de sub semnul integrală trebuie înlocuit cu $e^{x \ln a} = a^x$.

2.313

$$1. \quad \int \frac{dx}{a + be^{mx}} = \frac{1}{am} [mx - \ln(a + be^{mx})] \quad P(410)$$

$$2. \quad \int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x). \quad P(409)$$

$$2.314 \quad \int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \operatorname{arctg}\left(e^{mx} \sqrt{\frac{a}{b}}\right) [ab > 0]; \quad P(411)$$

$$= \frac{1}{2m\sqrt{-ab}} \ln \frac{b + e^{mx}\sqrt{-ab}}{b - e^{mx}\sqrt{-ab}} [ab < 0].$$

$$2.315 \quad \int \frac{dx}{\sqrt{a + be^{mx}}} = \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} [a > 0];$$

$$= \frac{2}{m\sqrt{-a}} \operatorname{arctg} \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} [a < 0].$$

2.32 Funcția exponențială și funcții raționale de x

2.321

$$1. \quad \int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$$

$$2. \quad \int x^n e^{ax} dx = e^{ax} \left(\frac{x^n}{a} + \sum_{k=1}^n (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k} \right)$$

2.322

$$1. \quad \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right).$$

$$2. \quad \int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$$

$$3. \quad \int x^3 e^{ax} dx = e^{ax} \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4} \right).$$

$$2.323 \quad \int P_m(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m (-1)^k \frac{P^{(k)}(x)}{a^k},$$

unde $P_m(x)$ este un polinom de gradul m în x , iar $P^{(k)}(x)$ derivata de ordinul k a lui $P_m(x)$ în raport cu x .

2.324

$$1. \quad \int \frac{e^{ax} dx}{x^m} = \frac{1}{m-1} \left[-\frac{e^{ax}}{x^{m-1}} + a \int \frac{e^{ax} dx}{x^{m-1}} \right].$$

$$2. \quad \int \frac{e^{ax}}{x^n} dx = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \operatorname{Ei}(ax)$$

2.325

$$1. \int \frac{e^{ax}}{x} dx = \text{Ei}(ax).$$

$$2. \int \frac{e^{ax}}{x^2} dx = -\frac{e^{ax}}{x} + a \text{Ei}(ax).$$

$$3. \int \frac{e^{ax}}{x^3} dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2} \text{Ei}(ax).$$

2.326

$$\int \frac{xe^{ax} dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$$

Funcții hiperbolice

Observația de la punctul 1.30 este valabilă și pentru integralele funcțiilor hiperbolice.

2.33 Forme care conțin $\text{sh } x$ și $\text{ch } x$.

2.331

$$1. \int \text{sh}^n x dx = \frac{\text{sh}^{n-1} x \text{ch } x}{n} - \frac{n-1}{n} \int \text{sh}^{n-2} x dx;$$

$$= \frac{\text{sh}^{n+1} x \text{ch } x}{n+1} - \frac{n+2}{n+1} \int \text{sh}^{n+2} x dx.$$

$$2. \int \text{ch}^n x dx = \frac{\text{ch}^{n-1} x \text{sh } x}{n} + \frac{n-1}{n} \int \text{ch}^{n-2} x dx;$$

$$= -\frac{\text{ch}^{n+1} x \text{sh } x}{n+1} + \frac{n+2}{n+1} \int \text{ch}^{n+2} x dx.$$

Pentru $n = -1$,

$$3. \int \frac{dx}{\text{sh } x} = \ln \text{th } \frac{x}{2}.$$

$$4. \int \frac{dx}{\text{ch } x} = \text{arctg}(\text{sh } x) = 2 \text{arctg}(e^x) = \arcsin(\text{th } x) = \text{gd } x.$$

2.332

$$1. \int \text{sh}^{2n} x dx = (-1)^n \binom{2n}{n} \frac{x}{2^{2n}} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\text{sh}(2n-2k)x}{2n-2k} \\ \text{(cf. 2.417 1.).} \quad \text{T (543)}$$

$$2. \int \text{sh}^{2n+1} x dx = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\text{ch}(2n-2k+1)x}{2n-2k+1} \\ \text{(cf. 2.417 2.).} \quad \text{T (544)}$$

$$3. \int \text{ch}^{2n} x dx = \binom{2n}{n} \frac{x}{2^{2n}} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\text{sh}(2n-2k)x}{2n-2k} \\ \text{(cf. 2.418 1.).} \quad \text{T (541)}$$

$$4. \int \text{ch}^{2n+1} x dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\text{sh}(2n-2k+1)x}{2n-2k+1} \\ \text{(cf. 2.418 2.).} \quad \text{T (542)}$$

2.333

$$1. \int \text{sh } x dx = \text{ch } x.$$

$$4. \int \text{ch } x dx = \text{sh } x.$$

$$2. \int \text{sh}^2 x dx = \frac{1}{2} (\text{sh } x \text{ch } x - x).$$

$$5. \int \text{ch}^2 x dx = \frac{1}{2} (\text{ch } x \text{sh } x + x).$$

$$3. \int \frac{dx}{\text{sh}^3 x} = -\text{cth } x.$$

$$6. \int \frac{dx}{\text{ch}^3 x} = \text{th } x.$$

2.334

1. $\int \operatorname{sh} mx \operatorname{sh} nx \, dx = \frac{1}{m^2 - n^2} [m \operatorname{sh} nx \operatorname{ch} mx - n \operatorname{ch} nx \operatorname{sh} mx].$
2. $\int \operatorname{ch} mx \operatorname{sh} nx \, dx = \frac{1}{m^2 - n^2} [m \operatorname{sh} nx \operatorname{sh} mx - n \operatorname{ch} nx \operatorname{ch} mx].$
3. $\int \operatorname{sh} x \operatorname{ch} x \, dx = \frac{1}{4} \operatorname{ch} 2x.$
4. $\int \operatorname{ch} mx \operatorname{ch} nx \, dx = \frac{1}{m^2 - n^2} [m \operatorname{sh} mx \operatorname{ch} nx - n \operatorname{sh} nx \operatorname{ch} mx]$

2.335

1. $\int \operatorname{sh} x \sin x \, dx = \frac{1}{2} (\operatorname{ch} x \sin x - \operatorname{sh} x \cos x).$
2. $\int \operatorname{sh} x \cos x \, dx = \frac{1}{2} (\operatorname{ch} x \cos x + \operatorname{sh} x \sin x).$
3. $\int \operatorname{ch} x \sin x \, dx = \frac{1}{2} (\operatorname{sh} x \sin x - \operatorname{ch} x \cos x).$
4. $\int \operatorname{ch} x \cos x \, dx = \frac{1}{2} (\operatorname{sh} x \cos x + \operatorname{ch} x \sin x).$

$$2.336 \quad \int \frac{dx}{\operatorname{ch} a + \operatorname{ch} x} = \operatorname{cosech} a \left[\ln \operatorname{ch} \frac{x+a}{2} - \ln \operatorname{ch} \frac{x-a}{2} \right];$$

$$= 2 \operatorname{cosech} a \operatorname{arth} \left(\operatorname{th} \frac{x}{2} \operatorname{th} \frac{a}{2} \right).$$

$$2.337 \quad \int \frac{dx}{\cos a + \operatorname{ch} x} = 2 \operatorname{cosec} a \operatorname{arctg} \left(\operatorname{th} \frac{x}{2} \operatorname{tg} \frac{a}{2} \right).$$

2.34 Forme care conțin $\operatorname{th} x$ și $\operatorname{cth} x$

2.341

1. $\int \operatorname{th}^n x \, dx = -\frac{\operatorname{th}^{n-1} x}{n-1} + \int \operatorname{th}^{n-2} x \, dx \quad (\text{cf. 2.431 1.}) \quad \text{T 545}$
2. $\int \operatorname{cth}^n x \, dx = -\frac{\operatorname{cth}^{n-1} x}{(n-1)} + \int \operatorname{cth}^{n-2} x \, dx \quad (\text{cf. 2.432 3.}) \quad \text{T 546}$

Pentru $n=1$,

3. $\int \operatorname{th} x \, dx = \ln \operatorname{ch} x \quad (\text{cf. 2.431 4.})$
4. $\int \operatorname{cth} x \, dx = \ln \operatorname{sh} x \quad (\text{cf. 2.432 4.})$

2.342

1. $\int \operatorname{th}^2 x \, dx = x - \operatorname{th} x.$
2. $\int \operatorname{cth}^2 x \, dx = x - \operatorname{cth} x.$

2.35 Funcții hiperbolice și puteri ale lui x

2.351

1. $\int x^n \operatorname{sh} x \, dx = x^n \operatorname{ch} x - n \int x^{n-1} \operatorname{ch} x \, dx =$
 $= x^n \operatorname{ch} x - n x^{n-1} \operatorname{sh} x + n(n-1) \int x^{n-2} \operatorname{sh} x \, dx \quad (\text{cf. 2.513 1.}) \quad \text{T 560}$
2. $\int x^n \operatorname{ch} x \, dx = x^n \operatorname{sh} x - n \int x^{n-1} \operatorname{sh} x \, dx =$
 $= x^n \operatorname{sh} x - n x^{n-1} \operatorname{ch} x + n(n-1) \int x^{n-2} \operatorname{ch} x \, dx \quad (\text{cf. 2.513 2.}) \quad \text{T 559}$

2.352

1. $\int x \operatorname{sh} x \, dx = x \operatorname{ch} x - \operatorname{sh} x.$
2. $\int x^2 \operatorname{sh} x \, dx = (x^2 + 2) \operatorname{ch} x - 2x \operatorname{sh} x.$
3. $\int x \operatorname{ch} x \, dx = x \operatorname{sh} x - \operatorname{ch} x.$
4. $\int x^2 \operatorname{ch} x \, dx = (x^2 + 2) \operatorname{sh} x - 2x \operatorname{ch} x.$

2.4-2.5 FUNCȚII TRIGONOMETRICE

2.40 Introducere

2.401 Integralele $\int R(\sin x, \cos x) dx$ pot fi reduse întotdeauna la integrale de funcții raționale, cu ajutorul substituției $t = \operatorname{tg} \frac{x}{2}$

2.402. Dacă funcția $R(\sin x, \cos x)$ satisface, totodată, relația

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

atunci este avantajos să folosim substituția $t = \cos x$;

2.403 dacă această funcție satisface relația

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

atunci este avantajos să folosim substituția $t = \sin x$;

2.404 dacă această funcție satisface relația

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

atunci este avantajos să folosim substituția $t = \operatorname{tg} x$.

2.41-2.42 Puteri ale sinusului și ale cosinusului

$$\begin{aligned}
 2.410 \quad \int \sin^p x \cos^q x \, dx &= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x \, dx; \\
 &= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx; \\
 &= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x \, dx; \\
 &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x \, dx; \\
 &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x \, dx; \\
 &= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x \, dx; \\
 &= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left\{ \sin^2 x - \frac{q-1}{p+q-2} \right\} + \\
 &+ \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x \, dx.
 \end{aligned}$$

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$$\begin{aligned}
 2.411 \quad \int \sin^m x \cos^{2n} x \, dx &= \frac{\sin^{m+1} x}{2n+m} \left\{ \cos^{2n-1} x + \right. \\
 &+ \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1) \cos^{2n-2k-1} x}{(2n+m-2)(2n+m-4)\dots(2n+m-2k)} \left. \right\} + \\
 &+ \frac{(2n-1)!!}{(2n+m)(2n+m-2)\dots(m+2)} \int \sin^m x \, dx.
 \end{aligned}$$

Această formulă este aplicabilă pentru orice m real, cu excepția următoarelor numere pare negative: $-2, -4, \dots, -2n$. Pentru m natural și $n=0$ avem

2.412

$$1. \int \sin^{2n} x \, dx = -\frac{\cos x}{2n} \left\{ \sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sin^{2n-2k-1} x \right\} + \frac{(2n-1)!!}{2^n \cdot n!} x$$

(vezi și 2.417 1.). T (232)

$$2. \int \sin^{2n+1} x \, dx = -\frac{\cos x}{2n+1} \left\{ \sin^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \sin^{2n-2k-2} x \right\}$$

(vezi și 2.417 2.). T (233)

$$2.413 \int \sin^m x \cos^{2n+1} x \, dx = \frac{\sin^{m+1} x}{2n+m+1} \left\{ \cos^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1) \dots (n-k+1) \cos^{2n-2k} x}{(2n+m-1)(2n+m-3) \dots (2n+m-2k+1)} \right\}.$$

Această formulă este aplicabilă pentru orice m real, cu excepția următoarelor numere impare negative: $-3, -5, \dots, -(2n+1)$.

$$2.414 \int \cos^m x \sin^{2n} x \, dx = -\frac{\cos^{m+1} x}{2n+m} \left\{ \sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \sin^{2n-2k-1} x}{(2n+m-2)(2n+m-4) \dots (2n+m-2k)} \right\} + \frac{(2n-1)!!}{(2n+m)(2n+m-2) \dots (m+2)} \int \cos^m x \, dx.$$

Această formulă este aplicabilă pentru orice m real, cu excepția următoarelor numere pare negative: $-2, \dots, -4, -2n$. Pentru m natural și $n=0$, avem

2.415

$$1. \int \cos^{2n} x \, dx = \frac{\sin x}{2n} \left\{ \cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \cos^{2n-2k-1} x}{2^k (n-1)(n-2) \dots (n-k)} \right\} + \frac{(2n-1)!!}{2^n \cdot n!} x$$

(vezi și 2.418 1.). T (230)

$$2. \int \cos^{2n+1} x \, dx = \frac{\sin x}{2n+1} \left\{ \cos^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} (n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \cos^{2n-2k-2} x \right\}.$$

(vezi și 2.418 2.). T (231)

$$2.416 \int \cos^m x \sin^{2n+1} x \, dx = -\frac{\cos^{m+1} x}{2n+m+1} \left\{ \sin^{2n-2} x + \sum_{k=1}^n \frac{2^k n(n-1) \dots (n-k+1) \sin^{2n-2k-2} x}{(2n+m-1)(2n+m-3) \dots (2n+m-2k+1)} \right\}.$$

Această formulă este aplicabilă pentru orice m real, cu excepția următoarelor numere negative: $-3, -5, \dots, -(2n+1)$.

2.417

$$1. \int \sin^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(vezi și 2.412 1.). T (226)

$$2. \int \sin^{2n+1} x \, dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$

(vezi și 2.412 2.). T (227)

2.418

$$1. \int \cos^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(vezi și 2.415 1.). T (224)

$$2. \int \cos^{2n+1} x \, dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}$$

(vezi și 2.415 2.). T (225)

2.419

$$1. \int \sin(px+q) \, dx = -\frac{1}{p} \cos(px+q).$$

$$2. \int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$3. \int \sin^3 x \, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x.$$

$$4. \int \cos(px+q) \, dx = \frac{1}{p} \sin(px+q).$$

$$5. \int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$6. \int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x.$$

$$7. \int \sin x \cos^2 x \, dx = -\frac{1}{4} \left\{ \frac{1}{3} \cos 3x + \cos x \right\} = -\frac{\cos^3 x}{3}.$$

$$8. \int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}.$$

$$9. \int \sin^2 x \cos x \, dx = -\frac{1}{4} \left\{ \frac{1}{3} \sin 3x - \sin x \right\} = \frac{1}{3} \sin^3 x.$$

$$10. \int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left\{ \frac{1}{4} \sin 4x - x \right\}.$$

$$11. \int \sin^2 x \cos^3 x \, dx = -\frac{1}{16} \left\{ \frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right\}.$$

$$12. \int \sin^3 x \cos x \, dx = \frac{1}{8} \left(\frac{1}{4} \cos 4x - \cos 2x \right) = \frac{1}{4} \sin^4 x.$$

$$13. \int \sin^3 x \cos^2 x \, dx = \frac{1}{16} \left(\frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right).$$

$$14. \int \sin^3 x \cos^3 x \, dx = \frac{1}{32} \left(\frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right).$$

$$2.421 \quad \int \frac{\sin^m x}{\cos^{2n} x} dx = \frac{\sin^{m+1} x}{2n-1} \left\{ \sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-m-2)(2n-m-4) \dots (2n-m-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right\} + \frac{(2n-m-2)(2n-m-4) \dots (-m+2)(-m)}{(2n-1)!!} \int \sin^m x dx.$$

Această formulă este aplicabilă pentru orice m real. $\int \sin^m x dx$, pentru m natural, vezi 2.412 și 2.417. Pentru m întreg și negativ, avem pentru această integrală

2.422

$$1. \quad \int \frac{dx}{\sin^{2n} x} = -\frac{\cos x}{2n-1} \left\{ \operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right\}. \quad T(242)$$

$$2. \quad \int \frac{dx}{\sin^{2n+1} x} = -\frac{\cos x}{2n} \left\{ \operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x \right\} + \frac{(2n-1)!!}{2^n \cdot n!} \ln \operatorname{tg} \frac{x}{2}. \quad T(243)$$

$$2.423 \quad \int \frac{\sin^m x}{\cos^{2n+1} x} dx = \frac{\sin^{m+1} x}{2n} \left\{ \sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-m-1)(2n-m-3) \dots (2n-m-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right\} + \frac{(2n-m-1)(2n-m-3) \dots (3-m)(1-m)}{2^n \cdot n!} \int \frac{\sin^m x}{\cos x} dx.$$

Această formulă este aplicabilă pentru orice m real. Pentru $n=0$ și m natural, avem

2.424

$$1. \quad \int \frac{\sin^{2n+1} x dx}{\cos x} = -\sum_{k=1}^n \frac{\sin^{2n-2k+2} x}{2n-2k+2} - \ln \cos x.$$

$$2. \quad \int \frac{\sin^{2n} x dx}{\cos x} = -\sum_{k=1}^n \frac{\sin^{2n-2k+1} x}{2n-2k+1} + \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$2.425 \quad \int \frac{\cos^m x dx}{\sin^{2n} x} = -\frac{\cos^{m+1} x}{2n-1} \left\{ \operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-m-2)(2n-m-4) \dots (2n-m-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right\} + \frac{(2n-m-2)(2n-m-4) \dots (2-m)(-m)}{(2n-1)!!} \int \cos^m x dx.$$

Această formulă este aplicabilă pentru orice m real. $\int \cos^m x dx$, pentru m natural, vezi 2.415 și 2.418. Pentru m întreg și negativ, avem pentru această integrală

2.426

$$1. \int \frac{dx}{\cos^{2n} x} = \frac{\sin x}{2n-1} \left\{ \sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right\}. \quad T(240)$$

$$2. \int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n} \left\{ \sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-2) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right\} + \frac{(2n-1)!!}{2^n \cdot n!} \ln \left(\frac{\pi}{4} + \frac{x}{2} \right). \quad T(241)$$

$$2.427 \quad \int \frac{\cos^m x dx}{\sin^{2n+1} x} = -\frac{\cos^{m+1} x}{2n} \left\{ \operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-m-1)(2n-m-3) \dots (2n-m-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x \right\} + \frac{(2n-m-1)(2n-m-3) \dots (3-m)(1-m)}{2^n \cdot n!} \int \frac{\cos^m x}{\sin x} dx.$$

Această formulă este aplicabilă pentru orice m real.
Pentru $n=0$ și m natural, avem

2.428

$$1. \int \frac{\cos^{2n+1} x dx}{\sin x} = \sum_{k=1}^n \frac{\cos^{2n-2k+2} x}{2n-2k+2} + \ln \sin x.$$

$$2. \int \frac{\cos^{2n} x dx}{\sin x} = \sum_{k=1}^n \frac{\cos^{2n-2k+1} x}{2n-2k+1} + \ln \operatorname{tg} \frac{x}{2}.$$

2.429

$$1. \int \frac{dx}{\sin x} = \ln \operatorname{tg} \frac{x}{2}.$$

$$2. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$3. \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$4. \int \frac{dx}{\cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) = \ln \sqrt{\frac{1+\sin x}{1-\sin x}}.$$

$$5. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$6. \int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$7. \int \frac{\sin x}{\cos x} dx = -\ln \cos x.$$

$$8. \int \frac{\sin^2 x}{\cos x} dx = -\sin x + \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

9. $\int \frac{\sin^2 x}{\cos x} dx = -\frac{\sin^2 x}{2} - \ln \cos x.$
10. $\int \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} :$
11. $\int \frac{\sin^2 x}{\cos^2 x} = \operatorname{tg} x - x.$
12. $\int \frac{\sin^2 x}{\cos^2 x} = \cos x + \frac{1}{\cos x}.$
13. $\int \frac{\sin x}{\cos^2 x} = \frac{1}{2 \cos^2 x}.$
14. $\int \frac{\sin^2 x}{\cos^2 x} = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$
15. $\int \frac{\sin^2 x}{\cos^2 x} = \frac{1}{2 \cos^2 x} + \ln \cos x.$
16. $\int \frac{\cos x}{\sin x} = \ln \sin x.$
17. $\int \frac{\cos^2 x}{\sin x} = \cos x + \ln \operatorname{tg} \frac{x}{2}.$
18. $\int \frac{\cos^2 x}{\sin x} = \frac{\cos^2 x}{2} + \ln \sin x.$
19. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}.$
20. $\int \frac{\cos^2 x}{\sin^2 x} dx = -\operatorname{ctg} x - x.$
21. $\int \frac{\cos^2 x}{\sin^2 x} dx = -\sin x - \frac{1}{\sin x}.$
22. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{2 \sin^2 x}.$
23. $\int \frac{\cos^2 x}{\sin^2 x} dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \operatorname{tg} \frac{x}{2}.$
24. $\int \frac{\cos^2 x}{\sin^2 x} dx = -\frac{1}{2 \sin^2 x} - \ln \sin x.$
25. $\int \frac{dx}{\sin x \cos x} = \ln \operatorname{tg} x.$
26. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \operatorname{tg} \frac{x}{2}.$
27. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{2 \cos^2 x} + \ln \operatorname{tg} x.$
28. $\int \frac{dx}{\sin^2 x \cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - \operatorname{cosec} x.$
29. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \operatorname{ctg} 2x.$
30. $\int \frac{dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{2 \cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right)$
31. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln \operatorname{tg} x.$
32. $\int \frac{dx}{\sin^2 x \cos^2 x} = -\frac{1}{\cos x} \left(\frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \operatorname{tg} \frac{x}{2}.$
33. $\int \frac{dx}{\sin^2 x \cos^2 x} = -\frac{2 \cos 2x}{\sin^2 2x} + 2 \ln \operatorname{tg} x.$

2.43 Puteri ale tangentei și ale cotangentei

2.431

$$1. \int \operatorname{tg}^n x \, dx = \frac{\operatorname{tg}^{n-1} x}{n-1} - \int \operatorname{tg}^{n-2} x \, dx. \quad \text{T (250)}$$

$$2. \int \operatorname{tg}^{2n} x \, dx = \sum_{k=1}^n \frac{(-1)^{k-1} \operatorname{tg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x. \quad \text{T (251)}$$

$$3. \int \operatorname{tg}^{2n+1} x \, dx = \sum_{k=1}^n (-1)^{k-1} \frac{\operatorname{tg}^{2n-2k+2} x}{2n-2k+2} + (-1)^{n-1} \ln \cos x. \quad \text{T (252)}$$

$$4. \int \operatorname{tg} x \, dx = -\ln \cos x.$$

$$5. \int \operatorname{tg}^2 x \, dx = \operatorname{tg} x - x.$$

$$6. \int \operatorname{tg}^2 x \, dx = \frac{1}{2} \operatorname{tg}^2 x + \ln \cos x.$$

2.432

$$1. \int \operatorname{ctg}^{2n+1} x \, dx = \sum_{k=1}^n (-1)^k \frac{\operatorname{ctg}^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln \sin x. \quad \text{T (254')}$$

$$2. \int \operatorname{ctg}^{2n} x \, dx = \sum_{k=1}^n (-1)^k \frac{\operatorname{ctg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x. \quad \text{T (253)}$$

$$3. \int \operatorname{ctg}^n x \, dx = \int \frac{dx}{\operatorname{tg}^n x} = -\frac{\operatorname{ctg}^{n-1} x}{(n-1)} - \int \operatorname{ctg}^{n-2} x \, dx. \quad \text{T (250')}$$

$$4. \int \frac{dx}{\operatorname{tg} x} = \int \operatorname{ctg} x \, dx = \ln \sin x.$$

$$5. \int \operatorname{ctg}^2 x \, dx = -\operatorname{ctg} x - x.$$

$$6. \int \operatorname{ctg}^2 x \, dx = -\frac{1}{2} \operatorname{ctg}^2 x - \ln \sin x.$$

2.44 Sinușii și cosinușii arcelor multiple

2.441

$$1. \int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \sin (n-1) x \, dx. \quad \text{P (366)}$$

$$2. \int \sin^m x \cos nx \, dx = \frac{\sin^m x \sin nx}{m+n} - \frac{m}{m+n} \int \sin^{m-1} x \sin (n-1) x \, dx. \quad \text{P (367)}$$

2.442

$$1. \int \sin (2n+1) x \sin^m x \, dx = (2n+1) \left\{ \int \sin^{m+1} x \, dx + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k+1)!} \int \sin^{2k+m+1} x \, dx \right\}. \quad \text{T (299)}$$

$$2. \int \sin 2nx \sin^m x \, dx = 2n \left\{ \frac{\sin^{m+2} x}{m+2} + \right. \\ \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots [4n^2-(2k)^2]}{(2k+1)!(2k+m+2)} \sin^{2k+m+2} x \right\}. \quad \text{T (303)}$$

$$3. \int \sin(2n+1)x \cos^m x \, dx = (-1)^{n+1} \left\{ \frac{\cos^{m+1} x}{m+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+m+1)} \cos^{2k+m+1} x \right\}. \quad T(295)$$

$$4. \int \sin 2nx \cos^m x \, dx = (-1)^n \left\{ \frac{\cos^{m+2} x}{m+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2)\dots[4n^2-(2k)^2]}{(2k+1)!(2k+m+2)} \cos^{2k+m+2} x \right\}. \quad T(297)$$

$$5. \int \cos(2n+1)x \sin^m x \, dx = \frac{\sin^{m+1} x}{m+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+m+1)} \sin^{2k+m+1} x \Big\}. \quad T(301)$$

$$6. \int \cos 2nx \sin^m x \, dx = \int \sin^m x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2 \cdot (4n^2-2^2)\dots[4n^2-(2k-2)^2]}{(2k)!} \int \sin^{2k+m} x \, dx. \quad T(300)$$

$$7. \int \cos(2n+1)x \cos^m x \, dx = (-1)^n (2n+1) \left\{ \int \cos^{m+1} x \, dx + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \int \cos^{2k+m+1} x \, dx \right\}. \quad T(293)$$

$$8. \int \cos 2nx \cos^m x \, dx = (-1)^n \left\{ \int \cos^m x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2 [4n^2-2^2]\dots[4n^2-(2k-2)^2]}{(2k)!} \int \cos^{2k+m} x \, dx \right\}. \quad T(294)$$

Formulele 2.442 3. și 2.442 5. sînt inaplicabile pentru $m = -3, -5, \dots, -(2n+1)$; formulele 2.442 2. și 2.442 4. sînt inaplicabile pentru $m = -2, -4, \dots, -2n$. În afară de excepțiile menționate, m poate avea orice valori reale în formulele 2.441 și 2.442.

2.443

$$1. \int \frac{\sin 2x}{\sin^n x} dx = -\frac{2}{(n-2)\sin^{n-2} x}.$$

Pentru $n=2$,

$$\int \frac{\sin 2x}{\sin^2 x} dx = 2 \ln \sin x.$$

$$2. \int \frac{\sin 2x}{\cos^n x} dx = \frac{2}{(n-2)\cos^{n-2} x}.$$

Pentru $n=2$,

$$\int \frac{\sin 2x}{\cos^2 x} dx = -2 \ln \cos x.$$

2.444

$$1. \int \frac{\cos 2x}{\sin x} dx = 2 \cos x + \ln \operatorname{tg} \frac{x}{2}.$$

$$2. \int \frac{\cos 2x}{\sin^3 x} dx = -\operatorname{ctg} x - 2x.$$

3. $\int \frac{\cos 2x dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \operatorname{tg} \frac{x}{2}.$
4. $\int \frac{\cos 2x dx}{\cos x} = 2 \sin x - \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$
5. $\int \frac{\cos 2x dx}{\cos^3 x} = 2x - \operatorname{tg} x.$
6. $\int \frac{\cos 2x dx}{\cos^3 x} = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$
7. $\int \frac{\sin 3x dx}{\sin x} = x + \sin 2x.$
8. $\int \frac{\sin 3x}{\sin^3 x} dx = 3 \ln \operatorname{tg} \frac{x}{2} + 4 \cos x.$
9. $\int \frac{\sin 3x}{\sin^3 x} dx = -3 \operatorname{ctg} x - 4x.$

2.445

$$1. \int \frac{\sin 3x}{\cos^n x} dx = \frac{4}{(n-3) \cos^{n-3} x} - \frac{1}{(n-1) \cos^{n-1} x}$$

Pentru $n=1$ și $n=3$,

$$\int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2 \cos^2 x} - 4 \ln \cos x.$$

$$\int \frac{\sin 3x}{\cos x} dx = 2 \sin^2 x + \ln \cos x.$$

$$2. \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}$$

Pentru $n=1$ și $n=3$,

$$\int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln \sin x.$$

$$\int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln \sin x.$$

$$3. \int \frac{\sin nx}{\cos^p x} dx = 2 \int \frac{\sin (n-1)x dx}{\cos^{p-1} x} - \int \frac{\sin (n-2)x dx}{\cos^p x}.$$

$$4. \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x.$$

$$5. \int \frac{\cos 3x}{\cos^3 x} dx = 4 \sin x - 3 \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$6. \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \operatorname{tg} x.$$

2.446

$$1. \int \sin (n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \sin nx. \quad \text{BH [71] (1)m}$$

$$2. \int \sin (n+1)x \cos^{n-1} x dx = -\frac{1}{n} \cos^n x \cos nx. \quad \text{BH [71] (2)m}$$

$$3. \int \cos (n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \cos nx. \quad \text{BH [71] (3)m}$$

$$4. \int \cos (n+1)x \cos^{n-1} x dx = \frac{1}{n} \cos^n x \sin nx. \quad \text{BH [71] (4)m}$$

$$5. \int \sin \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1} x dx = \frac{1}{n} \sin^n x \cos n \left(\frac{\pi}{2} - x \right). \quad \text{BH [71] (5)m}$$

$$6. \int \cos \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1} x dx = -\frac{1}{n} \sin^n x \sin n \left(\frac{\pi}{2} - x \right). \quad \text{BH [71] (6)m}$$

2.447

$$1. \int \frac{\sin^m x \, dx}{\sin (2n+1)x} = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \frac{2k+1}{2(2n+1)} \pi \ln \frac{\sin \left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[\frac{k+n+1}{2(2n+1)} \pi - \frac{x}{2} \right]}$$

[m natural $\leq 2n$]. T (378)

$$2. \int \frac{\sin^{2m} x \, dx}{\sin 2nx} = \frac{(-1)^n}{2n} \left\{ \ln \cos x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[m natural $\leq n$]. T (379)

$$3. \int \frac{\sin^{2m+1} x \, dx}{\sin 2nx} = \frac{(-1)^n}{2n} \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + \right. \\ \left. + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\operatorname{tg} \left(\frac{n+k}{4n} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{n-k}{4n} \pi - \frac{x}{2} \right) \right] \right\}$$

[m natural $< n$]. T (380)

$$4. \int \frac{\sin^{2m} x \, dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + \right. \\ \left. + \sum_{k=1}^{n'} (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\operatorname{tg} \left(\frac{2n+2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{2n-2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \right] \right\}$$

[m natural $\leq n$]. T (381)

$$5. \int \frac{\sin^{2m+1} x \, dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \cos x + \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m natural $\leq n$]. T (382) m

$$6. \int \frac{\sin^m x \, dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \frac{2k+1}{4n} \pi \ln \frac{\sin \left[\frac{2k-2n+1}{8n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+2n+1}{8n} \pi - \frac{x}{2} \right]}$$

[m natural $< 2n$]. T (377)

$$7. \int \frac{\cos^{2m+1} x \, dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln \sin x + \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m natural $\leq n$]. T (376)

$$8. \int \frac{\cos^{2m} x \, dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln \operatorname{tg} \frac{x}{2} + \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n+2} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n+2} \right) \right] \right\}$$

[m natural $\leq n$]. T (375)

$$9. \int \frac{\cos^{2m+1} x \, dx}{\sin 2nx} = \frac{1}{2n} \left\{ \ln \operatorname{tg} \frac{x}{2} + \right. \\ \left. + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n} \right) \right] \right\}$$

[m natural $< n$]. T (374)

$$10. \int \frac{\cos^{2m} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln \sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\} \\ [m \text{ natural} \leq n]. \quad T (373)$$

$$11. \int \frac{\cos^m x}{\cos nx} dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n} \pi \ln \frac{\sin \left[\frac{2k+1}{4n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+1}{4n} \pi - \frac{x}{2} \right]} \\ [m \text{ natural} \leq n] \quad T (372)$$

2.448

$$1. \int \sin px \sin qx dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)} \\ 2. \int \sin px \cos qx dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)} \\ 3. \int \cos px \cos qx dx = \frac{\sin(p+q)x}{2(p+q)} + \frac{\sin(p-q)x}{2(p-q)} \\ 4. \int \sin ax \sin bx \sin cx dx = -\frac{1}{4} \left\{ \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right\} \quad P (376)$$

$$5. \int \sin ax \cos bx \cos cx dx = -\frac{1}{4} \left\{ \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right\} \quad P (378)$$

$$6. \int \cos ax \sin bx \sin cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right\} \quad P (379)$$

$$7. \int \cos ax \cos bx \cos cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right\} \quad P (377)$$

2.449

$$1. \int \frac{\cos px + i \sin px}{\sin nx} dx = -2 \int \frac{z^{p+n+1}}{1-z^{2n}} dz \\ 2. \int \frac{\cos px + i \sin px}{\cos nx} dx = -2i \int \frac{z^{p+n-1}}{1+z^{2n}} dz \quad [z = \cos x + i \sin x]. \quad P (374) \quad P (373)$$

2.45 Funcții raționale de sinus și de cosinus

Forme care conțin $a + b \cos_{\sin} x$ și $\alpha + \beta \cos_{\sin} x$

2.451

$$1. \int (a + b \cos x)^n (\alpha + \beta \cos x) dx = \frac{\beta \sin x}{n+1} (a + b \cos x)^n + \\ + \int (a + b \cos x)^{n-1} \left[\alpha a + \frac{n}{n+1} b\beta + \left(\beta a + \frac{n}{n+1} a\beta \right) \cos x \right] dx. \quad T (279) \\ 2. \int (a + b \cos x)^n (\alpha + \beta \sin x) dx = -\frac{\beta}{b(n+1)} (a + b \cos x)^{n+1} + \\ + \alpha \int (a + b \cos x)^n dx. \quad T (226)m$$

$$3. \int (a + b \sin x)^n (a + \beta \cos x) dx = \frac{\beta}{b(n+1)} (a + b \sin x)^{n+1} + \alpha \int (a + b \sin x)^n dx. \quad T(226)m$$

$$4. \int (a + b \sin x)^n (a + \beta \sin x) dx = -\frac{\beta \cos x}{n+1} (a + b \sin x)^n + \int (a + b \sin x)^{n-1} \left[a\alpha + \frac{n}{n+1} b\beta + \left(b\alpha + \frac{n}{n+1} a\beta \right) \sin x \right] dx. \quad T(283)m$$

2.452

$$1. \int \frac{\alpha + \beta \sin x}{(a + b \sin x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(b\alpha - a\beta) \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(a\alpha - b\beta)(n-1) + (a\beta - b\alpha)(n-2) \sin x}{(a + b \sin x)^{n-1}} dx \right]. \quad T(358)m$$

Pentru $n=1$,

$$2. \int \frac{\alpha + \beta \sin x}{a + b \sin x} dx = \frac{\beta}{b} x + \frac{a\alpha - a\beta}{b} \int \frac{dx}{a + b \sin x} \quad (\text{vezi 2.452 10.}). \quad T(342)$$

$$3. \int \frac{\alpha + \beta \cos x}{(a + b \sin x)^n} dx = -\frac{\beta}{(n-1)b(a + b \sin x)^{n-1}} + \alpha \int \frac{dx}{(a + b \sin x)^n}. \quad (T 360)$$

Pentru $n=1$,

$$4. \int \frac{\alpha + \beta \cos x}{a + b \sin x} dx = \frac{\beta}{b} \ln(a + b \sin x) + \alpha \int \frac{dx}{a + b \sin x} \quad (\text{vezi 2.452 10.}). \quad T(344)$$

$$5. \int \frac{\alpha + \beta \sin x}{(a + b \cos x)^n} dx = \frac{\beta}{(n-1)b(a + b \cos x)^{n-1}} + \alpha \int \frac{dx}{(a + b \cos x)^n}. \quad T(355)$$

$$6. \int \frac{\alpha + \beta \sin x}{a + b \cos x} dx = -\frac{\beta}{b} \ln(a + b \cos x) + \alpha \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T(343)$$

$$7. \int \frac{(\alpha + \beta \cos x) dx}{(a + b \cos x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(a\beta - ab) \sin x}{(a + b \cos x)^{n-1}} + \int \frac{[(a\alpha - b\beta)(n-1) + (n-2)(a\beta - ba) \cos x] dx}{(a + b \cos x)^{n-1}} \right]. \quad T(353)$$

Pentru $n=1$,

$$8. \int \frac{(\alpha + \beta \cos x) dx}{a + b \cos x} = \frac{\beta x}{b} + \frac{a\alpha - a\beta}{b} \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T(341)$$

$$9. \int \frac{dx}{(a + b \sin x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left\{ \frac{b \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(n-1)a - (n-2)b \sin x}{(a + b \sin x)^{n-1}} dx \right\}. \quad T(359)$$

$$10. \int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{a \operatorname{tg} \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \quad [a^2 > b^2];$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \operatorname{tg} \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \operatorname{tg} \frac{x}{2} + b + \sqrt{b^2 - a^2}} \quad [a^2 < b^2].$$

$$11. \int \frac{dx}{(a + b \cos x)^n} = -\frac{1}{(n-1)(a^2 - b^2)} \left\{ \frac{b \sin x}{(a + b \cos x)^{n-1}} - \int \frac{(n-1)a - (n-2)b \cos x}{(a + b \cos x)^{n-1}} dx \right\}. \quad T(354)$$

$$12. \int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{\sqrt{a^2-b^2} \operatorname{tg} \frac{x}{2}}{a+b} \quad [a^2 > b^2];$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln \frac{\sqrt{b^2-a^2} \operatorname{tg} \frac{x}{2} + a+b}{\sqrt{b^2-a^2} \operatorname{tg} \frac{x}{2} - a-b} \quad [a^2 < b^2]. \quad \text{Fil 93 T (305)}$$

La integrarea funcțiilor de la punctele 2.452 10. și 2.452 12., nu avem voie să trecem prin punctele în care funcția de sub semnul integrală devine infinită, adică prin punctele $x = \arcsin\left(-\frac{b}{a}\right)$, în formula 2.452 10., și prin punctele $x = \arccos\left(-\frac{b}{a}\right)$, în formula 2.452 12.

2.453

Formulele 2.452 10. și 2.452 12. sînt inaplicabile pentru $a^2 = b^2$. În aceste cazuri, putem folosi, în locul lor, următoarele formule:

$$1. \int \frac{\alpha + \beta \sin x}{(1 + \sin x)^n} dx = -\frac{1}{2^{n-1}} \left\{ 2\beta \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} - \frac{x}{2} \right)}{2k+1} + \right.$$

$$\left. + (\alpha - \beta) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} - \frac{x}{2} \right)}{2k+1} \right\}. \quad \text{T (361)}$$

Pentru $n=1$,

$$2. \int \frac{\alpha + \beta \sin x}{1 + \sin x} dx = \beta x + (\alpha - \beta) \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right). \quad \text{T 250}$$

$$3. \int \frac{\alpha + \beta \sin x}{(1 - \sin x)^n} dx = -\frac{1}{2^{n-1}} \left\{ 2\beta \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} + \frac{x}{2} \right)}{2k+1} - \right.$$

$$\left. - (\alpha + \beta) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} + \frac{x}{2} \right)}{2k+1} \right\}. \quad \text{T (361)m}$$

Pentru $n=1$,

$$4. \int \frac{\alpha + \beta \sin x}{1 - \sin x} dx = -\beta x + (\alpha + \beta) \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right). \quad \text{T 250}$$

$$5. \int \frac{\alpha + \beta \cos x}{(1 + \cos x)^n} dx =$$

$$= \frac{1}{2^{n-1}} \left\{ 2\beta \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \frac{x}{2}}{2k+1} + (\alpha - \beta) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \frac{x}{2}}{2k+1} \right\}. \quad \text{T (356)}$$

Pentru $n=1$,

$$6. \int \frac{\alpha + \beta \cos x}{1 + \cos x} dx = \beta x + (\alpha - \beta) \operatorname{tg} \frac{x}{2}. \quad \text{T 248}$$

$$7. \int \frac{\alpha + \beta \cos x}{(1 - \cos x)^n} dx =$$

$$= \frac{1}{2^{n-1}} \left\{ 2\beta \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{ctg}^{2k+1} \frac{x}{2}}{2k+1} - (\alpha + \beta) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{ctg}^{2k+1} \frac{x}{2}}{2k+1} \right\}. \quad \text{T (356)}$$

Pentru $n=1$,

$$8. \int \frac{\alpha + \beta \cos x}{1 - \cos x} dx = -\beta x - (\alpha + \beta) \operatorname{ctg} \frac{x}{2}. \quad T\ 248$$

2.454

$$1. \int \frac{(\alpha + \beta \sin x) dx}{\sin x (a + b \sin x)} = \frac{\alpha}{a} \ln \operatorname{tg} \frac{x}{2} + \frac{\alpha\beta - ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T\ (348)$$

$$2. \int \frac{(\alpha + \beta \sin x) dx}{\sin x (a + b \cos x)} = \frac{\alpha}{a^2 - b^2} \left\{ a \ln \operatorname{tg} \frac{x}{2} + b \ln \frac{a + b \cos x}{\sin x} \right\} + \beta \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T\ (349)$$

Pentru $a^2 = b^2 (=1)$,

$$3. \int \frac{(\alpha + \beta \sin x) dx}{\sin x (1 + \cos x)} = \frac{\alpha}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} + \frac{1}{2(1 + \cos x)} \right\} + \beta \operatorname{tg} \frac{x}{2}.$$

$$4. \int \frac{(\alpha + \beta \sin x) dx}{\sin x (1 - \cos x)} = \frac{\alpha}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} - \frac{1}{2(1 - \cos x)} \right\} - \beta \operatorname{ctg} \frac{x}{2}.$$

$$5. \int \frac{(\alpha + \beta \sin x) dx}{\cos x (a + b \sin x)} = \frac{1}{a^2 - b^2} \left\{ (a\alpha - b\beta) \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - (ab - \alpha\beta) \ln \frac{a + b \sin x}{\cos x} \right\}. \quad T\ (346)$$

Pentru $a^2 = b^2 (=1)$,

$$6. \int \frac{(\alpha + \beta \sin x) dx}{\cos x (1 \pm \sin x)} = -\frac{\alpha \pm \beta}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{\alpha \mp \beta}{2(1 \pm \sin x)}.$$

$$7. \int \frac{(\alpha + \beta \sin x) dx}{\cos x (a + b \cos x)} = \frac{\alpha}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{\beta}{a} \ln \frac{a + b \cos x}{\cos x} - \frac{ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T\ (351)m$$

$$8. \int \frac{(\alpha + \beta \cos x) dx}{\sin x (a + b \sin x)} = \frac{\alpha}{a} \ln \operatorname{tg} \frac{x}{2} - \frac{\beta}{a} \ln \frac{a + b \sin x}{\sin x} - \frac{ab}{a} \int \frac{dx}{a + b \sin x} \quad (\text{vezi 2.452 10.}). \quad T\ (352)$$

$$9. \int \frac{(\alpha + \beta \cos x) dx}{\sin x (a + b \cos x)} = \frac{1}{a^2 - b^2} \left\{ (a\alpha - b\beta) \ln \operatorname{tg} \frac{x}{2} + (ab - \alpha\beta) \frac{a + b \cos x}{\sin x} \right\}. \quad T\ (345)$$

Pentru $a^2 = b^2 (=1)$,

$$10. \int \frac{(\alpha + \beta \cos x) dx}{\sin x (1 \pm \cos x)} = \pm \frac{\alpha \mp \beta}{2(1 \pm \cos x)} + \frac{\alpha \pm \beta}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$11. \int \frac{(\alpha + \beta \cos x) dx}{\cos x (a + b \sin x)} = \frac{\alpha}{a^2 - b^2} \left\{ a \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - b \ln \frac{a + b \sin x}{\cos x} \right\} + \beta \int \frac{dx}{a + b \sin x} \quad (\text{vezi 2.452 10.}). \quad T\ (350)$$

Pentru $a^2 = b^2 (=1)$,

$$12. \int \frac{(\alpha + \beta \cos x) dx}{\cos x (1 \pm \sin x)} = \frac{\alpha \pm \beta}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{\alpha \mp \beta}{2(1 \pm \sin x)}.$$

$$13. \int \frac{(\alpha + \beta \cos x) dx}{\cos x (a + b \cos x)} = \frac{\alpha}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{\alpha\beta - ba}{a} \int \frac{dx}{a + b \cos x} \quad (\text{vezi 2.452 12.}). \quad T\ (347)$$

Alte funcții raționale de sinus și cosinus

$$2.455 \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n}, \quad \text{unde } b = r \cos \alpha, \quad c = r \sin \alpha. \quad P\ (318)$$

$$\begin{aligned}
 2.456 \int \frac{dx}{a+b \cos x+c \sin x} &= \\
 &= \frac{2}{\sqrt{a^2-b^2-c^2}} \operatorname{arctg} \frac{(a-b) \operatorname{tg} \frac{x}{2}+c}{\sqrt{a^2-b^2-c^2}} \quad [a^2 > b^2+c^2]; \quad \text{T 253 F II 94} \\
 &= \frac{1}{\sqrt{b^2+c^2-a^2}} \ln \frac{(a-b) \operatorname{tg} \frac{x}{2}+c-\sqrt{b^2+c^2-a^2}}{(a-b) \operatorname{tg} \frac{x}{2}+c+\sqrt{b^2+c^2-a^2}} \quad [a^2 < b^2+c^2]; \quad \text{T 253 m} \\
 &= \frac{1}{c} \ln \left(a+c \operatorname{tg} \frac{x}{2} \right) \quad [a=b]; \quad \text{P (320)} \\
 &= \frac{2}{c+(a-b) \operatorname{tg} \frac{x}{2}} \quad [a^2=b^2+c^2]. \quad \text{T 253 m}
 \end{aligned}$$

$$2.457 \int \frac{dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{8} \right).$$

2.458

$$\begin{aligned}
 1. \int \frac{dx}{[a(1+\cos x)+c \sin x]^2} &= \\
 &= \frac{1}{c^3} \left[\frac{c(a \sin x-c \cos x)}{a(1+\cos x)+c \sin x} - a \ln \left(a+c \operatorname{tg} \frac{x}{2} \right) \right]. \quad \text{P (321)}
 \end{aligned}$$

$$\begin{aligned}
 2. \int \frac{dx}{a+b \cos^2 x+c \sin^2 x} &= \frac{\operatorname{sgn}(a+b)}{\sqrt{(a+b)(a+c)}} \operatorname{arctg} \left(\sqrt{\frac{a+c}{a+b}} \operatorname{tg} x \right) \\
 &\quad [(a+b)(a+c) > 0]; \\
 &= \frac{1}{2 \sqrt{-(a+b)(a+c)}} \ln \frac{\sqrt{-(a+b)}-\sqrt{a+c} \operatorname{tg} x}{\sqrt{-(a+b)}+\sqrt{a+c} \operatorname{tg} x} \\
 &\quad [(a+b)(a+c) < 0 \text{ și } a+b < 0]; \\
 &= \frac{1}{2 \sqrt{-(a+b)(a+c)}} \ln \frac{\sqrt{a+b}+\sqrt{-(a+c)} \operatorname{tg} x}{\sqrt{a+b}-\sqrt{-(a+c)} \operatorname{tg} x} \\
 &\quad [(a+b)(a+c) < 0 \text{ și } a+b > 0]. \quad \text{T (369)}
 \end{aligned}$$

$$\begin{aligned}
 3. \int \frac{dx}{a \sin^2 x+b \sin x \cos x+c \cos^2 x} &= \\
 &= \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{b+2a \operatorname{tg} x}{\sqrt{4ac-b^2}} \quad [4ac-b^2 > 0]; \quad \text{F II 92} \\
 &= \frac{1}{\sqrt{b^2-4ac}} \ln \frac{\sqrt{b^2-4ac}+b+2a \operatorname{tg} x}{\sqrt{b^2-4ac}-b-2a \operatorname{tg} x} \quad [4ac-b^2 < 0]; \\
 &= -\frac{2}{b+2a \operatorname{tg} x} \quad [4ac-b^2=0]. \quad \text{T (370)}
 \end{aligned}$$

$$4. \int \frac{\sin x \cos x dx}{a \cos^2 x+b \sin^2 x} = \frac{1}{2(b-a)} \ln (a \cos^2 x+b \sin^2 x).$$

2.46 Funcții algebrice de tangentă

2.461

$$1. \int \frac{\operatorname{tg} x dx}{1+m^2 \operatorname{tg}^2 x} = \frac{\ln (\cos^2 x+m^2 \sin^2 x)}{2(m^2-1)}. \quad \text{La 210 (10)}$$

$$2. \int \frac{\operatorname{tg} x - \operatorname{tg} x}{\operatorname{tg} x + \operatorname{tg} x} dx = \sin 2a \ln \sin (x+a) - x \cos 2a. \quad \text{La 210 (11) m}$$

$$3. \int \frac{\operatorname{tg} x dx}{a+b \operatorname{tg} x} = \frac{1}{a^2+b^2} \{bx - a \ln (a \cos x+b \sin x)\}. \quad \text{P (335)}$$

$$4. \int \frac{dx}{a+b \operatorname{tg}^2 x} = \frac{1}{a-b} \left[x - \sqrt{\frac{b}{a}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} \operatorname{tg} x \right) \right]. \quad \text{P (334)}$$

$$5. \int \frac{\operatorname{tg} x dx}{\sqrt{a+b \operatorname{tg}^2 x}} = \frac{1}{\sqrt{b-a}} \arccos \left(\frac{\sqrt{b-a}}{\sqrt{b}} \cos x \right).$$

P (333) vezi și T 298

2.47-2.48 Integrale reductibile la integrale eliptice și pseudoeliptice

2.470

$$1. \int \frac{d\varphi}{\sqrt{a+b \cos \varphi + c \sin \varphi}} = 2 \int \frac{d\psi}{\sqrt{a-p+2p \cos^2 \psi}} \\ \left[\varphi = 2\psi + \alpha, \operatorname{tg} \alpha = \frac{c}{b}, p = \sqrt{b^2 + c^2} \right].$$

$$2. \int \frac{d\varphi}{\sqrt{a+b \cos \varphi + c \sin \varphi + d \cos^2 \varphi + e \sin \varphi \cos \varphi + f \sin^2 \varphi}} = \\ = 2 \int \frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} \\ \left[\operatorname{tg} \frac{\varphi}{2} = x, A = a+b+d, B = 2c+2e, \right. \\ \left. C = 2a-2d+4f, D = 2c-2e, E = a-b+d \right].$$

Forme care conțin $\sqrt{1-k^2 \sin^2 x}$ Notații: $\Delta x = \sqrt{1-k^2 \sin^2 x}$; $k' = \sqrt{1-k^2}$.

2.471

$$1. \int (\Delta x)^n dx = \frac{n-1}{n} (2-k^2) \int (\Delta x)^{n-2} dx - \\ - \frac{n-2}{n} (1-k^2) \int (\Delta x)^{n-4} dx + \frac{k^2}{n} \sin x \cos x (\Delta x)^{n-2}. \quad \text{La 317 (7) m}$$

$$2. \int (\Delta x) dx = E(x, k).$$

$$3. \int (\Delta x)^2 dx = \left(1 - \frac{k^2}{2} \right) x + \frac{k^2}{2} \sin x \cos x.$$

$$4. \int (\Delta x)^3 dx = \frac{4-2k^2}{3} E(x, k) - \frac{k^2}{3} F(x, k) + \frac{k^2}{6} \sin 2x \Delta x.$$

2.472

$$1. \int \frac{dx}{(\Delta x)^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1) k'^2 (\Delta x)^{n-1}} + \\ + \frac{n-2}{n-1} \frac{2-k^2}{k'^2} \int \frac{dx}{(\Delta x)^{n-1}} - \frac{n-3}{n-1} \frac{1}{k'^2} \int \frac{dx}{(\Delta x)^{n-3}}. \quad \text{La 317 (8) m}$$

$$2. \int \frac{dx}{\Delta x} = F(x, k)$$

$$3. \int \frac{dx}{(\Delta x)^2} = \frac{1}{k'} \operatorname{arctg} (k' \operatorname{tg} x).$$

$$4. \int \frac{dx}{(\Delta x)^3} = \frac{1}{k'^2} E(x, k) - \frac{k^2}{k'^2} \frac{\sin x \cos x}{\Delta x}.$$

2.473

$$1. \int \frac{\sin^n x}{\Delta x} dx = \frac{\sin^{n-3} x}{(n-1) k^2} \cos x \Delta x + \\ + \frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta x} dx - \frac{n-3}{(n-1) k^2} \int \frac{\sin^{n-4} x}{\Delta x} dx. \quad \text{La 316 (1) m}$$

Pentru $n=1$,

$$2. \quad \int \frac{\sin x \, dx}{\Delta x} = \frac{1}{2k} \ln \frac{1 - \frac{k \cos x}{\Delta x}}{1 + \frac{k \cos x}{\Delta x}} = -\frac{1}{k} \ln (k \cos x + \Delta x).$$

P (327), E 210 (20)

$$3. \quad \int \frac{\sin^2 x}{\Delta x} dx = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k).$$

J 37 (120)

$$4. \quad \int \frac{\sin^3 x}{\Delta x} dx = \frac{\cos x \, \Delta x}{2k^2} + \frac{1+k^2}{4k^3} \ln \frac{\Delta x - k \cos x}{\Delta x + k \cos x}.$$

$$2.474 \quad \int \frac{\sin^3 x}{(\Delta x)^3} dx = \frac{1}{k'^2 k^2} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{k'^2} \frac{\sin x \cos x}{\Delta x}. \quad J 37 (127)$$

2.475

$$1. \quad \int \sin x \, \Delta x \, dx = -\frac{1}{2} \cos x \, \Delta x + \frac{k'^2}{4k} \ln \frac{\Delta x - k \cos x}{\Delta x + k \cos x}.$$

P (324) m

$$2. \quad \int \sin x (\Delta x)^3 dx = -\frac{1}{4} \cos x \, \Delta x^3 + \frac{3}{4} k'^2 \int \sin x \, \Delta x \, dx \text{ (vezi 2.475 1.)}.$$

P (325)

2.476

$$1. \quad \int \frac{\cos^n x}{\Delta x} dx = \frac{\cos^{n-3} x}{(n-1) k^2} \sin x \, \Delta x + \\ + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta x} dx + \frac{n-3}{n-1} \frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta x} dx.$$

La 316 (2)

Pentru $n=1$.

$$2. \quad \int \frac{\cos x \, dx}{\Delta x} = \frac{1}{k} \arcsin (k \sin x) = \frac{1}{k} \operatorname{arctg} \frac{k \sin x}{\Delta x}.$$

P (326) E 210 (20)

$$3. \quad \int \frac{\cos^3 x}{\Delta x} dx = \frac{1}{k^2} E(x, k) - \frac{k'^2}{k^2} F(x, k).$$

J 37 (121)

$$4. \quad \int \frac{\cos^3 x}{\Delta x} dx = \frac{\sin x \, \Delta x}{2k^2} + \frac{2k^2-1}{2k^2} \arcsin (k \sin x).$$

$$5. \quad \int \frac{\cos^3 x}{(\Delta x)^3} dx = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta x}.$$

J 38 (128)

$$2.477 \quad \int \cos x \, \Delta x \, dx = \frac{\sin x}{2} \Delta x + \frac{1}{2k} \arcsin (k \sin x).$$

P (323)

2.478

$$1. \quad \int \frac{\operatorname{tg}^n x}{\Delta x} dx = \frac{\operatorname{tg}^{n-3} x}{(n-1) k'^2} \frac{\Delta x}{\cos^2 x} - \\ - \frac{(n-2)(2-k^2)}{(n-1) k'^2} \int \frac{\operatorname{tg}^{n-2} x}{\Delta x} dx - \frac{(n-3)}{(n-1) k'^2} \int \frac{\operatorname{tg}^{n-4} x}{\Delta x} dx.$$

La 317 (3)

Pentru $n=1$,

$$2. \quad \int \frac{\operatorname{tg} x \, dx}{\Delta x} = \frac{1}{2k} \ln \frac{1 + \frac{k'}{\Delta x}}{1 - \frac{k'}{\Delta x}}.$$

P (328) E 210 (20)

$$3. \quad \int \frac{\operatorname{tg}^3 x}{\Delta x} dx = \frac{\operatorname{tg} x}{k'^2} \Delta x - \frac{1}{k'^2} E(x, k).$$

J 37 (122)

$$4. \quad \int \frac{\operatorname{tg}^3 x}{\Delta x} dx = \frac{1}{2k^2} \left[\frac{\Delta x'}{\cos^2 x} - \frac{1+k'^2}{2k'} \ln \frac{\Delta x + k'}{\Delta x - k'} \right].$$

2.479

$$1. \int \frac{\operatorname{ctg}^n x}{\Delta x} dx = -\frac{\operatorname{ctg}^{n-1} x}{n-1} \frac{\Delta x}{\cos^2 x} - \\ - \frac{n-2}{n-1} (2-k^2) \int \frac{\operatorname{ctg}^{n-2} x}{\Delta x} dx - \frac{n-3}{n-1} k'^2 \int \frac{\operatorname{ctg}^{n-4} x}{\Delta x} dx. \quad \text{La (317) (6)}$$

Pentru $n=1$

$$2. \int \frac{\operatorname{ctg} x}{\Delta x} dx = -\frac{1}{2} \ln \frac{1+\Delta x}{1-\Delta x}. \\ 3. \int \frac{\operatorname{ctg}^3 x}{\Delta x} dx = -\operatorname{ctg} x \Delta x - E(x, k). \quad \text{J 37 (125) m} \\ 4. \int \frac{\operatorname{ctg}^3 x}{\Delta^2} dx = -\frac{\Delta x}{2 \sin^2 x} + \frac{2-k^2}{4} \ln \frac{1+\Delta x}{1-\Delta x}.$$

2.481

$$1. \int \frac{(a+\sin x)^{n+3} dx}{\Delta x} = \frac{1}{(n+2)k^2} \left[(a+\sin x)^n \cos x \Delta x + \right. \\ + 2(2n+3)ak^2 \int \frac{(a+\sin x)^{n+2} dx}{\Delta x} + \\ + (n+1)(1+k^2-6a^2k^2) \int \frac{(a+\sin x)^{n+1} dx}{\Delta x} - \\ - a(2n+1)(1+k^2-2a^2k^2) \int \frac{(a+\sin x)^n dx}{\Delta x} - \\ \left. - n(1-a^2)(1-a^2k^2) \int \frac{(a+\sin x)^{n-1} dx}{\Delta x} \right]. \quad \text{E 209 (15)}$$

Pentru n natural, această integrală se poate reduce la următoarele trei integrale:

$$2. \int \frac{a+\sin x}{\Delta x} dx = aF(x, k) + \frac{1}{2k} \ln \frac{\Delta x - k \cos x}{\Delta x + k \cos x}. \quad \text{E 209 (19) m} \\ 3. \int \frac{(a+\sin x)^2}{\Delta x} dx = \frac{1+k^2a^2}{k^3} F(x, k) - \frac{1}{k^3} E(x, k) + \\ + \frac{a}{k} \ln \frac{\Delta x - k \cos x}{\Delta x + k \cos x}. \quad \text{E 210 (21) m} \\ 4. \int \frac{dx}{(a+\sin x) \Delta x} = \frac{1}{a} \Pi\left(x, -\frac{1}{a^2}, k\right) - \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta x}, \quad \text{E 210 (22)}$$

unde

$$5. \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta x} = \frac{1}{2 \sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2} \Delta x - \sqrt{1-k^2a^2} \cos x}{\sqrt{1-a^2} \Delta x + \sqrt{1-k^2a^2} \cos x}. \quad \text{E 214}$$

2.482

$$1. \int \frac{dx}{(a+\sin x)^n \Delta x} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta x}{(a+\sin x)^{n-1}} - \right. \\ - (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a+\sin x)^{n-1} \Delta x} - \\ - (n-2)(6a^2k^2-k^2-1) \int \frac{dx}{(a+\sin x)^{n-2} \Delta x} + \\ + (10-4n)ak^2 \int \frac{dx}{(a+\sin x)^{n-3} \Delta x} - \\ \left. - (n-3)k^2 \int \frac{dx}{(a+\sin x)^{n-4} \Delta x} \right]. \quad \text{E 209 (15)}$$

Pentru $n=1$, vezi 2.481 4. Pentru $n \neq 1$ și natural, această integrală se reduce la integralele

$$2. \int \frac{dx}{(a + \sin x)^2 \Delta x} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta x}{a + \sin a} - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x) \Delta x} + 2ak^2 \int \frac{(a + \sin x) dx}{\Delta x} + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta x} \right] \text{ (vezi 2.481).}$$

$$3. \int \frac{dx}{(a + \sin x)^3 \Delta x} = \frac{1}{2(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta x}{(a + \sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)^2 \Delta x} - (6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x) \Delta x} - 2ak^2 F(x, k) \right] \text{ (vezi 2.481 4., 2.482 2.).}$$

2.483

$$1. \int \frac{(b + \cos x)^{n+3} dx}{\Delta x} = \frac{1}{(n+2)k^2} \left[-k'^2(b + \cos x)^n \sin x \Delta x + 2(2n+3)bk^2 \int \frac{(b + \cos x)^{n+2} dx}{\Delta x} - (n+1)(k'^2 - k^2 + 6b^2k^2) \int \frac{(b + \cos x)^{n+1} dx}{\Delta x} + (2n+1)b(k'^2 - k^2 + 2b^2k^2) \int \frac{(b + \cos x)^n dx}{\Delta x} + n(1-b^2)(k'^2 + k^2b^2) \int \frac{(b + \cos x)^{n-1} dx}{\Delta x} \right]. \quad \text{E 209 (16)}$$

Pentru n natural, această integrală se poate reduce la următoarele trei integrale:

$$2. \int \frac{b + \cos x}{\Delta x} dx = b F(x, k) + \frac{1}{k} \arcsin(k \sin x). \quad \text{E 209 (19)}$$

$$3. \int \frac{(b + \cos x)^2}{\Delta x} dx = \frac{b^2k^2 - k'^2}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2b}{k} \arcsin(k \sin x). \quad \text{E 210 (21)}$$

$$4. \int \frac{dx}{(b + \cos x) \Delta x} = \frac{b}{b^2-1} \Pi\left(x, \frac{b}{b^2-1}, k\right) + \int \frac{\cos x dx}{(1-b^2-\sin^2 x) \Delta x}, \quad \text{E 210 (22)m}$$

unde

$$5. \int \frac{\cos x dx}{(1-b^2-\sin^2 x) \Delta x} = \frac{1}{2\sqrt{(1-b^2)(k'^2+k^2b^2)}} \ln \frac{\sqrt{1-b^2} \Delta x + k\sqrt{k'^2+k^2b^2} \sin x}{\sqrt{1-b^2} \Delta x - k\sqrt{k'^2+k^2b^2} \sin x}. \quad \text{E 211}$$

2.484

$$1. \int \frac{dx}{(b + \cos x)^n \Delta x} = \frac{1}{(n-1)(1-b^2)k'^2(k'^2+b^2k^2)} \left[-\frac{\sin x \Delta x}{(b + \cos x)^{n-1}} - (2n-3)(1-2k^2+2b^2k^2)b \int \frac{dx}{(b + \cos x)^{n-1} \Delta x} - (n-2)(2k^2-1-6b^2k^2) \int \frac{dx}{(b + \cos x)^{n-2} \Delta x} - (4n-10)bk^2 \int \frac{dx}{(b + \cos x)^{n-3} \Delta x} + (n-3)k^2 \int \frac{dx}{(b + \cos x)^{n-4} \Delta x} \right]. \quad \text{E 209 (16)}$$

Pentru $n=1$, vezi 2.483 4. Pentru $n \neq 1$ și natural, această integrală se reduce la următoarele integrale:

$$2. \int \frac{dx}{(b + \cos x)^3 \Delta x} = \frac{1}{(1-b^2)k'^2(k'^2 + b^2k^2)} \left[-\frac{\sin x \Delta x}{b + \cos x} - (1 - 2k^2 + 2b^2k^2) a \int \frac{dx}{(b + \cos x) \Delta x} + b k^2 \int \frac{b + \cos x}{\Delta x} dx - k^2 \int \frac{(b + \cos x)^2}{\Delta x} dx \right] \quad (\text{vezi 2.483}).$$

$$3. \int \frac{dx}{(b + \cos x)^3 \Delta x} = \frac{1}{2(1-b^2)k'^2(k'^2 + b^2k^2)} \left[-\frac{\sin x \Delta x}{(b + \cos x)^2} - 3b^2(1 - 2k^2 + 2k^2b^2) \int \frac{dx}{(b + \cos x)^2 \Delta x} - (2k^2 - 1 - 6b^2k^2) \int \frac{dx}{(b + \cos x) \Delta x} - 2bk^2 F(x, k) \right] \quad (\text{vezi 2.483 4., 2.484 2.}).$$

2.485

$$1. \int \frac{(c + \operatorname{tg} x)^{n+3} dx}{\Delta x} = \frac{1}{(n+2)k'^2} \left[\frac{(c + \operatorname{tg} x)^n \Delta x}{\cos^2 x} + 2(2n+3)ck'^2 \int \frac{(c + \operatorname{tg} x)^{n+2} dx}{\Delta x} - (n+1)(1+k'^2+6c^2k'^2) \int \frac{(c + \operatorname{tg} x)^{n+1} dx}{\Delta x} + (2n+1)c(1+k'^2+2c^2k'^2) \int \frac{(c + \operatorname{tg} x)^n dx}{\Delta x} - n(1+c^2)(1+k'^2c^2) \int \frac{(c + \operatorname{tg} x)^{n-1} dx}{\Delta x} \right]. \quad \text{E 209 (17)}$$

Pentru n natural, această integrală se poate reduce la următoarele trei integrale:

$$2. \int \frac{c + \operatorname{tg} x}{\Delta x} dx = cF(x, k) + \frac{1}{2k'} \ln \frac{\Delta x + k'}{\Delta x - k'}. \quad \text{E 209 (19)}$$

$$3. \int \frac{(c + \operatorname{tg} x)^3}{\Delta x} dx = \frac{1}{k'^2} \operatorname{tg} x \Delta x + c^2 F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{c}{k'} \ln \frac{\Delta x + k'}{\Delta x - k'}. \quad \text{E 210 (21)}$$

$$4. \int \frac{dx}{(c + \operatorname{tg} x) \Delta x} = \frac{c}{1+c^2} F(x, k) + \frac{1}{c(1+c^2)} \Pi\left(x, -\frac{1+c^2}{c^2}, k\right) - \int \frac{\sin x \cos x dx}{[c^2 - (1+c^2) \sin^2 x] \Delta x}, \quad \text{E 210 (22)}$$

unde

$$5. \int \frac{\sin x \cos x dx}{[c^2 - (1+c^2) \sin^2 x] \Delta x} = \frac{1}{2\sqrt{(1+c^2)(1+c^2k'^2)}} \ln \frac{\sqrt{1+c^2k'^2} + \sqrt{1+c^2} \Delta x}{\sqrt{1+c^2k'^2} - \sqrt{1+c^2} \Delta x}. \quad \text{E 211}$$

2.486

$$1. \int \frac{dx}{(c + \operatorname{tg} x)^n \Delta x} = \frac{1}{(n-1)(1+c^2)(1+k'^2c^2)} \left[-\frac{\Delta x}{(c + \operatorname{tg} x)^{n-1} \cos^2 x} + (2n-3)c(1+k'^2+2c^2k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^{n-1} \Delta x} - (n-2)(1+k'^2+6c^2k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^{n-2} \Delta x} + (4n-10)ck'^2 \int \frac{dx}{(c + \operatorname{tg} x)^{n-3} \Delta x} - (n-3)k'^2 \int \frac{dx}{(c + \operatorname{tg} x)^{n-4} \Delta x} \right]. \quad \text{E 209 (17)}$$

Pentru $n=1$, vezi 2.485 4. Pentru $n \neq 1$ și natural, această integrală se reduce la integralele

$$2. \int \frac{dx}{(c + \operatorname{tg} x)^2 \Delta x} = \frac{1}{(1+c^2)(1+k'^2 c^2)} \left[\frac{-\Delta x}{(c + \operatorname{tg} x) \cos^2 x} + \right. \\ \left. + c(1+k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x) \Delta x} - 2ck'^2 \int \frac{c + \operatorname{tg} x}{\Delta x} dx + k'^2 \int \frac{(c + \operatorname{tg} x)^2}{\Delta x} dx \right] \\ (\text{vezi 2.485}).$$

$$3. \int \frac{dx}{(c + \operatorname{tg} x)^3 \Delta x} = \frac{1}{2(1+c^2)(1+k'^2 c^2)} \left[\frac{-\Delta x}{(c + \operatorname{tg} x)^2 \cos^2 x} + \right. \\ \left. + 3c(1+k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^2 \Delta x} (1+k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x) \Delta x} + \right. \\ \left. + 2ck'^2 F(x, k) \right] (\text{vezi 2.485 4., 2.486 2.}).$$

2.487

$$1. P_n = \int \frac{(a + \sin^2 x)^n}{\Delta x} dx.$$

Formula de recurență

$$P_{n+2} = \frac{1}{(2n+3)k^2} \{ (a + \sin^2 x)^n \sin x \cos x \Delta x + (2n+2)(1+k^2+3ak^2) P_{n+1} - \\ - (2n+1)[1+2a(1+k^2)+3a^2 k^2] P_n + 2na(1+a)(1+k^2 a) P_{n-1} \}. \quad \text{E 211 (24)}$$

reduce această integrală, pentru n întreg, la integralele

$$2. P_1 \text{ vezi 2.472 2., 2.473 3.}$$

$$3. P_0 \text{ vezi 2.472 2.}$$

$$4. P_{-1} = \int \frac{dx}{(a + \sin^2 x) \Delta x} = \frac{1}{a} \Pi\left(x, \frac{1}{a}, k\right).$$

Pentru $a=0$,

$$5. \int \frac{dx}{\sin^2 x \Delta x} = F(x, k) - E(x, k) - \operatorname{ctg} x \Delta x.$$

J 37 (124)_m

$$6. T_n = \int \frac{dx}{(h + g \sin^2 x)^n \Delta x}$$

se calculează cu ajutorul formulei de recurență

$$T_{n+2} = \frac{1}{(2n+5)k^2} \left\{ \frac{-g^2 \sin x \cos x \Delta x}{(h + g \sin^2 x)^{n+1}} + 2(n+2)[g(1+k^2) + 3hk^2] T_{n+1} - \right. \\ \left. - (2n+3)[g^2 + 2hg(1+k^2) + 3h^2 k^2] T_{n+1} + 2(n+1)h(g+h)(g+hk^2) T_n \right\}.$$

E 207 (9)

2.488

$$1. Q_n = \int \frac{(b + \cos^2 x)^n}{\Delta x} dx.$$

Formula de recurență

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \{ (b + \cos^2 x)^n \sin x \cos x \Delta x - (2n+2)(1-2k^2-3bk^2) Q_{n+1} + \\ + (2n+1)[k'^2 + 2b(k'^2 - k^2) - 3b^2 k^2] Q_n - 2nb(1-b)(k'^2 - k^2 b) Q_{n-1} \}$$

reduce această integrală, pentru n întreg, la integralele

E 211 (25)

$$2. Q_1 \text{ vezi 2.472 2. și 2.475 3.}$$

$$3. Q_0 \text{ vezi 2.472 2.}$$

$$4. Q_{-1} = \int \frac{dx}{(b + \cos^2 x) \Delta x} = \frac{1}{b+1} \Pi\left(x, -\frac{1}{b+1}, k\right).$$

Pentru $b=0$,

$$5. \int \frac{dx}{\cos^2 x \Delta x} = \frac{\operatorname{tg} x \Delta x}{k'^2} + F(x, k) - \frac{1}{k'^2} E(x, k).$$

J 37 (123)

2.489

$$1. R_n = \int \frac{(c + \operatorname{tg}^2 x)^n dx}{\Delta x}.$$

Formula de recurență

$$R_{n+1} = \frac{1}{(2n+3)k'^2} \left\{ \frac{(c + \operatorname{tg}^2 x)^n \operatorname{tg} x \Delta x}{\cos^2 x} - (2n+2)(1+k'^2 - 3ck'^2) R_{n+1} + \right. \\ \left. + (2n+1)[1-2c(1+k'^2) + 3c^2k'^2] R_n + 2nc(1-c)(1-k'^2c) R_{n-1} \right\} \quad \text{E 211 (26)}$$

reduce această integrală, pentru n întreg, la integralele2. R_1 vezi 2.472 2., 2.478 3.3. R_0 vezi 2.472 2.

$$4. R_{-1} = \int \frac{dx}{(c + \operatorname{tg}^2 x) \Delta x} = \frac{1}{c-1} F(x, k) + \frac{1}{c(1-c)} \Pi\left(x, \frac{1-c}{c}, k\right).$$

Pentru $c=0$ vezi 2.476 1.

$$5. \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta x} \quad (\text{vezi 2.481 5.})$$

$$6. \int \frac{\cos x dx}{(b^2 - \sin^2 x) \Delta x} \quad (\text{vezi 2.483 5.})$$

$$7. \int \frac{\sin x \cos x dx}{[c^2 - (1+c^2) \sin^2 x] \Delta x} \quad (\text{vezi 2.485 5.})$$

2.49 Sinușii și cosinușii logaritmicilor

2.491

$$1. \int \sin \ln x dx = \frac{x}{2} (\sin \ln x - \cos \ln x). \quad \text{P (444)}$$

$$2. \int \cos \ln x dx = \frac{x}{2} (\sin \ln x + \cos \ln x). \quad \text{P (445).}$$

2.51 Forme care conțin $x^m \sin^n x$ și $x^m \cos^n x$ [$m > 0, n > 0$]

2.511

$$1. \int x^m \sin^n x dx = \frac{x^{m-1} \sin^{n-1} x}{n^2} \{m \sin x - nx \cos x\} + \\ + \frac{n-1}{n} \int x^m \sin^{n-2} x dx - \frac{m(m-1)}{n^2} \int x^{m-2} \sin^n x dx.$$

$$2. \int x^m \cos^n x dx = \frac{x^{m-1} \cos^{n-1} x}{n^2} \{m \cos x + nx \sin x\} + \\ + \frac{n-1}{n} \int x^m \cos^{n-2} x dx - \frac{m(m-1)}{n^2} \int x^{m-2} \cos^n x dx.$$

2.512

$$1. \int x^n \sin^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \\ + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cos(2m-2k)x dx \quad (\text{vezi 2.513 4.}). \quad \text{T 333}$$

$$2. \int x^n \sin^{2m+1} x dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sin(2m-2k+1)x dx \\ (\text{vezi 2.513 3.}). \quad \text{T 333}$$

$$3. \int x^n \cos^{2m} x \, dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \\ + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cos(2m-2k)x \, dx. \quad (\text{vezi 2.513 4.}) \quad T 333$$

$$4. \int x^n \cos^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cos(2m-2k+1)x \, dx \\ (\text{vezi 2.513 4.}) \quad T 333$$

2.513

$$1. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx; \\ = -x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x \, dx.$$

$$2. \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx; \\ = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx.$$

$$3. \int x^n \sin ax \, dx = - \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \cos\left(ax + \frac{1}{2} k\pi\right). \quad T(487)$$

$$4. \int x^n \cos ax \, dx = \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \sin\left(ax + \frac{1}{2} k\pi\right). \quad T(486)$$

2.514

$$1. \int x \sin x \, dx = \sin x - x \cos x.$$

$$2. \int x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x.$$

$$3. \int x^3 \sin x \, dx = (3x^2 - 6) \sin x - (x^3 - 6x) \cos x.$$

$$4. \int x \cos x \, dx = \cos x + x \sin x.$$

$$5. \int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x.$$

$$6. \int x^3 \cos x \, dx = (3x^2 - 6) \cos x + (x^3 - 6x) \sin x.$$

2.515

$$1. \int P_n(x) \sin mx \, dx = \\ = -\frac{\cos mx}{m} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

$$2. \int P_n(x) \cos mx \, dx = \\ = \frac{\sin mx}{m} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

În formulele 2.515, $P_n(x)$ este un polinom de gradul n , iar $P_n^{(k)}(x)$ derivata lui de ordinul k în raport cu x .

2.52 Forme care conțin $x^m \sin^n x$ și $x^m \cos^n x$ [$m < 0, n > 0$]

2.521

$$\begin{aligned}
 1. \quad \int \frac{\sin^n x \, dx}{x^m} &= -\frac{\sin^{n-1} x [(m-2) \sin x + nx \cos x]}{(m-1)(m-2)x^{m-1}} - \\
 &\quad - \frac{n^2}{(m-1)(m-2)} \int \frac{\sin^{n-2} x \, dx}{x^{m-2}} + \frac{n(n-1)}{(m-1)(m-2)} \int \frac{\sin^{n-2} x \, dx}{x^{m-2}} \quad T(496) \\
 2. \quad \int \frac{\cos^n x \, dx}{x^m} &= -\frac{\cos^{n-1} x \left\{ \frac{(m-2) \cos x - nx \sin x}{(m-1)(m-2)} \right\}}{x^{m-1}} - \\
 &\quad - \frac{n^2}{(m-1)(m-2)} \int \frac{\cos^{n-2} x \, dx}{x^{m-2}} + \frac{n(n-1)}{(m-1)(m-2)} \int \frac{\cos^{n-2} x \, dx}{x^{m-2}} \quad T(495)
 \end{aligned}$$

Pentru $m=1$ și $m=2$, avem

2.522

$$\begin{aligned}
 1. \quad \int \frac{\sin^{2m} x}{x} \, dx &= \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \operatorname{ci}((2m-2k)x). \\
 2. \quad \int \frac{\sin^{2m+1} x}{x} \, dx &= \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \operatorname{si}((2m-2k+1)x). \\
 3. \quad \int \frac{\cos^{2m} x}{x} \, dx &= \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \operatorname{ci}((2m-2k)x). \\
 4. \quad \int \frac{\cos^{2m+1} x}{x} \, dx &= \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \operatorname{ci}((2m-2k-1)x).
 \end{aligned}$$

2.523

$$\begin{aligned}
 1. \quad \int \frac{\sin^{2m} x}{x^2} \, dx &= -\binom{2m}{m} \frac{1}{2^{2m} x} + \\
 &\quad + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + \right. \\
 &\quad \left. + (2m-2k) \operatorname{si}((2m-2k)x) \right\}. \\
 2. \quad \int \frac{\sin^{2m+1} x}{x^2} \, dx &= \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \left\{ \frac{\sin(2m-2k+1)x}{x} - \right. \\
 &\quad \left. - (2m-2k+1) \operatorname{ci}((2m-2k+1)x) \right\}. \\
 3. \quad \int \frac{\cos^{2m} x}{x^2} \, dx &= -\binom{2m}{m} \frac{1}{2^{2m} x} - \\
 &\quad - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}((2m-2k)x) \right\}. \\
 4. \quad \int \frac{\cos^{2m+1} x}{x^2} \, dx &= -\frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cos(2m-2k+1)x}{x} + \right. \\
 &\quad \left. + (2m-2k+1) \operatorname{si}((2m-2k+1)x) \right\}.
 \end{aligned}$$

2.524

$$\begin{aligned}
 1. \quad \int \frac{\sin x \, dx}{x^n} &= -\frac{\sin x}{(n-1)x^{n-1}} + \frac{1}{n-1} \int \frac{\cos x \, dx}{x^{n-1}}; \\
 &= -\frac{\sin x}{(n-1)x^{n-1}} - \frac{\cos x}{(n-1)(n-2)x^{n-2}} - \frac{1}{(n-1)(n-2)} \int \frac{\sin x \, dx}{x^{n-2}}.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int \frac{\cos x \, dx}{x^n} &= -\frac{\cos x}{(n-1)x^{n-1}} - \frac{1}{n-1} \int \frac{\sin x \, dx}{x^{n-1}}; \\
 &= -\frac{\cos x}{(n-1)x^{n-1}} + \frac{\sin x}{(n-1)(n-2)x^{n-2}} - \frac{1}{(n-1)(n-2)} \int \frac{\cos x \, dx}{x^{n-2}}.
 \end{aligned}$$

2.525

$$\begin{aligned}
 1. \quad \int \frac{\sin x}{x^{2n}} \, dx &= \sin x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-1)(2n-2)\dots(2n-2k-1)x^{2n-2k-1}} + \\
 &+ \cos x \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-1)(2n-2)\dots(2n-2k)x^{2n-2k}} + \frac{(-1)^{n+1}}{(2n-1)!} \operatorname{ci}(x).
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int \frac{\sin x}{x^{2n+1}} \, dx &= \sin x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2n(2n-1)\dots(2n-2k)x^{2n-2k}} + \\
 &+ \cos x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2n(2n-1)\dots(2n-2k-1)x^{2n-2k-1}} + \frac{(-1)^n}{(2n)!} \operatorname{si}(x).
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \frac{\cos x}{x^{2n}} \, dx &= \cos x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-1)(2n-2)\dots(2n-2k-1)x^{2n-2k-1}} + \\
 &+ \sin x \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{(2n-1)(2n-2)\dots(2n-2k)x^{2n-2k}} + \frac{(-1)^n}{(2n-1)!} \operatorname{si}(x).
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int \frac{\cos x}{x^{2n+1}} \, dx &= \cos x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2n(2n-1)\dots(2n-2k)x^{2n-2k}} + \\
 &+ \sin x \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2n(2n-1)\dots(2n-2k-1)x^{2n-2k-1}} + \frac{(-1)^n}{(2n)!} \operatorname{ci}(x).
 \end{aligned}$$

2.526

$$1. \quad \int \frac{\sin x \, dx}{x} = \operatorname{si}(x). \text{ (sinus integral).}$$

$$2. \quad \int \frac{\sin x \, dx}{x^2} = -\frac{\sin x}{x} + \operatorname{ci}(x).$$

$$3. \quad \int \frac{\sin x \, dx}{x^3} = -\frac{\sin x}{2x^2} - \frac{\cos x}{2x} - \frac{1}{2} \operatorname{si}(x).$$

$$4. \quad \int \frac{\cos x \, dx}{x} = \operatorname{ci}(x) \text{ (cosinus integral).}$$

$$5. \quad \int \frac{\cos x \, dx}{x^2} = -\frac{\cos x}{x} - \operatorname{si}(x).$$

$$6. \quad \int \frac{\cos x \, dx}{x^3} = -\frac{\cos x}{2x^2} + \frac{\sin x}{2x} - \frac{1}{2} \operatorname{ci}(x).$$

2.53 Forme care conțin $x^m \sin^n x$ și $x^m \cos^n x$ [$m > 0, n < 0$]

2.531

$$\begin{aligned}
 1. \quad \int \frac{x^m \, dx}{\sin^n x} &= -\frac{x^{m-1}}{(n-1)(n-2)\sin^{n-1} x} \{m \sin x + (n-2)x \cos x\} + \\
 &+ \frac{n-2}{n-1} \int \frac{x^m \, dx}{\sin^{n-2} x} + \frac{m(m-1)}{(n-1)(n-2)} \int \frac{x^{m-2} \, dx}{\sin^{n-2} x}.
 \end{aligned}$$

$$2. \int \frac{x^n dx}{\cos^n x} = -\frac{x^{n-1}}{(n-1)(n-2)\cos^{n-1} x} \{m \cos x - (n-2)x \sin x\} + \\ + \frac{n-2}{n-1} \int \frac{x^n dx}{\cos^{n-2} x} + \frac{m(m-1)}{(n-1)(n-2)} \int \frac{x^{m-2} dx}{\cos^{n-2} x}$$

2.532

$$1. \int \frac{x dx}{\sin^{2n} x} = \\ = -\sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)(2n-2k+1)} \frac{\sin x + (2n-2k)x \cos x}{(2n-2k) \sin^{2n-2k+1} x} + \\ + \frac{2^{n-1} \cdot (n-1)!}{(2n-1)!} (\ln \sin x - x \operatorname{ctg} x).$$

$$2. \int \frac{x dx}{\sin^{2n+1} x} = \\ = -\sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)(2n-2k)} \frac{\sin x + (2n-2k-1)x \cos x}{(2n-2k-1) \sin^{2n-2k} x} + \\ + \frac{(2n-1)!}{2^n \cdot n!} \int \frac{x dx}{\sin x} \quad (\text{vezi 1.411 11. și 0.312}),$$

$$3. \int \frac{x dx}{\cos^{2n} x} = \\ = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)(2n-2k+1)} \frac{(2n-2k)x \sin x - \cos x}{(2n-2k) \cos^{2n-2k+1} x} + \\ + \frac{2^{n-1} \cdot (n-1)!}{(2n-1)!} (x \operatorname{tg} x + \ln \cos x).$$

$$4. \int \frac{x dx}{\cos^{2n+1} x} = \\ = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)(2n-2k)} \frac{(2n-2k+1)x \sin x - \cos x}{(2n-2k-1) \cos^{2n-2k} x} + \\ + \frac{(2n-1)!}{2^n \cdot n!} \int \frac{x dx}{\cos x} \quad (\text{vezi 1.411 9. și 0.312}).$$

$$\int \frac{x dx}{\sin x} \text{ și } \int \frac{x dx}{\cos x} \text{ vezi 1.411 9., 1.411 11., 0.312.}$$

2.533

$$1. \int \frac{x dx}{\sin^2 x} = -x \operatorname{ctg} x + \ln \sin x.$$

$$2. \int \frac{x dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x}{\sin x} dx \quad (\text{vezi 1.411 11., 0.312}).$$

$$3. \int \frac{x dx}{\cos^2 x} = x \operatorname{tg} x + \ln \cos x.$$

$$4. \int \frac{x dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x dx}{\cos x} \quad (\text{vezi 1.411 9. și 0.312}).$$

2.54 Funcții raționale de $\sin x$, $\cos x$ și x

$$1. \int \frac{x dx}{1 + \sin x} = -x \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \cos \left(\frac{\pi}{4} - \frac{x}{2} \right). \quad \text{P (329)}$$

$$2. \int \frac{x dx}{1 - \sin x} = x \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \sin \left(\frac{\pi}{4} - \frac{x}{2} \right). \quad \text{P (330)}$$

$$3. \int \frac{x dx}{1 + \cos x} = x \operatorname{tg} \frac{x}{2} + 2 \ln \cos \frac{x}{2}. \quad \text{P (331)}$$

$$4. \int \frac{x dx}{1 - \cos x} = -x \operatorname{ctg} \frac{x}{2} + 2 \ln \sin \frac{x}{2}. \quad \text{P (332)}$$

2.55 Funcții trigonometrice și funcția exponențială

$$\begin{aligned} 2.551 \quad & \int e^{ax} \sin^m x \cos^n x dx = \\ &= \frac{1}{(m+n)^2 + a^2} \left\{ e^{ax} \sin^m x \cos^{n-1} x (a \cos x + (m+n) \sin x) - \right. \\ & \quad \left. - ma \int e^{ax} \sin^{m-1} x \cos^{n-1} x dx + (n-1)(m+n) \int e^{ax} \sin^m x \cos^{n-2} x dx \right\}; \\ &= \frac{1}{(m+n)^2 + a^2} \left\{ e^{ax} \sin^{m-1} x \cos^n x (a \sin x - (m+n) \cos x) + \right. \\ & \quad \left. + na \int e^{ax} \sin^{m-1} x \cos^{n-1} x dx + (m-1)(m+n) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\}; \\ &= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \cos^{n-1} x \sin^{m-1} x (a \sin x \cos x + n \sin^2 x - m \cos^2 x)] + \right. \\ & \quad \left. + n(n-1) \int e^{ax} \sin^m x \cos^{n-2} x dx + m(m-1) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\}; \\ &= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x - m \cos^2 x)] + \right. \\ & \quad \left. + n(n-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x dx + (m-n)(m+n-1) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\}; \\ &= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x - m \cos^2 x)] + \right. \\ & \quad \left. + m(m-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x dx - \right. \\ & \quad \left. - (m-n)(m+n-1) \int e^{ax} \sin^m x \cos^{n-2} x dx \right\}. \end{aligned}$$

Pentru m și n pari, integrala $\int e^{ax} \sin^m x \cos^n x dx$ se reduce, cu ajutorul acestor formule, la integrala $\int e^{ax} dx$; dacă numai m sau numai n este par, atunci ea se reduce la integrale de forma $\int e^{ax} \cos^n x dx$ sau respectiv, $\int e^{ax} \sin^m x dx$.

2.552

$$\begin{aligned} 1. \quad & \int e^{ax} \sin^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[(a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + \right. \\ & \quad \left. + n(n-1) b^2 \int e^{ax} \sin^{n-2} bx dx \right]. \\ 2. \quad & \int e^{ax} \cos^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + \right. \\ & \quad \left. + n(n-1) b^2 \int e^{ax} \cos^{n-2} bx dx \right]. \end{aligned}$$

2.553

1. $\int e^{ax} \sin bx \, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}.$
2. $\int e^{ax} \sin^2 bx \, dx = \frac{e^{ax} \sin bx (a \sin bx - 2b \cos bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2) a}.$
3. $\int e^{ax} \cos bx \, dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}.$
4. $\int e^{ax} \cos^2 bx \, dx = \frac{e^{ax} \cos bx (a \cos bx + 2b \sin bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2) a}.$
5. $\int e^{ax} \sin bx \cos bx \, dx = \frac{e^{ax}}{8b^2 + 2a^2} (a \sin 2bx - 2b \cos 2bx).$

2.554

1. $\int \frac{e^{ax} dx}{\sin^n bx} = -\frac{e^{ax} [a \sin bx + (n-2) b \cos bx]}{(n-1)(n-2)b^2 \sin^{n-1} bx} + \frac{a^2 + (n-2)^2 b^2}{(n-1)(n-2)b^2} \int \frac{e^{ax} dx}{\sin^{n-2} bx}.$
T (530)m
2. $\int \frac{e^{ax} dx}{\cos^n bx} = -\frac{e^{ax} [a \cos bx - (n-2) b \sin bx]}{(n-1)(n-2)b^2 \cos^{n-1} bx} + \frac{a^2 + (n-2)^2 b^2}{(n-1)(n-2)b^2} \int \frac{e^{ax} dx}{\cos^{n-2} bx}.$
T (529)m

Prin aplicarea succesivă a formulelor 2.554, pentru n natural, ajungem la integrale de forma $\int \frac{e^{ax} dx}{\sin bx}$, $\int \frac{e^{ax} dx}{\sin^2 bx}$, $\int \frac{e^{ax} dx}{\cos bx}$, $\int \frac{e^{ax} dx}{\cos^2 bx}$, care nu se pot exprima cu ajutorul unei combinații finite de funcții elementare.

2.555

1. $\int e^{ax} \operatorname{tg}^n x \, dx = \frac{e^{ax}}{n-1} \operatorname{tg}^{n-1} x - \frac{a}{n-1} \int e^{ax} \operatorname{tg}^{n-1} x \, dx - \int e^{ax} \operatorname{tg}^{n-2} x \, dx.$
2. $\int e^{ax} \operatorname{ctg}^n x \, dx =$
 $= -\frac{e^{ax} \operatorname{ctg}^{n-1} x}{n-1} + \frac{a}{n-1} \int e^{ax} \operatorname{ctg}^{n-1} x \, dx - \int e^{ax} \operatorname{ctg}^{n-2} x \, dx.$

2.556

1. $\int e^{ax} \operatorname{tg} x \, dx = \frac{e^{ax} \operatorname{tg} x}{a} - \frac{1}{a} \int \frac{e^{ax} dx}{\cos^2 x}$ (vezi observația de la 2.554).
2. $\int e^{ax} \operatorname{tg}^2 x \, dx = \frac{e^{ax}}{a} (a \operatorname{tg} x - 1) - a \int e^{ax} \operatorname{tg} x \, dx$ (vezi 2.556 1).
3. $\int e^{ax} \operatorname{ctg} x \, dx = \frac{e^{ax} \operatorname{ctg} x}{a} + \frac{1}{a} \int \frac{e^{ax} dx}{\sin^2 x}$ (vezi observația de la 2.554).
4. $\int e^{ax} \operatorname{ctg}^2 x \, dx = -\frac{e^{ax}}{a} (a \operatorname{ctg} x + 1) + a \int e^{ax} \operatorname{ctg} x \, dx$ (vezi 2.556 3).

2.6 FUNCȚIA LOGARITMICĂ. FUNCȚII HIPERBOLICE INVERSE

2.61 Funcția logaritmică

1. $\int \ln^m x \, dx = x \ln^m x - m \int \ln^{m-1} x \, dx = x \sum_{k=0}^m (-1)^k m(m-1) \dots (m-k+1) \ln^{m-k} x.$
2. $\int \ln x \, dx = x \ln x - x.$

2.62-2.63 Funcția logaritmică și funcția putere

$$2.621 \quad \int x^n \ln^m x \, dx = \frac{x^{n+1} \ln^m x}{n+1} - \frac{m}{n+1} \int x^n \ln^{m-1} x \, dx.$$

Pentru $n = -1$,

$$\int \frac{\ln^m x \, dx}{x} = \frac{\ln^{m+1} x}{m+1}.$$

Pentru $n = -1$ și $m = -1$,

$$\int \frac{dx}{x \ln x} = \ln(\ln x).$$

$$2.622 \quad \int x^n \ln^m x \, dx = x^{n+1} \sum_{k=0}^m (-1)^k m(m-1)\dots(m-k+1) \frac{\ln^{m-k} x}{(n+1)^{k+1}}.$$

T(604)

2.623

$$1. \quad \int x^n \ln x \, dx = x^{n+1} \left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right].$$

$$2. \quad \int x^n \ln^2 x \, dx = x^{n+1} \left[\frac{\ln^2 x}{n+1} - \frac{2 \ln x}{(n+1)^2} + \frac{2}{(n+1)^3} \right].$$

$$3. \quad \int x^n \ln^3 x \, dx = x^{n+1} \left[\frac{\ln^3 x}{n+1} - \frac{3 \ln^2 x}{(n+1)^2} + \frac{6 \ln x}{(n+1)^3} - \frac{6}{(n+1)^4} \right].$$

2.624

$$1. \quad \int \frac{x^n dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n dx}{(\ln x)^{m-1}}.$$

Pentru $m = 1$,

$$2. \quad \int \frac{x^n dx}{\ln x} = \text{li}(x^{n+1}).$$

2.625

$$1. \quad \int (a+bx)^m \ln x \, dx = \frac{1}{(m+1)b} \left[(a+bx)^{m+1} \ln x - \int \frac{(a+bx)^{m+1} dx}{x} \right].$$

$$2. \quad \int (a+bx)^m \ln x \, dx = \frac{1}{(m+1)b} [(a+bx)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^m \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}.$$

Pentru $m = -1$, vezi 2.627 2.

2.626

$$1. \quad \int (a+bx) \ln x \, dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b} \right] \ln x - \left(ax + \frac{1}{4} bx^2 \right).$$

$$2. \quad \int (a+bx)^2 \ln x \, dx = \frac{1}{3b} [(a+bx)^3 - a^3] \ln x - \left(a^2 x + \frac{abx^2}{2} + \frac{b^2 x^3}{9} \right).$$

$$3. \quad \int (a+bx)^3 \ln x \, dx = \frac{1}{4b} [(a+bx)^4 - a^4] \ln x - \left(a^3 x + \frac{3}{4} a^2 bx^2 + \frac{1}{3} ab^2 x^3 + \frac{1}{16} b^3 x^4 \right).$$

2.627

$$1. \int \frac{\ln x \, dx}{(a+bx)^m} = \frac{1}{b(m-1)} \left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}} \right].$$

Pentru $m=1$,

$$2. \int \frac{\ln x \, dx}{a+bx} = \frac{1}{b} \ln x \ln(a+bx) - \frac{1}{b} \int \frac{\ln(a+bx) \, dx}{x} \quad (\text{vezi 2.628 2.}).$$

$$3. \int \frac{\ln x \, dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab} \ln \frac{x}{a+bx}.$$

$$4. \int \frac{\ln x \, dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b} \ln \frac{x}{a+bx}.$$

$$5. \int \frac{\ln x \, dx}{\sqrt{a+bx}} = \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a+bx} + \sqrt{a} \ln \frac{\sqrt{a+bx} + \sqrt{a}}{\sqrt{a+bx} - \sqrt{a}} \right\} \quad [a > 0];$$

$$= \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a+bx} + 2\sqrt{-a} \operatorname{arctg} \sqrt{\frac{a+bx}{-a}} \right\} \quad [a < 0].$$

2.628

$$1. \int x^m \ln(a+bx) \, dx = \frac{1}{m+1} \left[x^{m+1} \ln(a+bx) - b \int \frac{x^{m+1} \, dx}{a+bx} \right].$$

$$2. \int \frac{\ln(a+bx)}{x} \, dx \text{ nu se exprimă cu ajutorul unei combinații finite de}$$

funcții elementare; vezi 1.511 și 0.312.

2.629

$$1. \int x^m \ln(a+bx) \, dx = \frac{1}{m+1} \left[x^{m+1} - \frac{a^{m+1}}{b^{m+1}} \right] \ln(a+bx) +$$

$$+ \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^{k-1}}.$$

$$2. \int x \ln(a+bx) \, dx = \frac{1}{2} \left[x^2 - \frac{a^2}{b^2} \right] \ln(a+bx) - \frac{1}{2} \left[\frac{x^2}{2} - \frac{ax}{b} \right].$$

$$3. \int x^2 \ln(a+bx) \, dx = \frac{1}{3} \left[x^3 - \frac{a^3}{b^3} \right] \ln(a+bx) - \frac{1}{3} \left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2} \right].$$

$$4. \int x^3 \ln(a+bx) \, dx = \frac{1}{4} \left[x^4 - \frac{a^4}{b^4} \right] \ln(a+bx) -$$

$$- \frac{1}{4} \left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3} \right].$$

$$2.631 \quad \int x^{2n} \ln(x^2+a^2) \, dx = \frac{1}{2n+1} \left\{ x^{2n+1} \ln(x^2+a^2) + (-1)^n 2a^{2n+1} \operatorname{arctg} \frac{x}{a} - \right.$$

$$\left. - 2 \sum_{k=0}^n \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right\}.$$

$$2.632 \quad \int x^{2n+1} \ln(x^2+a^2) \, dx = \frac{1}{2n+1} \left\{ (x^{2n+2} + (-1)^n a^{2n+2}) \ln(x^2+a^2) + \right.$$

$$\left. + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} a^{2n-2k+2} x^{2k} \right\}.$$

2.633

$$1. \int \ln(x^2+a^2) \, dx = x \ln(x^2+a^2) - 2x + 2a \operatorname{arctg} \frac{x}{a}. \quad D(623)$$

$$2. \int x \ln(x^2+a^2) \, dx = \frac{1}{2} [(x^2+a^2) \ln(x^2+a^2) - x^2]. \quad D(623.1)$$

$$3. \int x^2 \ln(x^2 + a^2) dx = \frac{1}{3} \left[x^3 \ln(x^2 + a^2) - \frac{2}{3} x^3 + 2a^2 x - 2a^3 \operatorname{arctg} \frac{x}{a} \right]. \quad \text{D (623.2)}$$

$$4. \int x^3 \ln(x^2 + a^2) dx = \frac{1}{4} \left[(x^4 - a^4) \ln(x^2 + a^2) - \frac{x^4}{2} + a^2 x^2 \right]. \quad \text{D (623.3)}$$

$$5. \int x^4 \ln(x^2 + a^2) dx = \frac{1}{5} \left[x^5 \ln(x^2 + a^2) - \frac{2}{5} x^5 + \frac{2}{3} a^2 x^3 - 2a^4 x + 2a^5 \operatorname{arctg} \frac{x}{a} \right]. \quad \text{D (623.4)}$$

$$2.634 \quad \int x^{2n} \ln|x^2 - a^2| dx = \frac{1}{2n+1} \left\{ x^{2n+1} \ln|x^2 - a^2| + a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| - 2 \sum_{k=0}^n \frac{1}{2k+1} a^{2n-2k} x^{2k+1} \right\}.$$

$$2.635 \quad \int x^{2n+1} \ln|x^2 - a^2| dx = \frac{1}{2n+2} \left\{ (x^{2n+2} - a^{2n+2}) \ln|x^2 - a^2| - \sum_{k=1}^{n+1} \frac{1}{k} a^{2n-2k+2} x^{2k} \right\}.$$

2.636

$$1. \int \ln|x^2 - a^2| dx = x \ln|x^2 - a^2| - 2x + a \ln \left| \frac{x+a}{x-a} \right|. \quad \text{D (624)}$$

$$2. \int x \ln|x^2 - a^2| dx = \frac{1}{2} \left\{ (x^2 - a^2) \ln|x^2 - a^2| - x^2 \right\}. \quad \text{D (624.1)}$$

$$3. \int x^2 \ln|x^2 - a^2| dx = \frac{1}{3} \left\{ x^3 \ln|x^2 - a^2| - \frac{2}{3} x^3 - 2a^2 x + a^3 \ln \left| \frac{x+a}{x-a} \right| \right\}. \quad \text{D (624.2)}$$

$$4. \int x^3 \ln|x^2 - a^2| dx = \frac{1}{4} \left\{ (x^4 - a^4) \ln|x^2 - a^2| - \frac{x^4}{2} + a^2 x^2 \right\}. \quad \text{D (624.3)}$$

$$5. \int x^4 \ln|x^2 - a^2| dx = \frac{1}{5} \left\{ x^5 \ln|x^2 - a^2| - \frac{2}{5} x^5 - \frac{2}{3} a^2 x^3 - 2a^4 x + a^5 \ln \left| \frac{x+a}{x-a} \right| \right\}. \quad \text{D (624.4)}$$

2.64 Funcții hiperbolice inverse

2.641

$$1. \int \operatorname{Arsh} \frac{x}{a} dx = x \operatorname{Arsh} \frac{x}{a} - \sqrt{x^2 + a^2}. \quad \text{D (730)}$$

$$2. \int \operatorname{Arch} \frac{x}{a} dx = x \operatorname{Arch} \frac{x}{a} - \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} > 0 \right]; \\ = x \operatorname{Arch} \frac{x}{a} + \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} < 0 \right]. \quad \text{D (732)}$$

$$3. \int \operatorname{Arth} \frac{x}{a} dx = x \operatorname{Arth} \frac{x}{a} + \frac{a}{2} \ln(a^2 - x^2). \quad \text{D (734)}$$

$$4. \int \operatorname{Arcth} \frac{x}{a} dx = x \operatorname{Arcth} \frac{x}{a} + \frac{a}{2} \ln(x^2 - a^2). \quad \text{D (736)}$$

2.642

$$1. \int x \operatorname{Arsh} \frac{x}{a} dx = \left(\frac{x^2}{2} + \frac{a^2}{4} \right) \operatorname{Arsh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}. \quad \text{D (730.1)}$$

$$2. \int x \operatorname{Arch} \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{Arch} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} > 0 \right]; \\ = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{Arch} \frac{x}{a} + \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} < 0 \right]. \quad \text{D (732.1)}$$

2.7 FUNCȚII TRIGONOMETRICE INVERSE

2.71 Arcsinus și arccosinus

$$2.711 \quad \int \left(\arcsin \frac{x}{a} \right)^n dx = x \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arcsin \frac{x}{a} \right)^{n-2k} + \\ + \sqrt{a^2 - x^2} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^{k-1} \binom{n}{2k-1} \cdot (2k-1)! \left(\arcsin \frac{x}{a} \right)^{n-2k+1}.$$

$$2.712 \quad \int \left(\arccos \frac{x}{a} \right)^n dx = x \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arccos \frac{x}{a} \right)^{n-2k} + \\ + \sqrt{a^2 - x^2} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^k \binom{n}{2k-1} \cdot (2k-1)! \left(\arccos \frac{x}{a} \right)^{n-2k+1}.$$

2.713

$$1. \quad \int \arcsin \frac{x}{a} dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}.$$

$$2. \quad \int \left(\arcsin \frac{x}{a} \right)^3 dx = x \left(\arcsin \frac{x}{a} \right)^3 + 2 \sqrt{a^2 - x^2} \arcsin \frac{x}{a} - 2x.$$

$$3. \quad \int \left(\arcsin \frac{x}{a} \right)^3 dx = x \left(\arcsin \frac{x}{a} \right)^3 + 3 \sqrt{a^2 - x^2} \left(\arcsin \frac{x}{a} \right)^2 - \\ - 6x \arcsin \frac{x}{a} - 6 \sqrt{a^2 - x^2}.$$

2.714

$$1. \quad \int \arccos \frac{x}{a} dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2}.$$

$$2. \quad \int \left(\arccos \frac{x}{a} \right)^3 dx = x \left(\arccos \frac{x}{a} \right)^3 - 2 \sqrt{a^2 - x^2} \arccos \frac{x}{a} - 2x.$$

$$3. \quad \int \left(\arccos \frac{x}{a} \right)^3 dx = x \left(\arccos \frac{x}{a} \right)^3 - 3 \sqrt{a^2 - x^2} \arccos \frac{x}{a} - \\ - 6x \arccos \frac{x}{a} + 6 \sqrt{a^2 - x^2}.$$

2.72 Arcsecanta și arccosecanta, arctangenta și arccotangenta

2.721

$$1. \quad \int \operatorname{arccosec} \frac{x}{a} dx = \int \arcsin \frac{a}{x} dx = \\ = x \arcsin \frac{a}{x} + a \ln(x + \sqrt{x^2 - a^2}) \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right]; \\ = x \arcsin \frac{a}{x} - a \ln(x + \sqrt{x^2 - a^2}) \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right].$$

D(534)

$$2. \quad \int \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \\ = x \arccos \frac{a}{x} - a \ln(x + \sqrt{x^2 - a^2}) \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right]; \\ = x \arccos \frac{a}{x} + a \ln(x + \sqrt{x^2 - a^2}) \left[-\frac{\pi}{2} < \arccos \frac{a}{x} < 0 \right].$$

D(531)

2.722

$$1. \int \operatorname{arctg} \frac{x}{a} dx = x \operatorname{arctg} \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2). \quad D(525)$$

$$2. \int \operatorname{arccotg} \frac{x}{a} dx = x \operatorname{arccotg} \frac{x}{a} + \frac{a}{2} \ln(a^2 + x^2) \quad D(528)$$

2.73 Arcsinus, arccosinus și funcția algebrică

$$2.731 \quad \int x^n \arcsin \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}} \\ \text{(vezi 2.263 1., 2.264, 2.27).}$$

$$2.732 \quad \int x^n \arccos \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arccos \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}} \\ \text{(vezi 2.263 1., 2.264, 2.27).}$$

1. Pentru $n = -1$, aceste integrale (adică $\int \frac{\arcsin x}{x} dx$ și $\int \frac{\arccos x}{x} dx$) nu se exprimă cu ajutorul unei combinații finite de funcții elementare.

$$2. \int \frac{\arccos x}{x} dx = -\frac{\pi}{2} \ln \frac{1}{x} - \int \frac{\arcsin x}{x} dx.$$

2.733

$$1. \int x \arcsin \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arcsin \frac{x}{a} + \frac{x}{4} \sqrt{a^2 - x^2}.$$

$$2. \int x \arccos \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arccos \frac{x}{a} - \frac{x}{4} \sqrt{a^2 - x^2}.$$

2.734

$$1. \int \frac{1}{x^2} \arcsin \frac{x}{a} dx = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}.$$

$$2. \int \frac{1}{x^2} \arccos \frac{x}{a} dx = -\frac{1}{x} \arccos \frac{x}{a} + \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}.$$

$$2.735 \quad \int \frac{\arcsin x}{(a+bx)^2} dx = -\frac{\arcsin x}{b(a+bx)} - \frac{2}{b\sqrt{a^2-b^2}} \operatorname{arctg} \sqrt{\frac{(a-b)(1-x)}{(a+b)(1+x)}} \quad [a^2 > b^2]; \\ = -\frac{\arcsin x}{b(a+bx)} - \frac{1}{b\sqrt{b^2-a^2}} \ln \frac{\sqrt{(a+b)(1+x)} + \sqrt{(b-a)(1-x)}}{\sqrt{(a+b)(1+x)} - \sqrt{(b-a)(1-x)}} \quad [a^2 < b^2].$$

$$2.736 \quad \int \frac{x \arcsin x}{(1+cx^2)^2} dx = \frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{2c\sqrt{c+1}} \operatorname{arctg} \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \quad [c > -1]; \\ = -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \quad [c < -1].$$

2.737

$$1. \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx = x - \sqrt{1-x^2} \arcsin x.$$

$$2. \int \frac{x^2 \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^3}{4} - \frac{x}{2} \sqrt{1-x^2} \arcsin x + \frac{1}{4} (\arcsin x)^2.$$

$$3. \int \frac{x^3 \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3} (x^2 + 2) \sqrt{1-x^2} \arcsin x.$$

2.738

$$1. \int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln(1-x^2).$$

$$2. \int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1-x}{1+x}.$$

2.74 Arcsecanta, arccosecanta și puteri ale lui x

2.741

1. $\int x \operatorname{arcsec} \frac{x}{a} dx = \int x \arccos \frac{a}{x} dx =$
 $= \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} - a \sqrt{x^2 - a^2} \right\} \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right];$
 $= \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} + a \sqrt{x^2 - a^2} \right\} \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right].$
2. $\int x^2 \operatorname{arcsec} \frac{x}{a} dx = \int x^2 \arccos \frac{a}{x} dx =$
 $= \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} - \frac{a}{2} x \sqrt{x^2 - a^2} - \frac{a^3}{2} \ln (x + \sqrt{x^2 - a^2}) \right\}$
 $\left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right]; \quad D(531.1)$
 $= \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} + \frac{a}{2} x \sqrt{x^2 - a^2} + \frac{a^3}{2} \ln (x + \sqrt{x^2 - a^2}) \right\}$
 $\left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right]. \quad D(531.2)$
3. $\int x \operatorname{arccosec} \frac{x}{a} dx = \int x \arcsin \frac{a}{x} dx =$
 $= \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} + a \sqrt{x^2 - a^2} \right\} \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right];$
 $= \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} - a \sqrt{x^2 - a^2} \right\} \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right].$
 $D(534.2)$

2.75 Arctangenta, arccotangenta și funcția putere

$$2.751 \quad \int x^n \operatorname{arctg} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arctg} \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}.$$

2.752

$$1. \quad \int x^n \operatorname{arccotg} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arccotg} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}.$$

Pentru $n = -1$,

$\int \frac{\operatorname{arctg} x}{x} dx$ nu se poate exprima cu ajutorul unei combinații finite de funcții elementare.

$$2. \quad \int \frac{\operatorname{arccotg} x}{x} dx = \frac{\pi}{2} \ln x - \int \frac{\operatorname{arctg} x}{x} dx.$$

2.753

$$1. \quad \int x \operatorname{arctg} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{ax}{2}.$$

$$2. \quad \int x \operatorname{arccotg} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arccotg} \frac{x}{a} + \frac{ax}{2}.$$

$$2.754 \quad \int \frac{1}{x^2} \operatorname{arctg} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arctg} \frac{x}{a} - \frac{1}{2a} \ln \frac{a^2 + x^2}{x^2}.$$

2.755

$$1. \quad \int \frac{x \operatorname{arctg} x}{1+x^2} dx = \frac{1}{2} \operatorname{arctg} x \ln(1+x^2) + \int \ln \cos x dx.$$

$$2. \quad \int \frac{x^2 \operatorname{arctg} x}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\operatorname{arctg} x)^2.$$

$$3. \quad \int \frac{x^3 \operatorname{arctg} x}{1+x^2} dx = -\frac{1}{2} x + \frac{1}{2} (1+x^2) \operatorname{arctg} x - \int \frac{x \operatorname{arctg} x}{1+x^2} dx.$$

- 2.756 $\int \frac{\operatorname{arctg} x}{(a+\beta x)^2} dx = \frac{1}{a^2+\beta^2} \left\{ \ln \frac{a+\beta x}{\sqrt{1+x^2}} - \frac{\beta-ax}{a+\beta x} \operatorname{arctg} x \right\}.$
- 2.757 $\int \frac{x^4 \operatorname{arctg} x}{1+x^2} dx = -\frac{1}{6} x^3 + \frac{2}{3} \ln(1+x^2) + \left(\frac{x^3}{3} - x\right) \operatorname{arctg} x + \frac{1}{2} (\operatorname{arctg} x)^2.$
- 2.758 $\int \frac{x \operatorname{arctg} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \operatorname{arctg} x + \sqrt{2} \operatorname{arctg} \frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x.$
- 2.759 $\int \frac{\operatorname{arctg} x}{\sqrt{(a+bx^2)^3}} dx = \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}} \operatorname{arctg} \sqrt{\frac{a+bx^2}{b-a}} \quad [a < b];$
 $= \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} - \frac{1}{2a\sqrt{a-b}} \ln \frac{\sqrt{a+bx^2} - \sqrt{a-b}}{\sqrt{a+bx^2} + \sqrt{a-b}} \quad [a > b].$

2.8 FUNCȚII SPECIALE

2.81 Funcții eliptice Jacobi

2.811

1. $\int \operatorname{sn}^m u du = \frac{1}{m+1} \left[\operatorname{sn}^{m+1} u \operatorname{cn} u \operatorname{dn} u + \right.$
 $\left. + (m+2)(1+k^2) \int \operatorname{sn}^{m+2} u du - (m+3)k^2 \int \operatorname{sn}^{m+4} u du \right]. \quad \text{Si 259 P (567)}$
2. $\int \operatorname{cn}^m u du = \frac{1}{(m+1)k'^2} \left[-\operatorname{cn}^{m+1} u \operatorname{sn} u \operatorname{dn} u + \right.$
 $\left. + (m+2)(1-2k^2) \int \operatorname{cn}^{m+2} u du + (m+3)k^2 \int \operatorname{cn}^{m+4} u du \right]. \quad \text{P (568)}$
3. $\int \operatorname{dn}^m u du = \frac{1}{(m+1)k'^2} \left[k^2 \operatorname{dn}^{m+1} u \operatorname{sn} u \operatorname{cn} u + \right.$
 $\left. + (m+2)(2-k^2) \int \operatorname{dn}^{m+2} u du - (m+3) \int \operatorname{dn}^{m+4} u du \right]. \quad \text{P (569)}$

Aceste integrale se reduc la integralele

2.812

1. $\int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}; \quad \text{J 87 (164)}$
 $= \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u}. \quad \text{Si 266 (4)}$
2. $\int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u}. \quad \text{Si 266 (5)}$
3. $\int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \operatorname{arctg} \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u}; \quad \text{J 88 (166)}$
 $= \frac{1}{k'} \arccos \frac{\operatorname{cn} u}{\operatorname{dn} u}; \quad \text{JE 192}$
 $= \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}; \quad \text{Si 266 (6)}$
 $= \frac{1}{k'} \arcsin \frac{k' \operatorname{sn} u}{\operatorname{dn} u}. \quad \text{JE 192}$

2.813

1. $\int \operatorname{sn} u du = \frac{1}{k} \ln (\operatorname{dn} u - k \operatorname{cn} u); \quad \text{J 87 (161)}$
 $= \frac{1}{k} \operatorname{Arch} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1-k^2}; \quad \text{JE 192}$
 $= \frac{1}{k} \operatorname{Arsh} \left(k \frac{\operatorname{dn} u - \operatorname{cn} u}{1-k^2} \right); \quad \text{JE 192}$
 $= -\frac{1}{k} \ln (\operatorname{dn} u + k \operatorname{cn} u). \quad \text{Si 265 (1)}$

$$\begin{aligned}
 2. \quad \int \operatorname{cn} u \, du &= \frac{1}{k} \arccos(\operatorname{dn} u); & \text{J 87 (162)} \\
 &= \frac{i}{k} \ln(\operatorname{dn} u - ik \operatorname{sn} u); & \text{Si 265 (2) m, J 87 (162)} \\
 &= \frac{1}{k} \arcsin(k \operatorname{sn} u). & \text{JE 192}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \operatorname{dn} u \, du &= \arcsin(\operatorname{sn} u); & \text{J 87 (163)} \\
 &= \operatorname{am} u = i \ln(\operatorname{cn} u - i \operatorname{sn} u). & \text{Si 266 (3) J 87 (163)}
 \end{aligned}$$

2.814

$$\begin{aligned}
 1. \quad \int \operatorname{sn}^2 u \, du &= \frac{1}{k^2} [u - E(\operatorname{am} u, k)]. & \text{P (564)} \\
 2. \quad \int \operatorname{cn}^2 u \, du &= \frac{1}{k^2} [E(\operatorname{am} u, k) - k'^2 u]. & \text{P (565)} \\
 3. \quad \int \operatorname{dn}^2 u \, du &= E(\operatorname{am} u, k). & \text{P (566)}
 \end{aligned}$$

2.815

$$\begin{aligned}
 1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn} u} \, du &= \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u}; & \text{Si 266 (7)} \\
 &= \frac{1}{2k'} \ln \frac{k' + \operatorname{dn} u}{k' - \operatorname{dn} u}. & \text{J 88 (167)} \\
 2. \quad \int \frac{\operatorname{sn} u}{\operatorname{dn} u} \, du &= \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u}; & \text{Si 266 (8)} \\
 &= \frac{1}{kk'} \operatorname{arctg} \frac{k \operatorname{cn} u}{k'}. & \text{J 88 (169)} \\
 3. \quad \int \frac{\operatorname{cn} u}{\operatorname{sn} u} \, du &= \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u}; & \text{Si 266 (10)} \\
 &= \frac{1}{2} \ln \frac{\operatorname{dn} u - 1}{\operatorname{dn} u + 1}. & \text{J 88 (168)} \\
 4. \quad \int \frac{\operatorname{cn} u}{\operatorname{dn} u} \, du &= -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u}; & \text{Si 266 (9)} \\
 &= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u}. & \text{J 88 (171)} \\
 5. \quad \int \frac{\operatorname{dn} u}{\operatorname{cn} u} \, du &= \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u}; & \text{J 88 (172)} \\
 &= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}. & \text{JE 193} \\
 6. \quad \int \frac{\operatorname{dn} u}{\operatorname{sn} u} \, du &= \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}. & \text{J 87 (170)}
 \end{aligned}$$

2.816

$$\begin{aligned}
 1. \quad \int \operatorname{sn} u \operatorname{cn} u \, du &= -\frac{1}{k^2} \operatorname{dn} u. \\
 2. \quad \int \operatorname{sn} u \operatorname{dn} u \, du &= -\operatorname{cn} u. \\
 3. \quad \int \operatorname{cn} u \operatorname{dn} u \, du &= \operatorname{sn} u.
 \end{aligned}$$

2.817

$$\begin{aligned}
 1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u} \, du &= \frac{1}{k'^2} \frac{\operatorname{dn} u}{\operatorname{cn} u}. & \text{J 88 (173)} \\
 2. \quad \int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u} \, du &= -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u}. & \text{J 88 (175)} \\
 3. \quad \int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u} \, du &= -\frac{\operatorname{dn} u}{\operatorname{sn} u}. & \text{J 88 (174)}
 \end{aligned}$$

$$4. \int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{dn} u}. \quad \text{J 88 (177)}$$

$$5. \int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}. \quad \text{J 88 (176)}$$

$$6. \int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{cn} u}. \quad \text{J 88 (178)}$$

2.818

$$1. \int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u}. \quad \text{J 88 (183)}$$

$$2. \int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \ln \frac{\operatorname{dn} u}{\operatorname{cn} u}. \quad \text{J 88 (182)}$$

$$3. \int \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u}. \quad \text{J 88 (184)}$$

2.819

$$1. \int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du = \ln \operatorname{sn} u. \quad \text{J 88 (179)}$$

$$2. \int \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} du = \ln \frac{1}{\operatorname{cn} u}. \quad \text{J 88 (180)}$$

$$3. \int \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} du = -\frac{1}{k^2} \ln \operatorname{dn} u. \quad \text{J 88 (181)}$$

2.82 Integrale eliptice complete

2.821 Vezi și Anexa

$$1. \int \frac{K dk}{k^3} = -\frac{1}{k} E = \frac{k^2 D - K}{k}.$$

$$2. \int \frac{E}{k^3} dk = \frac{k'^2 K - 2E}{k}.$$

$$3. \int K k dk = E - k'^2 K = k^2 (K - D).$$

$$4. \int E k dk = \frac{1+k^2}{3} E - \frac{k'^2}{3} K.$$

$$5. \int \frac{E}{k'^2} dk = kK. \quad \text{Lo 1252}$$

$$6. \int \left(\frac{E}{k^2 k'^2} - \frac{K}{k^2} \right) dk = \frac{1}{k} (K - E).$$

$$7. \int \frac{K - D}{k'^2} dk = kD.$$

2.83 Funcții eliptice Weierstrass

2.831

$$1. \int \wp(u) du = -\zeta(u).$$

$$2. \int \wp^2(u) du = \frac{1}{6} \wp'(u) + \frac{1}{12} g_2 u. \quad \text{J 120 (192)}$$

$$3. \int \wp^3(u) du = \frac{1}{120} \wp'''(u) - \frac{3}{20} g_2 \zeta(u) + \frac{1}{10} g_3 u. \quad \text{J 120 (193)}$$

$$4. \int \frac{du}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left[2u\zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right]. \quad \text{J 120 (194)}$$

$$5. \int \frac{\alpha \wp(u) + \beta}{\gamma \wp(u) + \delta} du = \frac{\alpha u}{\gamma} - \frac{\alpha \delta - \beta \gamma}{\gamma^2 \wp'(v)} \left[\ln \frac{\sigma(u+v)}{\sigma(u-v)} - 2u\zeta(v) \right], \text{ unde } \wp'(v) = -\frac{\delta}{\gamma}. \quad \text{J 120 (195)}$$

2.84 Sinus integral și cosinus integral

2.841

$$1. \int \cos ax \operatorname{ci}(\beta x) dx = \frac{\sin ax \operatorname{ci}(\beta x)}{a} - \frac{\operatorname{si}(ax + \beta x) + \operatorname{si}(ax - \beta x)}{2a}. \quad \text{NI 49 (1)}$$

$$2. \int \sin ax \operatorname{ci}(\beta x) dx = -\frac{\cos ax \operatorname{ci}(\beta x)}{a} + \frac{\operatorname{ci}(ax + \beta x) + \operatorname{ci}(ax - \beta x)}{2a}. \quad \text{NI 49 (2)}$$

2.842

$$1. \int \cos ax \operatorname{si}(\beta x) dx = \frac{\sin ax \operatorname{si}(\beta x)}{a} + \frac{\operatorname{ci}(ax + \beta x) - \operatorname{ci}(ax - \beta x)}{2a}. \quad \text{NI 49 (3)}$$

$$2. \int \sin ax \operatorname{si}(\beta x) dx = -\frac{\cos ax \operatorname{si}(\beta x)}{a} + \frac{\operatorname{si}(ax + \beta x) - \operatorname{si}(ax - \beta x)}{2a}. \quad \text{NI 49 (4)}$$

2.843

$$1. \int \operatorname{ci}(ax) \operatorname{ci}(\beta x) dx = x \operatorname{ci}(ax) \operatorname{ci}(\beta x) + \frac{1}{2a} (\operatorname{si}(ax + \beta x) + \operatorname{si}(ax - \beta x)) + \frac{1}{2\beta} (\operatorname{si}(ax + \beta x) + \operatorname{si}(\beta x - ax)) - \frac{1}{a} \sin ax \operatorname{ci}(\beta x) - \frac{1}{\beta} \sin \beta x \operatorname{ci}(ax). \quad \text{NI 53 (5)}$$

$$2. \int \operatorname{si}(ax) \operatorname{si}(\beta x) dx = x \operatorname{si}(ax) \operatorname{si}(\beta x) - \frac{1}{2\beta} (\operatorname{si}(ax + \beta x) + \operatorname{si}(ax - \beta x)) - \frac{1}{2a} (\operatorname{si}(ax + \beta x) + \operatorname{si}(\beta x - ax)) + \frac{1}{a} \cos ax \operatorname{si}(\beta x) + \frac{1}{\beta} \cos \beta x \operatorname{si}(ax). \quad \text{NI 54 (6)}$$

$$3. \int \operatorname{si}(ax) \operatorname{ci}(\beta x) dx = x \operatorname{si}(ax) \operatorname{ci}(\beta x) + \frac{1}{a} \cos ax \operatorname{ci}(\beta x) - \frac{1}{\beta} \sin \beta x \operatorname{si}(ax) - \left(\frac{1}{2a} + \frac{1}{2\beta}\right) \operatorname{ci}(ax + \beta x) - \left(\frac{1}{2a} - \frac{1}{2\beta}\right) \operatorname{ci}(ax - \beta x). \quad \text{NI 54 (10)}$$

2.85 Funcții cilindrice

$$2.851 \quad \int J_p(x) dx = 2 \sum_{k=0}^{\infty} J_{p+2k+1}(x). \quad \text{JE 237 MO 30}$$

2.852

$$1. \int x^{p+1} Z_p(x) dx = x^{p+1} Z_{p+1}(x). \quad \text{W 146 (1)m}$$

$$2. \int x^{-p+1} Z_p(x) dx = -x^{-p+1} Z_{p-1}(x). \quad \text{W 146 (2)m}$$

2.853

$$\int \left[(\alpha^2 - \beta^2)x - \frac{p^2 - q^2}{x} \right] Z_p(ax) \mathfrak{B}_q(\beta x) dx = \beta x Z_p(ax) \mathfrak{B}_{q-1}(\beta x) - \alpha x Z_{p-1}(ax) \mathfrak{B}_q(\beta x) + (p - q) Z_p(ax) \mathfrak{B}_q(\beta x). \quad \text{JE 237, W 148 (7)m}$$

2.854

$$1. \int x Z_p(ax) \mathfrak{B}_p(\beta x) dx = \frac{\beta x Z_p(ax) \mathfrak{B}_{p-1}(\beta x) - \alpha x Z_{p-1}(ax) \mathfrak{B}_p(\beta x)}{\alpha^2 - \beta^2}. \quad \text{W 148 (8)m}$$

$$2. \int x [Z_p(ax)]^2 dx = \frac{x^2}{2} \{ [Z_p(ax)]^2 - Z_{p-1}(ax) Z_{p+1}(ax) \}. \quad \text{W 149 (11)m}$$

2.855

$$\int \frac{1}{x} Z_p(ax) \mathfrak{B}_q(ax) dx = \alpha x \frac{Z_{p-1}(ax) \mathfrak{B}_q(ax) - Z_p(ax) \mathfrak{B}_{q-1}(ax)}{p^2 - q^2} - \frac{Z_p(ax) \mathfrak{B}_q(ax)}{p + q}. \quad \text{W 149 (13)m}$$

2.856

Vezi și Anexa

$$1. \int Z_1(x) dx = -Z_0(x). \quad \text{JE 237}$$

$$2. \int x Z_0(x) dx = x Z_1(x). \quad \text{JE 237}$$

3. INTEGRALE DEFINITE ALE FUNCȚIILOR ELEMENTARE

3.0 INTRODUCERE¹

3.01 Teoreme cu caracter general

3.011 Fie $f(x)$ integrabilă² în cel mai mare dintre intervalele (p, q) , (p, r) , (r, q) . Atunci (independent de poziția relativă a punctelor p, q, r), ea este integrabilă și în celelalte două intervale, și avem egalitatea

$$\int_p^q f(x) dx = \int_p^r f(x) dx + \int_r^q f(x) dx. \quad \text{F II 126}$$

3.012 Teorema valorii medii (prima). Fie 1) $f(x)$ continuă și $g(x)$ integrabilă în intervalul (p, q) ; 2) $m \leq f(x) \leq M$; 3) $g(x)$ nu-și schimbă semnul în tot intervalul (p, q) . Atunci există cel puțin un punct ξ ($p \leq \xi \leq q$), pentru care

$$\int_p^q f(x) g(x) dx = f(\xi) \int_p^q g(x) dx. \quad \text{F II 132}$$

3.013 A doua teoremă a valorii medii. Dacă, în intervalul (p, q) [$p < q$], $f(x)$ este monoton descrescătoare și nenegativă, iar $g(x)$ este integrabilă, atunci există cel puțin un punct ξ [$p \leq \xi \leq q$], pentru care

$$1. \quad \int_p^q f(x) g(x) dx = f(p) \int_p^{\xi} g(x) dx.$$

Dacă, celelalte condiții ale teoremei 3.013 1. fiind păstrate, $f(x)$ este monoton crescătoare, atunci

$$2. \quad \int_p^q f(x) g(x) dx = f(q) \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q].$$

¹ Omitem definiția integralelor definite și a integralelor multiple, deoarece ele sînt foarte cunoscute și pot fi ușor găsite în orice manual. Dăm aici numai unele teoreme cu caracter general, care evaluează integrala dată sau o reduc la alta mai simplă.

² Funcția $f(x)$ se numește *integrabilă* în intervalul (p, q) , dacă există $\int_p^q f(x) dx$. Aici subînțelegem, de obicei, existența integralei în sensul lui Riemann. Dacă este vorba, însă, de existența integralei în sensul lui Stieltjes, Lebesgue etc., atunci vorbim despre integrabilitate în sensul lui Stieltjes, Lebesgue etc.

Dacă, în intervalul (p, q) [$p < q$], $f(x)$ este monotonă și $g(x)$ integrabilă, atunci

$$3. \quad \int_p^q f(x) g(x) dx = f(p) \int_p^{\xi} g(x) dx + f(q) \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q]$$

sau

$$4. \quad \int_p^q f(x) g(x) dx = A \int_p^{\xi} g(x) dx + B \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q],$$

unde A și B sînt două numere oarecare, satisfăcînd condițiile

$$\begin{aligned} A &\geq f(p+0) \text{ și } B \leq f(q-0) \quad [\text{dacă } f \text{ descrește}], \\ A &\leq f(p+0) \text{ și } B \geq f(q-0) \quad [\text{dacă } f \text{ crește}]; \end{aligned}$$

în particular,

$$5. \quad \int_p^q f(x) g(x) dx = f(p+0) \int_p^{\xi} g(x) dx + f(q-0) \int_{\xi}^q g(x) dx. \quad \text{F II 138}$$

3.02 Schimbarea de variabilă în integrala definită

$$3.020 \quad \int_a^b f(x) dx = \int_{\varphi}^{\psi} f[g(t)] g'(t) dt; \quad x = g(t).$$

Această formulă este valabilă în următoarele condiții:

1. $f(x)$ este continuă pe un segment oarecare $A \leq x \leq B$, care cuprinde vechile limite α și β .

2. Au loc egalitățile: $\alpha = g(\varphi)$, $\beta = g(\psi)$.

3. $g(t)$ și derivata ei $g'(t)$ sînt continue pe segmentul $\varphi \leq t \leq \psi$.

4. Când t variază de la φ pînă la ψ , $g(t)$ variază întotdeauna în același sens, de la $g(\varphi) = \alpha$ pînă la $g(\psi) = \beta$.

3.021 Integrala $\int_a^b f(x) dx$ poate fi transformată într-o altă integrală cu limitele φ și ψ , date, cu ajutorul substituției liniare

$$x = \frac{\beta - \alpha}{\psi - \varphi} t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi};$$

$$1. \quad \int_a^b f(x) dx = \frac{\beta - \alpha}{\psi - \varphi} \int_{\varphi}^{\psi} f\left(\frac{\beta - \alpha}{\psi - \varphi} t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi}\right) dt;$$

în particular, pentru $\varphi = 0$, $\psi = 1$:

$$2. \quad \int_a^b f(x) dx = (\beta - \alpha) \int_0^1 f((\beta - \alpha)t + \alpha) dt.$$

Pentru $\varphi = 0$, $\psi = \infty$,

$$3. \quad \int_a^b f(x) dx = (\beta - \alpha) \int_0^{\infty} f\left(\frac{\alpha + \beta t}{1+t}\right) \frac{dt}{(1+t)^2}.$$

¹ În cazul cînd ultima condiție nu este satisfăcută, segmentul $\varphi \leq x \leq \psi$ trebuie împărțit în porțiuni pentru care condiția este satisfăcută.

$$\int_a^b f(x) dx = \int_{\varphi_1}^{\varphi_2} f[g(t)] g'(t) dt + \int_{\varphi_2}^{\varphi_3} f[g(t)] g'(t) dt + \dots + \int_{\varphi_{n-1}}^{\varphi_n} f[g(t)] g'(t) dt.$$

3.022 Au loc, de asemenea, următoarele egalități:

$$1. \int_a^{\beta} f(x) dx = \int_a^{\beta} f(\alpha + \beta - x) dx$$

$$2. \int_0^{\beta} f(x) dx = \int_0^{\beta} f(\beta - x) dx.$$

$$3. \int_{-a}^a f(x) dx = \int_{-a}^a f(-x) dx.$$

3.03 Formule cu caracter general

3.031

1. Fie $f(x)$ o funcție integrabilă pe segmentul $[-p, p]$ și care satisface, pe acest segment, relația $f(-x) = f(x)$ (o astfel de funcție se numește *pară*); atunci

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx. \quad \text{F II 159}$$

2. Fie $f(x)$ o funcție integrabilă pe segmentul $[-p, p]$ și care satisface, pe acest segment, relația $f(-x) = -f(x)$ (o astfel de funcție se numește *impară*); atunci

$$\int_{-p}^p f(x) dx = 0. \quad \text{F II 159}$$

3.032

$$1. \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx,$$

unde $f(x)$ este o funcție integrabilă pe segmentul $[0, 1]$, F II 159

$$2. \int_0^{2\pi} f(p \cos x + q \sin x) dx = 2 \int_0^{\pi} f(\sqrt{p^2 + q^2} \cos x) dx,$$

unde $f(x)$ este o funcție integrabilă pe segmentul $[-\sqrt{p^2 + q^2}, \sqrt{p^2 + q^2}]$, F II 160

$$3. \int_0^{\frac{\pi}{2}} f(\sin 2x) \cos x dx = \int_0^{\frac{\pi}{2}} f(\cos^2 x) \cos x dx,$$

unde $f(x)$ este o funcție integrabilă pe segmentul $[0, 1]$. F II 161

3.033

1. Dacă $f(x + \pi) = f(x)$ și $f(-x) = f(x)$, atunci

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx.$$

2. Dacă $f(x+\pi)=-f(x)$ și $f(-x)=f(x)$, atunci

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) \cos x dx.$$

În formulele 3.033, se presupune că integralele din membrul întâi al formulelor există.

$$3.034 \quad \int_0^{\infty} \frac{f(px)-f(qx)}{x} dx = [f(0)-f(+\infty)] \ln \frac{q}{p},$$

dacă $f(x)$ este o funcție continuă pentru $x \geq 0$ și dacă există o limită finită $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$. F II 633

3.035

$$1. \quad \int_0^{\pi} \frac{f(\alpha + e^{ix}) + f(\alpha + e^{-ix})}{1 + 2p \cos x + p^2} dx = \frac{2\pi}{1-p^2} f(\alpha + p) \quad [|p| < 1]. \quad \text{La 230 (16)}$$

$$2. \quad \int_0^{\pi} \frac{1-p \cos x}{1-2p \cos x + p^2} \{f(\alpha + e^{ix}) + f(\alpha + e^{-ix})\} dx = \pi \{f(\alpha + p) + f(\alpha)\} \\ [|p| < 1]. \quad \text{B 169}$$

$$3. \quad \int_0^{\pi} \frac{f(\alpha + e^{-ix}) - f(\alpha + e^{ix})}{1-2p \cos x + p^2} \sin x dx = \frac{\pi}{pi} \{f(\alpha + p) - f(\alpha)\} \quad [|p| < 1].$$

B 169

În formulele 3.035, se presupune că funcția f este analitică într-un cerc închis de rază 1, cu centrul în punctul α .

3.036

$$1. \quad \int_0^{\pi} f\left(\frac{\sin^2 x}{1+2p \cos x + p^2}\right) dx = \int_0^{\pi} f(\sin^2 x) dx \quad [p^2 \geq 1]; \\ = \int_0^{\pi} f\left(\frac{\sin^2 x}{p^2}\right) dx \quad [p^2 < 1]. \quad \text{La 228 (6)}$$

$$2. \quad \int_0^{\pi} F^{(n)}(\cos x) \sin^{2n} x dx = (2n-1)! \int_0^{\pi} F(\cos x) \cos nx dx. \quad \text{B 174}$$

3.037 Dacă f este o funcție analitică într-un cerc de rază r și dacă

$$f[r(\cos x + i \sin x)] = f_1(r, x) + i f_2(r, x),$$

atunci

$$1. \quad \int_0^{\infty} \frac{f_1(r, x)}{p^2 + x^2} dx = \frac{\pi}{2p} f(re^{-p}). \quad \text{La 230 (19)}$$

$$2. \quad \int_0^{\infty} f_2(r, x) \frac{x dx}{p^2 + x^2} = \frac{\pi}{2} [f(re^{-p}) - f(0)]. \quad \text{La 230 (20)}$$

$$3. \int_0^{\infty} \frac{f_2(r, x)}{x} dx = \frac{\pi}{2} [f(r) - f(0)]. \quad \text{La 230 (21)}$$

$$4. \int_0^{\infty} \frac{f_2(r, x)}{x(p^2 + x^2)} dx = \frac{\pi}{2p^2} [f(r) - f(re^{-p})]. \quad \text{La 230 (22)}$$

$$3.038 \int_{-\infty}^{\infty} \frac{x dx}{\sqrt{1+x^2}} F(qx + p\sqrt{1+x^2}) = \int_{-\infty}^{\infty} F(p \operatorname{ch} x + q \operatorname{sh} x) \operatorname{sh} x dx = \\ = 2q \int_0^{\infty} F'(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2} \operatorname{ch} x) \operatorname{sh}^2 x dx$$

F este o funcție avînd derivata continuă în intervalul $(-\infty, \infty)$; toate integralele folosite sînt convergente]. Lo III 281 m, Lo III 391 m.

3.04 Integrale improprii

3.041 Fie funcția $f(x)$ definită în intervalul $(p, +\infty)$ și integrabilă în orice parte finită (p, P) a acestuia; atunci, prin definiție,

$$\int_p^{+\infty} f(x) dx = \lim_{P \rightarrow +\infty} \int_p^P f(x) dx,$$

dacă această limită există. În cazul cînd această limită există, se spune că integrala $\int_p^{+\infty} f(x) dx$ există sau converge. În caz contrar, se spune că integrala este divergentă.

3.042 Fie funcția $f(x)$ mărginită și integrabilă într-un interval oarecare $(p, q - \eta)$ ($0 < \eta < q - p$), dar nemărginită în orice interval $(q - \eta, q)$ situat la stînga punctului q . Punctul q se numește, în acest caz, *punct singular*. Atunci, prin definiție,

$$\int_p^q f(x) dx = \lim_{\eta \rightarrow 0} \int_p^{q-\eta} f(x) dx,$$

dacă această limită există. În acest caz, se spune că integrala $\int_p^q f(x) dx$ există sau converge.

3.043 Dacă este convergentă, nu numai integrala din $f(x)$, dar și integrala din $|f(x)|$, atunci se spune că integrala din $f(x)$ converge absolut.

3.044 Integrala $\int_p^{+\infty} f(x) dx$ converge absolut, dacă se poate determina un număr $\alpha > 1$, pentru care limita

$$\lim_{x \rightarrow +\infty} \{x^\alpha |f(x)|\}$$

există; dacă însă

$$\lim_{x \rightarrow +\infty} \{x |f(x)|\} = L > 0,$$

atunci integrala $\int_p^{+\infty} |f(x)| dx$ diverge.

3.045 Integrala $\int_p^q f(x) dx$, pentru care limita superioară q este un punct singular, converge absolut, dacă se poate determina un număr $\alpha < 1$, pentru care limita

$$\lim_{x \rightarrow q} [(q-x)^\alpha |f(x)|]$$

există; dacă însă

$$\lim_{x \rightarrow q} [(q-x) |f(x)|] = L > 0,$$

atunci integrala $\int_p^q f(x) dx$ este divergentă.

3.046 Fie funcțiile $f(x)$ și $g(x)$ definite în intervalul $(p, +\infty)$, $f(x)$ fiind integrabilă în orice interval finit (p, P) . Dacă integrala

$$\int_p^P f(x) dx$$

este o funcție de P mărginită, iar $g(x)$ o funcție monotonă, și anume $g(x) \rightarrow 0$ pentru $x \rightarrow +\infty$, atunci integrala

$$\int_p^{+\infty} f(x) g(x) dx$$

converge.

F II 577

3.05 Valori principale ale integralelor improprii .

3.051 Să presupunem că funcția $f(x)$ are un punct singular r în interiorul intervalului (p, q) , în care este definită, și că este integrabilă în orice parte a acestui interval care nu conține punctul r . Atunci, prin definiție,

$$\int_p^q f(x) dx = \lim_{\substack{\eta \rightarrow 0 \\ \eta' \rightarrow 0}} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta'}^q f(x) dx \right\},$$

limita trebuind să existe pentru trecerea la limită independentă în raport cu η și η' . Dacă această limită nu există, dar există limita

$$\lim_{\eta \rightarrow 0} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta}^q f(x) dx \right\},$$

atunci această limită se numește *valoarea principală a integralei improprii* $\int_p^q f(x) dx$ și se spune că integrala $\int_p^q f(x) dx$ există ca *valoare principală*.

F II 603

3.052 Fie funcția $f(x)$, continuă în intervalul (p, q) și care se anulează numai într-un singur punct r din interiorul acestui interval. Să presupunem că derivata întâi $f'(x)$ există în vecinătatea punctului r și fie $f'(r) \neq 0$, iar în punctul r există derivata a doua $f''(r)$. Atunci

$$\int_p^q \frac{dx}{f'(x)}$$

este divergentă, dar există ca valoare principală.

F II 605

3.053 Integrala divergentă a unei funcții pozitive nu poate exista ca valoare principală. F II 605

3.054 Să presupunem că funcția $f(x)$ nu are puncte singulare în intervalul $(-\infty, +\infty)$. Atunci, prin definiție,

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{\substack{p \rightarrow -\infty \\ Q \rightarrow +\infty}} \int_p^Q f(x) dx,$$

limita trebuind să existe pentru trecerea la limită independentă în raport cu P și cu Q . Dacă această limită nu există, dar există limita

$$\lim_{P \rightarrow +\infty} \int_{-P}^{+P} f(x) dx,$$

atunci această limită se numește valoarea principală a integralei improprie.

$$\int_{-\infty}^{+\infty} f(x) dx. \quad \text{F II 607}$$

3.055 Pentru o funcție pară, valoarea principală a integralei improprie există, numai în cazul când această integrală converge (în sensul obișnuit). F II 607

3.1 FUNCȚII PUTERE ȘI FUNCȚII ALGEBRICE

3.11 Funcții raționale

$$1. \quad \int_{-\infty}^{\infty} \frac{p+qx}{r^2+2rx \cos \lambda + x^2} dx = \frac{\pi}{r \sin \lambda} (p - qr \cos \lambda) \quad (\text{valoarea principală})^1. \quad \text{BH [22] (14)}$$

3.12 Radicali de ordinul doi din polinoame de gradul întâi și doi

$$1. \quad \int_0^1 \frac{1}{1-2x \cos \lambda + x^2} \frac{dx}{\sqrt{x}} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin k\lambda}{2k-1}. \quad \text{BH [10] (17)}$$

$$2. \quad \int_0^1 \frac{1}{q-px} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \quad [0 < p < q]. \quad \text{BH [10] (9)}$$

$$3. \quad \int_0^1 \frac{dx}{1-2rx+r^2} \sqrt{\frac{1 \mp x}{1 \pm x}} = \pm \frac{\pi}{4r} \mp \frac{1}{r} \frac{1 \mp r}{1 \pm r} \arctg \frac{1+r}{1-r}.$$

Li [14] (5) Li [14] (16)

¹ Lucrarea de față dă valorile integralelor convergente proprii și improprie, precum și valorile principale ale integralelor divergente (vezi 3.05), dacă acestea există. Valorile principale ale integralelor divergente nu sînt scoase în evidență nicăieri, cu excepția acestor formule.

În secțiunile 3, 4 și 5, numerele naturale sînt notate cu primele litere ale alfabetului: $a, b, c, \dots, h$. Celelalte litere indică, în general, numere reale, iar atunci cînd există indicațiile corespunzătoare, și numere complexe. Litera k (atunci cînd nu servește ca indice de sumare) indică un număr situat pe segmentul $[0,1]$. Această notație este folosită în integralele care se reduc la integrale eliptice. Aici numărul $\sqrt{1-k^2}$ se notează cu k' .

3.13 Radicali de ordinul doi din polinoame de gradul trei

3.131

$$1. \int_u^{\infty} \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[3]{3}} F\left(\arccos \frac{u-1-\sqrt[3]{3}}{u-1+\sqrt[3]{3}}, \sin 15^\circ\right)^* \quad \text{J 65 (282)}$$

$$2. \int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[3]{3}} F\left(\arccos \frac{\sqrt[3]{3}+1-u}{\sqrt[3]{3}-1+u}, \sin 15^\circ\right)^* \quad \text{J 65 (283)}$$

$$3. \int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} F\left(\arccos \frac{\sqrt[3]{3}-1+u}{\sqrt[3]{3}+1-u}, \sin 75^\circ\right)^* \quad \text{J 65 (284)}$$

$$4. \int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} F\left(\arccos \frac{1-u-\sqrt[3]{3}}{1-u+\sqrt[3]{3}}, \sin 75^\circ\right)^* \quad \text{J 66 (285)}$$

3.132

$$1. \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi \sqrt[3]{3} \sqrt[3]{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^2 \quad \text{MO 9}$$

$$2. \int_0^1 \frac{x dx}{\sqrt{1-x^3}} = \frac{1}{\pi} \frac{\sqrt[3]{3}}{\sqrt[3]{4}} \left\{ \Gamma\left(\frac{2}{3}\right) \right\}^2 \quad \text{MO 9}$$

3.133 Dacă $\gamma < \beta < \alpha$, atunci

$$1. \int_u^{\infty} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}} F\left(\arcsin \sqrt{\frac{\alpha-\gamma}{u-\gamma}}, \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right) \quad [\alpha < u]. \quad \text{P (542)}$$

$$2. \int_u^{\infty} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}} F\left(\arcsin \sqrt{\frac{u-\alpha}{u-\beta}}, \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right) \quad [\alpha < u]. \quad \text{P (543)}$$

$$3. \int_u^{\alpha} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}} F\left(\arcsin \sqrt{\frac{\alpha-u}{\alpha-\beta}}, \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right) \quad [\beta < u < \alpha]. \quad \text{P (544)}$$

$$4. \int_{\beta}^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}} F\left(\arcsin \sqrt{\frac{(\alpha-\gamma)(u-\beta)}{(\alpha-\beta)(u-\gamma)}}, \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right) \quad [\beta < u < \alpha]. \quad \text{P (545)}$$

$$* \frac{1}{\sqrt[3]{3}} = 0,759836 \dots; \sin 15^\circ = \frac{\sqrt{2-\sqrt[3]{3}}}{2} = 0,258819 \dots; \sin 75^\circ = \frac{\sqrt{2+\sqrt[3]{3}}}{2} = 0,965926 \dots$$

$$\begin{aligned}
 5. \quad & \int_u^\beta \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)}} = \\
 & = \frac{2}{\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{(a-\gamma)(\beta-u)}{(\beta-\gamma)(a-u)}}, \sqrt{\frac{\beta-\gamma}{a-\gamma}}\right) \quad [\gamma < u < \beta]. \quad P(546)
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \int_\gamma^u \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)}} = \\
 & = \frac{2}{\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{u-\gamma}{\beta-\gamma}}, \sqrt{\frac{\beta-\gamma}{a-\gamma}}\right) \quad [\gamma < u < \beta]. \quad P(547)
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int_u^\gamma \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}} = \\
 & = \frac{2}{\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{\gamma-u}{\beta-u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) \quad [u < \gamma]. \quad P(548)
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}} = \\
 & = \frac{2}{\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{a-\gamma}{a-u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) \quad [u < \gamma]. \quad P(549)
 \end{aligned}$$

3.134

$$\begin{aligned}
 1. \quad & \int_u^\infty \frac{dx}{\sqrt{(a+x)(\beta+x)(\gamma+x)}} = \frac{2}{\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) \\
 & \quad [\gamma < \beta < \alpha]. \quad MO 136
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_u^\infty \frac{dx}{(a+x)\sqrt{(a+x)(\beta+x)(\gamma+x)}} = \\
 & = \frac{2}{(a-\beta)\sqrt{a-\gamma}} \left\{ F\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) - \right. \\
 & \quad \left. - E\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) \right\} \quad [\gamma < \beta < \alpha]. \quad MO 136
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_u^\infty \frac{dx}{(\beta+x)\sqrt{(a+x)(\beta+x)(\gamma+x)}} = \\
 & = \frac{2\sqrt{a-\gamma}}{(a-\beta)(\beta-\gamma)} E\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) - \\
 & - \frac{2}{(a-\beta)\sqrt{a-\gamma}} F\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) - \frac{2\sqrt{\gamma+u}}{(\beta-\gamma)\sqrt{(a+u)(\beta+u)}} \\
 & \quad [\gamma < \beta < \alpha]. \quad MO 136
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_u^\infty \frac{dx}{(\gamma+x)\sqrt{(a+x)(\beta+x)(\gamma+x)}} = \frac{2}{\beta-\gamma} \sqrt{\frac{\beta+u}{(a+u)(\gamma+u)}} - \\
 & - \frac{2}{(\beta-\gamma)\sqrt{a-\gamma}} E\left(\arcsin \sqrt{\frac{a-\gamma}{a+u}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right) \quad [\gamma < \beta < \alpha]. \quad MO 137
 \end{aligned}$$

3.135

$$1. \int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2F(\arcsin \sqrt{u}, k), \quad P(532)$$

$$2. \int_0^u \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} = 2F(\arccos \sqrt{u}, k), \quad P(533)$$

$$3. \int_0^u \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2F\left(\arcsin \frac{\sqrt{1-u}}{k}, k\right), \quad P(534)$$

$$4. \int_0^u \frac{dx}{\sqrt{(1+x)(1+k'^2x)}} = 2F(\operatorname{arctg} \sqrt{u}, k), \quad [0 < u < 1]. \quad P(535)$$

3.136

$$1. \int_0^u \frac{dx}{\sqrt{x[1+x^2+2(k'^2-k^2)x]}} = F(2 \operatorname{arctg} \sqrt{u}, k), \quad JE(150)$$

$$2. \int_0^1 \frac{dx}{\sqrt{x[k'^2(1+t^2)+2(1+k^2)x]}} = F\left(\frac{\pi}{2} - 2 \operatorname{arctg} \sqrt{u}, k\right) \quad JE(150)$$

3.137

$$1. \int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \operatorname{arctg} \sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right) \quad [\alpha < u],$$

$$2. \int_u^\alpha \frac{dx}{\sqrt{(\alpha-x)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \operatorname{arctg} \sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right) \quad [u < \alpha],$$

E 25

unde $p = \sqrt{(m-\alpha)^2 + n^2}$.

3.138

$$1. \int_1^u \frac{dx}{(1-x)^2} \sqrt{2(1+k^2)-k'^2\left(x+\frac{1}{x}\right)} = E\left(2 \operatorname{arctg} \sqrt{u}-\frac{\pi}{2}, k\right) \quad [1 < u]. \quad JE(150)$$

$$2. \int_0^u \frac{\sqrt{1+x^2+2(k'^2-k^2)x}}{(1+x^2)\sqrt{x}} dx = E(2 \operatorname{arctg} \sqrt{u}, k) \quad [0 < u < 1]. \quad JE(150)$$

3.14 Radicali de ordinul doi din polinoame de gradul patru

3.141

$$1. \int_u^4 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F\left(\arccos \frac{u\sqrt{2}}{\sqrt{1+u^4}}, \sin 80^\circ 7' 15''\right)^1 \quad J66(286)$$

$$2. \int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F\left(\arccos \frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2}\right). \quad J66(287)$$

¹ $\sin 80^\circ 7' 15'' = 2\sqrt{2}(\sqrt{2}-1) = 0,985171\dots$

$$3. \int_0^u \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F \left(\arccos \frac{1-u^2}{1+u^2}, \frac{\sqrt{2}}{2} \right). \quad \text{J 66 (288)}$$

$$4. \int_1^u \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}} F \left(\arccos \frac{1}{u}, \frac{\sqrt{2}}{2} \right). \quad \text{J 66 (289)}$$

$$5. \int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}} F \left(\arccos u, \frac{\sqrt{2}}{2} \right). \quad \text{J 66 (290)}$$

$$3.142 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}} \left\{ \Gamma \left(\frac{1}{4} \right) \right\}^2. \quad \text{MO 9}$$

3.143 Dacă $\delta < \gamma < \beta < \alpha$, atunci

$$1. \int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\beta-\delta)(u-\alpha)}{(\alpha-\delta)(u-\beta)}}, \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}} \right)$$

$$[\alpha < u]. \quad \text{P (550)}$$

$$2. \int_u^\alpha \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\beta-\delta)(\alpha-u)}{(\alpha-\beta)(u-\delta)}}, \sqrt{\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}} \right)$$

$$[\alpha > u > \beta]. \quad \text{P (551)}$$

$$3. \int_\beta^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\alpha-\gamma)(u-\beta)}{(\alpha-\beta)(u-\gamma)}}, \sqrt{\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}} \right)$$

$$[\alpha > u > \beta]. \quad \text{P (552)}$$

$$4. \int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\alpha-\gamma)(\beta-u)}{(\beta-\gamma)(\alpha-u)}}, \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}} \right)$$

$$[\gamma < u < \beta]. \quad \text{P (553)}$$

$$5. \int_\gamma^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\beta-\delta)(u-\gamma)}{(\beta-\gamma)(u-\delta)}}, \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}} \right)$$

$$[\gamma < u < \beta]. \quad \text{P (554)}$$

$$6. \int_u^\gamma \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} F \left(\arcsin \sqrt{\frac{(\beta-\delta)(\gamma-u)}{(\gamma-\delta)(\beta-u)}}, \sqrt{\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\gamma)}} \right)$$

$$[\delta < u < \gamma]. \quad \text{P (555)}$$

$$7. \int_{\delta}^u \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}} =$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} F\left(\arcsin \sqrt{\frac{(a-\gamma)(u-\delta)}{(\gamma-\delta)(a-u)}}, \sqrt{\frac{(a-\beta)(\gamma-\delta)}{(a-\gamma)(\beta-\delta)}}\right)$$

$$[\delta < u < \gamma]. \quad P(556)$$

$$8. \int_u^{\delta} \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(\delta-x)}} =$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} F\left(\arcsin \sqrt{\frac{(a-\gamma)(\delta-u)}{(a-\delta)(\gamma-u)}}, \sqrt{\frac{(\beta-\gamma)(a-\delta)}{(a-\gamma)(\beta-\delta)}}\right)$$

$$[u < \delta]. \quad P(557)$$

3.144

$$1. \int_0^u \frac{dx}{\sqrt{(x^2+\alpha^2)(x^2+\beta^2)}} = \frac{1}{\alpha} F\left(\operatorname{arctg} \frac{u}{\beta}, \frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right)$$

$$[\alpha > \beta > 0]. \quad J 62(258)$$

$$2. \int_u^{\infty} \frac{dx}{\sqrt{(\beta^2+x^2)(\gamma^2+x^2)}} = \frac{1}{\gamma} F\left(\operatorname{arcctg} \frac{u}{\gamma}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right)$$

$$[\gamma > \beta > 0]. \quad J 63(259)$$

$$3. \int_0^x \frac{dx}{\sqrt{(a^2-x^2)(\beta^2+x^2)}} = \frac{1}{\sqrt{a^2+\beta^2}} F\left(\arcsin \frac{u\sqrt{a^2+\beta^2}}{\alpha\sqrt{\beta^2+u^2}}, \frac{\alpha}{\sqrt{a^2+\beta^2}}\right)$$

$$[\alpha > u > 0]. \quad J 63(260)$$

$$4. \int_u^{\beta} \frac{dx}{\sqrt{(a^2+x^2)(\beta^2-x^2)}} = \frac{1}{\sqrt{a^2+\beta^2}} F\left(\arccos \frac{u}{\beta}, \frac{\beta}{\sqrt{a^2+\beta^2}}\right)$$

$$[\beta > u > 0]. \quad J 63(261)$$

$$5. \int_{\beta}^u \frac{dx}{\sqrt{(a^2+x^2)(x^2-\beta^2)}} = \frac{1}{\sqrt{a^2+\beta^2}} F\left(\arccos \frac{\beta}{u}, \frac{\alpha}{\sqrt{a^2+\beta^2}}\right)$$

$$[u > \beta > 0]. \quad J 63(262)$$

$$6. \int_u^{\infty} \frac{dx}{\sqrt{(a^2+x^2)(x^2-\beta^2)}} = \frac{1}{\sqrt{a^2+\beta^2}} F\left(\arcsin \frac{\sqrt{a^2+\beta^2}}{\sqrt{a^2+u^2}}, \frac{\alpha}{\sqrt{a^2+\beta^2}}\right)$$

$$[u > \beta > 0]. \quad J 63(263)$$

$$7. \int_0^u \frac{dx}{\sqrt{(a^2-x^2)(\beta^2-x^2)}} = \frac{1}{\alpha} F\left(\arcsin \frac{u}{\beta}, \frac{\beta}{\alpha}\right)$$

$$[\alpha > \beta > u > 0]. \quad J 63(264)$$

$$8. \int_u^{\alpha} \frac{dx}{\sqrt{(a^2-x^2)(\gamma^2-x^2)}} = \frac{1}{\gamma} F\left(\arcsin \frac{\gamma\sqrt{a^2-u^2}}{\alpha\sqrt{\gamma^2-u^2}}, \frac{\alpha}{\gamma}\right)$$

$$[\gamma > \alpha > u > 0]. \quad J 63(265)$$

$$9. \int_{\beta}^u \frac{dx}{\sqrt{(x^2-\beta^2)(\gamma^2-x^2)}} = \frac{1}{\gamma} F\left(\arcsin \frac{\gamma\sqrt{u^2-\beta^2}}{u\sqrt{\gamma^2-\beta^2}}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right)$$

$$[\gamma > u > \beta > 0]. \quad J 63(266)$$

$$10. \int_u^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-\beta^2)}} = \frac{1}{a} F\left(\arcsin \sqrt{\frac{a^2-u^2}{a^2-\beta^2}}, \sqrt{\frac{a^2-\beta^2}{a^2}}\right) \\ [\alpha > u > \beta > 0]. \quad \text{J 63 (267)}$$

$$11. \int_\gamma^u \frac{dx}{\sqrt{(x^2-a^2)(x^2-\gamma^2)}} = \frac{1}{\gamma} F\left(\arccos \sqrt{\frac{\gamma^2-a^2}{u^2-a^2}}, \frac{a}{\gamma}\right) \\ [u > \gamma > a > 0]. \quad \text{J 64 (268)}$$

$$12. \int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-\beta^2)}} = \frac{1}{a} F\left(\arcsin \frac{a}{u}, \frac{\beta}{a}\right) \\ [u > a > \beta > 0]. \quad \text{J 64 (269)}$$

3.145

$$1. \int_0^u \sqrt{\frac{\gamma^2+x^2}{(\beta^2+x^2)^3}} dx = \frac{\gamma}{\beta^3} E\left(\operatorname{arctg} \frac{u}{\beta}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right) \\ [\gamma > \beta > 0]. \quad \text{J 64 (270)}$$

$$2. \int_u^\infty \sqrt{\frac{\beta^2+x^2}{(\gamma^2+x^2)^3}} dx = \frac{1}{\gamma} E\left(\operatorname{arctg} \frac{u}{\gamma}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right) \\ [\gamma > \beta > 0]. \quad \text{J 64 (271)}$$

$$3. \int_0^u \frac{dx}{\sqrt{(a^2-x^2)(\beta^2+x^2)^3}} = \frac{1}{\beta^3 \sqrt{a^2+\beta^2}} E\left(\arcsin \frac{u \sqrt{a^2+\beta^2}}{a \sqrt{\beta^2+u^2}}, \frac{a}{\sqrt{a^2+\beta^2}}\right). \\ \text{J 64 (272)}$$

$$4. \int_u^a \sqrt{\frac{\beta^2+x^2}{a^2-x^2}} dx = \sqrt{a^2+\beta^2} E\left(\arccos \frac{u}{a}, \frac{a}{\sqrt{a^2+\beta^2}}\right). \quad \text{J 64 (273) m}$$

$$5. \int_\beta^u \frac{1}{x^3} \sqrt{\frac{x^2+a^2}{x^2-\beta^2}} dx = \frac{\sqrt{a^2+\beta^2}}{\beta^3} E\left(\arccos \frac{\beta}{u}, \frac{a}{\sqrt{a^2+\beta^2}}\right). \quad \text{J 64 (274)}$$

$$6. \int_u^\infty x^2 \sqrt{\frac{x^2+a^2}{x^2-\beta^2}} dx = \sqrt{a^2+\beta^2} E\left(\arcsin \frac{\sqrt{a^2+\beta^2}}{\sqrt{a^2+u^2}}, \frac{a}{\sqrt{a^2+\beta^2}}\right). \\ \text{J 64 (275)}$$

$$7. \int_0^u \sqrt{\frac{\gamma^2-x^2}{a^2-x^2}} dx = \gamma E\left(\arcsin \frac{u}{a}, \frac{a}{\gamma}\right) \quad [\gamma > a > 0]. \quad \text{J 64 (276)}$$

3.146

$$1. \int_u^a \frac{dx}{\sqrt{(a^2-x^2)(\gamma^2-x^2)^3}} = \frac{1}{\gamma(\gamma^2-a^2)} E\left(\arcsin \frac{\gamma \sqrt{a^2-u^2}}{a \sqrt{\gamma^2-u^2}}, \frac{a}{\gamma}\right) \\ [\gamma > a > 0]. \quad \text{J 65 (277)}$$

$$2. \int_\beta^u \frac{dx}{x^3 \sqrt{(x^2-\beta^2)(\gamma^2-x^2)}} = \frac{1}{\beta^2 \gamma} E\left(\arcsin \frac{\gamma}{u} \sqrt{\frac{u^2-\beta^2}{\gamma^2-\beta^2}}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right) \\ [\gamma > u > \beta > 0]. \quad \text{J 65 (278)}$$

$$3. \int_u^\gamma \frac{x^3 dx}{\sqrt{(x^2-\beta^2)(\gamma^2-x^2)}} = \gamma E\left(\arcsin \sqrt{\frac{\gamma^2-u^2}{\gamma^2-\beta^2}}, \frac{\sqrt{\gamma^2-\beta^2}}{\gamma}\right) \\ [\gamma > u > \beta > 0]. \quad \text{J 65 (279) m}$$

$$4. \int_{\gamma}^u \frac{dx}{\sqrt{(x^2-\gamma^2)(x^2-\alpha^2)}} = \frac{1}{\gamma^2-\alpha^2} E\left(\arccos \sqrt{\frac{\gamma^2-\alpha^2}{u^2-\alpha^2}}, \frac{\alpha}{\gamma}\right) \\ [u > \gamma > \alpha > 0]. \quad J 65 (280)$$

$$5. \int_u^{\infty} \sqrt{\frac{x^2-\alpha^2}{x^2-\gamma^2}} \frac{dx}{x^2} = \frac{1}{\gamma} \left(\arcsin \frac{\gamma}{u}, \frac{\alpha}{\gamma}\right) \quad [\gamma > \alpha]. \quad J 65 (281)$$

3.147

$$1. \int_a^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \\ = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arctg} \sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2+(a-\beta)^2}{pq}}\right) \quad [\beta < \alpha < u]. \quad E 24$$

$$2. \int_{\beta}^u \frac{dx}{\sqrt{(a-x)(x-\beta)[(x-m)^2+n^2]}} = \\ = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arctg} \sqrt{\frac{q(a-u)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2+(a-\beta)^2}{pq}}\right) \quad [\beta < u < \alpha]. \quad E 24$$

$$3. \int_u^{\beta} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \\ = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arctg} \sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2+(a-\beta)^2}{pq}}\right) \quad [u < \beta < \alpha], \quad E 24$$

unde $(m-\alpha)^2+n^2=p^2$, $(m-\beta)^2+n^2=q^2$ ¹

3.148 Fie

$$(m_1-m)^2+(n_1+n)^2=p^2, \quad (m_1-m)^2+(n_1-n)^2=p_1^2,$$

$$\operatorname{ctg} \alpha = \sqrt{\frac{(p+p_1)^2-4n^2}{4n^2-(p-p_1)^2}};$$

atunci

$$\int_{m-n \operatorname{tg} \alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2][(x-m_1)^2+n_1^2]}} = \frac{2}{p+p_1} F\left(\alpha + \operatorname{arctg} \frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right) \\ [m-n \operatorname{tg} \alpha < u < m+n \operatorname{ctg} \alpha]. \quad E 25$$

3.149

$$1. \int_0^1 \frac{x^2 dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2} \left\{ \sqrt{\frac{1+k^2}{2}} - E\left(\frac{\pi}{4}, \sqrt{1-k^2}\right) \right\}. \quad BH [14] (9)$$

$$2. \int_0^1 \frac{1}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4} \sqrt{2} K\left(\frac{\sqrt{2}}{2}\right). \quad BH [13] (6)$$

$$3. \int_0^1 \frac{x^2}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}. \quad BH [13] (7)$$

$$4. \int_0^1 \frac{x^4}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4} \sqrt{2} K\left(\frac{\sqrt{2}}{2}\right). \quad BH [13] (8)$$

¹ Pentru $\alpha+\beta=2m$, formulele 3.147 nu sînt valabile; atunci putem aplica substituția $x-m=z$, care duce la una dintre formulele 3.144.

3.15-3.18 Puteri ale lui x și puteri ale binomului $\alpha + \beta x$

$$3.151 \quad \int_0^1 \frac{x^{p-1}}{1+x} dx = \beta(p) \quad [\operatorname{Re}(p) > 0]. \quad \text{WW II 39 BH [2] (1)}$$

$$3.152 \quad \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi} \quad [0 < p < 1]. \quad \text{F II 718 F II 737}$$

$$3.153 \quad \int_0^\infty \frac{x^{p-1}}{1-x} dx = \pi \operatorname{ctg} p\pi \quad [0 < p < 1]. \quad \text{D (856.5)}$$

3.154

$$1. \quad \int_1^\infty \frac{dx}{(r-qx)(x-1)^p} = -\frac{\pi}{q} \operatorname{cosec} p\pi \left(\frac{q}{q-r}\right)^p \quad [r < q, q > 0, 0 < p < 1]. \quad \text{Li [23] (5)}$$

$$2. \quad \int_{-\infty}^1 \frac{dx}{(r-qx)(1-x)^p} = \frac{\pi}{q} \operatorname{cosec} p\pi \left(\frac{q}{r-q}\right)^p \quad [r > q, 0 < q, 0 < p < 1]. \quad \text{Li [24] (10)}$$

3.155

$$1. \quad \int_q^\infty \frac{(x-q)^{p-1} dx}{x-r} = \frac{\pi (q-r)^{p-1}}{\sin p\pi} \quad [q > r; p < 1]. \quad \text{Li [24] (8)}$$

$$2. \quad \int_{-\infty}^q \frac{(q-x)^{p-1}}{(r-x)} dx = \frac{\pi (r-q)^{p-1}}{\sin \pi p} \quad [q < r; p < 1]. \quad \text{Li [24] (8)}$$

$$3.156 \quad \int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \pi \sec p\pi \quad \left[-\frac{1}{2} < p < \frac{1}{2}\right]. \quad \text{BH [23] (7)}$$

$$3.157 \quad \int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x^2} = \frac{1-2p}{2} \pi \sec p\pi \quad \left[-\frac{1}{2} < p < \frac{3}{2}\right]. \quad \text{BH [23] (8)}$$

$$3.158 \quad \int_0^\infty \frac{x^{p-1} dx}{x^2 + 2x \cos \lambda + 1} = \pi \frac{\sin [(1-p)\lambda]}{\sin \lambda \sin p\pi} \quad [0 < p < 1, \lambda^2 < \pi^2]. \quad \text{F II 738}$$

$$3.159 \quad \int_0^1 \frac{(1-x)^{p-1}}{\sqrt{x}} dx = 2^{2p-1} B(p, p) = B\left(\frac{1}{2}, p\right) \quad [\operatorname{Re}(p) > 0]. \quad \text{F II 784}$$

$$3.161 \quad \int_0^1 \frac{x^{q-1}-1}{x-1} dx = \psi(q) + C \quad [\operatorname{Re}(q) > 0]. \quad \text{F II 796 WW II 37}$$

$$3.162 \quad \int_0^\infty \{(1+x)^{-1} - (1+x)^{-q}\} \frac{dx}{x} = \psi(q) + C \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 37}$$

$$3.163 \quad \int_0^\infty \{(s+px)^{-r} - (s+qx)^{-r}\} \frac{dx}{x} = s^{-r} \ln \frac{q}{p} \quad [r > 0]. \quad \text{BH [18] (14)}$$

$$3.164 \quad \int_0^{\infty} \frac{(1+px)^{-r} \pm (1+qx)^{-r}}{2} x^{s-1} dx =$$

$$= \pm \frac{(pq)^{\frac{s}{2}} \Gamma(s) \Gamma(r-s)}{\Gamma(r)} \frac{\cos \left[s \arccos \left(\frac{p+q}{2\sqrt{pq}} \right) \right]}{\sin \left[s \arccos \left(\frac{p+q}{2\sqrt{pq}} \right) \right]} [s < r]. \quad \text{BH [19] (9) BH [19] (10)}$$

$$3.165 \quad \int_0^1 \frac{x^{p-1} - x^r}{1-x} dx = \pi \operatorname{ctg} p\pi \quad [p^2 < 1]. \quad \text{BH [4] (4)}$$

$$3.166 \quad \int_0^1 \frac{x^p - x^q}{1-x} dx = \frac{1}{p} - \pi \operatorname{ctg} p\pi \quad [p^2 < 1]. \quad \text{BH [4] (3)}$$

$$3.167 \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x} dx = \psi(q) - \psi(p) \quad [p > 0, q > 0]. \quad \text{F II 815}$$

$$3.168 \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{1-x} dx = \pi (\operatorname{ctg} \pi p - \operatorname{ctg} \pi q) \quad [p > 0, q > 0]. \quad \text{F II 718}$$

$$3.169 \quad \int_0^{\infty} \frac{x^p - x^q}{x-1} \frac{dx}{x+r} = \frac{\pi}{1+r} \left(\frac{r^p - \cos p\pi}{\sin p\pi} - \frac{r^q - \cos q\pi}{\sin q\pi} \right)$$

$$[p^2 < 1, r > 0, q^2 < 1]. \quad \text{BH [19] (2)}$$

$$3.171 \quad \int_p^q (q-x)^{r-1} (x-p)^{s-1} dx = (q-p)^{r+s-1} B(r, s)$$

$$[0 < p < q; \operatorname{Re}(r) > 0, \operatorname{Re}(s) > 0]. \quad \text{MO 7}$$

$$3.172 \quad \int_0^{\infty} \frac{x^{p-1} dx}{(1+qx)^{p+r}} = \frac{\Gamma(p) \Gamma(r)}{q^p \Gamma(p+r)} = \frac{1}{q^p} B(p, r)$$

$$[p > 0, q > 0, r > 0]. \quad \text{F II 774 m BH [16] (10)}$$

$$3.173 \quad \int_0^{\infty} x^a (p+qx)^{-b-\frac{1}{2}} dx = \frac{2^{a+1} \cdot a! q^{-a-1} p^{a-b+\frac{1}{2}}}{(2b-1)(2b-3) \dots (2b-2a-1)}$$

$$\left[a < b - \frac{1}{2}, pq > 0 \right]. \quad \text{BH [21] (2)}$$

$$3.174 \quad \int_0^1 \frac{x^{q-1} dx}{(1-x)^q (1+px)} = \frac{\pi}{(1+p)^q} \operatorname{cosec} q\pi \quad [0 < q < 1, p > -1]. \quad \text{BH [5] (1)}$$

$$3.175 \quad \int_0^1 \frac{x^{\frac{q}{2}} dx}{[(1-x)(1-p^2x)]^{\frac{q+1}{2}}} = \frac{(1-p)^{-q} - (1+p)^{-q}}{2pq \sqrt{\pi}} \Gamma\left(\frac{q+2}{2}\right) \Gamma\left(\frac{1-q}{2}\right)$$

$$p^2 < 1, -2 < q < 1]. \quad \text{BH [12] (32)}$$

$$3.176 \quad \int_0^1 \frac{(1-x)^a dx}{(1+x)^{a+1} (1-x)} = \frac{1}{2^{a+1}} \sum_{k=1}^a \frac{2^k}{k}. \quad \text{BH [5] (3)}$$

$$3.177 \quad \int_0^1 [(1+x)^{p-1} (1-x)^{q-1} + (1+x)^{q-1} (1-x)^{p-1}] dx =$$

$$= 2^{p+q-1} B(p, q) [p > 0, q > 0] \quad \text{BH [1] (15)}$$

3.178

$$1. \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx = B(p, q) \quad [\operatorname{Re}(p) > 0; \operatorname{Re}(q) > 0]. \quad \text{F II 775}$$

$$2. \int_1^\infty \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx = B(p, q) \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0].$$

$$3.179 \quad \int_0^\infty \left[\frac{x^q}{(1+x)^{q+1}} - \frac{x^p}{(1+x)^{p+1}} \right] dx = \psi(p+1) - \psi(q+1) \\ [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0]. \quad \text{BH [19] (13)}$$

$$3.181 \quad \int_0^1 x^{q-1} (1-x)^{r-q-1} (1-xz)^{-p} dx = B(q, r-q) F(p, q; r; z) \\ [\operatorname{Re}(q) > 0, \operatorname{Re}(r-q) > 0, |z| < 1]. \quad \text{WW II 79}$$

$$3.182 \quad \int_p^q \frac{(q-x)^{r-1} (x-p)^{s-1}}{|x-u|^{r+s}} dx = \frac{(q-p)^{r+s-1}}{|q-u|^r |p-u|^s} B(r, s) \\ [\operatorname{Re}(r) > 0, \operatorname{Re}(s) > 0, 0 < u < p < q \text{ sau } 0 < p < q < u]. \quad \text{MO 7}$$

3.183

$$1. \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p (1+qx)^p} = \\ = \frac{2\Gamma\left(p+\frac{1}{2}\right) \Gamma(1-p)}{\sqrt{\pi}} \cos^{2p}(\operatorname{arctg} \sqrt{q}) \frac{\sin[(2p-1) \operatorname{arctg}(\sqrt{q})]}{(2p-1) \sin[\operatorname{arctg}(\sqrt{q})]} \\ \left[-\frac{1}{2} < p < 1, q > 0 \right]. \quad \text{BH [11] (1)}$$

$$2. \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p (1-qx)^p} = \\ = \frac{\Gamma\left(p+\frac{1}{2}\right) \Gamma(1-p)}{\sqrt{\pi}} \frac{(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2p}}{(2p-1) \sqrt{q}} \\ \left[-\frac{1}{2} < p < 1, 0 < q < 1 \right]. \quad \text{BH [11] (2)}$$

$$3.184 \quad \int_0^1 \frac{x^{\alpha-1} (1-x)^{\beta-1} dx}{[\alpha x + \beta(1-x) + \gamma]^{p+q}} = (\alpha + \gamma)^{-p} (\beta + \gamma)^{-q} B(p, q) \\ [\alpha \geq 0, \beta \geq 0, \gamma > 0, p > 0, q > 0]. \quad \text{F II 787}$$

$$3.185 \quad \int_q^p \frac{(x-q)^{r-1} (p-x)^{s-1}}{(t+x)^{r+s}} dx = \frac{(p-q)^{r+s-1}}{(p+t)^r (q+t)^s} B(r, s) \\ [p > q, t > -q, \operatorname{Re}(r) > 0, \operatorname{Re}(s) > 0]. \quad \text{BH [24] (7)}$$

$$3.186 \quad \int_0^\infty \left[\frac{q^p x^{p-1}}{(1+qx)^p} - \frac{(1+qx)^{p-1}}{q^{p-1} x^p} \right] dx = \pi \operatorname{ctg} p\pi \quad [0 < p < 1, q > 0]. \quad \text{BH [18] (13)}$$

3.19 Puteri ale polinoamelor și ale funcțiilor putere

3.191

$$1. \int_0^1 x^{2q-1} (1-x^2)^{p-1} dx = \frac{1}{2} B(p, q) \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0].$$

$$2. \int_0^{\infty} \frac{x^{2p-1}}{(1+x^2)^{p+q}} dx = \frac{1}{2} B(p, q) \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0].$$

$$3. \int_0^{\infty} \frac{dx}{(p+x^2)^{a+1}} = \frac{(2a-1)!!}{2^{a+1} \cdot a!} \frac{1}{p^a} \frac{\pi}{\sqrt{p}}. \quad \text{F II 743}$$

$$4. \int_0^p (p^2 - x^2)^{\frac{2a-1}{2}} dx = p^{2a} \frac{(2a-1)!!}{2^{a+1} \cdot a!} \pi. \quad \text{F II 156}$$

$$5. \int_0^{\infty} \frac{x^{p+t} dx}{(q+rx+sx^2)^{p+\frac{1}{2}}} = \frac{1}{\Gamma(p+\frac{1}{2})} \left(\frac{q}{s}\right)^{\frac{t}{2}} \sqrt{\frac{\pi}{s}} \times \\ \times \sum_{k=0}^{\infty} \frac{(t+k)(t+k-1)\dots(t-k+1)}{(4\sqrt{qs})^k k!} \cdot \frac{\Gamma(p-k)}{(r+2\sqrt{qs})^{p-k}}. \quad \text{BH [21] (22)}$$

$$3.192 \quad \int_{-1}^1 \frac{(1+x)^{2p-1} (1-x)^{2q-1}}{(1+x^2)^{p+q}} dx = 2^{p+q-2} B(p, q) \quad [p > 0, q > 0]. \quad \text{F II 787}$$

$$3.193 \quad \int_0^1 (1-\sqrt{x})^{p-1} dx = \frac{2}{p(p+1)} \quad [p > 0]. \quad \text{BH [7] (7)}$$

$$3.194 \quad \int_0^1 (1-x^p)^{-\frac{1}{q}} dx = \frac{1}{p} B\left(\frac{1}{p}, 1-\frac{1}{q}\right) \quad [p > 0, q > 1].$$

3.195

$$1. \int_0^{\infty} \frac{x^{g-1} dx}{(p+qx^c)^{h+1}} = \frac{\Gamma\left(h+1-\frac{g}{c}\right) \Gamma\left(\frac{p}{qc}+g\right) \Gamma\left(\frac{g}{c}\right)}{c^{1-g} p^{h+1} \Gamma\left(\frac{p}{qc}\right) \Gamma(h+1)} \quad \left[0 < \frac{g}{c} < h+1\right].$$

BH [17] (22)m

$$2. \int_0^1 (1-x^r)^{p-1} x^{q-1} dx = \frac{1}{r} B\left(p, \frac{q}{r}\right) \\ \left[\operatorname{Re}(r) > 0, \operatorname{Re}\left(\frac{q}{r}\right) > 0, \operatorname{Re}(p) > 0\right]. \quad \text{F II 787}$$

$$3. \int_0^1 \frac{x^{p-1}}{1+x^q} dx = \frac{1}{q} \beta\left(\frac{p}{q}\right) \quad [p > 0]. \quad \text{BH [2] (13)}$$

3.196

$$1. \int_0^1 \left[\frac{x^{a-1}}{1-\sqrt[p]{x}} - \frac{px^{pa-1}}{1-x} \right] dx = p \ln p \quad [p > 0]. \quad \text{BH [13] (9)}$$

$$2. \int_0^1 \left[\frac{a}{1-x} - \frac{x^{p-1}}{1-\sqrt[p]{x}} \right] dx = aC + \sum_{k=1}^a \psi\left(p+\frac{a-k}{a}\right) \quad [p > 0]. \quad \text{BH [13] (10)}$$

3.197

$$1. \int_0^{\infty} \frac{1-x^q}{1-x^p} x^{p-1} dx = \frac{\pi \sin \frac{q\pi}{r}}{r \sin \frac{p\pi}{r} \sin \left(\frac{p+q}{r} \pi \right)} [r < p+q, p > 0]. \text{ BH [17] (12)}$$

$$2. \int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [0 < p < q]. \quad \text{BH [2] (14)}$$

$$3. \int_0^1 \frac{x^{q-1} - x^{p-1}}{1-x^q} dx = \frac{1}{q} \left\{ C + \psi \left(\frac{p}{q} \right) \right\} \quad [q > 0, p > 0]. \quad \text{BH [2] (17)}$$

$$4. \int_{-\infty}^{\infty} \frac{x^{2a} - x^{2b}}{1-x^{2c}} dx = \frac{\pi}{c} \left[\operatorname{ctg} \frac{2a+1}{2c} \pi - \operatorname{ctg} \frac{2b+1}{2c} \pi \right] [a < c, b < c]. \quad \text{F II 640}$$

$$3.198 \quad \int_0^{\infty} \frac{x^{p-1}(1-x)}{1-x^a} dx = \frac{\pi}{a} \sin \frac{\pi}{a} \operatorname{cosec} \frac{p\pi}{a} \operatorname{cosec} \frac{(p+1)\pi}{a} [0 < p < a]. \quad \text{BH [20] (13)}$$

$$3.199 \quad \int_{-\infty}^{\infty} \frac{x^{2a} dx}{x^{4b} + 2x^{2b} \cos \lambda + 1} = \frac{\pi}{b} \frac{\sin \left(1 - \frac{2a+1}{2b} \right) \lambda}{\sin \frac{2a+1}{2b} \pi \sin \lambda} \quad [a < b, \lambda^2 < \pi^2]. \quad \text{F II 642}$$

3.2 FUNCȚIA EXPONENȚIALĂ

3.21 Funcția exponențială

3.211

$$1. \int_0^{\infty} \frac{dx}{1+e^{px}} = \frac{1}{p} \ln 2. \quad \text{Lo III 284m}$$

$$2. \int_0^{\infty} \frac{e^{-px}}{1+e^{-x}} dx = \beta(p) \quad [\operatorname{Re}(p) > 0]. \quad \text{MO (4)}$$

$$3. \int_{-\infty}^{\infty} \frac{e^{-px}}{1+e^{-qx}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [p^2 < q^2]. \quad \text{BH [28] (7)}$$

$$4. \int_0^{\infty} \frac{e^{-qx} - e^{-px}}{1-e^{-x}} dx = \psi(p) - \psi(q) \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0]. \quad \text{BH [27] (8)}$$

$$5. \int_0^{\infty} \frac{e^{-x} - e^{-qx}}{1-e^{-x}} dx = \psi(q) + C \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 37}$$

$$3.212 \quad \int_0^{\infty} e^{-px} (1-e^{-x})^q dx = B(p, q+1) \\ [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > -1] \quad (\text{vezi și 5.183.}). \quad \text{Li [26] (13)}$$

3.213

1. $\int_0^{\infty} e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \Phi(qu) \quad [q > 0].$
2. $\int_0^{\infty} e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \quad [q > 0].$ F II 624
3. $\int_{-\infty}^{\infty} e^{-qx - x^2} dx = \frac{e^{-q^2/4}}{q+2} \sum_{k=0}^{\infty} (-1)^k \frac{2^k (2k-1)!!}{(q+2)^{2k}} \quad [q \neq -2].$ BH [29] (4)
4. $\int_{-\infty}^{\infty} e^{-px^2 \pm qx} dx = e^{\frac{q^2}{4p}} \sqrt{\frac{\pi}{p}} \quad [p > 0].$ BH [28] (1)
5. $\int_0^{\infty} e^{-p^2 x^2 - qx} dx = \frac{\sqrt{\pi}}{2p} e^{-\frac{q^2}{4p^2}} \left[1 - \Phi\left(\frac{q}{2p}\right) \right].$ NI 27 (1)m

3.214

1. $\int_0^{\infty} e^{-\frac{1}{x^2}} dx = \sqrt{\pi}.$ Li 247 (26)
2. $\int_0^{\infty} e^{-px^2 - \frac{q}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{p}} e^{-2\sqrt{pq}} \quad [p > 0, q > 0].$ F II 644
3. $\int_{-\infty}^{\infty} e^{-(x - \frac{q}{x})^{2a}} dx = \frac{1}{a} \Gamma\left(\frac{1}{2a}\right).$ BH [28] (6)

3.215

1. $\int_0^{\infty} e^{-x^p} dx = \frac{1}{p} \Gamma\left(\frac{1}{p}\right) \quad [\operatorname{Re} p > 0].$ BH [26] (4)
 2. $\int_0^{\infty} e^{-p e^{bx}} dx = -\frac{1}{b} \operatorname{Ei}(-p).$ Li [26] (5)
 3. $\int_{-\infty}^{+\infty} e^{-e^x} e^{qx} dx = \Gamma(q) \quad [\operatorname{Re}(q) > 0].$ NH 145 (14)
- 3.216 $\int_{-\infty}^{\infty} e^{-\sinh x - px} dx = 2K_p(z) \quad \left[|\arg z| < \frac{\pi}{2} \right].$ W 201 (7)

3.217

1. $\int_0^{\pi} e^{2 \cos x} dx = \pi I_0(2).$ BH [277] (2)m
2. $\int_0^{\frac{\pi}{2}} e^{-q \operatorname{tg} x} dx = \operatorname{ci}(q) \sin q - \operatorname{si}(q) \cos q.$ BH [271] (2)m

3.218

$$\int_0^1 e^{-px \ln x} dx = \int_0^1 x^{-px} dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k}.$$

BH [29] (1)

3.22 Polinoame în e^{px} și funcția putere

3.221

$$1. \int_{-\infty}^u \frac{e^x}{x} dx = \text{li}(e^u) = \text{Ei}(u). \quad \text{F II 95}$$

$$2. \int_1^{\infty} \frac{e^{-px}}{x} dx = -\text{Ei}(-p) \quad [\text{Re}(p) > 0]. \quad \text{BH [104] (10)}$$

3.222

$$1. \int_u^{\infty} \frac{e^{-px}}{q+x} dx = -e^{pq} \text{Ei}[-p(q+u)] \quad [\text{Re}(p) > 0]. \quad \text{JE 99}$$

$$2. \int_0^{\infty} \frac{e^{-px}}{q+x} dx = -e^{pq} \text{Ei}(-pq) \quad [p > 0, q > 0]$$

(vezi și 5.211 1.). Li 281 (25)

$$3. \int_0^{\infty} \frac{e^{-x}}{(q+x)^2} dx = e^q \text{Ei}(-q) + \frac{1}{q}. \quad \text{Li 281 (28)}$$

$$4. \int_0^1 \frac{xe^x}{(1+x)^2} dx = \frac{e}{2} - 1. \quad \text{BH [80] (6)}$$

3.223

$$1. \int_0^{\infty} e^{-px} \frac{dx}{q^2+x^2} = \frac{1}{q} \{ \text{ci}(pq) \sin pq - \text{si}(pq) \cos pq \} \quad [p > 0]$$

(vezi și 5.211 2.). BH [91] (7)

$$2. \int_0^{\infty} e^{-px} \frac{x dx}{q^2+x^2} = -\text{ci}(pq) \cos(pq) - \text{si}(pq) \sin pq \quad [p > 0]$$

(vezi și 5.211 3.) BH [91] (8)

$$3. \int_0^{\infty} e^{-px} \frac{dx}{q^2-x^2} = \frac{1}{2q} [e^{-pq} \text{Ei}(pq) - e^{pq} \text{Ei}(-pq)] \quad [p > 0]. \quad \text{BH [91] (14)}$$

$$4. \int_0^{\infty} e^{-px} \frac{x dx}{q^2-x^2} = \frac{1}{2} [e^{-pq} \text{Ei}(pq) + e^{pq} \text{Ei}(-pq)] \quad [p > 0]. \quad \text{BH [91] (15)}$$

3.224

$$1. \int_0^u \frac{e^{-qx}}{\sqrt{qx}} dx = \sqrt{\frac{\pi}{q}} \Phi(\sqrt{qu}).$$

$$2. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{p}} \quad [p > 0]. \quad \text{BH [98] (10)}$$

$$3. \int_{-1}^{\infty} \frac{e^{-qx}}{\sqrt{1+x}} dx = e^q \sqrt{\frac{\pi}{q}} \quad [q > 0]. \quad \text{BH [104] (16)}$$

$$4. \int_1^{\infty} \frac{e^{-\frac{x}{p}}}{\sqrt{x-1}} dx = \frac{\sqrt{p\pi}}{\sqrt{e}} \quad [p > 0]. \quad \text{BH [104] (11)}$$

$$5. \int_{-1}^1 \frac{e^{2x}}{\sqrt{1-x^2}} dx = \pi I_0(2). \quad (\text{cf. 3.217 1}).$$

3.225

$$1. \int_u^\infty \frac{e^{-x}}{x^q} dx = u^{-\frac{q}{2}} e^{-\frac{u}{2}} W_{-\frac{q}{2}, \frac{1}{2}-\frac{q}{2}}(u) \quad [u > 0]. \quad \text{WW II 160}$$

$$2. \int_u^\infty \frac{e^{-px}}{x^{a+1}} dx = (-1)^{a+1} \frac{p^a \text{Ei}(-pu)}{a!} + \frac{e^{-pu}}{u^a} \sum_{k=0}^{a-1} \frac{(-1)^k (pu)^k}{a(a-1)\dots(a-k)} \quad [p > 0]. \quad \text{NI 21 (3)}$$

$$3. \int_0^\infty e^{-qx} x^{p-1} dx = q^{-p} \Gamma(p) \quad [\text{Re}(p) > 0, \text{Re}(q) > 0] \quad (\text{vezi și 5.213}).$$

F II 779

$$4. \int_1^\infty \frac{e^{-px}}{x^{a+1}} dx = \frac{(-p)^a}{a!} \left(C + \ln p - \sum_{k=1}^a \frac{1}{k} \right) - \sum_{k=1}^a \frac{(-p)^{k-1}}{(k-1)! (a-k+1)} + \sum_{k=2}^\infty \frac{(-p)^{a+k-1}}{(a+k)! (k-1)} \quad [p > 0]. \quad \text{BH [104] (12)}$$

3.226

$$1. \int_0^\infty \frac{e^{-px}}{(q+x)^a} dx = (-1)^a \frac{p^{a-1}}{(a-1)!} e^{pq} \text{Ei}(-pq) + \frac{q^{1-a}}{(a-1)!} \sum_{k=1}^{a-1} (a-k-1)! (-pq)^{k-1} \quad [p > 0, q > 0]. \quad \text{BH [92] (2)}$$

$$2. \int_0^\infty e^{-px} (1+x)^{-q} dx = p^{\frac{q}{2}-1} e^{\frac{p}{2}} W_{-\frac{q}{2}, \frac{1}{2}-\frac{q}{2}}(p) \quad (\text{vezi și 5.215}).$$

WW II 160

3.227

$$1. \int_0^\infty e^{-x} \frac{x^a}{1+x^b} dx = \sum_{k=0}^\infty (-1)^k (a+bk)! \quad \text{BH [91] (26)}$$

$$2. \int_0^\infty e^{-x} x^q \left(1 + \frac{x}{z}\right)^p dx = e^{\frac{z}{2}} z^{\frac{q-p}{2}} \Gamma(q+1) W_{\frac{p-q}{2}, \frac{p+q+1}{2}}(z) \quad \left[\text{Re}(q) \geq -1, |\arg z| < \frac{3}{2}\pi, p+q \text{ neîntreg} \right] \quad \text{WW II 143}$$

$$3. \int_{-1}^1 e^{-ix} (1-x^2)^p dx = \sqrt{\pi} \left(\frac{z}{2}\right)^{-p-\frac{1}{2}} \Gamma(p+1) I_{p+\frac{1}{2}}(z) \quad [\text{Re}(p) \geq -1]. \quad \text{W 190 (2)}$$

$$4. \int_1^\infty e^{-ix} (x^2-1)^p dx = \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{-p-\frac{1}{2}} K_{p+\frac{1}{2}}(z) \Gamma(p+1) \quad \left[|\arg z| < \frac{\pi}{2}, \text{Re}(p) > -1 \right]. \quad \text{W 190 (4)}$$

3.228

$$1. \int_0^{\infty} \left(\frac{1}{1+x} - e^{-x} \right) \frac{dx}{x} = 0. \quad \text{F II 795 F II 802}$$

$$2. \int_0^{\infty} \left(e^{-qx} - \frac{1}{1+px} \right) \frac{dx}{x} = \ln \frac{p}{q} - 0 \quad [p > 0, q > 0] \quad \text{BH [92] (10)}$$

$$3. \int_0^{\infty} \left[e^{-x} - \frac{1}{(1+x)^p} \right] \frac{dx}{x} = \psi(p) \quad [\operatorname{Re}(p) > 0]. \quad \text{NH 184 (7)}$$

$$4. \int_0^{\infty} x^{p-1} \left[e^{-x} - \sum_{k=1}^{a-1} (-1)^k \frac{x^k}{k!} \right] dx = \Gamma(p) \quad [-a < p < -a+1]. \quad \text{F II 805}$$

$$3.229 \int_0^{\infty} \frac{e^{-px} - e^{-qx}}{x} dx = \ln \frac{q}{p} \quad [p > 0, q > 0]. \quad \text{F II 634}$$

3.23- 3.23-3.24 Funcții raționale de e^{px} și funcția putere

$$3.231 \int_0^{\ln 2} \frac{x dx}{1-e^{-x}} = \frac{\pi^2}{12}. \quad \text{BH [104] (5)}$$

3.232

$$1. \int_{-\infty}^{\infty} \frac{x dx}{p^2 e^{2x} + q^2 e^{-x}} = \frac{\pi}{2pq} \ln \frac{q}{p}. \quad \text{BH [101] (1)}$$

$$2. \int_{-\infty}^{\infty} \frac{x dx}{p^2 e^{2x} - q^2 e^{-x}} = \frac{\pi^2}{4qp}. \quad \text{Li [101] (2)}$$

3.233

$$1. \int_0^{\infty} \frac{x^{p-1}}{e^x + 1} dx = (1 - 2^{1-p}) \Gamma(p) \zeta(p) \quad [\operatorname{Re}(p) > 0]. \quad \text{F II 792 WW 46}$$

$$2. \int_0^{\infty} \frac{x^{p-1}}{e^x - 1} dx = \Gamma(p) \zeta(p) \quad [\operatorname{Re}(p) > 0]. \quad \text{F II 792m}$$

$$3. \int_0^{\infty} \frac{x^{2a-1}}{e^{px} - 1} dx = (-1)^{a-1} \left(\frac{2\pi}{p} \right)^{2a} \frac{B_{2a}}{4a}. \quad \text{F II 721m}$$

$$4. \int_0^{\infty} \frac{x^{p-1}}{e^{rx} - q} dx = \frac{1}{q^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{q^k}{k^p} \quad [p > 0]. \quad \text{BH [83] (5)}$$

$$3.234 \int_0^{\ln 2} \frac{x dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2. \quad \text{BH [104] (7)}$$

3.235

$$1. \int_0^{\infty} \frac{x dx}{e^x + e^{-x} - 1} = 1,471\,953\,619\,4\dots \quad \text{Li [88] (1)}$$

$$2. \int_0^{\infty} \frac{x e^{-x} dx}{e^x + e^{-x} - 1} = 0,311\,821\,131\,9\dots \quad \text{Li [88] (2)}$$

$$3.236 \quad \int_0^{\infty} \frac{x e^{-2ax} dx}{1+e^x} = -\frac{\pi^2}{12} + \sum_{k=1}^{2a-1} \frac{(-1)^{k-1}}{k^2}. \quad \text{BH 82 (5)}$$

3.237

$$1. \quad \int_0^{\infty} \frac{x^{a-1} e^{-qx} dx}{1+e^x} = \Gamma(a) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(q+k)^a} \quad [a > 0, q > -1]. \quad \text{BH [83] (9)}$$

$$2. \quad \int_0^{\infty} \frac{x^{p-1} e^{-qx} dx}{1-e^{-x}} = \zeta(p, q) \Gamma(p) \quad [\operatorname{Re}(q) > 0, \operatorname{Re}(p) > 1]. \quad \text{F II 792 WW II 45}$$

$$3.238 \quad \int_0^{\infty} \frac{e^{-qx} + e^{(q-p)x}}{1-e^{-px}} x dx = \left(\frac{\pi}{p} \operatorname{cosec} \frac{q\pi}{p} \right)^2 \quad [0 < q < p]. \quad \text{BH [82] (8)}$$

3.239

$$1. \quad \int_0^{\infty} \frac{e^{-qx} - e^{-px}}{e^{-x} + 1} \frac{dx}{x} = \ln \frac{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q+1}{2}\right)} \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0]. \quad \text{BH [93] (6)}$$

$$2. \quad \int_0^{\infty} \frac{1 - e^{(1-q)x}}{e^x + 1} \frac{dx}{x} = \ln \left[\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \sqrt{\pi} \right] \quad [\operatorname{Re}(q) > 0]. \quad \text{BH [93] (4)}$$

$$3. \quad \int_0^{\infty} \frac{e^{-px} - e^{(p-q)x}}{e^{-qx} + 1} \frac{dx}{x} = \ln \operatorname{ctg} \frac{p\pi}{2q} \quad [0 < p < q]. \quad \text{BH [93] (7)}$$

$$4. \quad \int_0^{\infty} \frac{\{1 - e^{(q-p)x}\}^2}{e^{2x} - e^{(q-2p)x}} \frac{dx}{x} = \ln \operatorname{cosec} \frac{q\pi}{2p} \quad [0 < q < p]. \quad \text{BH [95] (6)}$$

$$3.241 \quad \int_0^{\infty} \left(\frac{1}{e^x - 1} + \frac{1}{2} - \frac{1}{x} \right) \frac{e^{-qx}}{x} dx = \ln \Gamma(q) - \left(q - \frac{1}{2} \right) \ln q + q - \frac{1}{2} \ln 2\pi \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 23}$$

$$3.242 \quad \int_0^{\infty} \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi. \quad \text{BH [94] (6)}$$

3.243

$$1. \quad \int_0^{\infty} \left(\frac{e^{-x}}{x} + \frac{e^{-qx}}{e^{-x} - 1} \right) dx = \psi(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 20}$$

$$2. \quad \int_0^{\infty} \left(\frac{e^{-qx}}{1 - e^{-x}} - \frac{e^{-px}}{x} \right) dx = \ln p - \psi(q) \quad [p > 0, q > 0]. \quad \text{BH [94] (3)}$$

$$3.244 \quad \int_0^{\infty} \left(q - \frac{1 - e^{-qx}}{1 - e^{-x}} \right) \frac{e^{-x}}{x} dx = \ln \Gamma(q+1) \quad [\operatorname{Re}(q) > -1]. \quad \text{WW II 24}$$

$$3.245 \quad \int_0^{\infty} \left[1 - e^{-x} - \frac{(1 - e^{-qx})(1 - e^{-px})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(p, q) \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0]. \quad \text{BH [94] (12)}$$

$$3.246 \quad \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{1 + e^{-\frac{x}{2}}} \right) \frac{e^{-x}}{x} dx = \frac{1}{2} \ln \frac{\pi}{4}. \quad \text{BH [94] (5)}$$

$$3.247 \quad \int_0^{\infty} \frac{x^q e^{-rx}}{(1 - p e^{-rx})^2} dx = \frac{\Gamma(q+1)}{p r^{q+1}} \sum_{k=1}^{\infty} \frac{p^k}{k^q} \quad [p < 1, q > -1, r > 0]. \quad \text{BH [85] (13)}$$

3.25 Funcții algebrice de e^{px} și funcția putere

$$3.251 \quad \int_0^{\infty} e^{-x} x \sqrt{1 - e^{-x}} dx = \frac{4}{3} \left(\frac{4}{3} - \ln 2 \right). \quad \text{BH [99] (1)}$$

3.252

$$1. \quad \int_0^{\infty} \frac{x dx}{\sqrt{e^x - 1}} = 2\pi \ln 2. \quad \text{F II 643 m BH [99] (4)}$$

$$2. \quad \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{2} (2 \ln 2 - 1). \quad \text{BH [99] (6)}$$

$$3.253 \quad \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^{2x} - 1}} = 1 - \ln 2. \quad \text{BH [99] (8)}$$

3.254

$$1. \quad \int_0^{\infty} \frac{x e^x dx}{[p^2 e^x - (p^2 - q^2)] \sqrt{e^x - 1}} = \frac{2\pi}{pq} \ln \frac{p+q}{p}. \quad \text{BH [99] (16)}$$

$$2. \quad \int_0^{\infty} \frac{x e^x dx}{[p^2 e^x - (p^2 + q^2)] \sqrt{e^x - 1}} = \frac{2\pi}{pq} \operatorname{arctg} \frac{q}{p}. \quad \text{BH [99] (17)}$$

3.26 Puteri ale binoamelor de tipul $\alpha e^{px} + \beta e^{qx}$ și funcția putere

$$3.261 \quad \int_0^{\ln 2} (e^x - 1)^{q-1} x e^x dx = \frac{1}{q} \left[\ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{q+k+1} \right] \quad [q > -1]. \quad \text{BH [104] (4)}$$

$$3.262 \quad \int_0^{\infty} e^{-x} (1 - e^{-2x})^{a-\frac{1}{2}} dx = \frac{(2a-1)!!}{2^{2a+1} \cdot a!} \pi [C + \psi(a+1) + 2 \ln 2]. \quad \text{BH [99] (3)}$$

$$3.263 \quad \int_{-\infty}^{\infty} \frac{x^2 dx}{(q + e^x)^{p+1}} = \frac{1}{pq^p} [\ln q - C - \psi(p)] \quad [q > 0];$$

$$= \frac{1}{pq^p} \left(\ln q - \sum_{k=1}^{p-1} \frac{1}{k} \right) \text{ [pentru } p \text{ întreg, } q > 0]. \quad \text{BH [101] (11)}$$

$$3.264 \quad \int_{-\infty}^{\infty} \frac{x e^x dx}{(q + e^x)^{b+\frac{1}{2}}} = \frac{2}{(2b+1)q^{b+\frac{1}{2}}} \left[\ln(4q) - \sum_{k=1}^{b-1} \frac{1}{k} - 2 \sum_{k=b}^{2b-1} \frac{1}{k} \right] \quad [q > 0].$$

BH [101] (12)

$$3.265 \quad \int_{-\infty}^{\infty} \frac{x dx}{(q^2 e^x + e^{-x})^p} = -\frac{1}{2q^p} \frac{\left[\Gamma\left(\frac{p}{2}\right) \right]^2}{\Gamma(p)} \ln q \quad [q > 0, p > 0]. \text{ BH [101] (14)}$$

$$3.266 \quad \int_{-\infty}^{\infty} \frac{x e^x dx}{(q^2 + r^2 e^{2x})^p} = \frac{\Gamma\left(p - \frac{1}{2}\right) \sqrt{\pi}}{4q^{2p-1} r \Gamma(p)} \left[2 \ln \frac{q}{r} - C - \psi\left(p - \frac{1}{2}\right) \right] \\ [p > 0, q > 0, r > 0]. \text{ BH [101] (13)}$$

$$3.267 \quad \int_{-\infty}^{\infty} \frac{q^2 e^x - e^{-x}}{(q^2 e^x + e^{-x})^{p+1}} x^2 dx = -\frac{1}{q^p} \frac{\left[\Gamma\left(\frac{p}{2}\right) \right]^2}{\Gamma(p+1)} \ln q \quad [p > 0, q > 0].$$

BH [102] (5)

3.27-3.29 Funcția exponențială în care argumentul este funcție de x și funcția putere

$$3.271 \quad \int_0^{\infty} e^{-px^2} x^{2a} dx = \frac{(2a-1)!!}{2(2p)^a} \sqrt{\frac{\pi}{p}} \quad [p > 0]. \text{ F II 743}$$

$$3.272 \quad \int_0^{\infty} e^{-px^2} x^{2a+1} dx = \frac{a!}{2p^{a+1}} \quad [p > 0]. \text{ BH [81] (7)}$$

$$3.273 \quad \int_u^{\infty} \frac{e^{-p^2 x^2}}{x^{2a}} dx = \frac{(-1)^a p^{2a-1} 2^{a-1} \sqrt{\pi} [1 - \Phi(pu)]}{(2a-1)!!} + \\ + \frac{e^{-p^2 u^2}}{2u^{2a-1}} \sum_{k=0}^{a-1} \frac{(-1)^k 2^{k+1} (pu)^{2k}}{(2a-1)(2a-3) \dots (2a-2k-1)}. \text{ NI 21 (4)}$$

$$3.274 \quad \int_0^{\infty} \frac{e^{-x^2 q^2} dx}{p^2 + x^2} = [1 - \Phi(pq)] \frac{\pi}{2p} e^{p^2 q^2} \quad [\operatorname{Re}(q^2) > 0]. \text{ NI 19 (13)}$$

$$3.275 \quad \int_0^{\infty} \frac{x e^{-q^2 x^2} dx}{\sqrt{p^2 + x^2}} = [1 - \Phi(pq)] \frac{\sqrt{\pi}}{2q} e^{p^2 q^2} \quad [\operatorname{Re}(q^2) > 0]. \text{ NI 19 (11)}$$

$$3.276 \quad \int_{p\sqrt{2}}^{\infty} \frac{e^{-x^2}}{\sqrt{x^2 - p^2}} \frac{dx}{x} = \frac{\pi}{4p} [1 - \Phi(p)]^2. \text{ NI 33 (17)}$$

$$3.277 \quad \int_0^{\infty} \frac{e^{-p^2 x^2} - e^{-q^2 x^2}}{x^2} dx = \sqrt{\pi} (q - p). \text{ F II 645}$$

$$3.278 \quad \int_0^{\infty} (e^{-x^2} - e^{-x}) \frac{dx}{x} = \frac{C}{2}. \text{ BH [89] (5)}$$

$$3.279 \quad \int_{-\infty}^{\infty} e^{-px^2 + 2qx} x^{a+1} dx = \frac{1}{2^a p} \sqrt{\frac{\pi}{p}} \frac{d^a (q e^{\frac{q^2}{p}})}{dq^a} \quad [p > 0]. \text{ BH [100] (8)}$$

$$3.281 \quad \int_0^1 \frac{e^{\frac{1-x}{2}-x^2}}{x(1-x)} dx = \phi(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{BH [80] (7)}$$

$$3.282 \quad \int_{-\infty}^{\infty} e^{-\frac{z^2}{2x}} x^{-p} dx = 2 \left(\frac{z}{2}\right)^{1-p} K_{p-1}(z) \quad \left[|\arg z| < \frac{\pi}{2}, \operatorname{Re}(z^2) > 0\right].$$

W 203 (15)

$$3.283 \quad \int_0^{\infty} e^{-\frac{1+x^2}{2qx}} \sqrt{x} dx = \int_0^{\infty} e^{-\frac{1+x^2}{2qx}} \frac{dx}{x^2 \sqrt{x}} = (1+q) \frac{\sqrt{2q\pi}}{\sqrt{e}} \quad [q > 0]. \quad \text{BH [98] (3)}$$

$$3.284 \quad \int_0^{\infty} e^{-\frac{1+x^2}{2qx}} \frac{dx}{\sqrt{x}} = \int_0^{\infty} e^{-\frac{1+x^2}{2qx}} \frac{dx}{x \sqrt{x}} = \frac{\sqrt{2q\pi}}{\sqrt{e}} \quad [q > 0]. \quad \text{BH [98] (12)}$$

$$3.285 \quad \int_0^{\infty} e^{-x^2} x^{(b+\frac{1}{2})a-1} dx = \frac{(2b-1)!!}{2^b a} \sqrt{\pi}. \quad \text{BH [98] (6)}$$

$$3.286 \quad \int_0^{\infty} \left(e^{-x^{2a}} - \frac{1}{1+x^2}\right) \frac{dx}{x} = -\frac{C}{2a}. \quad \text{BH [92] (13)}$$

$$3.287 \quad \int_0^{\infty} e^{-x^q} x^p dx = \frac{1}{q} \Gamma\left(\frac{p+1}{q}\right) \quad [p > -1, q > 0]. \quad \text{BH [81] (8)}$$

$$3.288 \quad \int_0^{\infty} (e^{-qx^r} - e^{-px^r}) \frac{dx}{x} = \frac{1}{r} \ln \frac{p}{q} \quad [p > 0, q > 0]. \quad \text{BH [89] (3)}$$

$$3.289 \quad \int_0^{\infty} (e^{-xp} - e^{-xq}) \frac{dx}{x} = \frac{p-q}{pq} C \quad [p > 0, q > 0]. \quad \text{BH [89] (9)}$$

$$3.291 \quad \int_{-\infty}^{\infty} e^{-qx^2} x e^x dx = -\frac{1}{q} (C + \ln q) \quad [q > 0]. \quad \text{BH [100] (13)}$$

$$3.292 \quad \int_{-\infty}^{\infty} e^{-qx^{2a}} x e^x dx = -\frac{1}{4} [C + \ln(4q)] \sqrt{\frac{\pi}{q}} \quad [q > 0]. \quad \text{BH [100] (14)}$$

3.3 FUNCȚII HIPERBOLICE

3.31 Funcții hiperbolice

3.311

$$1. \quad \int_0^{\infty} \frac{\operatorname{sh} px}{\operatorname{sh} qx} dx = \frac{\pi}{2q} \operatorname{tg} \frac{p\pi}{2q} \quad [q > p]. \quad \text{BH [27] (10)}$$

$$2. \quad \int_0^{\infty} \frac{\operatorname{ch} px}{\operatorname{ch} qx} dx = \frac{\pi}{2q} \sec \frac{p\pi}{2q} \quad [q > p]. \quad \text{BH [27] (4)}$$

$$3.312 \quad \int_0^{\infty} \frac{\operatorname{ch} px - \cos \lambda}{\operatorname{ch} qx - \cos \mu} dx = \frac{\pi}{q} \frac{\sin\left(p \frac{\pi - \mu}{q}\right)}{\sin \mu \sin \frac{p\pi}{q}} - \frac{\pi - \mu}{q} \frac{\cos \lambda}{\sin \mu} \quad [q > p]. \quad \text{BH [27] (24)}$$

$$3.313 \quad \int_0^{\infty} \frac{\operatorname{ch} 2px}{\operatorname{ch}^{2q} x} dx = 2^{2(q-1)} B(q+p, q-p) \quad [\operatorname{Re} q > |\operatorname{Re}(p)|]. \quad \text{BH [27] (18)}$$

$$3.314 \quad \int_0^{\frac{\pi}{2}} \frac{\operatorname{sh}[(r-p)\operatorname{tg} x]}{\operatorname{sh}(r\operatorname{tg} x)} dx = \pi \sum_{k=1}^{\infty} \frac{1}{k\pi + r} \sin \frac{pk\pi}{r} \quad [p^2 < r^2]. \quad \text{BH [274] (13)}$$

3.32 Funcții hiperbolice și funcția putere

3.321

$$1. \quad \int_0^1 \sqrt{1-x^2} \operatorname{ch} px dx = \frac{\pi}{2p} I_1(p). \quad \text{W 95 (9)}$$

$$2. \quad \int_0^1 \frac{\operatorname{ch} px}{\sqrt{1-x^2}} dx = \frac{\pi}{2} I_0(p). \quad \text{W 95 (9)}$$

3.322

$$1. \quad \int_0^{\infty} \frac{x dx}{\operatorname{ch} x} = 2G = \pi \ln 2 - 4L\left(\frac{\pi}{4}\right) \quad \text{Lo III 225 (103a) BH [84] (1)m}$$

$$2. \quad \int_0^{\infty} \frac{1}{\operatorname{ch} px} \frac{dx}{q^2 + x^2} = \frac{1}{q} \beta\left(\frac{pq}{\pi} + \frac{1}{2}\right) \quad [p > 0, q > 0]. \quad \text{BH [97] (4)m}$$

$$3. \quad \int_0^{\infty} \frac{x}{\operatorname{sh} px} \frac{dx}{q^2 + x^2} = \frac{\pi}{2pq} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{pq + k\pi} \quad [p > 0, q > 0], \quad \text{BH [97] (16)m}$$

3.323

$$1. \quad \int_0^{\infty} \frac{\sqrt{x} dx}{\operatorname{ch} x} = \sqrt{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{(2k+1)^2}}. \quad \text{BH [98] (7)m}$$

$$2. \quad \int_0^{\infty} \frac{dx}{\sqrt{x} \operatorname{ch} x} = 2 \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}. \quad \text{BH [98] (25)m}$$

3.324

$$1. \quad \int_0^{\infty} \frac{x^{q-1}}{\operatorname{sh} px} dx = \frac{2^q - 1}{2^{q-1} p^q} \Gamma(q) \zeta(q) \quad [\operatorname{Re}(q) > 1, p > 0]. \quad \text{WW II 46}$$

$$2. \quad \int_0^{\infty} \frac{x^{2a-1}}{\operatorname{sh} px} dx = \frac{2^{2a} - 1}{4a} \left(\frac{\pi}{p}\right)^{2a} |B_{2a}|. \quad \text{WW I 169m}$$

$$3. \quad \int_0^{\infty} \frac{x^{2a}}{\operatorname{ch} px} dx = \frac{8}{p^{2a+1}} (2a)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2a+1}}. \quad \text{BH [102] (15)m}$$

3.325

$$1. \quad \int_0^{\infty} \left(\frac{1}{\operatorname{sh} x} - \frac{1}{x}\right) \frac{dx}{x} = -\ln 2. \quad \text{BH [94] (10)m}$$

$$2. \quad \int_0^{\infty} \left(\frac{q}{\operatorname{sh} qx} - \frac{p}{\operatorname{sh} px}\right) \frac{dx}{x} = (p-q) \ln 2 \quad [pq > 0]. \quad \text{BH [94] (11)m}$$

$$3. \int_0^{\infty} \frac{x^q dx}{\operatorname{ch} x + \cos \lambda} = \frac{2\Gamma(q+1)}{\sin \lambda} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\lambda}{k^{q+1}} \quad [q > -1]. \quad \text{BH [96] (14)}m$$

$$4. \int_0^u \frac{x dx}{\operatorname{ch} 2x - \cos 2\alpha} = \frac{1}{2} \operatorname{cosec} 2\alpha [L(\theta + \alpha) - L(\theta - \alpha) - 2L(\alpha)]$$

$$[\theta = \operatorname{arctg}(\operatorname{tg} u \operatorname{ctg} \alpha), \alpha \neq \alpha\pi] \quad \text{Lo III 402}$$

$$5. \int_0^{\infty} \frac{x dx}{\operatorname{ch} 2x + \cos 2\alpha} = \frac{\alpha \ln 2 - L(\alpha)}{\sin \alpha \cos \alpha} \quad \text{Lo III 402}$$

3.326

$$1. \int_0^{\infty} \frac{\operatorname{sh} qx}{\operatorname{ch} px} \frac{dx}{x} = \ln \operatorname{tg} \left(\frac{p+q}{4p} \pi \right) \quad [p^2 > q^2] \quad (\text{vezi și 3.637 3.}).$$

$$\text{BH [95] (3)}m$$

$$2. \int_0^{\infty} \frac{\operatorname{sh}(r-p)x}{\operatorname{sh} rx} \frac{dx}{q^2 + x^2} = \frac{\pi}{q} \sum_{k=1}^{\infty} \frac{1}{qr + k\pi} \sin \frac{kp\pi}{r} \quad [p^2 < r^2]. \quad \text{BH [97] (18)}m$$

$$3. \int_0^{\infty} \frac{\operatorname{ch}(r-p)x}{\operatorname{sh} rx} \frac{x dx}{q^2 + x^2} = \frac{\pi}{2qr} + \pi \sum_{k=1}^{\infty} \frac{1}{qr + k\pi} \cos \frac{kp\pi}{r} \quad [p^2 < r^2].$$

$$\text{BH [97] (19)}m$$

$$4. \int_0^{\infty} \frac{x^p \operatorname{sh} qx}{\operatorname{sh} x} dx = \Gamma(p+1) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-q-1)^{p+1}} - \frac{1}{(2k+q-1)^{p+1}} \right\}$$

$$[p > -1, 0 < q < 1]. \quad \text{BH [95] (10)}m$$

$$5. \int_0^{\infty} \frac{\operatorname{sh}^2 qx}{x \operatorname{sh} x} dx = -\frac{1}{2} \ln \cos q\pi \quad \left[0 < q < \frac{1}{2} \right]. \quad \text{BH [130] (12)}m$$

$$6. \int_0^u \frac{x \operatorname{ch} x dx}{\operatorname{ch} 2x - \cos 2\alpha} = \frac{1}{2} \operatorname{cosec} \alpha \left\{ L\left(\frac{\theta+\alpha}{2}\right) - L\left(\frac{\theta-\alpha}{2}\right) + L\left(\pi - \frac{\psi+\alpha}{2}\right) + \right.$$

$$\left. + L\left(\frac{\psi-\alpha}{2}\right) - 2L\left(\frac{\alpha}{2}\right) - 2L\left(\frac{\pi-\alpha}{2}\right) \right\}$$

$$\left[\operatorname{tg} \frac{\theta}{2} = \operatorname{th} \frac{u}{2} \operatorname{cth} \frac{\alpha}{2}, \operatorname{tg} \frac{\psi}{2} = \operatorname{ctg} \frac{u}{2} \operatorname{ctg} \frac{\alpha}{2}; \alpha \neq \alpha\pi \right]. \quad \text{Lo III 288}m$$

$$7. \int_0^{\infty} \frac{x \operatorname{ch} x dx}{\operatorname{ch} 2x - \cos 2\alpha} = \operatorname{cosec} \alpha \left\{ \frac{\pi}{2} \ln 2 - L\left(\frac{\alpha}{2}\right) - L\left(\frac{\pi-\alpha}{2}\right) \right\} \quad [\alpha \neq \alpha\pi].$$

Lo III 403

3.327

$$1. \int_0^{\infty} \frac{x^p}{\operatorname{sh}^2 qx} dx = \frac{4\Gamma(p+1)}{q^{p+1}} \zeta(p) \quad [p > 1, q > 0]. \quad \text{BH [85] (12)}m$$

$$2. \int_0^{\infty} \frac{x^p}{\operatorname{ch}^2 qx} dx = \frac{\Gamma(p+1)}{2q^{p-1} q^{p+1}} (1 - 2^{1-p}) \zeta(p) \quad [p > 0, q > 0]. \quad \text{BH [85] (11)}m$$

Pentru $p=1$

$$\int_0^{\infty} \frac{x dx}{\operatorname{ch}^2 qx} = \frac{\ln 2}{q^2}. \quad \text{Lo III 397}$$

$$3. \int_0^{\infty} \frac{x \operatorname{sh} qx}{\operatorname{ch}^3 qx} dx = \frac{\pi}{2q^2}. \quad \text{BH [86] (8)m}$$

$$4. \int_0^{\infty} \frac{\operatorname{th} x}{\operatorname{ch} 2x} \frac{dx}{x} = \ln 2. \quad \text{BH [93] (17)m}$$

$$5. \int_0^{\infty} \frac{x^{2a+1} \operatorname{sh} x dx}{(\operatorname{ch} x - \cos 2p\pi)^2} = \frac{2 \cdot (2a+1)!}{\sin 2p\pi} \sum_{k=1}^{\infty} \frac{\sin 2kp\pi}{k^{2a+1}}. \quad \text{BH [88] (14)m}$$

$$3.328 \int_0^{\infty} \frac{x \operatorname{sh} qx}{\operatorname{ch}^{2p+1} qx} dx = \frac{\sqrt{\pi}}{4pq^2} \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)} \quad [p > 0]. \quad \text{LI [86] (9)}$$

3.329

$$1. \int_0^{\infty} \frac{dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^q} = \mathfrak{D}_{q-1}(z) \quad [q > -1; \text{prin alegerea corespunzătoare a}$$

unei ramuri de același semn a funcției de sub semnul integrală, această formulă este valabilă pentru orice valori ale lui z , tăiat de la -1 până la $+1$, dacă $q < 0$; dacă $q > 0$, atunci această formulă încetează de a mai fi valabilă în punctele în care numitorul se anulează]. CH 425 WW II 113

$$2. \int_0^{\infty} \frac{\operatorname{ch} ax dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^q} = e^{-a\pi i} \frac{\Gamma(q-a)}{\Gamma(q)} \hat{\mathfrak{D}}_{q-1}^a(z) \quad [\operatorname{Re}(q) > a]. \quad \text{WW II 134}$$

3.33 Funcții hiperbolice și funcția exponențială

$$3.331 \int_0^{\infty} \frac{\operatorname{sh} px}{\operatorname{sh} qx} e^{-qx} dx = \frac{1}{p} - \frac{\pi}{2q} \operatorname{ctg} \frac{p\pi}{2q} \quad [0 < p < 2q]. \quad \text{BH [27] (9)m}$$

$$3.332 \int_0^{\infty} \left\{ -\frac{\operatorname{sh} qx}{\operatorname{sh} \frac{x}{2}} + 2qe^{-x} \right\} \frac{dx}{x} = 2 \ln \Gamma\left(q + \frac{1}{2}\right) + \ln \cos \pi q - \ln \pi$$

$$\left[q^2 < \frac{1}{2} \right]. \quad \text{WW II 24}$$

3.333

$$1. \int_0^{\infty} \frac{\operatorname{sh} px}{e^{qx} + 1} dx = \frac{\pi}{2q} \operatorname{cosec} \frac{p}{q} \pi - \frac{1}{2p} \quad [q > p, q > 0]. \quad \text{BH [27] (3)}$$

$$2. \int_0^{\infty} \frac{\operatorname{sh} px}{e^{qx} - 1} dx = \frac{1}{2p} - \frac{\pi}{2q} \operatorname{ctg} \frac{p}{q} \pi \quad [q > p, q > 0]. \quad \text{BH [27] (9)}$$

$$3.334 \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} x} dx = \beta\left(\frac{p+1}{2}\right) \quad [\operatorname{Re}(p) > 0]. \quad \text{MO 8m}$$

$$3.335 \int_0^{\infty} e^{-q^2 x^2} \operatorname{ch} 2px dx = \frac{\sqrt{\pi}}{2q} e^{\frac{p^2}{q^2}}. \quad \text{F II 720}$$

$$3.336 \int_0^{\infty} e^{-z \operatorname{ch} x} \operatorname{ch} px dx = K_p(z) \quad \left[|\arg z| < \frac{\pi}{2} \right]. \quad \text{W 201 (5)}$$

3.34 Funcții hiperbolice, funcția putere și funcția exponențială

$$3.341 \quad \int_0^{\infty} \frac{x e^{-x} dx}{(q^2 + x^2) \operatorname{sh} x} = \ln \frac{q}{\pi} - \frac{\pi}{2q} - \psi\left(\frac{q}{\pi}\right) \quad [\operatorname{Re}(q) > 0]. \quad \text{BH [97] (20)}_m$$

$$3.342 \quad \int_0^{\infty} x e^{-x} \operatorname{cth} \frac{x}{2} dx = \frac{\pi^2}{3} - 1. \quad \text{BH [82] (6)}_m$$

$$3.343 \quad \int_0^{\infty} \frac{x^q e^{-\frac{x}{2}} \operatorname{ch} \frac{x}{2}}{\operatorname{ch} x - \cos \lambda} dx = \frac{2\Gamma(q+1)}{\cos \frac{\lambda}{2}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos\left(k - \frac{1}{2}\right)\lambda}{k^{q+1}} \quad [q > -1].$$

Li [96] (5)_m

3.344

$$1. \quad \int_0^{\infty} \frac{\operatorname{sh}^2 qx}{\operatorname{sh} x} \frac{e^{-x}}{x} dx = \frac{1}{2} \ln(q\pi \operatorname{cosec} q\pi) \quad [q < 1]. \quad \text{BH [95] (7)}_m$$

$$2. \quad \int_0^{\infty} \frac{\operatorname{th} \frac{x}{2}}{\operatorname{ch} x} \frac{e^{-2x}}{x} dx = \ln \frac{\pi^2}{8}. \quad \text{BH [93] (18)}_m$$

3.4-3.5 FUNCȚII TRIGONOMETRICE

3.41 Funcții algebrice de $\sin x$, $\cos x$ și $\operatorname{tg} x$

$$3.411 \quad \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x dx = \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}} x dx = \frac{1}{6\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right).$$

$$3.412 \quad \int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2} K\left(\sin \frac{\pi - 2u}{4}\right). \quad \text{BH [74] (11)}$$

3.413

$$1. \quad \int_0^{\pi} \frac{dx}{\sqrt{1 \pm 2p \cos x + p^2}} = 2K(p) \quad [p^2 < 1]. \quad \text{BH [67] (5)}$$

$$2. \quad \int_0^{\pi} \frac{\sin x dx}{\sqrt{1 - 2p \cos x + p^2}} = 2 \quad [p^2 \leq 1];$$

$$= \frac{2}{p} \quad [p^2 > 1]. \quad \text{BH [67] (6)}$$

$$3. \quad \int_0^{\pi} \frac{\cos x dx}{\sqrt{1 - 2p \cos x + p^2}} = \frac{2}{p} [K(p) - E(p)] \quad [p^2 < 1]. \quad \text{BH [67] (7)}$$

3.414

$$1. \quad \int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} E\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} K\left(\frac{\sqrt{2}}{2}\right). \quad \text{BH [60] (2)}$$

$$2. \quad \int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} \left[K\left(\frac{\sqrt{2}}{2}\right) - E\left(\frac{\sqrt{2}}{2}\right) \right]. \quad \text{BH [60] (3)}$$

3.415

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \operatorname{arctg} p. \quad \text{BH [60] (5)}$$

$$2. \int_0^{\frac{\pi}{2}} \operatorname{tg}^2 x \sqrt{1-p^2 \sin^2 x} \, dx = \infty. \quad \text{BH [53] (8)}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{p^2 \cos^2 x + q^2 \sin^2 x}} = \frac{1}{p} K\left(\frac{\sqrt{p^2 - q^2}}{p}\right) \quad [0 < q < p]. \quad \text{F II 165}$$

3.416

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1 - \cos^2 \beta \cos^2 x} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= \frac{1}{\sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - KE(\beta, k') - EF(\beta, k') + KF(\beta, k') \right\}. \quad \text{MO 138}$$

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1 - (1 - k'^2 \sin^2 \beta) \sin^2 x} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= \frac{1}{k'^2 \sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - KE(\beta, k') - EF(\beta, k') + KF(\beta, k') \right\}. \quad \text{MO 138}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1 - k^2 \sin^2 \beta \sin^2 x} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{KE(\beta, k) - EF(\beta, k')}{k^2 \sin \beta \cos \beta \sqrt{1 - k^2 \sin^2 \beta}}. \quad \text{MO 138}$$

3.42-3.43 Puteri de $\sin x$, $\cos x$ și $\operatorname{tg} x$

3.421

$$1. \int_0^{\frac{\pi}{2}} \sin^{p-1} x \, dx = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right) \quad [p > 0]. \quad \text{BH [40] (3)}$$

$$2. \int_0^{\frac{\pi}{2}} \cos^{p-1} x \, dx = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right) \quad [p > 0]. \quad \text{BH [41] (3)m}$$

$$3. \int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \, dx = \frac{\pi}{2} \sec \frac{p\pi}{2} \quad [p^2 < 1]. \quad \text{BH [42] (1)}$$

$$4. \int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \, dx = \frac{1}{2} \beta\left(\frac{p+1}{2}\right) \quad [p > -1]. \quad \text{BH [34] (1)}$$

3.422

$$1. \int_0^{\frac{\pi}{4}} \operatorname{tg}^p x \sin^2 x \, dx = \frac{1+p}{4} \beta\left(\frac{p+1}{2}\right) - \frac{1}{4} \quad [p > -1]. \quad \text{BH[34] (4)}$$

$$2. \int_0^{\frac{\pi}{4}} \operatorname{tg}^p x \cos^2 x \, dx = \frac{1-p}{4} \beta\left(\frac{p+1}{2}\right) + \frac{1}{4} \quad [p > -1]. \quad \text{BH[34] (5)}$$

3.423

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^p x}{\sin x + \cos x} \frac{dx}{\sin x} = \pi \operatorname{cosec} p\pi \quad [0 < p < 1]. \quad \text{BH[49] (1)}$$

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^p x}{\sin x - \cos x} \frac{dx}{\sin x} = -p \operatorname{ctg} p\pi \quad [0 < p < 1]. \quad \text{BH[49] (2)}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{p+\frac{1}{2}} x}{\sin x + \cos x} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{p-\frac{1}{2}} x}{\sin x + \cos x} \frac{dx}{\cos x} = \pi \sec p\pi \quad \left[p^2 < \frac{1}{4}\right].$$

BH[61] (1) BH[61] (2)

$$4. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm(p-1)} x \, dx}{\cos^2 x - \sin^2 x} = \pm \frac{\pi}{2} \operatorname{ctg} \frac{p\pi}{2} \quad [0 < p < 2]. \quad \text{BH[45] (27) BH[45] (29)}$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^r x \, dx}{p^2 \sin^2 x + q^2 \cos^2 x} = \frac{\pi}{2} q^{r-1} p^{-r-1} \sec \frac{r\pi}{2} \quad [-1 < r < 1]. \quad \text{BH[47] (19)}$$

$$6. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^p x \, dx}{1 + 2 \cos \lambda \sin x \cos x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^p x \, dx}{1 + 2 \cos \lambda \sin x \cos x} = \frac{\pi \sin p\lambda}{\sin p\pi \sin \lambda}$$

 $[p^2 < 1; \lambda^2 < \pi^2]. \quad \text{BH[47] (4) BH[49] (3)}$

$$7. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^p x \, dx}{1 + \sin x \cos x} = \frac{1}{3} \left[\psi\left(\frac{p+2}{3}\right) - \psi\left(\frac{p+1}{3}\right) \right] \quad [p > -1]. \quad \text{BH[36] (3)}$$

$$8. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^p x \, dx}{1 - \sin x \cos x} = \frac{1}{3} \left[\beta\left(\frac{p+2}{3}\right) + \beta\left(\frac{p+1}{3}\right) \right] \quad [p > -1]. \quad \text{BH[36] (4)m}$$

$$9. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^q x \, dx}{(\sin x + \cos x)^2} = \frac{q\pi}{\sin q\pi} \quad [q^2 < 1]. \quad \text{BH[56] (9)m}$$

$$3.424 \quad \int_0^{\pi} \frac{\sin^{2a} x}{\sqrt{1-p^2 \sin^2 x}} dx = \frac{\pi}{2^a} \sum_{k=0}^{\infty} \frac{(2k-1)!! (2a+2k-1)!!}{2^{2k} \cdot k! (a+k)!} p^{2k} \quad [p^2 < 1]. \quad \text{Li [67] (2)}$$

$$= \frac{(2a-1)!! \pi}{2^a \sqrt{1-p^2}} \sum_{k=0}^{\infty} \frac{[(2k-1)!!]^2}{2^{2k} \cdot k! (a+k)!} \left(\frac{p^2}{p^2-1}\right)^k \quad \left[p^2 < \frac{1}{2}\right].$$

Li [67] (2)

$$3.425 \quad \int_0^{\pi} \frac{dx}{(p+q \cos x)^{a+1}} = \frac{\pi}{2^a (p+q)^a \sqrt{p^2-q^2}} \sum_{k=0}^a \frac{(2a-2k-1)!! (2k-1)!!}{(a-k)! k!} \left(\frac{p+q}{p-q}\right)^k$$

$$[p^2 > q^2]. \quad \text{Li [64] (14)}$$

3.426

$$1. \quad \int_0^{\pi} (z + \sqrt{z^2-1} \cos x)^q dx = \pi \mathfrak{P}_q(z)$$

$$\left[\operatorname{Re}(z) > 0; \arg(z + \sqrt{z^2-1} \cos x) = \arg z \text{ pentru } x = \frac{\pi}{2} \right]. \quad \text{Sm III 482}$$

$$2. \quad \int_0^{\pi} \frac{dx}{(z + \sqrt{z^2-1} \cos x)^q} = \pi \mathfrak{P}_{q-1}(z)$$

$$\left[\operatorname{Re}(z) > 0; \arg(z + \sqrt{z^2-1} \cos x) = \arg z \text{ pentru } x = \frac{\pi}{2} \right]. \quad \text{WW II 106}$$

3.427

$$1. \quad \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad [\operatorname{Re}(p) > -1, \operatorname{Re}(q) > -1].$$

F II 788m

$$2. \quad \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^p x}{\cos^p x} dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^p x}{\sin^p x} dx = \frac{\Gamma(p) \Gamma\left(\frac{1}{2}-p\right)}{2^p p \sqrt{\pi}} \sin \frac{p\pi}{2} \quad \left[0 < p < \frac{1}{2}\right].$$

BH [55] (12)

3.428

$$1. \quad \int_0^{\pi} \frac{\sin^{r-1} x dx}{(p+q \cos x)^r} = \frac{2^{r-1}}{\sqrt{(p^2-q^2)^r}} B\left(\frac{r}{2}, \frac{r}{2}\right) \quad [r > 0, 0 < q < p]. \quad \text{F II 790m}$$

$$2. \quad \int_0^{\pi} \frac{\cos^q x}{(p+q \cos x)^{a+1}} dx =$$

$$= \frac{\pi}{2^a (p+q)^a \sqrt{p^2-q^2}} \sum_{k=0}^a (-1)^k \frac{(2a-2k-1)!! (2k-1)!!}{(a-k)! k!} \left(\frac{p+q}{p-q}\right)^k \quad [p^2 > q^2].$$

Li [64] (16)

$$3.429 \quad \int_0^{\frac{\pi}{2}} \frac{\sin^{2r-1} x \cos^{2s-1} x dx}{(p^2 \sin^2 x + q^2 \cos^2 x)^{r+s}} = \frac{1}{2p^2 r q^{2s}} B(r, s) \quad [r > 0, s > 0]. \quad \text{BH [48] (28)}$$

3.431

$$1. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^p x + \operatorname{ctg}^p x) dx = \frac{\pi}{2} \sec \frac{p\pi}{2} \quad [p^2 < 1]. \quad \text{BH [35] (9)}$$

$$2. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^p x - \operatorname{ctg}^p x) \operatorname{tg} x dx = \frac{1}{p} - \frac{\pi}{2} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 2]. \quad \text{BH [35] (15)}$$

3.432

$$1. \int_0^{\frac{\pi}{2}} (\sin^p x - \sin^q x) \frac{dx}{\cos x} = \frac{1}{2} \left[\psi \left(\frac{q+1}{2} \right) - \psi \left(\frac{p+1}{2} \right) \right] \\ [p > -1, q > -1]. \quad \text{BH [46] (2)}$$

$$2. \int_0^{\frac{\pi}{2}} (\cos^p x - \sec^p x) \frac{dx}{\sin x} = -\frac{\pi}{2} \operatorname{tg} \frac{p\pi}{2} \quad [-1 < p < 1]. \quad \text{BH [46] (3)}$$

3.433

$$1. \int_0^{\frac{\pi}{4}} \frac{(\operatorname{tg}^p x + \operatorname{ctg}^p x) dx}{1 + 2 \cos \lambda \sin x \cos x} = \pi \operatorname{cosec} \lambda \operatorname{cosec} p\pi \sin p\lambda \\ [\lambda \neq a\pi, -1 < p < 1]. \quad \text{BH [36] (6)}$$

$$2. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{p-1} x + \operatorname{ctg}^p x}{\sin x + \cos x} \frac{dx}{\cos x} = \pi \operatorname{cosec} p\pi \quad [0 < p < 1]. \quad \text{BH [37] (3)}$$

$$3. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^p x - \operatorname{ctg}^p x}{\sin x + \cos x} \frac{dx}{\cos x} = \frac{1}{p} - \pi \operatorname{cosec} p\pi \quad [-1 < p < 1]. \quad \text{BH [37] (4)}$$

$$4. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^q x - \operatorname{tg}^p x}{\cos x - \sin x} \frac{dx}{\cos x} = \psi(1+p) - \psi(1+q) \quad [p > -1, q > -1]. \quad \text{BH [37] (5)}$$

$$5. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{p-1} x - \operatorname{ctg}^p x}{\cos x - \sin x} \frac{dx}{\cos x} = \pi \operatorname{ctg} p\pi \quad [0 < p < 1]. \quad \text{BH [37] (7)}$$

$$6. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^p x - \operatorname{ctg}^p x}{\cos x - \sin x} \frac{dx}{\cos x} = \pi \operatorname{ctg} p\pi - \frac{1}{p} \quad [-1 < p < 1]. \quad \text{BH [37] (8)}$$

$$7. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^q x - \operatorname{tg}^p x}{\cos x - \sin x} \frac{dx}{\sin x} = \phi(p) - \phi(q) \quad [p > 0, q > 0]. \quad \text{BH [37] (10)}$$

3.434

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^p x - 2 \cos \lambda + \operatorname{cosec}^p x}{\sin^q x + 2 \cos \mu + \operatorname{cosec}^q x} \frac{dx}{\operatorname{tg} x} = \frac{\pi}{q} \operatorname{cosec} \mu \operatorname{cosec} \frac{p\pi}{q} \sin \frac{p\pi}{q} - \frac{p}{q} \operatorname{cosec} \mu \cos \lambda$$

[$q > p > 0$ sau $q < p < 0$ sau $p > 0, q < 0$ și $p + q < 0$
sau $p < 0, q > 0$ și $p + q > 0$]. BH[50] (15)

$$2. \int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^p x + \operatorname{ctg}^p x)^q} \frac{dx}{\operatorname{tg} x} = \int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^p x + \operatorname{ctg}^p x)^q} \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2q+1} p} \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)}$$

[$q > 0$]. BH [49] (25) BH [49] (26)

3.435

$$1. \int_0^{2\pi} (p \sin x + q \cos x)^{2a} dx = \frac{(2a-1)!!}{2^{2a-1} \cdot a!} \pi (p^2 + q^2)^a. \quad \text{BH [68] (8)}$$

$$2. \int_0^{2\pi} (p \sin x + q \cos x)^{2a+1} dx = 0. \quad \text{BH [68] (9)}$$

3.436

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^p x}{(\cos x - \sin x)^{p+1}} \frac{dx}{\cos x} = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0]. \quad \text{BH [37] (16)}$$

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^p x}{(\cos x - \sin x)^p} \frac{dx}{\cos^2 x} = p\pi \operatorname{cosec} p\pi \quad [-1 < p < 1]. \quad \text{BH [37] (17)}$$

$$3. \int_0^{\frac{\pi}{4}} \frac{\sin^p x}{(\cos x - \sin x)^p} \frac{dx}{\sin x \cos x} = \pi \operatorname{cosec} p\pi \quad [0 < p < 1]. \quad \text{Li [37] (20)m}$$

3.437

$$1. \int_0^{\frac{\pi}{2}} (\sec x - 1)^p \sin x \, dx = p\pi \operatorname{cosec} p\pi \quad [-1 < p < 1]. \quad \text{BH [55] (13)}$$

$$2. \int_0^{\frac{\pi}{2}} (\sec x - 1)^p \operatorname{tg} x \, dx = \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^p \frac{dx}{\operatorname{tg} x} = -\pi \operatorname{cosec} p\pi$$

[$-1 < p < 0$]. BH [46] (4) BH [46] (6)

3.438

$$1. \int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^p \frac{dx}{\sin x \cos x} = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0]. \quad \text{BH [38] (22)m}$$

$$2. \int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^p \frac{dx}{\cos^2 x} = p\pi \operatorname{cosec} p\pi \quad [-1 < p < 1]. \quad \text{BH [38] (11)m}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{(1+\operatorname{tg} x)^q - 1}{(1+\operatorname{tg} x)^{p+q}} \frac{dx}{\sin x \cos x} = \psi(p+q) - \psi(p) [p > 0, q > 0]. \text{ BH [49] (29)}$$

$$3.439 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)^{\cos 2x} dx = \frac{\pi}{2 \sin(\pi \cos^2 \alpha)}. \quad \text{F II 788}$$

3.44 Funcții trigonometrice ale unghiului dublu

3.441

$$1. \int_0^{\frac{\pi}{4}} (\sec^{\frac{1}{2}} 2x - 1) \frac{dx}{\operatorname{tg} x} = \ln 2. \quad \text{BH [38] (23)}$$

$$2. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^2 x dx}{\sqrt{1-k^2 \sin^2 2x}} = \sqrt{1-k^2} - E(k) + \frac{1}{2} K(k). \quad \text{BH [39] (2)}$$

$$3. \int_0^u \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}} dx = \frac{\pi}{2} (1 - \cos u) \quad \left[u^2 < \frac{\pi^2}{4} \right]. \quad \text{Li [74] (6)}$$

3.442

$$1. \int_0^{\frac{\pi}{4}} \frac{\cos^q 2x}{\cos^{2(q+1)} x} dx = 2^{2q} \frac{[\Gamma(q+1)]^2}{\Gamma(2q+2)} \quad [q > -1]. \quad \text{BH [35] (1)}$$

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^{2p-2} x}{\cos^p 2x} dx = \frac{\Gamma(2p-1) \Gamma(1-p)}{2^{2p-1} \Gamma(p)} = \frac{\Gamma\left(p - \frac{1}{2}\right) \Gamma(1-p)}{2 \sqrt{\pi}} \quad \left[\frac{1}{2} < p < 1 \right]. \quad \text{BH [35] (4)}$$

$$3. \int_0^{\frac{\pi}{4}} \frac{\sin^{2a-1} x}{\cos^{2a+2b} x} \cos^{b-\frac{1}{2}} 2x dx = \frac{2^{a-1} \cdot (a-1)! (2b-1)!!}{(2a+2b-1)!!}. \quad \text{BH [38] (6)}$$

$$4. \int_0^{\frac{\pi}{4}} \frac{\sin^{2a} x}{\cos^{2a+2b+1} x} \cos^{b-\frac{1}{2}} 2x dx = \frac{(2a-1)!! (2b-1)!!}{2^{a+b+1} \cdot (a+b)!} \pi. \quad \text{BH [38] (7)}$$

3.443

$$\int_0^{\frac{\pi}{4}} \frac{\sin^{p-1} 2x dx}{(\cos x + \sin x)^{2p}} = \frac{1}{2^{p+1}} \frac{\Gamma(p) \sqrt{\pi}}{\Gamma\left(p + \frac{1}{2}\right)} \quad [p > 0]. \quad \text{BH [37] (1)}$$

3.444

$$1. \int_0^{\frac{\pi}{4}} (\cos^{p-1} 2x - \sec^p 2x) \operatorname{ctg} x dx = \frac{\pi}{2} \operatorname{ctg} p\pi \quad [0 < p < 1]. \quad \text{BH [35] (7)}$$

$$2. \int_0^{\frac{\pi}{4}} (\cos^{p-1} 2x + \sec^p 2x) \operatorname{tg} x \, dx = \frac{\pi}{2} \operatorname{cosec} p\pi \quad [0 < p < 1]. \quad \text{BH [35] (8)}$$

$$3.445 \quad \int_0^{\frac{\pi}{4}} \frac{\cos^p 2x - \sec^p 2x}{\operatorname{tg} x} \, dx = -\frac{1}{2p} + \frac{\pi}{2} \operatorname{ctg} p\pi \quad [-1 < p < 1]. \quad \text{BH [35] (14)}$$

3.45 Funcții trigonometrice ale arcelor multiple

3.451

$$1. \int_0^{\pi} \frac{\cos ax \, dx}{1 + p \cos x} = \frac{\pi}{\sqrt{1-p^2}} \left(\frac{\sqrt{1-p^2}-1}{q} \right)^a \quad [p^2 < 1]. \quad \text{BH [64] (12)}$$

$$2. \int_0^{\pi} \frac{\cos ax \, dx}{1 - 2p \cos x + p^2} = \frac{\pi p^a}{1-p^2} \quad [p^2 < 1];$$

$$= \frac{\pi p^{-a}}{p^2-1} \quad [p^2 > 1]. \quad \text{BH [65] (3)}$$

$$3. \int_0^{\pi} \frac{\sin ax \sin x \, dx}{1 - 2p \cos x + p^2} = \frac{\pi}{2} p^{a-1} \quad [p^2 < 1];$$

$$= \frac{\pi}{2p^{a+1}} \quad [p^2 > 1]. \quad \text{BH [65] (4)}$$

$$4. \int_0^{\pi} \frac{\cos ax \cos x \, dx}{1 - 2p \cos x + p^2} = \frac{\pi}{2} \frac{1+p^2}{1-p^2} p^{a-1} \quad [p^2 < 1];$$

$$= \frac{\pi}{2p^{a+1}} \frac{p^2+1}{p^2-1} \quad [p^2 > 1]. \quad \text{BH [65] (5)}$$

3.452

$$1. \int_u^{\pi} \frac{\sin \left(a + \frac{1}{2} \right) x \, dx}{\sqrt{2} (\cos u - \cos x)} = \frac{\pi}{2} P_a(\cos u). \quad \text{WW II 108}$$

$$2. \int_0^u \frac{\cos \left(q + \frac{1}{2} \right) x \, dx}{\sqrt{2} (\cos x - \cos u)} = \frac{\pi}{2} P_q(\cos u). \quad \text{F II 684 WW II 108}$$

$$3.453 \quad \int_0^{\frac{\pi}{2}} \left(\frac{\sin ax}{\sin x} \right)^2 \, dx = \frac{\pi}{2} a. \quad \text{F II 145}$$

3.454

$$1. \int_0^{\frac{\pi}{2}} \cos^a x \sin ax \, dx = \frac{1}{2^{a+1}} \sum_{k=1}^a \frac{2^k}{k}. \quad \text{F II 153}$$

$$2. \int_0^{\frac{\pi}{2}} \cos^a x \cos ax \, dx = \frac{\pi}{2^{a+1}}. \quad \text{F II 153}$$

$$3. \int_0^{\pi} \sin^q x \sin px \, dx = \frac{\pi q}{2^{q-2} (q^2 - p^2)} \frac{\sin \frac{p\pi}{2}}{B \left(\frac{q+p}{2}, \frac{q-p}{2} \right)}$$

$$[q > -1, q+p > 0, q-p > 0]. \quad \text{BH [62] (9)m}$$

$$4. \int_0^{\pi} \sin^q x \cos px \, dx = \frac{\pi q}{2^{q-1}(q^2 - p^2)} \frac{\cos \frac{p\pi}{2}}{B\left(\frac{q+p}{2}, \frac{q-p}{2}\right)} \quad [q > -1, q+p > 0, q-p > 0]. \quad \text{BH [62] (10)}$$

$$5. \int_0^{\frac{\pi}{2}} \cos^q x \cos px \, dx = \frac{\pi q}{2^{q-1}(q^2 - p^2)} \frac{1}{B\left(\frac{q+p}{2}, \frac{q-p}{2}\right)} \quad [q > -1, q+p > 0, q-p > 0]. \quad \text{BH [41] (21)m}$$

3.455

$$1. \int_0^{\frac{\pi}{2}} \sin^{p-1} x \cos^{q-1} x \sin(p+q)x \, dx = B(p, q) \sin \frac{p\pi}{2} \quad [p > 0, q > 0]. \quad \text{BH [42] (23) F II 814m}$$

$$2. \int_0^{\frac{\pi}{2}} \sin^{p-1} x \cos^{q-1} x \cos(p+q)x \, dx = B(p, q) \cos \frac{p\pi}{2} \quad [p > 0, q > 0]. \quad \text{BH [42] (24) BH II 814m}$$

$$3. \int_0^{\frac{\pi}{2}} \cos^{p-2} x \sin px \operatorname{ctg}^q x \, dx = \frac{\pi \operatorname{cosec} \frac{q\pi}{2}}{2(p+q-1)B(p, q)} \quad [0 < q, p > 0, p+q > 1]. \quad \text{BH [45] (23)}$$

$$4. \int_0^{\frac{\pi}{2}} \cos^{p-2} x \cos px \operatorname{ctg}^q x \, dx = \frac{\pi \sec \frac{q\pi}{2}}{2(p+q-1)B(p, q)} \quad [0 < q, p > 0, p+q > 1]. \quad \text{BH [45] (24)}$$

3.456

$$1. \int_0^{2\pi} (1 - \cos x)^a \sin ax \, dx = 0. \quad \text{BH [68] (10)}$$

$$2. \int_0^{2\pi} (1 - \cos x)^a \cos ax \, dx = (-1)^a \frac{\pi}{2^{a-1}}. \quad \text{BH [68] (11)}$$

$$3. \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos x)^q \cos ax \, dx = \frac{\pi}{(q+1)(q+2)\dots(q+a)} \mathfrak{B}_q^a(z) \quad [\operatorname{Re}(z) > 0, \arg(z + \sqrt{z^2 - 1} \cos x) = \arg z \text{ pentru } x = \frac{\pi}{2}, \text{ atunci } z \text{ se află în afara segmentului } (-1, 1) \text{ de pe axa reală}]. \quad \text{Sm III 483 (15)}$$

3.457

$$1. \int_0^{\pi} (1 - 2p \cos x + p^2)^a \cos bx \, dx = (-p)^b \pi \sum_{k=0}^a \binom{a}{k} \binom{a}{b+k} p^{2k}. \quad \text{BH [63] (3)}$$

$$\begin{aligned}
 2. \quad \int_0^\pi \frac{\cos bx \, dx}{(1-2p \cos x + p^2)^{a+1}} &= \frac{\pi p^b}{(1-p^2)^{2a+1}} \frac{a+1}{a-b+1} \sum_{k=0}^a \binom{a}{k} \binom{a}{b+k} p^{2k} \\
 &\quad [p^2 < 1]; \\
 &= \frac{\pi p^{-b}}{(p^2-1)^{2a+1}} \frac{a+1}{a-b+1} \sum_{k=0}^a \binom{a}{k} \binom{a}{b+k} p^{2(a-k)} \\
 &\quad [p^2 > 1]. \quad \text{BH [66] (5)}
 \end{aligned}$$

3.458

$$\begin{aligned}
 1. \quad \int_0^{\frac{\pi}{2}} \frac{\cos^a x \cos ax \, dx}{1-2q \cos 2x + q^2} &= \frac{\pi}{2(1-q^2)} \left(\frac{1+q}{2} \right)^a \quad [q^2 < 1]. \quad \text{BH [50] (5)} \\
 2. \quad \int_0^{\frac{\pi}{2}} \frac{\cos^a x \sin ax \sin 2x \, dx}{1-2q \cos 2x + q^2} &= \frac{\pi}{4q} \left[\left(\frac{1+q}{2} \right)^a - \frac{1}{2^a} \right] \quad [q^2 < 1]. \quad \text{BH [50] (6)}
 \end{aligned}$$

3.459

$$\begin{aligned}
 1. \quad \int_0^{\frac{\pi}{2}} \cos^{p+q} x \sin px \sin qx \, dx &= \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} \quad [p+q > -1]. \\
 &\quad \text{BH [42] (16)} \\
 2. \quad \int_0^{\frac{\pi}{2}} \cos^{p+q} x \cos px \cos qx \, dx &= \frac{\pi}{2^{p+q+2}} \left[2 + \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} \right] [p+q > -1]. \\
 &\quad \text{BH [42] (17)}
 \end{aligned}$$

3.46 Funcții trigonometrice de puteri ale lui x

$$3.461 \quad \int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}. \quad \text{F II 743}$$

Pentru integralele cosinuşilor al căror argument sînt polinoame speciale, vezi 4.9.

3.47 Funcții trigonometrice de funcții hiperbolice

3.471

$$\begin{aligned}
 1. \quad \int_0^\infty \sin(p \operatorname{ch} x) \, dx &= \frac{\pi}{2} J_0(p) \quad [p > 0]. \quad \text{MO 36} \\
 2. \quad \int_0^\infty \cos(p \operatorname{ch} x) \, dx &= -\frac{\pi}{2} N_0(p) \quad [p > 0]. \quad \text{MO 37}
 \end{aligned}$$

$$3.472 \quad \int_0^\infty \cos(p \operatorname{sh} x) \operatorname{ch} qx \, dx = \cos \frac{q\pi}{2} K_q(p) \quad [p > 0, -1 < \operatorname{Re}(q) < 1].$$

W 202 (13)

$$3.473 \quad \int_0^\pi \cos(z \operatorname{ch} x) \sin^{2p} x \, dx = \sqrt{\pi} \left(\frac{z}{2} \right)^{-p} \Gamma\left(p + \frac{1}{2}\right) I_p(z) \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right].$$

WW II 203

3.48 Funcții trigonometrice de funcții trigonometrice

3.481

1. $\int_0^{\pi} \sin(z \sin x) \sin 2ax \, dx = 0.$ BH [63] (7)
2. $\int_0^{\frac{\pi}{2}} \sin(z \sin x) \sin [(2a+1)x] \, dx = \frac{\pi}{2} J_{2a+1}(z).$ W 30 (6)
3. $\int_0^{\pi} \sin(z \sin x) \sin [(2a+1)x] \, dx = \pi J_{2a+1}(z).$ W 30 (6)
4. $\int_0^{\frac{\pi}{2}} \sin(z \cos x) \cos [(2a+1)x] \, dx = (-1)^a \frac{\pi}{2} J_{2a+1}(z).$ W 30 (8)

3.482

1. $\int_0^{\pi} \cos(z \sin x) \cos [(2a+1)x] \, dx = 0.$ BH [63] (10)
2. $\int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos 2ax \, dx = \frac{\pi}{2} J_{2a}(z).$ W 30 (7)
3. $\int_0^{\pi} \cos(z \sin x) \cos 2ax \, dx = \pi J_{2a}(z).$ W 30 (7)
4. $\int_0^{\frac{\pi}{2}} \cos(z \cos x) \cos 2ax \, dx = (-1)^a \frac{\pi}{2} J_{2a}(z).$ W 30 (9)

3.483

1. $\int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos^{2p} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{z}{2}\right)^{-p} \Gamma\left(p + \frac{1}{2}\right) J_p(z) \quad \left[\operatorname{Re}(p) > -\frac{1}{2}\right].$
W 35
2. $\int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos^{2a} x \, dx = \frac{(2a-1)!! \pi}{2z^a} J_a(z).$ F II 486
3. $\int_0^{\frac{\pi}{2}} \cos(z \cos x) \sin^{2p} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{z}{2}\right)^{-p} \Gamma\left(p + \frac{1}{2}\right) J_p(z) \quad \left[\operatorname{Re}(p) > -\frac{1}{2}\right].$
WW II 178
4. $\int_0^{\pi} \cos(z \cos x) \sin^{2p} x \, dx = \sqrt{\pi} \left(\frac{z}{2}\right)^{-p} \Gamma\left(p + \frac{1}{2}\right) J_p(z) \quad [\operatorname{Re}(p) > 0].$
WW II 179

3.484

$$1. \int_0^{\frac{\pi}{2}} \cos(p \operatorname{tg} x - qx) \cos^{q-2} x \, dx = \frac{\pi e^{-p} p^{q-1}}{\Gamma(q)} \quad [\operatorname{Re}(q) > 1]. \quad \text{NH 157 (14)}$$

$$2. \int_0^{\frac{\pi}{2}} \sin(p \operatorname{tg} x - qx) \sin^{q-2} x \, dx = 0 \quad [\operatorname{Re}(q) > 0]. \quad \text{NH 157 (15)}$$

3.485

$$\int_0^{\pi} \cos(ax - z \sin x) \, dx = \pi J_0(z). \quad \text{WW II 172}$$

3. 3.49 Funcții trigonometrice de funcții trigonometrice inverse

3.491

$$1. \int_0^{\infty} \cos^{p+1}\left(\operatorname{arctg} \frac{x}{q}\right) \sin\left[(p+1) \operatorname{arctg} \frac{x}{q}\right] \sin x \, dx = \frac{\pi q^{p+1} e^{-q}}{2\Gamma(p+1)}. \quad \text{BH [347] (7)}$$

$$2. \int_0^{\infty} \cos^{p+1}\left(\operatorname{arctg} \frac{x}{q}\right) \cos\left[(p+1) \operatorname{arctg} \frac{x}{q}\right] \cos x \, dx = \frac{\pi q^{p+1} e^{-q}}{2\Gamma(p+1)}. \quad \text{BH [347] (8)}$$

Integralele cosinurilor al căror argument sînt polinoame speciale, vezi 4.9

3.51-3.55 Funcții trigonometrice și funcții putere

3.511

$$1. \int_0^{\infty} \frac{x \sin x}{z^2 + x^2} \, dx = \frac{\pi}{2} e^{-z'} \quad [z' = z \operatorname{sgn} \operatorname{Re}(z)]. \quad \text{NI 24 (12)}$$

$$2. \int_0^{\infty} \frac{z' \sin x}{z^2 + x^2} \, dx = \frac{1}{2} [e^{-z'} \operatorname{Ei}(z') - e^{z'} \operatorname{Ei}(-z')] \quad [z' = z \operatorname{sgn} \operatorname{Re}(z)].$$

NI 24 (11)

$$3. \int_0^{\infty} \frac{x \cos x}{z^2 + x^2} \, dx = -\frac{1}{2} [e^{z'} \operatorname{Ei}(-z') + e^{-z'} \operatorname{Ei}(z')] \quad [z' = z \operatorname{sgn} \operatorname{Re}(z)].$$

NI 24 (10)

$$4. \int_0^{\infty} \frac{z' \cos x}{z^2 + x^2} \, dx = \frac{\pi}{2} e^{-z'} \quad [z' = z \operatorname{sgn} \operatorname{Re}(z)]. \quad \text{NI 24 (12)}$$

3.512

$$1. \int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}. \quad \text{F II 632}$$

$$2. \int_0^{\infty} \frac{\sin^2 x}{x} \, dx = \frac{\pi}{4}. \quad \text{Lo V}$$

3.513

$$1. \int_0^{\frac{\pi}{2}} \frac{x dx}{\sin x} = 2G. \quad \text{BH [206] (1)}$$

$$2. \int_0^{\frac{\pi}{4}} x \operatorname{tg} x dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} G. \quad \text{BH [204] (1)}$$

$$3. \int_0^{\frac{\pi}{2}} x \operatorname{tg} x dx = \infty. \quad \text{BH [205] (2)}$$

$$4. \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \operatorname{tg} x dx = \pi \ln 2. \quad \text{BH [218] (12)}$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg} x}{x} dx = \frac{\pi}{2}. \quad \text{Lo V}$$

$$6. \int_0^{\frac{\pi}{4}} x \operatorname{ctg} x dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G. \quad \text{BH [204] (2)}$$

$$7. \int_0^{\frac{\pi}{2}} x \operatorname{ctg} x dx = \frac{\pi}{2} \ln 2. \quad \text{F II 623}$$

$$8. \int_0^{\frac{\pi}{2}} x \operatorname{tg}^2 x dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2} \ln 2. \quad \text{BH [204] (3)}$$

3.514

$$1. \int_0^{\pi} \frac{x dx}{p \pm \cos x} = \frac{\pi^2}{2\sqrt{p^2-1}} \pm \frac{4}{\sqrt{p^2-1}} \sum_{k=0}^{\infty} \frac{(p - \sqrt{p^2-1})^{2k+1}}{(2k+1)^2} \quad [\rho > 1]. \quad \text{Li [219] (2)}$$

$$2. \int_0^{\pi} \frac{x \sin x}{p + \cos x} dx = -\pi \ln \{2(1-p)\} \quad [p^2 < 1];$$

$$= \pi \ln \frac{p + \sqrt{p^2-1}}{2(p-1)} \quad [p^2 > 1]. \quad \text{BH [219] (4)}$$

$$3. \int_0^{\pi} \frac{x \sin x dx}{1-p \cos x} = \frac{\pi}{p} \ln \frac{2(1+p)}{1+\sqrt{1-p^2}} \quad [p^2 < 1];$$

$$= \frac{\pi}{p} \ln \frac{2p}{1+p} \quad [p^2 > 1]. \quad \text{BH [219] (5)}$$

$$4. \int_0^{\pi} \frac{x \cos x dx}{1+p \sin x} = \frac{2\pi}{p} \ln \frac{2}{1+\sqrt{1+2p}}. \quad \text{BH [219] (15)}$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x dx = \mp \frac{\pi}{4} \ln 2 - G. \quad \text{BH [207] (8) BH [207] (9)}$$

3.515

$$1. \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}. \quad \text{F II 157}$$

$$2. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x}{1 + q \cos^2 x} dx = \frac{\pi}{q} \ln \frac{1 + \sqrt{1+q}}{2}. \quad \text{BH [207] (10)}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{x dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 \pm G. \quad \text{BH [208] (16) BH [208] (17)}$$

$$4. \int_0^{\frac{\pi}{4}} \frac{x}{\sin x + \cos x} \frac{dx}{\cos x} = \frac{\pi}{8} \ln 2. \quad \text{BH [204] (28)}$$

$$5. \int_0^{\frac{\pi}{4}} \frac{x}{\sin x + \cos x} \frac{dx}{\sin x} = -\frac{\pi}{8} \ln 2 + G. \quad \text{BH [204] (29)}$$

$$6. \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} \frac{x dx}{\cos^2 x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2. \quad \text{BH [204] (30)}$$

3.516

$$1. \int_0^{\infty} \sin x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = E(k). \quad \text{BH [154] (8)}$$

$$2. \int_0^{\infty} \sin x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = E(k). \quad \text{BH [154] (20)}$$

$$3. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = E(k). \quad \text{BH [154] (9)}$$

$$4. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = E(k). \quad \text{BH [154] (21)}$$

3.517

$$1. \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \sin^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \sin^2 x}} \frac{dx}{x}; \quad \text{BH [183] (4) BH [183] (5)}$$

$$= \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right).$$

$$\text{BH [183] (9) BH [183] (10)}$$

$$2. \int_u^{\frac{\pi}{2}} \frac{x \cos x dx}{\sqrt{\sin^2 x - \sin^2 u}} = \frac{\pi}{2} \ln(1 + \cos u). \quad \text{BH [226] (4)}$$

$$3. \int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x}; \quad \text{BH [183] (12) BH [183] (13)}$$

$$= \int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = K(k). \quad \text{BH [183] (21) BH [183] (22)}$$

$$4. \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{1}{2k^2} [-\pi k' + 2E(k)]. \quad \text{BH [211] (1)}$$

$$5. \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{2k^2} [\pi - 2E(k)]. \quad \text{BH [214] (1)}$$

$$6. \int_0^{\alpha} \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 \alpha - \sin^2 x}} = \frac{\pi \sin^2 \frac{\alpha}{2}}{\cos^2 \alpha}. \quad \text{Lo III 284}$$

$$7. \int_0^{\beta} \frac{x \sin x dx}{(1 - \sin^2 \alpha \sin^2 x) \sqrt{\sin^2 \beta - \sin^2 x}} =$$

$$\frac{\pi \ln \frac{\cos \alpha + \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}{2 \cos \beta \cos^2 \frac{\alpha}{2}}}{2 \cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}. \quad \text{Lo III 284}$$

$$8. \int_0^{\beta} \frac{x \sin x dx}{(\sin^2 \alpha - \sin^2 x) \sqrt{\sin^2 \beta - \sin^2 x}} = \frac{\pi \left(\alpha - \arccos \frac{\cos \alpha}{\cos \beta} \right)}{2 \cos \beta \sqrt{\sin^2 \beta - \sin^2 \alpha}} \quad [\alpha > \beta > 0].$$

Lo III 284

3.518

$$1. \int_0^{\pi} x \sin^q x dx = \frac{\pi^2}{2^{q+1}} \frac{\Gamma(q+1)}{\left[\Gamma\left(\frac{q}{2}+1\right) \right]^2} \quad [q > -1]. \quad \text{BH [218] (7)}$$

$$2. \int_0^{\infty} \frac{\sin^p x}{x} dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right) [p \text{ este o fracție rațională,}$$

cu numărătorul și numitorul impari, $p > 0$].

Lo V F II 808

$$3. \int_0^{\infty} \sin^{2a} x \frac{dx}{q^2 + x^2} = \frac{(-1)^a}{2^{2a+1}} \frac{\pi}{q} \left[(e^q - e^{-q})^{2a} - \right.$$

$$\left. - e^{2aq} \sum_{k=0}^a (-1)^k \binom{2a}{k} e^{-2qk} + e^{-2aq} \sum_{k=0}^a (-1)^k \binom{2a}{k} e^{2kq} \right]. \quad \text{BH [160] (12)}$$

$$4. \int_0^{\infty} \sin^{2a+1} x \frac{dx}{q^2 + x^2} =$$

$$= \frac{(-1)^{a-1}}{2^{2a+2}} \frac{1}{q} \left\{ e^{(2a+1)q} \sum_{k=0}^{2a+1} (-1)^k \binom{2a+1}{k} e^{-2kq} \operatorname{Ei}[q(2k-2a-1)] + \right.$$

$$\left. + e^{-(2a+1)q} \sum_{k=0}^{2a+1} (-1)^{k-1} \binom{2a+1}{k} e^{2kq} \operatorname{Ei}[q(2a+1-2k)] \right\}. \quad \text{BH [160] (14)}$$

$$5. \int_{a\pi}^{c\pi} x \cos^{2b} x \, dx = \frac{(2b-1)!!}{2^{b+1} \cdot b!} \cdot (c^2 - a^2) \pi^2. \quad \text{BH [226] (3)}$$

$$6. \int_0^{\infty} \cos^{2a} x \frac{dx}{q^2 + x^2} = \frac{1}{2^{2a+1}} \frac{\pi}{q} \binom{2a}{a} + 2^{-2a} \frac{\pi}{q} \sum_{k=1}^a \binom{2a}{k+a} e^{-2kq}. \quad \text{BH [160] (16)}$$

$$7. \int_0^{\infty} \cos^{2a+1} x \frac{dx}{q^2 + x^2} = \frac{1}{2^{2a+1}} \frac{\pi}{q} \sum_{k=1}^a \binom{2a+1}{k+a+1} e^{-(2k+1)q}. \quad \text{BH [160] (17)}$$

$$8. \int_0^{\infty} \cos^{2a+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2a}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^a (-1)^k \binom{2a+1}{a+k+1} \frac{1}{\sqrt{2k+1}}. \quad \text{BH [177] (8)}$$

3.519

$$1. \int_0^{\frac{\pi}{2}} \frac{x \sin^p x}{\operatorname{tg} x} dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p} B\left(\frac{p+1}{2}, \frac{p+1}{2}\right) \quad [p > -1]. \quad \text{BH [206] (11)}$$

$$2. \int_0^{\frac{\pi}{2}} x \cos^p x \operatorname{tg} x \, dx = \frac{\pi}{2^{p+1} \cdot p} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2} [p > -1]. \quad \text{BH [205] (3)}$$

$$3. \int_0^{\infty} \sin^{2a+1} x \cos^{2b} x \frac{dx}{x} = \int_0^{\infty} \sin^{2a+1} x \cos^{2b-1} x \frac{dx}{x} = \frac{(2a-1)!!}{2^{2a+b+1} \cdot (a+b)!} \pi. \quad \text{BH [151] (24) BH [151] (25)}$$

$$4. \int_0^{\frac{\pi}{4}} \frac{x \sin^{2a} x \, dx}{\cos^{2a+2} x} = \frac{1}{2(2a+1)} \left[\frac{\pi}{2} + (-1)^{a-1} \ln 2 + \sum_{k=0}^{a-1} \frac{(-1)^{k-1}}{a-k} \right]. \quad \text{BH [204] (16)}$$

$$5. \int_0^{\frac{\pi}{4}} \frac{x \sin^{2a-1} x}{\cos^{2a+1} x} dx = \frac{\pi}{8a} (1 - \cos a\pi) + \frac{1}{2a} \sum_{k=0}^{a-1} \frac{(-1)^{k-1}}{2a-2k-1}. \quad \text{BH [204] (17)}$$

$$6. \int_0^{\infty} (1 - \cos^{2a-1} x) \frac{dx}{x^2} = \int_0^{\infty} (1 - \cos^{2a} x) \frac{dx}{x^2} = \frac{a\pi}{2^{2a}} \binom{2a}{a}. \quad \text{BH [158] (7) BH [158] (8)}$$

3.521

$$1. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \sin^2 2x} \frac{dx}{x} = E(k). \quad \text{BH [154] (10)}$$

$$2. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \cos^2 2x} \frac{dx}{x} = E(k). \quad \text{BH [154] (22)}$$

$$3. \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \sin^2 2x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \cos^2 2x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right) \quad \text{BH [183] (6) BH [183] (11)}$$

$$4. \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \frac{dx}{x} = K(k).$$

BH [183] (14) BH [183] (23)

$$5. \int_0^{\infty} \frac{\sin x}{(1-2p \cos 2x + p^2)^{a+1}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{(1-2p \cos 2x + p^2)^{a+1}} \frac{dx}{x} = \text{BH [187] (14)}$$

$$= \int_0^{\infty} \frac{\operatorname{tg} x}{(1-2p \cos 4x + p^2)^{a+1}} \frac{dx}{x} = \text{BH [187] (15)}$$

$$= \frac{\pi}{2(1-p^2)^{2a+1}} \sum_{k=0}^a \binom{a}{k}^2 p^{2k} [p^2 < 1]. \text{ BH [187] (16)}$$

3.522

$$1. \int_{-\infty}^{\infty} \frac{\sin px}{x} dx = -\operatorname{si}(pu).$$

NI 18

$$2. \int_0^{\infty} \sin px \frac{dx}{x} = \frac{\pi}{2} \quad [p > 0];$$

$$= 0 \quad [p = 0];$$

$$= -\frac{\pi}{2} \quad [p < 0].$$

F II 645

$$3. \int_0^1 \frac{\sin px}{x} dx = \operatorname{si}(p) + \frac{\pi}{2}.$$

BH [149] (5)

$$4. \int_0^{\infty} \frac{\sin px}{q+x} dx = \sin pq \operatorname{ci}(pq) - \cos pq \operatorname{si}(pq).$$

BH [160] (1)

$$5. \int_0^{\infty} \frac{\sin px}{q-x} dx = \sin pq \operatorname{ci}(pq) - \cos pq [\operatorname{si}(pq) + \pi].$$

BH [161] (1)

3.523

$$1. \int_0^{\infty} \sin px \frac{dx}{q^2+x^2} = \frac{1}{2q} [e^{-pq} \operatorname{Ei}(pq) - e^{pq} \operatorname{Ei}(-pq)].$$

BH [160] (3)

$$2. \int_0^{\infty} \sin px \frac{x dx}{q^2+x^2} = \frac{1}{2} \pi e^{-pq} \quad [p > 0, q > 0].$$

F II 750 F II 741

$$3. \int_0^{\infty} \sin px \frac{dx}{q^2-x^2} = \frac{1}{q} \left[\operatorname{ci}(pq) \sin pq - \operatorname{si}(pq) \cos pq - \frac{\pi}{2} \cos pq \right].$$

BH [161] (3)

$$4. \int_0^{\infty} \sin px \frac{x dx}{q^2-x^2} = -\frac{\pi}{2} \cos pq, \quad [p > 0, q > 0].$$

F II 647

$$5. \int_0^{\infty} \frac{\sin px}{x(q^2-x^2)} dx = \frac{\pi}{2q^2} (1 - \cos pq).$$

BH [172] (4)

$$6. \int_0^{\infty} \frac{x \sin px}{(q^2+x^2)^2} dx = \frac{\pi p}{4q} e^{-pq}.$$

BH [170] (3)

$$7. \int_0^{\infty} \sin px \frac{x dx}{(q^2 + x^2)(r^2 + x^2)} = \frac{\pi}{2(q^2 - r^2)} (e^{-pr} - e^{-pq}). \quad \text{BH [174] (1)}$$

$$8. \int_0^{\infty} \sin px \frac{x dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2(q^2 - r^2)} (\cos pq - \cos pr). \quad \text{BH [174] (3)}$$

3.524

$$1. \int_0^{\infty} \frac{\sin px}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2p}}. \quad \text{BH [177] (1)}$$

$$2. \int_0^1 \sin px \sqrt{1-x^2} dx = \sum_{k=0}^{\infty} (-1)^k \frac{p^{2k+1}}{(2k+3)[(2k+1)!!]^2} \quad [p > 0]. \quad \text{BH [149] (6)}$$

$$3. \int_0^1 \sin px \frac{dx}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{p^{2k+1}}{[(2k+1)!!]^2} \quad [p > 0]. \quad \text{BH [149] (9)}$$

$$4. \int_1^{\infty} \sin px \frac{dx}{\sqrt{x^2-1}} = \frac{\pi}{2} J_0(p) \quad [p > 0]. \quad \text{W 200 (14)}$$

$$5. \int_0^1 \frac{x \sin px}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_1(p). \quad \text{W 30 (6)}$$

$$3.525 \quad \int_0^{\infty} x^{p-1} \sin qx dx = q^{-p} \Gamma(p) \sin \frac{p\pi}{2} \quad [q > 0, 0 < \operatorname{Re}(p) < 1].$$

F II 809m NH 153 (5)

3.526

$$1. \int_0^{\infty} \frac{\cos px}{x} dx = \infty. \quad \text{BH [151] (2)}$$

$$2. \int_{-\infty}^{\infty} \frac{\cos px}{x} dx = -\operatorname{ci}(pu). \quad \text{NI 18 (6)}$$

$$3. \int_0^{\infty} \frac{\cos px}{x} dx = \operatorname{ci}(p) \quad (\text{cf. 3.526 2.}).$$

$$4. \int_0^{\infty} \frac{\cos px}{q+x} dx = -\cos pq \operatorname{ci}(pq) - \sin pq \operatorname{si}(pq). \quad \text{BH [160] (2)}$$

$$5. \int_0^{\infty} \frac{\cos px}{q-x} dx = \cos pq \operatorname{ci}(pq) + \sin pq \operatorname{si}(pq) + \pi \sin pq. \quad \text{BH [161] (2)}$$

3.527

$$1. \int_0^{\infty} \cos px \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} e^{-pq} \quad [p > 0, q > 0]. \quad \text{B II 741 BH [160] (5)}$$

$$2. \int_0^{\infty} \cos px \frac{x dx}{q^2 + x^2} = -\frac{1}{2} \{e^{pq} \operatorname{Ei}(-pq) + e^{-pq} \operatorname{Ei}(pq)\}. \quad \text{BH [160] (6)}$$

$$3. \int_0^{\infty} \cos px \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \sin pq. \quad \text{BH [161] (5)}$$

$$4. \int_0^{\infty} \cos px \frac{x dx}{q^2 - x^2} = \operatorname{ci}(pq) \cos pq + \operatorname{si}(pq) \sin pq + \frac{\pi}{2} \sin pq. \quad \text{BH [161] (6)}$$

$$5. \int_0^{\infty} \cos px \frac{1-x^2}{(1+x^2)^2} dx = \frac{\pi p}{2} e^{-p}. \quad \text{BH [43] (10)m}$$

$$6. \int_0^{\infty} \cos px \frac{dx}{(q^2 + x^2)(r^2 + x^2)} = \frac{\pi}{2qr(q^2 - r^2)} (qe^{-pr} - re^{-pq}). \quad \text{BH [175] (1)}$$

$$7. \int_0^{\infty} \cos px \frac{dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2qr(q^2 - r^2)} (q \sin pr - r \sin pq). \quad \text{BH [175] (3)}$$

3.528

$$1. \int_0^{\infty} \frac{\cos px}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2p}}. \quad \text{BH [177] (2)}$$

$$2. \int_0^1 \cos px \sqrt{1-x^2} dx = \frac{\pi}{2p} J_1(p). \quad \text{Ku 65 (6)m}$$

$$3. \int_0^1 \frac{\cos px}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_0(p). \quad \text{W 30 (7)m}$$

$$4. \int_1^{\infty} \frac{\cos px}{\sqrt{x^2-1}} dx = -\frac{\pi}{2} N_0(p). \quad \text{W 200 (15) Ku 89 (28)m}$$

$$5. \int_0^{\infty} \frac{\cos px}{\sqrt{q^2+x^2}} dx = K_0(pq) \quad (\text{cf. 3.529 4.}). \quad \text{W 191 (1)}$$

3.529

$$1. \int_0^{2a\pi} x^b \cos cx dx = -\sum_{k=0}^{b-1} \frac{k!}{c^{k+1}} \binom{b}{k} (2a\pi)^{b-k} \cos \frac{k+1}{2} \pi. \quad \text{BH [226] (2)}$$

$$2. \int_0^{\infty} x^{q-1} \cos px dx = p^{-q} \cos \frac{\pi q}{2} \Gamma(q) \quad [p > 0, 0 < \operatorname{Re}(q) < 1].$$

F II 809m NH 154 (4)

$$3. \int_0^1 (1-x^2)^p \cos 2zx dx = \frac{\sqrt{\pi}}{2} z^{-p-\frac{1}{2}} \Gamma(p+1) J_{p+\frac{1}{2}}(2z) \quad [\operatorname{Re}(p) > -1].$$

WW II 178m

$$4. \int_0^{\infty} \frac{\cos px dx}{(q^2+x^2)^{r+1}} = \sqrt{\pi} \left(\frac{p}{2q}\right)^{r+\frac{1}{2}} \frac{1}{\Gamma(r+1)} K_{r+\frac{1}{2}}(pq)$$

$$\left[\operatorname{Re}(r) > -1; p > 0; |\arg q| < \frac{\pi}{2} \right]. \quad \text{W 191 (1)}$$

$$3.531 \quad \int_0^{\infty} \frac{\sin^2 x}{x^2} \cos 2px \, dx = \frac{\pi}{2} (1-p) \quad [0 < p \leq 1];$$

$$= 0 \quad [p > 1].$$

F III 648

3.532

$$1. \quad \int_0^{\infty} \sin^{2a+1} x \sin 2ax \frac{dx}{q^2+x^2} = \frac{(-1)^a}{2^{2a+1}} \frac{\pi}{q} (e^q - e^{-q}) [(1 - e^{-2q})^{2a} - 1].$$

BH [162] (17)

$$2. \quad \int_0^{\infty} \sin^{2a} x \sin 2ax \frac{x \, dx}{q^2+x^2} = \frac{(-1)^a \pi}{2^{2a+1}} [(1 - e^{-2q})^{2a} - 1].$$

BH [162] (14)

$$3. \quad \int_0^{\infty} \sin^{2a-1} x \sin [(2a-1)x] \frac{dx}{q^2+x^2} = \frac{(-1)^{a-1}}{2^{2a}} \frac{\pi}{q} (1 - e^{-2q})^{2a-1} \quad \text{BH [162] (11)}$$

$$4. \quad \int_0^{\infty} \sin^{2a} x \sin [(2a-1)x] \frac{x \, dx}{q^2+x^2} = \frac{(-1)^a \pi}{2^{2a+1}} e^q [(1 - e^{-2q})^{2a} - 1]. \quad \text{BH [162] (13)}$$

$$5. \quad \int_0^{\infty} \sin^{2a-1} x \sin [(2a+1)x] \frac{dx}{q^2+x^2} = \frac{(-1)^{a-1}}{2^{2a}} \frac{\pi}{q} e^{-2q} (1 - e^{-2q})^{2a-1}.$$

BH [162] (12)

$$6. \quad \int_0^{\infty} \sin^{2a} x \sin [(2a+2)x] \frac{x \, dx}{q^2+x^2} = \frac{(-1)^a \pi}{2^{2a+1}} e^{-2q} (1 - e^{-2q})^{2a}. \quad \text{BH [162] (15)}$$

3.533

$$1. \quad \int_0^{\frac{\pi}{2}} x \cos^{p-1} x \sin qx \, dx = \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+q+1}{2}\right) - \psi\left(\frac{p-q+1}{2}\right)}{\Gamma\left(\frac{p+q+1}{2}\right) \Gamma\left(\frac{p-q+1}{2}\right)}$$

$$[p > 0; -(p+1) < q < p+1]. \quad \text{BH [205] (6)}$$

$$2. \quad \int_0^{\infty} \cos^a x \sin [(a-1)x] \frac{x \, dx}{q^2+x^2} = \frac{\pi}{2^{a+1}} e^q (1 + e^{-2q})^a. \quad \text{BH [163] (7)}$$

$$3. \quad \int_0^{\infty} \cos^{a-1} x \sin [(a+1)x] \frac{x \, dx}{q^2+x^2} = \frac{\pi}{2^a} e^{-2q} (1 + e^{-2q})^{a-1}. \quad \text{BH [163] (6)}$$

$$4. \quad \int_0^{\infty} \cos^a x \sin [(a+1)x] \frac{x \, dx}{q^2+x^2} = \frac{\pi}{2^{a+1}} e^{-q} (1 + e^{-2q})^a. \quad \text{BH [163] (10)}$$

$$5. \quad \int_0^{\infty} \cos^a x \sin ax \frac{dx}{q^2+x^2} = \frac{1}{2^{a+1}q} \sum_{k=1}^a \binom{a}{k} [e^{-2kq} \text{Ei}(2kq) - e^{2kq} \text{Ei}(-2kq)].$$

$$\text{BH [163] (8)}$$

3.534

$$1. \quad \int_0^{\infty} \sin^{2a} x \cos 2ax \frac{dx}{q^2+x^2} = \frac{(-1)^a}{2^{2a+1}} \frac{\pi}{q} (1 - e^{-2q})^{2a}. \quad \text{BH [162] (26)}$$

$$2. \int_0^{\infty} \sin^{2a+1} x \cos 2ax \frac{x dx}{q^2 + x^2} = \frac{(-1)^{a-1} \pi}{2^{2a+1}} e^{-q} [(1 - e^{-2q})^{2a+1} - 1]. \text{ BH [162] (29)}$$

$$3. \int_0^{\infty} \sin^{2a-1} x \cos [(2a-1)x] \frac{x dx}{q^2 + x^2} = \frac{(-1)^a \pi}{2^{2a}} [(1 - e^{-2q})^{2a-1} - 1].$$

BH [162] (23)

$$4. \int_0^{\infty} \sin^{2a} x \cos [(2a-1)x] \frac{dx}{q^2 + x^2} = \frac{(-1)^a \pi}{2^{2a}} \frac{1}{q} \operatorname{sh} q [(1 - e^{-2q})^{2a-1} - 1].$$

BH [162] (25)

$$5. \int_0^{\infty} \sin^{2a-1} x \cos [(2a+1)x] \frac{x dx}{q^2 + x^2} = \frac{(-1)^a \pi}{2^{2a}} e^{-2q} (1 - e^{-2q})^{2a-1}. \text{ BH [162] (24)}$$

$$6. \int_0^{\infty} \sin^{2a} x \cos [(2a+2)x] \frac{dx}{q^2 + x^2} = \frac{(-1)^a \pi}{2^{2a+1}} e^{-2q} (1 - e^{-2q})^{2a}. \text{ BH [162] (27)}$$

3.535

$$1. \int_0^{\infty} \cos^a x \cos [(a-1)x] \frac{dx}{q^2 + x^2} = \frac{\pi}{2^{a+1} q} e^q (1 + e^{-2q})^a. \text{ BH [163] (15)}$$

$$2. \int_0^{\infty} \cos^{a-1} x \cos [(a+1)x] \frac{dx}{q^2 + x^2} = \frac{\pi}{2^a q} e^{-2q} (1 + e^{-2q})^{a-1}. \text{ BH [163] (14)}$$

$$3. \int_0^{\infty} \cos^a x \cos [(a+1)x] \frac{dx}{q^2 + x^2} = \frac{\pi}{2^{a+1} q} e^{-q} (1 + e^{-2q})^a. \text{ BH [163] (17)}$$

$$3.536 \int_0^{\infty} \frac{\cos [(2a-1)x]}{\cos x} \left(\frac{\sin x}{x} \right)^{2a} dx = (-1)^{a-1} \frac{2^{2a-1}}{(2a)!} 2^{2a-1} \pi B_{2a}. \text{ BH [180] (15)}$$

3.537

$$1. \int_0^{\infty} \frac{\cos 2ax}{\cos x} \sin^{2a} x \frac{dx}{x^b} = 0 \quad \left[a > \frac{b-1}{2}, \quad b > 0 \right]. \text{ BH [180] (16)}$$

$$2. \int_0^{\infty} \frac{\cos 2ax}{\cos x} \sin^{2a+1} x \frac{dx}{x^b} = 0 \quad \left[a > \frac{b-2}{2}, \quad b > 0 \right]. \text{ BH [180] (17)}$$

$$3.538 \int_0^{\infty} \frac{\cos px - \cos qx}{x} dx = \ln \frac{q}{p} \quad [p > 0, q > 0]. \text{ F II 635}$$

3.539

$$1. \int_0^{\infty} \frac{p \sin qx - q \sin px}{x^2} dx = pq \ln \frac{p}{q} \quad [p > 0, q > 0]. \text{ F II 647}$$

$$2. \int_0^{\infty} (\cos qx - \cos px) \frac{dx}{x^3} = \frac{1}{2} (p - q) \pi. \text{ F II 645}$$

3.541

$$1. \int_0^{\infty} \frac{\sin px \sin qx}{x} dx = \ln \sqrt{\frac{p+q}{|p-q|}} \quad [p \neq q]. \text{ F II 647}$$

$$2. \int_0^{\infty} \sin qx \sin px \frac{dx}{x^2} = \frac{1}{2} p\pi \quad [p \leq q];$$

$$= \frac{1}{2} q\pi \quad [p > q]. \quad \text{BH [157] (1)}$$

$$3. \int_0^{\infty} \sin px \sin rx \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} e^{-pq} \operatorname{sh} rq \quad [0 < r \leq p]. \quad \text{BH [162] (1)}$$

$$4. \int_0^{\infty} \sin px \sin rx \frac{dx}{q^2 - x^2} = -\frac{\pi}{2q} \cos pq \sin qr \quad [p > r];$$

$$= -\frac{\pi}{4q} \sin 2pq \quad [p = r];$$

$$= -\frac{\pi}{2q} \sin pq \cos qr \quad [p < r]. \quad \text{BH [166] (1)}$$

3.542

$$1. \int_0^{\infty} \frac{\sin px \cos qx}{x} dx = \frac{\pi}{2} \quad [0 < q < p];$$

$$= \frac{\pi}{4} \quad [0 < p = q];$$

$$= 0 \quad [0 < p < q]. \quad \text{F II 645}$$

$$2. \int_0^{\infty} \sin px \cos rx \frac{dx}{q^2 + x^2} =$$

$$= -\frac{1}{4q} e^{-pq} \{e^{rq} \operatorname{Ei} [q(p-r)] + e^{-rq} \operatorname{Ei} [q(p+r)]\} -$$

$$- \frac{1}{4q} e^{pq} \{e^{rq} \operatorname{Ei} [-q(p+r)] + e^{-rq} \operatorname{Ei} [q(r-p)]\}. \quad \text{BH [162] (3)}$$

$$3. \int_0^{\infty} \sin px \cos rx \frac{x dx}{q^2 - x^2} = -\frac{\pi}{2} \cos pq \cos qr \quad [p > r];$$

$$= -\frac{\pi}{4} \cos 2pq \quad [p = r];$$

$$= \frac{\pi}{2} \sin pq \sin qr \quad [p < r]. \quad \text{BH [166] (2)}$$

$$4. \int_0^{\infty} \cos px \cos rx \frac{dx}{q^2 + x^2} = \frac{\pi}{q} e^{-pq} \operatorname{ch} rq \quad [0 < r \leq p]. \quad \text{BH [163] (1)}$$

$$5. \int_0^{\infty} \cos px \cos rx \frac{x dx}{q^2 + x^2} =$$

$$= \frac{1}{4} e^{pq} \{e^{rq} \operatorname{Ei} [-q(p+r)] + e^{-rq} \operatorname{Ei} [q(r-p)]\} -$$

$$- \frac{1}{4} e^{-pq} \{e^{rq} \operatorname{Ei} [q(p-r)] - e^{-rq} \operatorname{Ei} [q(p+r)]\} \quad [p \geq r];$$

$$= \infty \quad [p = r]. \quad \text{BH [163] (2)}$$

$$6. \int_0^{\infty} \cos px \cos rx \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \sin pq \cos qr \quad [p > r];$$

$$= \frac{\pi}{4q} \sin 2pq \quad [p = r];$$

$$= \frac{\pi}{2q} \cos pq \sin qr \quad [p < r]. \quad \text{BH [166] (3)}$$

$$7. \int_0^{\infty} \frac{1 - \cos px}{x} \cos qx \, dx = \ln \frac{\sqrt{p^2 - q^2}}{q} \quad [p \neq q]. \quad \text{F II 647}$$

3.543

$$1. \int_0^{\infty} \frac{\sin px \sin qx \sin rx}{x} \, dx = \frac{\pi}{4} \quad [p < q + r]$$

$$= \frac{\pi}{8} \quad [p = q + r] \quad \begin{cases} p > q > 0 \\ p > r > 0 \end{cases}$$

$$= 0 \quad [p > q + r] \quad \text{F II 645}$$

$$2. \int_0^{\infty} \sin^2 qx \sin^2 px \frac{dx}{x^2} = \frac{1}{4} p\pi \quad [p \leq q];$$

$$= \frac{1}{4} q\pi \quad [p > q]. \quad \text{BH [157] (3)}$$

$$3. \int_0^{\infty} \sin^2 qx \sin^2 px \frac{dx}{x^2} = \frac{1}{6} p^2 \pi (3q - p) \quad [p \leq q];$$

$$= \frac{1}{6} q^2 \pi (3p - q) \quad [p > q]. \quad \text{BH [157] (27)}$$

$$4. \int_0^{\infty} \frac{dx}{x^{a+1}} \prod_{k=1}^a \sin p_k x = \frac{\pi}{2} \prod_{k=1}^a p_k \quad \left[p_0 > \sum_{k=1}^a p_k, p_k > 0 (k=1, 2, \dots, a) \right].$$

F II 646

3.544

$$1. \int_0^{\infty} \frac{1}{\cos px} \frac{dx}{q^2 + x^2} = \int_0^{\infty} \frac{x}{\sin px} \frac{dx}{q^2 - x^2} = \int_0^{\infty} \frac{x}{\operatorname{tg} px} \frac{dx}{q^2 - x^2} = \infty.$$

BH [191] (1) BH [191] (16) BH [191] (17)

$$2. \int_0^{\infty} \operatorname{tg} px \frac{dx}{x} = \frac{\pi}{2}. \quad \text{Lo V}$$

$$3. \int_0^{\alpha} \frac{x \, dx}{\cos(\alpha - x) \cos(\alpha + x)} =$$

$$= \operatorname{cosec} 2\alpha \left\{ u \ln \frac{\cos(\alpha - u)}{\cos(\alpha + u)} - L(\alpha + u) - L(\alpha - u) + 2L(\alpha) \right\}$$

$[\alpha \neq \alpha\pi; |u| < \left| \frac{\pi}{2} - |\alpha_0| \right|]$, unde α_0 este valoarea argumentului α , redusă cu ajutorul perioadei π la intervalul $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)]$. Lo III 288

3.545

$$1. \int_0^{\infty} \frac{1}{1 - 2p \cos rx + p^2} \frac{dx}{q^2 + x^2} = \frac{\pi}{2q(1 - p^2)} \frac{1 + pe^{-qr}}{1 - pe^{-qr}} \quad [p^2 < 1]. \quad \text{BH [192] (1)}$$

$$2. \int_0^{\infty} \frac{1}{1 - 2p \cos rx + p^2} \frac{dx}{q^2 - x^2} = \frac{p\pi}{q(1 - p^2)} \frac{\sin qr}{1 - 2p \cos qr + p^2} \quad [p^2 > 1].$$

BH [193] (1)

$$3. \int_0^{\infty} \frac{\sin srx}{1 - 2p \cos rx + p^2} \frac{x \, dx}{q^2 + x^2} = \frac{\pi}{2} \frac{e^{-sqr} - p^s}{(1 - pe^{-qr})(1 - pe^{qr})} \quad [p^2 < 1]. \quad \text{BH [192] (8)}$$

$$4. \int_0^{\infty} \frac{\sin rx}{1-2p \cos rx + p^2} \frac{x dx}{q^2 + x^2} = \frac{\pi}{2p} \frac{1}{pe^{qr} - 1} \quad [p^2 > 1]. \quad \text{BH [192] (2)}$$

$$5. \int_0^{\infty} \frac{\sin srx}{1-2p \cos rx + p^2} \frac{x dx}{q^2 - x^2} = \frac{\pi}{2} \frac{p^s - \cos sqr}{1-2p \cos qr + p^2} \quad [p^2 < 1]. \quad \text{BH [193] (5)}$$

$$6. \int_0^{\infty} \frac{\cos srx}{1-2p \cos rx + p^2} \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \frac{1}{(1-pe^{-qr})(1-pe^{qr})} \left[e^{-sqr} - \frac{2p^{s+1}}{1-p^2} \operatorname{sh} qr \right] \\ [p^2 < 1]. \quad \text{BH [192] (9)}$$

$$7. \int_0^{\infty} \frac{\cos srx}{1-2p \cos rx + p^2} \frac{dx}{q^2 - x^2} = \frac{\pi}{2q(1-p^2)} \frac{(1-p^2) \sin sqr + 2p^{s+1} \sin qr}{1-2p \cos qr + p^2} \\ [p^2 < 1]. \quad \text{BH [193] (9)}$$

$$8. \int_0^{\infty} \frac{(1-r \cos \beta x) dx}{(1+x^2)(1-2r \cos \beta x + r^2)} = \frac{\pi}{2} \frac{e^{\beta}}{e^{\beta} - r} \quad [r^2 < 1]. \quad \text{F II 719}$$

3.546

$$1. \int_0^{\infty} \frac{\sin^{2a} px}{x} dx = \infty. \quad \text{BH [151] (3)}$$

$$2. \int_0^{\infty} \frac{\sin^{2a} px}{\sqrt{x}} dx = \infty. \quad \text{BH [177] (5)}$$

$$3. \int_0^{\infty} \sin^{2a+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2a}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^a (-1)^k \binom{2a+1}{a+k+1} \frac{1}{\sqrt{2k+1}}. \quad \text{BH [177] (7)}$$

3.547

$$1. \int_0^{\infty} \cos^s rx \sin srx \frac{dx}{x} = \frac{\pi}{2^{s+1}} (2^s - 1) \quad [s > -1]. \quad \text{BH [153] (1)}$$

$$2. \int_0^{\infty} \cos^s rx \sin qx \frac{dx}{x} = \frac{\pi}{2} \quad [q > rs; s > -1]. \quad \text{BH [153] (11)}$$

$$3. \int_0^{\infty} \cos^a sx \sin asx \frac{x dx}{q^2 + x^2} = \frac{\pi}{2^{a+1}} [(1 + e^{-2as})^a - 1]. \quad \text{BH [163] (9)}$$

$$4. \int_0^{\infty} \cos^a sx \cos asx \frac{dx}{q^2 + x^2} = \frac{\pi}{2^{a+1}q} (1 + e^{-2as})^a. \quad \text{BH [163] (16)}$$

$$5. \int_0^{\infty} \cos^a sx \cos asx \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \cos^a qs \sin aqs. \quad \text{BH [166] (11)}$$

$$6. \int_0^{\infty} \cos^a sx \sin asx \frac{x dx}{q^2 - x^2} = \frac{\pi}{2} (2^{-a} - \cos^a qs \cos aqs). \quad \text{BH [166] (10)}$$

3.548

$$1. \int_0^{\infty} \cos^s rx \sin qx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [q > rs, s > -1]. \quad \text{BH [153] (12)}$$

$$2. \int_0^{\infty} \cos^s rx \sin srx \cos x \frac{dx}{x} = \frac{\pi}{2^{s+1}} (2^s - 1) \quad [s > -1]. \quad \text{BH [153] (2)}$$

$$3. \int_0^{\infty} \cos^s rx \operatorname{tg} qx \frac{dx}{x} = \frac{\pi}{2} \quad [s > -1]. \quad \text{BH [151] (26)}$$

3.549

$$1. \int_0^{\infty} (\sin^{2a} qx - \sin^{2a} px) \frac{dx}{x} = \frac{(2a-1)!!}{2^a \cdot a!} \ln \frac{p}{q} \quad [p > 0; q > 0]. \quad \text{F II 651}$$

$$2. \int_0^{\infty} (\cos^{2a} px - \cos^{2a} qx) \frac{dx}{x} = \left(1 - \frac{(2a-1)!!}{2^a \cdot a!}\right) \ln \frac{q}{p} \quad [p > 0, q > 0]. \quad \text{F II 651}$$

$$3. \int_0^{\infty} (\cos^{2a+1} qx - \cos^{2a+1} px) \frac{dx}{x} = \ln \frac{p}{q} \quad [p > 0, q > 0]. \quad \text{F II 651}$$

$$4. \int_0^{\infty} (\cos^a px \cos apx - \cos^a qx \cos aqx) \frac{dx}{x} = \left(1 - \frac{a}{2^a}\right) \ln \frac{q}{p} \quad [pq > 0].$$

BH [155] (8)

$$5. \int_0^{\infty} \operatorname{tg}^r px \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \sec \frac{r\pi}{2} \operatorname{th}^r pq \quad [r^2 < 1]. \quad \text{BH [160] (19)}$$

$$6. \int_0^1 \frac{x dx}{\cos rx \cos [r(1-x)]} = \frac{1}{r} \operatorname{cosec} r \ln \sec r \quad \left[r < \frac{\pi}{2}\right]. \quad \text{BH [149] (20)}$$

$$3.551 \quad \int_0^1 \left(\cos qx - \cos \frac{q}{x}\right) \frac{dx}{1-x^2} = \frac{1}{2} \pi \sin q. \quad \text{BH [149] (11)}$$

3.552

$$1. \int_0^{\infty} \sin x \sin \frac{p^2}{x} \frac{dx}{x} = \frac{\pi}{2} N_0(2p) + K_0(2p). \quad \text{W 204 (3)}$$

$$2. \int_0^{\infty} \cos x \cos \frac{p^2}{x} \frac{dx}{x} = -\frac{\pi}{2} N_0(2p) + K_0(2p). \quad \text{W 204 (4)}$$

$$3.553 \quad \int_0^{\infty} (\cos x^p - \cos x^q) \frac{dx}{x} = \frac{p-q}{pq} C. \quad \text{BH [155] (13)}$$

3.554

$$1. \int_0^{\frac{\pi}{2}} \sin(q \operatorname{ctg} x) \frac{x}{\sin^2 x} dx = \frac{1-e^{-q}}{2q} \pi. \quad \text{Li [206] (14)}$$

$$2. \int_0^{\frac{\pi}{2}} \cos(q \operatorname{tg} x) \frac{x}{\sin 2x} dx = -\frac{\pi}{4} \operatorname{Ei}(-q). \quad \text{BH [206] (15)}$$

3.56 Funcții trigonometrice și funcția exponențială

3.561

$$1. \int_0^{\infty} \frac{e^{-px^2} dx}{1-2r \cos x + r^2} = \frac{1}{1-r^2} \left(\frac{1}{2} + \sum_{k=1}^{\infty} r^k e^{-\frac{k^2}{4p}} \right) \sqrt{\frac{\pi}{p}} \quad [p > 0]. \quad \text{BH [266] (1)}$$

$$2. \int_0^{\frac{\pi}{2}} e^{-\frac{\operatorname{cosec} x}{q}} \frac{\operatorname{ctg} x dx}{\sqrt{\sin x (1-\sin x)}} = \int_0^{\frac{\pi}{2}} e^{-\frac{\sec x}{q}} \frac{\operatorname{tg} x dx}{\sqrt{\cos x (1-\cos x)}} = \frac{\sqrt{q\pi}}{\sqrt{e}} \quad [q > 0]. \quad \text{BH [276] (7), BH [276] (8)}$$

3.562

$$1. \int_0^{\pi} e^{ax} \sin^b x dx = \frac{\pi}{2^b} \frac{e^{\frac{a\pi}{2}} \cdot b!}{\Gamma\left(\frac{a+bi}{2}+1\right) \Gamma\left(\frac{a-bi}{2}+1\right)}. \quad \text{BH [277] (1)}$$

$$2. \int_0^{\infty} e^{-px} \sin^{2a} x dx = \frac{(2a)!}{p(p^2+2^2)(p^2+4^2) \dots [p^2+(2a)^2]} \quad [p > 0]. \quad \text{F II 615}$$

$$3. \int_0^{\frac{\pi}{2}} e^{-px} \sin^{2a} x dx = \frac{(2a)!}{(2^2+p^2)(4^2+p^2) \dots [(2a)^2+p^2]} \times \\ \times \frac{1}{p} \left\{ 1 - e^{-\frac{p\pi}{2}} \left[1 + \frac{p^2}{2!} + \frac{p^2(2^2+p^2)}{4!} + \dots + \frac{p^2(2^2+p^2) \dots [(2a-2)^2+p^2]}{(2a)!} \right] \right\} \\ [p > 0]. \quad \text{BH [270] (4)}$$

$$4. \int_0^{\infty} e^{-px} \sin^{2a+1} x dx = \frac{(2a+1)!}{(p^2+1^2)(p^2+3^2) \dots [p^2+(2a+1)^2]} \quad [p > 0] \\ (\text{cf. 5.225 4.}). \quad \text{F II 615}$$

$$5. \int_0^{\frac{\pi}{2}} e^{-px} \sin^{2a+1} x dx = \frac{(2a+1)!}{(1^2+p^2)(3^2+p^2) \dots [(2a+1)^2+p^2]} \times \\ \times \left\{ 1 - pe^{-\frac{p\pi}{2}} \left[1 + \frac{1^2+p^2}{3!} + \dots + \frac{(1^2+p^2)(3^2+p^2) \dots [(2a-1)^2+p^2]}{(2a+1)!} \right] \right\} \\ [p > 0]. \quad \text{BH [270] (5)}$$

$$6. \int_0^{\infty} e^{-px} \cos^{2a} x dx = \frac{(2a)!}{p(p^2+2^2)(p^2+4^2) \dots [p^2+(2a)^2]} \times \\ \times \left\{ 1 + \frac{p^2}{2!} + \frac{p^2(p^2+2^2)}{4!} + \dots + \frac{p^2(p^2+2^2)(p^2+4^2) \dots [p^2+(2a-2)^2]}{(2a)!} \right\} \\ [p > 0]. \quad \text{BH [262] (3)}$$

$$7. \int_0^{2\pi} e^{-px} \cos^{2a} x \, dx = \frac{(2a)!}{(2^2 + p^2)(4^2 + p^2) \dots [(2a)^2 + p^2]} \times \\ \times \frac{1}{p} \left\{ -e^{-\frac{p\pi}{2}} + 1 + \frac{p^2}{2!} + \frac{p^2(2^2 + p^2)}{4!} + \dots + \frac{p^2(2^2 + p^2) \dots [(2a-2)^2 + p^2]}{(2a)!} \right\} \\ [p > 0]. \quad \text{BH [270] (6)}$$

$$8. \int_0^{\pi} e^{-px} \cos^{2a+1} x \, dx = \frac{p \cdot (2a+1)!}{(p^2 + 1^2)(p^2 + 3^2) \dots [p^2 + (2a+1)^2]} \times \\ \times \left\{ 1 + \frac{p^2 + 1^2}{3!} + \frac{(p^2 + 1^2)(p^2 + 3^2)}{5!} + \dots + \frac{(p^2 + 1^2)(p^2 + 3^2) \dots [p^2 + (2a-1)^2]}{(2a+1)!} \right\} \\ [p > 0]. \quad \text{BH [262] (4)}$$

$$9. \int_0^{2\pi} e^{-px} \cos^{2a+1} x \, dx = \frac{(2a+1)!}{(1^2 + p^2)(3^2 + p^2) \dots [(2a+1)^2 + p^2]} \times \\ \times \left\{ e^{-\frac{p\pi}{2}} + p \left[1 + \frac{1^2 + p^2}{3!} + \dots + \frac{(1^2 + p^2)(3^2 + p^2) \dots [(2a-1)^2 + p^2]}{(2a+1)!} \right] \right\} \\ [p > 0]. \quad \text{BH [270] (7)}$$

3.563

$$1. \int_0^{2\pi} \frac{e^{-p \operatorname{tg} x}}{\sin 2x \pm q \cos 2x \pm q} \, dx = -\frac{1}{2} e^{\pm pq} \operatorname{Ei}(\mp pq). \quad \text{BH [273] (11)}$$

$$2. \int_0^{2\pi} \frac{e^{-p \operatorname{ctg} x}}{\sin 2x \pm q \cos 2x \mp q} \, dx = -\frac{1}{2} e^{\mp pq} \operatorname{Ei}(\pm pq). \quad \text{BH [273] (12)}$$

$$3. \int_0^{2\pi} e^{-p^2 \operatorname{tg} x} \frac{\sin \frac{x}{2} \sqrt{\cos x}}{\sin 2x} \, dx = \left[C(x) - \frac{1}{2} \right]^2 + \left[S(x) - \frac{1}{2} \right]^2. \quad \text{NI 33 (18) m}$$

3.564

$$1. \int_0^{\frac{\pi}{q}} \frac{\sin px}{e^{qx} + 1} \, dx = \frac{1}{2p} - \frac{1}{2q} \frac{\pi}{\operatorname{sh} \frac{p\pi}{q}} \quad [q > p, q > 0]. \quad \text{BH [264] (1) F II 722 m}$$

$$2. \int_0^{\frac{\pi}{q}} \frac{\sin px}{e^{qx} - 1} \, dx = \frac{\pi}{2q} \operatorname{cth} \frac{p\pi}{q} - \frac{1}{2p} \quad [q > p, q > 0]. \quad \text{BH [264] (2) F II 722 m}$$

$$3. \int_0^{\frac{\pi}{2}} e^{-x^2} \sin 2px \, dx = \frac{\sqrt{\pi}}{2} e^{-p^2} \Phi(p) \quad [p > 0]. \quad \text{F II 720 F II 747}$$

$$4. \int_0^{\frac{\pi}{2}} e^{-px^2} \sin qx \, dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{q^k}{(2k)!} \frac{1}{p^k} \quad [p > 0]. \quad \text{BH [263] (1)}$$

$$5. \int_0^{\frac{\pi}{2}} e^{-px^2} \cos qx \, dx = \frac{1}{2} e^{-\frac{q^2}{4p}} \sqrt{\frac{\pi}{p}} \quad [p > 0, q > 0]. \quad \text{F II 720 m BH [23] (2)}$$

3.565

$$1. \int_0^{\infty} e^{-px} \sin(qx + \lambda) dx = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda) \quad [p > 0]. \quad \text{BH [261] (3)}$$

$$2. \int_0^{\infty} e^{-px} \cos(qx + \lambda) dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda) \quad [p > 0]. \quad \text{BH [261] (4)}$$

$$3. \int_{-\infty}^{\infty} e^{-q^2 x^2} \sin[p(x + \lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda. \quad \text{BH [269] (2)}$$

$$4. \int_{-\infty}^{\infty} e^{-q^2 x^2} \cos[p(x + \lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda. \quad \text{BH [269] (3)}$$

$$5. \int_0^{2\pi x} \frac{\cos^p x \sin(z - px)}{\sin^{2p+2} x} e^{-2x \cot x} dx = \frac{1}{2} \sqrt{\frac{\pi}{2z}} (2z)^{-p} \Gamma(p+1) J_{p+\frac{1}{2}}(z) \\ \left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}(p) > -1 \right]. \quad \text{WW II 183}$$

3.566

$$1. \int_0^{\infty} e^{-px^2} \sin qx^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{-p + \sqrt{p^2 + q^2}}{p^2 + q^2}} \quad [p > 0, q > 0]. \quad \text{F II 750}$$

$$2. \int_0^{\infty} e^{-px^2} \cos qx^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2 + q^2}}{p^2 + q^2}} \quad [p > 0, q > 0]. \quad \text{F II 750}$$

$$3. \int_0^{\infty} e^{-\frac{p^2}{x^2}} \sin(2q^2 x^2) dx = \frac{\sqrt{\pi}}{4q} e^{-2pq} (\sin 2pq + \cos 2pq). \quad \text{BH [268] (12)}$$

$$4. \int_0^{\infty} e^{-\frac{p^2}{x^2}} \cos(2q^2 x^2) dx = \frac{\sqrt{\pi}}{4q} e^{-2pq} (\cos 2pq - \sin 2pq). \quad \text{BH [268] (13)}$$

$$3.567 \quad \int_0^{\infty} [e^{-x} \cos(p\sqrt{x}) + pe^{-x^2} \sin px] dx = 1. \quad \text{Li [268] (3)}$$

3.568

$$1. \int_0^{2\pi} e^{-p \cos x} \sin(p \sin x) dx = \operatorname{Ei}(-p) - \operatorname{ci}(p). \quad \text{NI 13 (27)}$$

$$2. \int_0^{2\pi} e^{-p \cos x} \cos(p \sin x) dx = -\operatorname{si}(p). \quad \text{NI 13 (26)}$$

$$3. \int_0^{\pi} e^{p \cos x} \sin(p \sin x) \frac{dx}{\sin x} = \pi \operatorname{sh} p. \quad \text{BH [278] (1)}$$

$$4. \int_0^{\pi} e^{p \cos x} \cos(p \sin x) \frac{dx}{\cos x} = \infty. \quad \text{BH [278] (2)}$$

$$5. \int_0^{\frac{\pi}{2}} e^{p \cos 2x} \sin(p \sin 2x) \operatorname{tg} x \, dx = \frac{\pi}{2} (1 - e^{-p}). \quad \text{BH [271] (8)}$$

$$6. \int_0^{\frac{\pi}{2}} e^{q \cos 2x} \sin(q \sin 2x) \frac{dx}{\operatorname{tg} x} = \frac{\pi}{2} (e^q - 1). \quad \text{BH [272] (5)}$$

$$7. \int_0^{\pi} e^{p \cos x} \sin(p \sin x) \sin ax \, dx = \int_0^{\pi} e^{p \cos x} \cos(p \sin x) \cos ax \, dx = \\ = \frac{\pi}{2} \frac{p^a}{a!}. \quad \text{BH [277] (7) BH [277] (8)}$$

3.569

$$1. \int_0^{\pi} e^{r(\cos px + \cos qx)} \sin(r \sin px) \sin(r \sin qx) \, dx = \\ = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(pk+1) \Gamma(qk+1)} r^{(p+q)k}. \quad \text{BH [277] (14)}$$

$$2. \int_0^{\pi} e^{r(\cos px + \cos qx)} \cos(r \sin px) \cos(r \sin qx) \, dx = \\ = \frac{\pi}{2} \left(2 + \sum_{k=1}^{\infty} \frac{r^{(p+q)k}}{\Gamma(pk+1) \Gamma(qk+1)} \right). \quad \text{BH [277] (15)}$$

$$3. \int_0^{\pi} e^{q \cos x} \frac{\sin rx}{1 - 2p^r \cos rx + p^{2r}} \sin(q \sin x) \, dx = \frac{\pi}{2pr} \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \quad [p^2 < 1]. \\ \text{BH [278] (15)}$$

$$4. \int_0^{\pi} e^{q \cos x} \frac{1 - p^r \cos rx}{1 - 2p^r \cos rx + p^{2r}} \cos(q \sin x) \, dx = \frac{\pi}{2} \left[2 + \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \right] \\ [p^2 < 1]. \quad \text{BH [278] (16)}$$

3.57 Funcții trigonometrice, funcția exponențială și funcția putere

$$3.571 \int_0^{\infty} (1 - e^{-x}) \cos x \frac{dx}{x} = \ln \sqrt{2}. \quad \text{F II 745}$$

$$3.572 \int_0^{\infty} \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{\cos x}{x} \right) dx = \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C. \quad \text{NI 65 (8)}$$

3.573

$$1. \int_0^{\infty} x e^{-px} \sin qx \, dx = \frac{2pq}{(p^2 + q^2)^2} \quad [p > 0]. \quad \text{BH [361] (1)}$$

$$2. \int_0^{\infty} x^2 e^{-px} \sin qx \, dx = \frac{2q(3p^2 - q^2)}{(p^2 + q^2)^3} \quad [p > 0]. \quad \text{BH [361] (2)}$$

$$3. \int_0^{\infty} e^{-px} \sin qx \frac{dx}{x} = \arctg \frac{q}{p} \quad [p > 0]. \quad \text{BH [365] (1)}$$

$$4. \int_0^{\infty} x^{r-1} e^{-px} \sin qx dx = \frac{\Gamma(r)}{\sqrt{(p^2 + q^2)^r}} \sin \left(r \arctg \frac{q}{p} \right) \\ [p > 0, q \geq 0, \operatorname{Re}(r) > 0] \quad (\text{vezi și 5.229 4.}). \quad \text{F II 812 WW II 36}$$

3.574

$$1. \int_0^{\infty} \frac{e^{-qx} - e^{-rx}}{x} \sin px dx = \arctg \frac{(r-q)p}{p^2 + qr} \quad [p > 0, q > 0]. \quad \text{BH [367] (3)}$$

$$2. \int_0^{\infty} \frac{e^{-px} - e^{-qx}}{x^2} \sin rx dx = \frac{r}{2} \ln \frac{q^2 + r^2}{p^2 + r^2} + q \arctg \frac{r}{q} - p \arctg \frac{q}{p} \\ [p > 0, q > 0]. \quad \text{BH [368] (21)}$$

$$3. \int_0^{\infty} e^{-2x^2} x^{2a+1} \sin(2qx) dx = (-1)^a 2^{-2a-3} \sqrt{2\pi} e^{-\frac{q^2}{4}} D_{2a+1}(q). \quad \text{WW II 162}$$

$$4. \int_0^{\infty} x e^{-p^2 x^2} \sin qx dx = \frac{q \sqrt{\pi}}{4p^3} \exp \left(-\frac{q^2}{4p^2} \right). \quad \text{BH [362] (1)}$$

3.575

$$1. \int_0^{\infty} x e^{-px} \cos qx dx = \frac{p^2 - q^2}{(p^2 + q^2)^2} \quad [p > 0]. \quad \text{BH [361] (5)}$$

$$2. \int_0^{\infty} x^2 e^{-px} \cos qx dx = \frac{2p(p^2 - 3q^2)}{(p^2 + q^2)^3} \quad [p > 0]. \quad \text{BH [361] (6)}$$

$$3. \int_0^{\infty} x^{r-1} e^{-px} \cos qx dx = \frac{\Gamma(r)}{\sqrt{(p^2 + q^2)^r}} \cos \left(r \arctg \frac{q}{p} \right) \\ [p > 0, q \geq 0, \operatorname{Re}(r) > 0] \quad (\text{vezi și 5.229 5.}). \quad \text{F II 812 WW II 36}$$

$$4. \int_0^{\infty} \frac{e^{-qx} - e^{-rx}}{x} \cos px dx = \frac{1}{2} \ln \frac{p^2 + r^2}{p^2 + q^2} \quad [q > 0, r > 0]. \quad \text{BH [367] (4)}$$

$$5. \int_0^{\infty} e^{-2x^2} x^{2a} \cos 2qx dx = (-1)^a 2^{-2a-2} \sqrt{2\pi} e^{-\frac{q^2}{4}} D_{2a}(q). \quad \text{WW II 162}$$

$$6. \int_0^{\infty} x e^{-p^2 x^2} \cos qx dx = \frac{1}{2p^2} - \frac{q}{4p^3} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{q}{p} \right)^{2k+1} \quad \text{BH [362] (2)}$$

3.576

$$1. \int_0^{\infty} (1 - \cos qx) e^{-px} \frac{dx}{x} = \ln \frac{\sqrt{p^2 + q^2}}{p} \quad [p > 0]. \quad \text{F II 744}$$

$$2. \int_0^{\infty} \frac{\sin px - \sin qx}{x} e^{-rx} dx = \arctg \frac{(p-q)r}{pq + r^2} \quad [r > 0]. \quad \text{BH [367] (7)}$$

$$3. \int_0^{\infty} \frac{\cos qx - \cos rx}{x} e^{-rx} dx = \frac{1}{2} \ln \frac{q^2 + r^2}{p^2 + r^2} \quad [r > 0]. \quad \text{F II 748}$$

$$4. \int_0^{\infty} \frac{\cos qx - \cos rx}{x^2} e^{-px} dx = \frac{p}{2} \ln \frac{p^2 + q^2}{p^2 + r^2} + r \operatorname{arctg} \frac{r}{p} - q \operatorname{arctg} \frac{q}{p}$$

BH [368] (20)

[$p > 0, r > 0$].

3.577

$$1. \int_0^{\infty} e^{-px} \sin qx \sin rx \frac{dx}{x} = \frac{1}{4} \ln \frac{p^2 + (q+r)^2}{p^2 + (q-r)^2} \quad [p > 0]. \quad \text{BH [365] (5)}$$

$$2. \int_0^{\infty} e^{-px} \sin rx \cos qx \frac{dx}{x} = \frac{1}{2} \operatorname{arctg} \frac{2pr}{p^2 + q^2 - r^2} \quad [p > 0]. \quad \text{F II 744}$$

$$3. \int_0^{\infty} e^{-px} \frac{\sin rx}{x} \frac{\sin sx}{x} dx = \frac{r+s}{2} \operatorname{arctg} \frac{r+s}{p} - \frac{r-s}{2} \operatorname{arctg} \frac{r-s}{p} +$$

$$+ \frac{p}{4} \ln \frac{p^2 + (r-s)^2}{p^2 + (r+s)^2} \quad [p > 0, r > 0, s > 0]. \quad \text{F II 744}$$

$$4. \int_0^{\infty} x e^{-p^2 x^2} \operatorname{tg} qx dx - \frac{q}{p^2} \sqrt{\pi} \sum_{k=1}^{\infty} (-1)^k k e^{-\frac{k^2 q^2}{p^2}}$$

BH [362] (15)

$$3.578 \int_0^{\infty} e^{p \cos x} \sin (p \sin x) \frac{dx}{x} = \frac{\pi}{2} (e^p - 1). \quad \text{F II 725}$$

3.579

$$1. \int_0^{\infty} (1+x^2)^{-\frac{p}{2}} \sin (p \operatorname{arctg} x) \frac{dx}{e^{\pi x} + 1} = \frac{1}{2(p-1)} - 2^{-p} \zeta(p) \quad [p > 1]. \text{WW II 62}$$

$$2. \int_0^{\infty} (q^2 + x^2)^{-\frac{p}{2}} \sin \left(p \operatorname{arctg} \frac{x}{q} \right) \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \zeta(p, q) - \frac{q^{1-p}}{2(p-1)}$$

[$\operatorname{Re}(p) > 1, \operatorname{Re}(q) > 0$]. WW II 50

$$3. \int_0^{\infty} e^{-px^2} (1+x^2)^{q-\frac{1}{2}} \cos [2px + (2q-1) \operatorname{arctg} x] dx =$$

$$= \frac{1}{2} p^{-q} e^{-p} \sin \pi q \Gamma(q) \quad [p > 0, \operatorname{Re}(q) > 0]. \text{WW II 19}$$

3.58 Funcții trigonometrice și funcții hiperbolice

3.581

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{ch} [(r-p) \operatorname{tg} x]}{\operatorname{sh} (r \operatorname{tg} x)} \operatorname{tg} x dx = \frac{\pi}{2r} + \pi \sum_{k=1}^{\infty} \frac{1}{k\pi + r} \cos \frac{pk\pi}{r} \quad [p^2 \leq r^2].$$

BH [274] (14)

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{sh} (q \operatorname{tg} x)}{\operatorname{ch} (p \operatorname{tg} x)} \frac{dx}{\sin 2x} = \frac{1}{2} \ln \operatorname{tg} \left(\frac{p+q}{4p} \pi \right) \quad [p^2 > q^2]. \quad \text{BH [275] (12)}$$

3.582

$$1. \int_0^{\infty} \frac{\sin px}{\operatorname{sh} qx} dx = \frac{\pi}{2q} \operatorname{th} \frac{p\pi}{2q}. \quad \text{F III 650m BH [264] (6)}$$

$$2. \int_0^{\infty} \frac{\cos px}{\operatorname{ch} qx} dx = \frac{\pi}{2q} \frac{1}{\operatorname{ch} \frac{p\pi}{2q}}. \quad \text{BH [264] (14) m}$$

$$3. \int_0^{\infty} \frac{\cos px}{\operatorname{sh}^2 qx} dx = \frac{\pi p}{q^2 \operatorname{sh} \frac{p\pi}{2q}}. \quad \text{BH [3] (13) m}$$

$$4. \int_0^{\infty} \frac{\operatorname{sh} px}{\operatorname{ch} qx} \sin rx dx = \frac{\pi}{q} \frac{\operatorname{sh} \frac{r\pi}{2q} \sin \frac{p\pi}{2q}}{\operatorname{ch} \frac{r\pi}{q} + \cos \frac{p\pi}{q}} \quad [p^2 < q^2]. \quad \text{BH [265] (2)}$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} px}{\operatorname{sh} qx} \sin rx dx = \frac{\pi}{2q} \frac{\operatorname{sh} \frac{r\pi}{q}}{\operatorname{ch} \frac{r\pi}{q} + \cos \frac{p\pi}{q}} \quad [p^2 < q^2]. \quad \text{BH [265] (4)}$$

$$6. \int_0^{\infty} \frac{\operatorname{sh} px}{\operatorname{sh} qx} \cos rx dx = \frac{\pi}{2q} \frac{\sin \frac{p\pi}{q}}{\operatorname{ch} \frac{r\pi}{q} + \cos \frac{p\pi}{q}} \quad [p^2 < q^2]. \quad \text{BH [265] (7)}$$

$$7. \int_0^{\infty} \frac{\operatorname{ch} px}{\operatorname{ch} qx} \cos rx dx = \frac{\pi}{q} \frac{\operatorname{ch} \frac{r\pi}{2q} \cos \frac{p\pi}{2q}}{\operatorname{ch} \frac{r\pi}{q} + \cos \frac{p\pi}{q}} \quad [p^2 < q^2]. \quad \text{BH [265] (6)}$$

3.583

$$1. \int_0^{\infty} \frac{\operatorname{sh} rx \sin qx}{\operatorname{ch} x + \cos p} dx = \frac{\pi [\operatorname{sh} q (\pi + p) \sin r (\pi - p) - \operatorname{sh} q (\pi - p) \sin r (\pi + p)]}{\sin p (\operatorname{ch} 2q\pi - \cos 2r\pi)} \quad [r^2 < 1]. \quad \text{BH [267] (2)}$$

$$2. \int_0^{\infty} \frac{\operatorname{ch} rx \cos qx}{\operatorname{ch} x + \cos p} dx = \frac{\pi [\operatorname{ch} q (\pi + p) \cos r (\pi - p) - \operatorname{ch} q (\pi - p) \cos r (\pi + p)]}{\sin p (\operatorname{ch} 2q\pi - \cos 2r\pi)} \quad [r^2 < 1]. \quad \text{BH [267] (6)}$$

$$3. \int_0^{\infty} \frac{\operatorname{sh} px \sin qx}{\operatorname{ch} 2px + \cos 2qx} dx = \frac{\pi}{4} \frac{q}{p^2 + q^2}. \quad \text{BH [267] (7)}$$

$$4. \int_0^{\infty} \frac{\operatorname{ch} px \cos qx}{\operatorname{ch} 2px + \cos 2qx} dx = \frac{\pi}{4} \frac{p}{p^2 + q^2}. \quad \text{BH [267] (8)}$$

3.584

$$1. \int_0^{\frac{\pi}{2}} \frac{dx}{\operatorname{ch} (\operatorname{tg} x) \cos x \sqrt{\sin 2x}} = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}. \quad \text{BH [276] (13)}$$

$$2. \int_0^{\pi} \sin^p x \operatorname{ch} (z \cos x) dx = \sqrt{\pi} \left(\frac{z}{2} \right)^{-\frac{p}{2}} \Gamma \left(\frac{p+1}{2} \right) J_{\frac{p}{2}}(z) \quad [\operatorname{Re}(p) > -1]. \quad \text{WW II 188 m}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^q x}{\operatorname{ch}(\operatorname{tg} x) + \cos \lambda} \frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin \lambda} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\lambda}{k^q} \quad [q > 0]. \quad \text{BH [275] (20)}$$

3.585

$$1. \int_0^{\infty} \cos(p \operatorname{sh} x) \operatorname{ch} qx \, dx = \cos \frac{q\pi}{2} K_q(p) \\ [p > 0; -1 < \operatorname{Re}(q) < 1]. \quad \text{W 202 (13)}$$

$$2. \int_0^{\infty} \sin\left(p \operatorname{ch} x - \frac{q\pi}{2}\right) \operatorname{ch} qx \, dx = \frac{\pi}{2} J_q(p) \\ [p > 0; -1 < \operatorname{Re}(q) < 1]. \quad \text{W 199 (12)}$$

$$3. \int_0^{\infty} \cos\left(p \operatorname{ch} x - \frac{q\pi}{2}\right) \operatorname{ch} qx \, dx = -\frac{\pi}{2} N_q(p) \\ [p > 0; -1 < \operatorname{Re}(q) < 1]. \quad \text{W 199 (13)}$$

$$4. \int_0^{\frac{\pi}{2}} \cos(p \sin x) \operatorname{ch}(q \cos x) \, dx = \frac{\pi}{2} J_0(\sqrt{p^2 - q^2}). \quad \text{MO 40}$$

3.59 Funcții trigonometrice, funcții hiperbolice și funcții algebrice

$$1. \int_0^{\infty} \frac{\sin px}{\operatorname{ch} qx} \frac{dx}{x} = 2 \operatorname{arctg} \left(\exp \frac{p\pi}{2q} \right). \quad \text{BH [387] (1)}$$

$$2. \int_0^{\infty} \frac{1 - \cos px}{\operatorname{sh} qx} \frac{dx}{x} = \ln \left(\operatorname{ch} \frac{p\pi}{2q} \right). \quad \text{BH [387] (2)m}$$

$$3. \int_0^1 \frac{\cos px \operatorname{ch}(q \sqrt{1-x^2})}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} J_0(\sqrt{p^2 - q^2}). \quad \text{MO (40)}$$

3.6-3.7 FUNCȚIA LOGARITMICĂ

3.61 Funcția logaritmică

3.611

$$1. \int_0^u \frac{dx}{\ln x} = \operatorname{li} u. \quad \text{F III 653 F II 606}$$

$$2. \int_0^{\infty} \frac{dx}{\ln \frac{1}{x}} = -\infty. \quad \text{BH [33] (9)}$$

3.612

$$1. \int_0^1 \frac{dx}{q \pm \ln x} = \pm e^{\mp q} \operatorname{Ei}(\pm q). \quad \text{BH [31] (4) BH [31] (5)}$$

$$2. \int_0^1 \frac{dx}{(p + \ln x)^2} = e^{-p} \operatorname{Ei}(p) - \frac{1}{p}. \quad \text{La 280 (24) m}$$

$$3.613 \quad \int_0^1 \left(\ln \frac{1}{x} \right)^p dx = \Gamma(p+1) \quad [\operatorname{Re}(p) > -1]. \quad \text{F II 778}$$

3.614

$$1. \quad \int_0^1 \ln x \ln(1+x) dx = 2 - 2 \ln 2 - \frac{\pi^2}{12}. \quad \text{F II 684}$$

$$2. \quad \int_0^1 \ln x \ln(1-x) dx = 2 - \frac{\pi^2}{6}. \quad \text{BH [30] (7)}$$

$$3. \quad \int_0^1 \ln \frac{1-px}{1-p} \frac{dx}{\ln x} = - \sum_{k=1}^{\infty} \frac{\ln(1+k)}{k} p^k \quad [p < 1]. \quad \text{BH [31] (3)}$$

3.615

$$1. \quad \int_0^u \ln \sin x dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right). \quad \text{Lo III 186 (15)}$$

$$2. \quad \int_0^{\frac{\pi}{4}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 - \frac{1}{2} G. \quad \text{BH [285] (1)}$$

$$3. \quad \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{2} \ln 2. \quad \text{F II 629}$$

$$4. \quad \int_0^{\pi} \ln \sin x dx = -\pi \ln 2. \quad \text{F II 643}$$

$$5. \quad \int_0^{\frac{\pi}{2}} (\ln \sin x)^2 dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]. \quad \text{BH [305] (19)}$$

$$6. \quad \int_0^u \ln \cos x dx = -L(u). \quad \text{Lo III 184 (10)}$$

$$7. \quad \int_0^{\frac{\pi}{4}} \ln \cos x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} G. \quad \text{BH [286] (1)}$$

$$8. \quad \int_0^{\pi} \ln \cos^2 x dx = -2\pi \ln 2. \quad \text{BH [330] (22)}$$

3.616

$$1. \quad \int_0^{\frac{\pi}{2}} \ln(1 \pm \cos x) dx = -\frac{\pi}{2} \ln 2 \pm 2G. \quad \text{BH [308] (5) BH [308] (6)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + 2p \sin x + p^2) dx = \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1) \cdot (2k+1)!} \left(\frac{2p}{1+p^2}\right)^{2k+1} \quad [p \leq 1].$$

BH [308] (24)

$$3. \int_0^{\pi} \ln(1 - 2p \cos x + p^2) dx = 0 \quad [p^2 < 1];$$

$$= 2\pi \ln p \quad [p^2 > 1].$$

F II 142 F II 163 F II 688

$$4. \int_0^{\frac{\pi}{2}} \ln(p^2 - \sin^2 x) dx = -2\pi \ln 2 \quad [p^2 \leq 1];$$

$$= 2\pi \ln \frac{p + \sqrt{p^2 - 1}}{2} \quad [p > 1]. \quad \text{F II 644 F II 687}$$

$$5. \int_0^u \ln(1 - \sin^2 \alpha \sin^2 x) dx = (\pi - 2\theta) \ln \operatorname{ctg} \frac{\alpha}{2} + 2u \ln \left(\frac{1}{2} \sin \alpha\right) -$$

$$-\frac{\pi}{2} \ln 2 + L(\theta + u) - L(\theta - u) + L\left(\frac{\pi}{2} - 2u\right)$$

$$\left[\operatorname{ctg} \theta = \cos \alpha \operatorname{tg} u; \quad -\pi \leq \alpha \leq \pi, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \right]. \quad \text{Lo III 287}$$

$$6. \int_0^{\frac{\pi}{2}} \ln[1 - \cos^2 x (\sin^2 \alpha - \sin^2 \beta \sin^2 x)] dx =$$

$$= \pi \ln \left[\frac{1}{2} \left(\cos^2 \frac{\alpha}{2} + \sqrt{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}} \right) \right] \quad [\alpha > \beta > 0]. \quad \text{Lo III 283}$$

3.617

$$1. \int_0^u \ln \operatorname{tg} x dx = L(u) + L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right). \quad \text{Lo III 186 (16)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + \operatorname{tg} x) dx = \frac{\pi}{4} \ln 2 + G. \quad \text{BH [308] (9)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln(p \operatorname{tg} x) dx = \frac{\pi}{2} \ln p. \quad \text{BH [307] (2)}$$

3.618

$$1. \int_0^u \ln\left(1 - \frac{\sin^2 x}{\sin^2 \alpha}\right) dx = -u \ln \sin^2 \alpha - L\left(\frac{\pi}{2} - \alpha + u\right) + L\left(\frac{\pi}{2} - \alpha - u\right)$$

$$\left[-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad |\sin u| \leq |\sin \alpha| \right]. \quad \text{Lo III 287}$$

$$2. \int_0^{\frac{\pi}{2}} \ln \frac{1 + \sin \alpha \cos^2 x}{1 - \sin \alpha \cos^2 x} dx = \pi \ln \frac{1 + \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \pi \ln \operatorname{tg} \frac{\pi - \alpha}{4} \quad [\alpha > 0]. \quad \text{Lo III 283}$$

$$3. \int_0^{\frac{\pi}{2}} \ln(\sin \alpha \sin x + \sqrt{1 - \cos^2 \alpha \sin^2 x}) dx = \frac{\pi}{2} \ln 2 - 2L\left(\frac{\alpha}{2}\right) - 2L\left(\frac{\pi - \alpha}{2}\right).$$

Lo III 290

$$4. \int_0^u \ln(\cos x + \sqrt{\cos^2 x - \cos^2 \alpha}) dx = -\left(\frac{\pi}{2} - x - \varphi\right) \ln \cos \alpha + \\ + \frac{1}{2} L(u + \varphi) - \frac{1}{2} L(u - \varphi) - L(\varphi) \left[\cos \varphi = \frac{\sin u}{\sin \alpha}; 0 \leq u \leq \alpha \leq \frac{\pi}{2} \right]. \quad \text{Lo III 290}$$

$$5. \int_0^a \ln(\cos x + \sqrt{\cos^2 x - \cos^2 \alpha}) dx = -\left(\frac{\pi}{2} - a\right) \ln \cos \alpha. \quad \text{Lo III 285}$$

$$6. \int_0^{\beta} \ln \frac{\sin \beta + \sin \alpha \cos x \sqrt{\sin^2 \beta - \sin^2 x}}{\sin \beta - \sin \alpha \cos x \sqrt{\sin^2 \beta - \sin^2 x}} dx = \\ = \pi \ln \left\{ \operatorname{tg} \frac{\alpha}{2} \sin \beta + \sqrt{\operatorname{tg}^2 \frac{\alpha}{2} \sin^2 \beta + 1} \right\} \quad [\alpha > 0, \beta > 0]. \quad \text{Lo III 283}$$

3.619

$$1. \int_0^1 \ln \ln \frac{1}{x} dx = -C. \quad \text{F II 807}$$

$$2. \int_0^1 \left(\ln \frac{1}{x}\right)^{p-1} \ln \ln \frac{1}{x} dx = \phi(p) \Gamma(p) \quad [p > 0]. \quad \text{BH [30] (10)}$$

3.62-3.65 $\ln x$ și funcția putere

3.621

$$1. \int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12}. \quad \text{F II 483}$$

$$2. \int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6}. \quad \text{F II 715}$$

3.622

$$1. \int_0^1 \frac{\ln x}{1+x^2} dx = -G. \quad \text{F II 614 F II 482}$$

$$2. \int_0^{\infty} \frac{\ln x}{q^2 + x^2} dx = \frac{\pi}{2q} \ln q \quad [q > 0] \quad (\text{vezi și 3.661}).$$

$$3. \int_0^1 \frac{\ln x dx}{1-x^2} = -\frac{\pi^2}{8}. \quad \text{BH [108] (11)}$$

$$4. \int_0^1 \frac{x}{1-x^2} \ln x dx = -\frac{\pi^2}{24}.$$

$$5. \int_0^{\infty} \frac{\ln x}{(q+x)^2} dx = \frac{\ln q}{q} \quad [0 < q < 1]. \quad \text{BH [139] (1)}$$

$$6. \int_0^1 \frac{\ln x \, dx}{1+x+x^2} = -0,781\,302\,412\,9\dots \quad \text{Li [113] (1)}$$

$$7. \int_0^1 \frac{\ln x \, dx}{1-x+x^2} = -1,171\,953\,619\,35\dots \quad \text{Li [113] (3)}$$

$$8. \int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = -0,157\,660\,149\,15\dots \quad \text{Li [113] (2)}$$

$$9. \int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = -0,311\,821\,131\,9\dots \quad \text{Li [113] (4)}$$

3.623

$$1. \int_0^\infty \frac{\ln x}{(1+x^2)^2} \, dx = \ln 2. \quad \text{BH [144] (18)}$$

$$2. \int_0^\infty \frac{\ln x \, dx}{(q^2 + p^2 x^2)(1+x^2)} = \frac{p\pi}{2q(p^2 - q^2)} \ln \frac{q}{p} \quad [pq > 0]. \quad \text{BH [317] (16) } m$$

$$3. \int_0^\infty \frac{x^2 \ln x \, dx}{(q^2 + p^2 x^2)(1+x^2)} = \frac{q\pi}{2p(p^2 - q^2)} \ln \frac{p}{q} \quad [pq > 0]. \quad \text{BH [317] (15) } m$$

3.624

$$1. \int_0^\infty \frac{1+x}{(1-x)^2} \frac{\ln x}{\sqrt{x}} \, dx = 0. \quad \text{BH [142] (2)}$$

$$2. \int_0^\infty \frac{1-x}{(1+x)^2} \frac{\ln x}{\sqrt{x}} \, dx = -2\pi. \quad \text{BH [142] (1)}$$

$$3. \int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x \, dx}{\sqrt{1-x^2}} = -\frac{\pi}{4} \ln 2 - \frac{1}{2} G, \quad \text{BH [145] (1)}$$

$$4. \int_0^1 \frac{\ln x \, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2. \quad \text{F II 614 F II 643}$$

3.625

$$1. \int_1^\infty (x-1)^{p-1} \ln x \, dx = \frac{\pi}{p} \operatorname{cosec} \pi p \quad [-1 < p < 0]. \quad \text{BH [289] (12) } m$$

$$2. \int_0^\infty \frac{\ln x \, dx}{(q+x)^{p+1}} = \frac{1}{pq^p} (\ln q - C - \psi(p))$$

[Re(p) > 0, q ≠ 0, n - q nu este un număr natural]. NÍ 68 (7)

3.626

$$1. \int_0^1 \ln x \frac{x^{2a} \, dx}{1+x} = -\frac{1}{12} \pi^2 + \sum_{k=1}^{2a} \frac{(-1)^{k-1}}{k^2}. \quad \text{BH [108] (4)}$$

$$2. \int_0^1 \ln x \frac{x^{2a-1} \, dx}{1+x} = \frac{1}{12} \pi^2 + \sum_{k=1}^{2a-1} \frac{(-1)^k}{k^2}. \quad \text{BH [108] (5)}$$

3. $\int_0^1 \frac{x^{p-1} \ln x}{1-x} dx = -\zeta(2, p) \quad [p > 0].$ BH [108] (8)
4. $\int_0^\infty \frac{x^{p-1} \ln x}{x+q} dx = \pi q^{p-1} \operatorname{cosec} p\pi (\ln q - \pi \operatorname{ctg} p\pi) \quad [0 < p < 1].$ BH [135] (1)
5. $\int_0^\infty \frac{x^{p-1} \ln x}{1-x^2} dx = -\frac{\pi^2}{4} \operatorname{cosec}^2 \frac{p\pi}{2} \quad [0 < p < 1].$ (vezi și 3.629 2.).

3.627

1. $\int_0^1 x^{2a-1} \sqrt{1-x^2} \ln x dx = -\frac{2^{a-1} \cdot (a-1)!}{(2a+1)!} \left\{ \frac{1}{2a+1} - \ln 2 + \sum_{k=1}^{2a-1} \frac{(-1)^{k-1}}{k} \right\}.$ BH [117] (5)
2. $\int_0^1 x^{2a} \sqrt{1-x^2} \ln x dx = -\frac{(2a-1)!}{2^{a+1} \cdot (a+1)!} \left\{ \frac{1}{2a+2} + \ln 2 + \sum_{k=1}^{2a} \frac{(-1)^k}{k} \right\}.$ Li [117] (4)
3. $\int_0^1 \frac{x^{2a-1}}{\sqrt{1-x^2}} \ln x dx = \frac{2^{a-1} \cdot (a-1)!}{(2a-1)!} \left\{ \ln 2 + \sum_{k=1}^{2a-1} \frac{(-1)^k}{k} \right\}.$ BH [118] (6) m
4. $\int_0^1 \frac{x^{2a}}{\sqrt{1-x^2}} \ln x dx = \frac{(2a-1)!}{2^a \cdot a!} \frac{\pi}{2} \left\{ -\ln 2 - \sum_{k=1}^{2a} \frac{(-1)^k}{k} \right\}.$ BH [118] (5) m

3.628

1. $\int_0^1 x^{p-1} (1-x)^{q-1} \ln \frac{1}{x} dx = B(p, q) [\psi(p+q) - \psi(p)] \quad [p > 0, q > 0].$ BH [107] (5)
2. $\int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}} \ln x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi \quad [0 < p < 1].$ BH [319] (10) m

3.629

1. $\int_0^\infty \ln x \frac{x^{p-1} dx}{1+x^q} = -\left(\frac{\pi}{q}\right)^2 \cos \frac{p\pi}{q} \operatorname{cosec}^2 \frac{p\pi}{q} \quad [0 < p < q].$ BH [135] (7)
2. $\int_0^\infty \ln x \frac{x^{p-1}}{1-x^q} dx = -\left(\frac{\pi}{q}\right)^2 \operatorname{cosec}^2 \frac{p\pi}{q} \quad [q > p > 0].$ BH [135] (8)
3. $\int_0^1 \ln x \frac{1-x^2}{1+x^{2p}} x^{p-2} dx = -\left(\frac{\pi}{2p}\right)^2 \sin \frac{\pi}{2p} \sec^2 \frac{\pi}{2p} \quad [p > 1].$ BH [108] (13)
4. $\int_0^1 \ln x \frac{1+x^2}{1-x^{2p}} x^{p-2} dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2 \frac{\pi}{2p} \quad [p > 1].$ BH [108] (14)
5. $\int_0^\infty \ln x \frac{dx}{(q^2 + r^2 x^2)^p} = \frac{\sqrt{\pi} \Gamma\left(p - \frac{1}{2}\right)}{4q^{2p-1} r \Gamma(p)} \left[2 \ln \frac{q}{2r} - C - \psi\left(p - \frac{1}{2}\right) \right]$
 $[p > 0, q > 0, r > 0].$ BH [139] (3)

$$3.631 \quad \int_0^1 \frac{(\ln x)^2}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \left\{ (\ln 2)^2 + \frac{\pi^2}{12} \right\}. \quad \text{BH [118] (13)}$$

3.632

$$1. \quad \int_0^1 \frac{1-x}{1+x} \frac{1}{1+x^2} \frac{dx}{\ln x} = -\frac{1}{2} \ln 2. \quad \text{BH [130] (16)}$$

$$2. \quad \int_0^1 \frac{1-x}{1+x} \frac{x^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi}. \quad \text{BH [130] (17)}$$

$$3.633 \quad \int_0^1 (1-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \ln(1+k) \quad [p \geq 1]. \quad \text{BH [123] (2)}$$

3.634

$$1. \quad \int_0^1 \left\{ \frac{x^{q-1}}{x-1} - \frac{1}{\ln x} \right\} dx = \psi(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 21}$$

$$2. \quad \int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = \ln p - \psi(q) \quad [p > 0, q > 0]. \quad \text{BH [127] (17)}$$

$$3.635 \quad \int_0^1 \left\{ \frac{x^q - x}{x-1} - x(q-1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{WW II 38}$$

3.636

$$1. \quad \int_0^1 (x^{p-1} - x^{q-1}) \frac{dx}{\ln x} = \ln \frac{p}{q} \quad [p > 0, q > 0]. \quad \text{F II 647}$$

$$2. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{1+x} \frac{dx}{\ln x} = \ln \frac{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q+1}{2}\right)} \quad [p > 0, q > 0] \quad \text{F II 816}$$

$$3. \quad \int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \ln \operatorname{tg} \frac{p\pi}{2} \quad [0 < p < 1]. \quad \text{F II 816}$$

3.637

$$1. \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{1+x^r} \frac{dx}{\ln x} = \ln \left(\operatorname{tg} \frac{p\pi}{2r} \operatorname{ctg} \frac{q\pi}{2r} \right) \quad [r > p > 0, r > q]. \quad \text{BH [143] (2)}$$

$$2. \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{1-x^r} \frac{dx}{\ln x} = \ln \left(\sin \frac{p\pi}{r} \operatorname{cosec} \frac{q\pi}{r} \right) \quad [r > p > 0, r > q]. \quad \text{BH [143] (4)}$$

$$3. \quad \int_0^1 \frac{1-x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1} dx}{\ln x} = \ln \operatorname{tg} \frac{q\pi}{4p} \quad [0 < q < p] \quad (\text{vezi și } 3.326 \text{ 1.}).$$

BH [128] (6)

3.638

$$1. \quad \int_0^1 (x^p - 1)^a x^{q-1} \frac{dx}{\ln x} = \sum_{k=0}^a (-1)^k \binom{a}{k} \ln [q + (a-k)p] \quad [p > 0, q > 0].$$

BH [123] (12)

$$2. \int_0^1 \frac{(1-x^p)^a}{1-x} \frac{x^{q-1} dx}{\ln x} = \sum_{k=0}^a (-1)^{k-1} \ln \Gamma[(a-k)p+q] \quad [p > 0, q > 0].$$

BH [127] (13)

3.639

$$1. \int_0^1 \frac{x^\alpha (1-x^\beta) (1-x^\gamma)}{(1-x) \ln x} dx = \ln \frac{\Gamma(\alpha+\gamma+1) \Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1) \Gamma(\alpha+\beta+\gamma+1)}$$

$$[\alpha > -1, \alpha+\beta > -1, \alpha+\gamma > -1, \alpha+\beta+\gamma > -1]. \quad \text{F II 815}$$

$$2. \int_0^1 \frac{(1-x^\alpha) (1-x^\beta) (1-x^\gamma)}{(1-x) \ln x} dx = \ln \frac{\Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(\gamma+1) \Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+1) \Gamma(\alpha+\gamma+1) \Gamma(\beta+\gamma+1)}$$

$$[\alpha > -1, \beta > -1, \gamma > -1, \alpha+\beta > -1, \alpha+\gamma > -1, \beta+\gamma > -1, \alpha+\beta+\gamma > -1].$$

F II 815

$$3.641 \quad \int_1^{\infty} \frac{dx}{x^2 (\ln p - \ln x)} = \frac{1}{p} \operatorname{li}(p). \quad \text{La 281 (30)}$$

$$3.642 \quad \int_1^{\infty} \frac{dx}{x^{p+1} (q \pm \ln x)} = \mp e^{\pm pq} \operatorname{Ei}(\mp pq) \quad (\text{vezi și 6.212 7}).$$

Li [144] (11) Li [144] (12)

$$3.643 \quad \int_0^1 \left\{ x^q - \frac{1}{1-p \ln x} \right\} \frac{dx}{x \ln x} = \ln \frac{p}{q} + C \quad [pq > 0]. \quad \text{Li [126] (8)}$$

$$3.644 \quad \int_0^1 \frac{1}{q^2 + \ln^2 x} \frac{dx}{1+x^2} = \frac{1}{2q} \beta\left(\frac{2q+\pi}{2\pi}\right) \quad \left[q > -\frac{\pi}{2}\right]. \quad \text{BH [129] (9)}$$

$$3.645 \quad \int_0^1 \frac{x}{1-x^2} \frac{\ln x dx}{q^2 + \ln^2 x} = \frac{1}{2} \left[\ln \frac{\pi}{q} + \frac{\pi}{2q} + \psi\left(\frac{q}{\pi}\right) \right] \quad [q > 0]. \quad \text{BH [129] (14)}$$

$$3.646 \quad \int_0^1 \frac{x^p - x^{-p}}{x^2 - 1} \frac{dx}{q^2 + \ln^2 x} = \frac{2\pi}{q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kp\pi}{2q+k\pi} \quad [p^2 < 1]. \quad \text{BH [132] (13)}m$$

$$3.647 \quad \int_0^1 \frac{x^{p-1}}{\sqrt{\ln \frac{1}{x}}} dx = \int_1^{\infty} \frac{dx}{x^{p+1} \sqrt{\ln x}} = \sqrt{\frac{\pi}{p}} \quad [p > 0].$$

BH [144] (10) BH [133] (1)

$$3.648 \quad \int_0^{\frac{1}{e}} \frac{\sqrt{x} dx}{x \sqrt{\ln \frac{1}{x} - 1}} = \frac{\sqrt{q\pi}}{\sqrt{e}} \quad [q > 0]. \quad \text{BH [145] (4)}$$

$$3.649 \quad \int_p^q \frac{dx}{x \sqrt{\ln \frac{x}{p} \ln \frac{q}{x}}} = \pi \quad [pq > 0]. \quad \text{BH [145] (37)}$$

3.651

$$1. \int_0^1 \frac{(\ln x)^{2a-1}}{1-x} dx = -\frac{1}{a} 2^{2a-2} \pi^{2a} |B_{2a}|. \quad \text{BH [110] (5)}$$

$$2. \int_0^{\infty} (\ln x)^{2a+1} \frac{dx}{1+x^2} = 0. \quad \text{BH [135] (2)}$$

$$3. \int_0^{\infty} (\ln x)^{2a} \frac{dx}{1+x^2} = 2 \cdot (2a)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2a+1}}. \quad \text{BH [135] (3)}$$

$$4. \int_0^{\infty} (\ln x)^{2a} \frac{dx}{1-x^2} = 0. \quad \text{BH [312] (7)m}$$

$$5. \int_0^{\infty} (\ln x)^{2a-1} \frac{dx}{1-x^2} = \frac{1-2^{2a}}{2a} \pi^{2a} |B_{2a}|. \quad \text{BH [312] (6)m}$$

$$6. \int_0^1 (\ln x)^{2a} \frac{dx}{1-x^2} = \frac{2^{2a+1}-1}{2^{a+1}} (2a)! \zeta(2a+1). \quad \text{BH [110] (12)}$$

$$7. \int_0^1 (\ln x)^{2a-1} \frac{dx}{1-x^2} = \frac{1-2^{2a}}{4a} \pi^{2a} |B_{2a}|. \quad \text{BH [290] (17)m}$$

$$8. \int_0^1 (\ln x)^{2a-1} \frac{x dx}{1-x^2} = -\frac{1}{4a} \pi^{2a} |B_{2a}|. \quad \text{BH [290] (19)m}$$

$$9. \int_0^1 (\ln x)^{2a} \frac{1+x^2}{(1-x^2)^2} dx = \frac{2^{2a}-1}{2} \pi^{2a} |B_{2a}|. \quad \text{BH [296] (17)m}$$

3.652

$$1. \int_0^1 (\ln x)^{2a} \frac{dx}{1+x^2-2x \cos 2p\pi} = \frac{(2a)!}{\sin 2p\pi} \sum_{k=1}^{\infty} \frac{\sin 2k p \pi}{k^{2a+1}}. \quad \text{BH [113] (9)}$$

$$2. \int_0^1 \left(\ln \frac{1}{x}\right)^{r-1} \frac{\cos \lambda - px}{1+p^2 x^2 - 2px \cos \lambda} x^{q-1} dx = \Gamma(r) \sum_{k=1}^{\infty} \frac{p^{k-1} \cos k \lambda}{(q+k-1)^r} \\ [r > 0, q > 0]. \quad \text{BH [113] (11)}$$

3.653

$$1. \int_1^{\infty} (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p) \quad [p > -1]. \quad \text{BH [144] (1)}$$

$$2. \int_0^1 x^{p-1} \left(\ln \frac{1}{x}\right)^q dx = \frac{1}{p^{q+1}} \Gamma(q+1) \quad [p > 0, q > -1]. \\ \text{F II 743 m, Li [124] (17)m}$$

$$3. \int_0^1 \left(\ln \frac{1}{x}\right)^a \frac{x^p}{1+x} dx = a! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(p+k)^{a+1}} \quad [p > -1]. \quad \text{BH [110] (4)}$$

$$4. \int_0^1 \left(\ln \frac{1}{x}\right)^a \frac{x^p}{1-x} dx = a! \zeta(a+1, p+1) \quad [p > -1]. \quad \text{BH [110] (7)}$$

3.654

$$1. \int_0^1 \left(\ln \frac{1}{x}\right)^a \frac{x^q}{1-x^2} dx = a! \sum_{k=0}^{\infty} \frac{1}{(2k+q+1)^{a+1}} = \frac{a!}{2^{a+1}} \zeta\left(a+1, \frac{q+1}{2}\right) \\ [q > -1]. \quad \text{BH [310] (11)}$$

$$2. \int_0^1 \frac{x^q - x^{-q}}{1-x^2} \left(\ln \frac{1}{x}\right)^p dx = \Gamma(p+1) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}} \right\} \\ [p > -1, q^2 < 1]. \quad \text{Li [326] (12)m}$$

3.655

$$1. \int_0^1 \left(\ln \frac{1}{x}\right)^b (1+x^q)^a x^{p-1} dx = b! \sum_{k=0}^a \binom{a}{k} \frac{1}{(p+kq)^{b+1}} \quad [p > 0]. \\ \text{BH [107] (6)}$$

$$2. \int_0^1 \left(\ln \frac{1}{x}\right)^h (1-x^b)^{\frac{c}{b}-1} x^{q-1} dx = \\ = h! \left\{ \frac{1}{q^{h+1}} + \sum_{k=1}^{\infty} \frac{(kb-c) [(k-1)b-c] \dots (b-c)}{k! b^k (q+bk)^{h+1}} \right\} \quad \left[q > 0; h + \frac{c}{b} > 0 \right]. \\ \text{BH [118] (14)}$$

$$3. \int_0^1 \left(\ln \frac{1}{x}\right)^a \frac{x^{p-1}}{1+x^q} dx = a! \sum_{k=0}^{\infty} \frac{(-1)^k}{(p+kq)^{a+1}} \quad [p > 0]. \quad \text{BH [110] (14)}$$

$$4. \int_0^1 \left(\ln \frac{1}{x}\right)^a \frac{x^{p-1}}{1-x^q} dx = \frac{a!}{q^{a+1}} \zeta\left(a+1, \frac{p}{q}\right) \quad [p > 0]. \quad \text{BH [110] (15)}$$

$$5. \int_0^1 \left(\ln \frac{1}{x}\right)^{p-1} \frac{x^{r-1}}{1-qx^r} dx = \frac{1}{q^{r^p}} \Gamma(p) \sum_{k=1}^{\infty} \frac{q^k}{k^p} \quad [r > 0, q < 1; p > 0]. \\ \text{Li [110] (8)}$$

$$3.656 \int_0^1 \left[\left(\ln \frac{1}{x}\right)^{q-1} - x^{p-1} (1-x)^{q-1} \right] dx = \\ = \frac{\Gamma(q)}{\Gamma(p+q)} [\Gamma(p+q) - \Gamma(p)] \quad [p > 0, q > 0]. \quad \text{BH [107] (8)}$$

$$3.657 \int_0^1 x^{p-1} \frac{dx}{(q+\ln x)^a} = \frac{p^{a-1}}{(a-1)!} e^{-pq} \text{Ei}(pq) - \\ - \frac{1}{(a-1)! q^{a-1}} \sum_{k=1}^{a-1} (a-k-1)! (pq)^{k-1} \quad [p > 0, q > 0]. \quad \text{BH [125] (21)}$$

$$3.658 \int_0^1 \left[x - \frac{1}{(1-\ln x)^q} \right] \frac{dx}{x \ln x} = -\psi(q) \quad [q > 0]. \quad \text{BH [126] (5)}$$

3.66-3.67 Logaritmul unui polinom și funcția putere

3.661

$$1. \int_0^1 \ln(1+x) \frac{dx}{x} = \frac{1}{12} \pi^2. \quad \text{F II 483}$$

$$2. \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2. \quad \text{F II 158}$$

$$3. \int_0^{\infty} \ln(1+x) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 + G. \quad \text{BH [136] (1)}$$

$$4. \int_0^1 \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2. \quad \text{BH [114] (4)}$$

$$5. \int_0^\infty \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{6}. \quad \text{BH [141] (9)m}$$

$$6. \int_0^1 \ln(1+x) \frac{1+x^2}{q^2+x^2} \frac{dx}{1+q^2x^2} = \\ = \frac{1}{2q(1+q^2)} \left[\frac{\pi}{2} \ln(1+q^2) - 2 \operatorname{arctg} q \cdot \ln q \right] \quad [q > 0]. \quad \text{Li [114] (11)}$$

$$7. \int_0^1 \frac{\ln(1+x)}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2 + 2G. \quad \text{BH [120] (1)}$$

3.662

$$1. \int_0^1 \ln\left(\frac{1}{2}+x\right) x^{q-1} dx = \frac{1}{q} \left[\ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{q+k+1} \right] \quad [q > 0]. \quad \text{BH [106] (4)}$$

$$2. \int_0^\infty \ln(1+x) x^{q-1} dx = \frac{\pi}{q} \operatorname{cosec} q\pi \quad [-1 < q < 0] \quad (\text{cf. 3.625 1.}).$$

$$3. \int_0^1 [\ln(1+x)]^a (1+x)^r dx = (-1)^{a-1} \frac{a!}{(r+1)^{a+1}} + \\ + 2^{r+1} \sum_{k=0}^a \frac{(-1)^k a! (\ln 2)^{a-k}}{(a-k)! (r+1)^{k+1}}. \quad \text{Li [106] (34)m}$$

3.663

$$1. \int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}. \quad \text{F II 714}$$

$$2. \int_0^{\frac{1}{2}} \ln(1-x) \frac{dx}{x} = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}. \quad \text{BH [145] (2)}$$

$$3. \int_0^1 \frac{\ln(1-x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - G. \quad \text{BH [114] (17)}$$

$$4. \int_1^\infty \frac{\ln(x-1)}{1+x^2} dx = \frac{\pi}{8} \ln 2 \quad (\text{cf. 3.667 1.}).$$

$$5. \int_0^1 \frac{\ln(1-x)}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2 - 2G. \quad \text{BH [120] (3)}$$

3.664

$$1. \int_0^1 \ln(1-x) x^{a-1} dx = -\frac{1}{a} \sum_{k=1}^a \frac{1}{k}. \quad \text{BH [106] (7)}$$

$$2. \int_0^\infty \ln(1-x^2) x^p dx = \frac{2\pi}{p+1} \operatorname{ctg} p\pi \quad [-3 < p < -1]. \quad \text{BH [134] (13)m}$$

$$3. \int_0^1 [\ln(1-x)]^a (1-x)^r dx = (-1)^a \frac{a!}{(r+1)^{a+1}} \quad [r > -1]. \quad \text{BH [106] (35)m}$$

3.665

$$1. \int_0^1 \ln(1+x^2) x^{q-1} dx = \frac{1}{q} \left[\ln 2 - 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{q+2k+2} \right] \quad [q > -2].$$

BH [106] (12)

$$2. \int_0^{\frac{\pi}{2}} \ln(1+x^2) x^{q-1} dx = \frac{\pi}{q} \operatorname{cosec} \frac{q\pi}{2} \quad [-2 < q < 0]. \quad \text{BH [311] (4)m}$$

$$3. \int_0^{\frac{\pi}{2}} \ln(1+x^2) \frac{dx}{(x+p)^2} = \frac{2p}{1+p^2} \left(\frac{\pi}{2p} + \ln p \right) \quad [p > 0]. \quad \text{BH [319] (6)m}$$

$$4. \int_0^1 \frac{\ln(1-x^2)}{x} dx = -\frac{\pi^2}{12}.$$

$$5. \int_0^1 \ln(1-x^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} \cdot K(k) - \frac{\pi}{2} K(k'). \quad \text{BH [120] (12)}$$

$$6. \int_0^1 \left(\ln \frac{1}{1-x^2} \right)^a x^{2q-1} dx = \frac{a!}{2} \zeta[a+1, q+1] \quad [-1 < q < 0].$$

BH [311] (15)m

$$7. \int_0^1 \ln(1-x^2) [px^{p-1} - qx^{q-1}] dx =$$

$$= \psi\left(\frac{1}{2}p+1\right) - \psi\left(\frac{1}{2}q+1\right) \quad [p > -2, q > -2]. \quad \text{BH [106] (15)}$$

$$3.666 \quad \int_0^1 (\ln x)^{2a} \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^{2a+2}}{2(a+1)(2a+1)} |B_{2a+2}|. \quad \text{BH [309] (5)m}$$

3.667

$$1. \int_0^1 \ln \frac{1-x}{x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2. \quad \text{BH [115] (5)}$$

$$2. \int_0^1 \ln \frac{1+x}{1-x} \frac{dx}{1+x^2} = G. \quad \text{BH [115] (17)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln \left(\frac{1+x}{1-x} \right)^2 \frac{dx}{x(1+x^2)} = \frac{\pi^2}{2}. \quad \text{BH [141] (13)}$$

3.668

$$1. \int_0^1 \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2. \quad \text{BH [115] (7)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \pi \ln 2. \quad \text{BH [137] (8)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln \frac{1+x^2}{x} \frac{dx}{1-x^2} = 0. \quad \text{BH [137] (9)}$$

$$4. \int_0^1 \ln \frac{1+x^2}{x} x^{2a} dx = \frac{1}{2a+1} \left\{ (-1)^a \frac{\pi}{2} + \ln 2 - \frac{1}{2a+1} + 2 \sum_{k=0}^{a-1} \frac{(-1)^k}{2a-2k-1} \right\}. \quad \text{BH [294] (8)}$$

$$5. \int_0^1 \ln \frac{1+x^2}{x} x^{2a-1} dx = \frac{1}{2a} \left\{ (-1)^{a+1} \ln 2 + \ln 2 - \frac{1}{2a} + (-1)^{a+1} \sum_{k=1}^{a-1} \frac{(-1)^k}{k} \right\}. \quad \text{BH [294] (9)m}$$

$$6. \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2a-1} dx}{1 \pm x} = \frac{1}{2a} \ln 2 + \frac{1}{4a^2} - \frac{1}{2a} \sum_{k=0}^\infty \frac{(-1)^k}{2a+k+1}. \quad \text{BH [137] (1) BH [137] (2)}$$

$$7. \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2a} dx}{1 \pm x} = \pm \frac{1}{4a^2} \left(2a \ln 2 + 1 + 2a \sum_{k=0}^\infty \frac{(-1)^{k-1}}{2a+k+1} \right). \quad \text{BH [137] (3) BH [137] (4)}$$

$$8. \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2a-1} dx}{1 \pm x^2} = \frac{1}{2a} \ln 2 + \frac{1}{4a^2} + \frac{1}{2a} \sum_{k=0}^\infty \frac{(-1)^{k-1}}{2a+k+1}. \quad \text{BH [137] (10) BH [137] (11)}$$

$$9. \int_0^1 \ln \frac{1-x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2. \quad \text{BH [115] (9)}$$

$$10. \int_0^1 \sqrt{1-x^2} \ln \left(\frac{1-x^2}{x^2} \right) dx = \pi. \quad \text{F II 643 m}$$

$$3.669 \int_0^1 \ln \frac{1+x^2}{1+x} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 - G. \quad \text{BH [115] (18)}$$

$$3.671 \int_0^\infty \frac{\ln px}{q^2+x^2} dx = \frac{\pi}{2q} \ln pq \quad [p > 0, q > 0]. \quad \text{BH [135] (4)}$$

3.672

$$1. \int_0^1 \ln(p+x) \frac{dx}{p+x^2} = \frac{1}{2\sqrt{p}} \operatorname{arctg}(\sqrt{p}) \ln[(1+p)p] \quad [p > 0]. \quad \text{BH [114] (20)}$$

$$2. \int_0^1 \ln(1+px) \frac{dx}{1+px^2} = \frac{1}{2\sqrt{p}} \operatorname{arctg}(\sqrt{p}) \ln(1+p) \quad [p > 0]. \quad \text{BH [114] (21)}$$

$$3. \int_0^\infty \frac{\ln(px+q)}{(1+x)^2} dx = \frac{1}{p-q} (p \ln p - q \ln q) \quad [p > 0, q > 0]. \quad \text{BH [139] (8)}$$

$$4. \int_0^{\infty} x^p \ln(1+qx) dx = -\frac{\pi}{q^{p+1}(p+1)} \operatorname{cosec} p\pi \quad [-2 < p < -1, q > 0].$$

BH [134] (3)

$$5. \int_0^{\infty} \frac{q \ln(1+px) - p \ln(1+qx)}{x^2} dx = pq \ln \frac{q}{p} \quad \{p > 0, q > 0\}. \quad \text{F II 647}$$

$$6. \int_0^1 \left(\ln \frac{1}{x}\right)^p \ln(1-qx) \frac{dx}{x} = -\Gamma(p+1) \sum_{k=1}^{\infty} \frac{q^k}{k^{p+2}} \quad [p > -1, q < 1].$$

BH [116] (7)m

3.673

$$1. \int_0^{\infty} \frac{\ln(p^2+x^2)}{q^2+x^2} dx = \frac{\pi}{q} \ln(p+q) \quad [p+q > 0, q > 0]. \quad \text{BH [136] (13)}$$

$$2. \int_0^1 \ln(\cos^2 \lambda - x^2 \sin^2 \lambda) \frac{dx}{1-x^2} = -\lambda^2. \quad \text{BH [114] (27)}$$

$$3. \int_0^{\infty} \frac{\ln(1+p^2x^2)}{r^2+q^2x^2} dx = \int_0^{\infty} \frac{\ln(p^2+x^2)}{q^2+r^2x^2} dx = \frac{\pi}{qr} \ln \frac{q+pr}{q} \quad [qr > 0, p > 0].$$

F II 745 m BH [318] (1)m BH [318] (4)m

$$4. \int_0^{\infty} \frac{\ln(p^2+x^2)}{q^2-r^2x^2} dx = -\frac{\pi}{qr} \operatorname{arctg} \frac{q}{pr} \quad [p > 0, qr > 0]. \quad \text{BH [136] (15)m}$$

$$5. \int_0^1 \ln(1+2x \cos \lambda + x^2) \frac{dx}{x} = \frac{1}{6} \pi^2 - \frac{1}{2} \lambda^2. \quad \text{BH [114] (34)}$$

3.674

$$1. \int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{1-x^2}} dx = \pi \ln \frac{1+k'}{2} \quad [k^2 < 1]. \quad \text{LI [119] (1)}$$

$$2. \int_0^1 \ln(1 \pm kx^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2 \pm 2k}{\sqrt{k}} K(k) - \frac{\pi}{8} K(k').$$

BH [120] (8) BH [120] (14)

$$3. \int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}} dx = \ln k' \cdot K(k). \quad \text{BH [119] (27)}$$

$$4. \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = (1+k'^2) K(k) - (2 - \ln k') E(k).$$

BH [119] (3)

$$5. \int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}} \ln(1-k^2x^2) dx = \frac{1}{k^2} (1+k'^2 - k'^2 \ln k') K(k) -$$

$$-(2 - \ln k') E(k). \quad \text{BH [119] (7)}$$

3.675

$$1. \int_0^{\infty} \frac{\ln(1+p^2x^2) - \ln(1+q^2x^2)}{x^2} dx = \pi(p-q) \quad [p > 0, q > 0]. \quad \text{F II 645}$$

$$2. \int_0^{\infty} \ln(1+q^2x^2) \ln(1+p^2x^2) \frac{dx}{x^2} = 2\pi [(p+q) \ln(p+q) - p \ln p - q \ln q] \\ [0 < p < 1, 0 < q < 1]. \quad \text{BH [134] (20)}$$

$$3. \int_0^1 \ln(1-2px \cos \lambda + p^2x^2) \left(\ln \frac{1}{x}\right)^r \frac{dx}{x} = -2\Gamma(r+1) \sum_{k=1}^{\infty} \frac{p^k \cos k\lambda}{k^{r+1}} \\ [r > -1]. \quad \text{Li [116] (8)}$$

$$4. \int_0^1 \frac{\ln(1-x^2) dx}{x(q^2 + \ln^2 x)} = -\frac{\pi}{q} \ln \Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q} \ln 2q + \ln \frac{q}{\pi} - 1 \quad [q > 0].$$

Li [327] (12)*m*

3.676

$$1. \int_p^q \ln \frac{q+x}{p+x} \frac{dx}{x} = \frac{1}{2} \left(\ln \frac{q}{p}\right)^2 \quad [pq > 0]. \quad \text{BH [145] (33)}$$

$$2. \int_0^1 \ln \frac{px+q}{qx+p} \frac{dx}{(1+x)^2} = \frac{1}{p-q} \left[(p+q) \ln \frac{p+q}{2} - p \ln p - q \ln q \right] \\ [pq > 0]. \quad \text{BH [115] (16)}$$

$$3. \int_0^{\infty} \ln \frac{px+q}{qx+p} \frac{dx}{(1+x)^2} = 0 \quad [pq > 0]. \quad \text{BH [139] (23)}$$

$$4. \int_0^1 \ln \left(\frac{1+qx}{1-qx}\right) \frac{dx}{x\sqrt{1-x^2}} = \pi \arcsin q \quad [q^2 \leq 1]. \quad \text{BH [122] (2)}$$

3.677

$$1. \int_0^{\infty} \ln \frac{p^2+x^2}{x^2} \frac{dx}{1+q^2x^2} = \frac{\pi}{q} \ln(1+pq) \quad [pq > -1, q > 0]. \quad \text{BH [138] (9)}$$

$$2. \int_0^{\infty} \ln \frac{1+p^2x^2}{x^2} \frac{dx}{q^2+x^2} = \frac{\pi}{q} \ln \frac{1+pq}{q} \quad [pq > -1, q > 0]. \quad \text{BH [138] (6)}$$

$$3. \int_0^{\infty} \ln \frac{1+p^2x^2}{x^2} \frac{dx}{q^2-x^2} = \frac{\pi}{q} \operatorname{arctg} pq \quad [q > 0]. \quad \text{BH [138] (8)}$$

$$4. \int_0^1 \ln \left(\frac{1+qx^2}{1-qx^2}\right) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1+q}}{1+\sqrt{1-q}} \quad [q^2 < 1]. \quad \text{BH [122] (12)}$$

$$5. \int_0^{\infty} \ln(1+q^2x^2) \ln\left(r^2 + \frac{1}{x^2}\right) \frac{dx}{x^2} = 2\pi [(q+r) \ln(q+r) - r \ln r - q] \\ [r > 0, q+r > 0]. \quad \text{BH [134] (22)}$$

3.678

$$1. \int_0^1 \ln \{ \sqrt{1+kx} + \sqrt{1-kx} \} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \\ = \frac{1}{4} \ln(4k) K(k) + \frac{\pi}{8} K(k'). \quad \text{BH [121] (8)}$$

$$2. \int_0^1 \ln \{ \sqrt{1+kx} - \sqrt{1-kx} \} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} =$$

$$= \frac{1}{4} \ln(4k) K(k) + \frac{3}{8} \pi K(k'). \quad \text{BH [121] (9)}$$

$$3. \int_0^1 \ln \{ 1 + \sqrt{1-k^2x^2} \} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} =$$

$$= \frac{1}{2} \ln k \cdot K(k) + \frac{\pi}{4} K(k'). \quad \text{BH [121] (6)}$$

$$4. \int_0^1 \ln \{ 1 - \sqrt{1-k^2x^2} \} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} =$$

$$= \frac{1}{2} \ln k \cdot K(k) - \frac{3}{4} \pi K(k'). \quad \text{BH [121] (7)}$$

3.679

$$1. \int_0^1 \ln \frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}} \frac{dx}{1-x} = \pi \arcsin p \quad [p^2 < 1]. \quad \text{BH [115] (29)}$$

$$2. \int_0^1 \ln \frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \pi F(\arcsin q, k') \quad [q^2 < 1].$$

BH [122] (15)

$$3. \int_0^\infty \ln \frac{\sqrt{1+x^2}+p}{\sqrt{1+x^2}-p} \frac{dx}{\sqrt{1+x^2}} = \pi \arcsin p \quad [p^2 < 1]. \quad \text{BH [142] (11)}$$

$$4. \int_0^1 \ln \left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \frac{x dx}{1-x^2} = \frac{1}{2} \pi^2. \quad \text{BH [115] (31)}$$

3.68-3.69 Logaritmul de argument mai complicat și funcția putere

$$3.681 \quad \int_0^\infty \ln \frac{p+qe^{-rx}}{p+qe^{-sx}} \frac{dx}{x} = \ln \left(1 + \frac{q}{p} \right) \ln \frac{s}{r} \quad [p > 0, q > 0, r > 0, s > 0].$$

F II 635

$$3.682 \quad \int_0^\pi x \ln \sin x \, dx = -\frac{\pi^2}{2} \ln 2. \quad \text{F II 643}$$

$$3.683 \quad \int_0^\pi \frac{\ln(1+p \cos x)}{\cos x} \, dx = \pi \arcsin p \quad [p^2 < 1]. \quad \text{F II 484}$$

$$3.684 \quad \int_0^\infty \frac{\ln \cos^2 x}{x^2} \, dx = -\pi. \quad \text{F II 686}$$

$$3.685 \quad \int_0^\infty \ln(1 \pm 2p \cos \beta x + p^2) \frac{dx}{q^2 + x^2} = \frac{\pi}{q} \ln(1 \pm pe^{-\beta q}) \quad [p^2 < 1];$$

$$= \frac{\pi}{q} \ln(p \pm e^{-\beta q}) \quad [p^2 > 1]. \quad \text{F II 718m}$$

$$3.686 \quad \int_1^{\infty} \ln \ln x \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \left\{ \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{2\pi} \right\}. \quad \text{Li [148] (1)}$$

$$3.687 \quad \int_1^{\infty} \ln \ln x \frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}} \ln \left\{ \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \sqrt[3]{2\pi} \right\}. \quad \text{Li [148] (2)}$$

$$3.688 \quad \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+2x \cos \lambda + x^2} = \frac{\pi}{2} \operatorname{cosec} \lambda \ln \frac{(2\pi)^{\frac{\lambda}{\pi}} \Gamma\left(\frac{1}{2} + \frac{\lambda}{2\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2\pi}\right)} \quad [\lambda^2 < \pi^2]. \quad \text{BH [147] (9)}$$

$$3.689 \quad \int_0^1 \frac{x^{p-1} \ln \ln \frac{1}{x}}{\sqrt{\ln \frac{1}{x}}} dx = -\sqrt{\frac{\pi}{p}} (C + 2 \ln 2 + \ln p) \quad [p > 0]. \quad \text{BH [304] (28)}$$

$$3.691 \quad \int_0^1 \ln(q - \ln x) x^{p-1} dx = \frac{1}{p} [\ln q - e^{pq} \operatorname{Ei}(-pq)] \quad [p > 0].$$

BH [107] (23)

$$3.692 \quad \int_0^{\frac{1}{e}} \ln \left(2 \ln \frac{1}{x} - 1 \right) \frac{x^{q-1} dx}{\ln x} = -\frac{1}{2} [\operatorname{Ei}(-q)]^2 \quad [q > 0]. \quad \text{BH [145] (5)}$$

3.71 Funcția logaritmică și funcția exponențială

3.711

$$1. \quad \int_u^{\infty} e^{-x} \ln x \, dx = e^{-u} \ln u - \operatorname{Ei}(-u). \quad \text{La 281 (33)}$$

$$2. \quad \int_0^{\infty} e^{-px} \ln x \, dx = -\frac{1}{p} (C + \ln p) \quad [p > 0]. \quad \text{BH [256] (2)}$$

$$3. \quad \int_1^{\infty} e^{-qx} \ln x \, dx = -\frac{1}{q} \operatorname{Ei}(-q) \quad [q > 0]. \quad \text{NI 32 (12)}$$

$$3.712 \quad \int_0^{\infty} \ln x \frac{dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}} \ln \left\{ \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \sqrt[3]{2\pi} \right\}. \quad \text{Li [260] (3)}$$

$$3.713 \quad \int_0^{\infty} e^{-px^2} \ln x \, dx = -\frac{1}{4} (C + \ln p + 2 \ln 2) \sqrt{\frac{\pi}{p}} \quad [p > 0].$$

F II 807m BH [256] (8)

3.714

$$1. \int_0^{\infty} e^{-x} \ln^2 x \, dx = \Gamma''(1) = C^2 + \frac{\pi^2}{6}. \quad \text{F II 808}$$

$$2. \int_0^{\infty} e^{-x^2} \ln^2 x \, dx = \frac{\sqrt{\pi}}{8} \left[(C + 2 \ln 2)^2 + \frac{\pi^2}{2} \right]. \quad \text{F II 808}$$

3.715

$$\int_0^{\infty} \frac{e^{-x}}{\ln x} \, dx = 0. \quad \text{BH [260] (9)}$$

3.716

$$\int_0^{\infty} \left\{ (q-1)e^{-x} + \frac{(1+x)^{-q} - (1+x)^{-1}}{\ln(1+x)} \right\} \frac{dx}{x} = \ln \Gamma(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{WWII 24}$$

3.717

$$\int_0^{\pi} e^{p \cos x} [\ln(2p \sin^2 x) + C] \, dx = -\pi K_0(p). \quad \text{W 95 (16)}$$

3.72 Funcția logaritmică, funcția exponențială și funcția putere

3.721

$$\int_1^{\infty} \frac{e^{-px} \ln x}{1+x} \, dx = \frac{1}{2} e^p [\operatorname{Ei}(-p)]^2 \quad [\operatorname{Re}(p) > 0]. \quad \text{NI 32 (10)}$$

3.722

$$\int_0^{\infty} \frac{e^{-qx}}{\sqrt{x}} \ln x \, dx = -(C + \ln q + 2 \ln 2) \sqrt{\frac{\pi}{q}} \quad [q > 0] \quad (\text{cf. 3.713}).$$

3.723

$$1. \int_0^{\infty} x^{p-1} e^{-qx} \ln x \, dx = \frac{1}{q^p} \Gamma(p) [\psi(p) - \ln q] \quad [q > 0, p > 0]. \quad \text{BH [353] (1)}$$

$$2. \int_0^{\infty} (x-q) x^{q-1} e^{-x} \ln x \, dx = \Gamma(q) \quad [\operatorname{Re}(q) > 0]. \quad \text{NH 173 (7)}$$

3.724

$$\int_0^{\infty} \frac{e^{-px} \ln(1+x^2)}{x} \, dx = [\operatorname{ci}(p)]^2 + [\operatorname{si}(p)]^2 \quad [\operatorname{Re}(p) > 0]. \quad \text{NI 32 (11)m}$$

3.73 Funcția logaritmică și funcții hiperbolice

3.731

$$1. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} x} \, dx = \pi \ln \left\{ \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{2\pi} \right\}. \quad \text{LI [260] (1)m}$$

$$2. \int_0^{\infty} \frac{\ln x \, dx}{\operatorname{ch} x + \cos \lambda} = \pi \operatorname{cosec} \lambda \ln \frac{(2\pi)^{\frac{\lambda}{\pi}} \Gamma\left(\frac{\pi+\lambda}{2\pi}\right)}{\Gamma\left(\frac{\pi-\lambda}{2\pi}\right)} \quad [\lambda^2 < \pi^2]. \quad \text{BH [257] (7)m}$$

$$3. \int_0^{\infty} \frac{\ln x}{\operatorname{ch}^2 \frac{x}{2}} \, dx = 2 \left[\psi\left(\frac{1}{2}\right) + \ln 2\pi \right]. \quad \text{BH [257] (4)m}$$

$$3.732 \quad \int_0^{\infty} \frac{\ln(x^2 + q^2)}{\operatorname{ch} \frac{\pi x}{2}} dx = 4 \ln \left\{ \frac{2\Gamma\left(\frac{q+3}{4}\right)}{\Gamma\left(\frac{q+1}{4}\right)} \right\} \quad [q > -1]. \quad \text{BH [258] (11)m}$$

$$3.733 \quad \int_0^{\infty} \ln(\cos^2 \lambda + e^{-2x} \sin^2 \lambda) \frac{dx}{\operatorname{sh} x} = -2\lambda^2. \quad \text{BH [259] (10)m}$$

$$3.734 \quad \int_0^{\infty} \ln \operatorname{cth} x \frac{dx}{\operatorname{ch} x} = \frac{\pi}{2} \ln 2. \quad \text{BH [259] (16)m}$$

3.74 Funcția logaritmică, funcții hiperbolice și funcția putere

$$3.741 \quad \int_0^{\infty} \frac{\ln x}{\sqrt{x} \operatorname{ch} x} dx = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} \{\ln(2k+1) + 2\ln 2 + C\}. \quad \text{BH [147] (4)}$$

3.75-3.77 Funcția logaritmică și funcții trigonometrice

3.751

$$1. \quad \int_0^1 \sin px \ln x \, dx = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{p^{2k+1}}{(2k+1)! \cdot (2k+1)!}. \quad \text{BH [284] (1)}$$

$$2. \quad \int_0^1 \cos px \ln x \, dx = -\frac{1}{p} \left[\operatorname{Si}(x) + \frac{\pi}{2} \right] \quad [p > 0]. \quad \text{BH [284] (2)}$$

$$3. \quad \int_0^{\infty} \cos px \ln \frac{1+x^2}{x^2} dx = \frac{\pi}{p} (1 - e^{-p}). \quad [p \geq 0]. \quad \text{F III 648}$$

3.752

$$1. \quad \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{(\sin x + p \cos x)^2} = \frac{1}{p(1+p^2)} \left(\ln p - \frac{p}{2} \right) \quad [p > 0]. \quad \text{BH [319] (1)}$$

$$2. \quad \int_0^{\frac{\pi}{2}} \ln \cos x \frac{dx}{(\sin x + p \cos x)^2} = -\frac{p}{1+p^2} \left(\ln p + \frac{\pi}{2p} \right) \quad [p > 0]. \quad \text{BH [319] (6)}$$

$$3. \quad \int_0^{\frac{\pi}{2}} \ln \sin x \frac{\sin x}{\sqrt{1+\sin^2 x}} dx = -\frac{\pi}{8} \ln 2. \quad \text{BH [322] (1)}$$

$$4. \quad \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = -\frac{1}{2} \ln k \cdot K(k) - \frac{\pi}{4} K(k'). \quad \text{BH [322] (3)}$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln \cos x}{\sqrt{1+\cos^2 x}} dx = -\frac{\pi}{8} \ln 2. \quad \text{BH [322] (6)}$$

$$6. \int_0^{\frac{\pi}{2}} \frac{\ln \cos x dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{1}{2} K(k) \ln \frac{k'}{k} - \frac{\pi}{4} K(k'). \quad \text{BH [322] (9)}$$

3.753

$$1. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{2a} x dx = \frac{(2a-1)!!}{2^{a+1} \cdot a!} \pi \left\{ \sum_{k=1}^{2a} \frac{(-1)^{k+1}}{k} - \ln 2 \right\}. \quad \text{F II 811}$$

$$2. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{2a-1} x dx = \frac{2^{a-1} (a-1)!}{(2a-1)!!} \left[\sum_{k=1}^{2a-1} \frac{(-1)^k}{k} + \ln 2 \right]. \quad \text{BH [305] (13)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln \sin x \cos^{2a} x dx = -\frac{(2a-1)!!}{2^{a+1} \cdot a!} \pi [C + \psi(a+1) + 2 \ln 2]. \quad \text{BH [305] (14)}$$

$$4. \int_0^{\frac{\pi}{2}} \ln \cos x \sin^{2a} x dx = -\frac{(2a-1)!!}{2^{a+1} \cdot a!} \frac{\pi}{2} \{C + 2 \ln 2 + \psi(a+1)\}. \quad \text{BH [306] (8)}$$

$$5. \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{2a} x dx = -\frac{(2a-1)!!}{2^a \cdot a!} \frac{\pi}{2} \left\{ \ln 2 + \sum_{k=1}^{2a} \frac{(-1)^k}{k} \right\}. \quad \text{BH [306] (10)}$$

$$6. \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{2a-1} x dx = \frac{2^{a-1} \cdot (a-1)!}{(2a-1)!!} \left[\ln 2 + \sum_{k=1}^{2a-1} \frac{(-1)^k}{k} \right]. \quad \text{BH [306] (9)}$$

3.754

$$1. \int_0^{\frac{\pi}{4}} \ln \sin x \frac{\sin^{2a} x}{\cos^{2a+3} x} dx = -\frac{1}{2a+1} \left[\frac{1}{2} \ln 2 + (-1)^a \frac{\pi}{4} + \sum_{k=0}^{a-1} \frac{(-1)^k}{2a-2k-1} \right]. \quad \text{BH [288] (1)}$$

$$2. \int_0^{\frac{\pi}{4}} \ln \sin x \frac{\sin^{2a-1} x}{\cos^{2a+1} x} dx = \frac{1}{4a} \left[-\ln 2 + (-1)^a \ln 2 + \sum_{k=1}^{a-1} \frac{(-1)^k}{a-k} \right]. \quad \text{Li [288] (2)}$$

$$3. \int_0^{\frac{\pi}{4}} \ln \cos x \frac{\sin^{2a} x}{\cos^{2a+3} x} dx = \frac{1}{2a+1} \left[-\frac{1}{2} \ln 2 + (-1)^{a+1} \frac{\pi}{4} + \sum_{k=0}^a \frac{(-1)^{k-1}}{2a-2k+1} \right]. \quad \text{BH [288] (10)}$$

$$4. \int_0^{\frac{\pi}{4}} \ln \cos x \frac{\sin^{2a-1} x}{\cos^{2a+1} x} dx = \frac{1}{4a} \left[-\ln 2 + (-1)^a \ln 2 + \sum_{k=0}^{a-1} \frac{(-1)^k}{a-k} \right].$$

Li [288] (11)

3.755

$$1. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{\sin^{p-1} x}{\cos^{p+1} x} dx = -\frac{\pi}{2p} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 2]. \quad \text{BH [310] (4)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln \cos x \frac{\cos^{p-1} x}{\sin^{p+1} x} dx = -\frac{\pi}{2p} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 2]. \quad \text{BH [311] (5)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{\operatorname{tg}^{p-1} x \sin 2x} = \frac{1}{4} \frac{\pi}{p-1} \sec \frac{p\pi}{2} \quad [p^2 < 1]. \quad \text{BH [310] (3)}$$

$$4. \int_0^{\frac{\pi}{2}} \ln \cos x \frac{\operatorname{tg}^{p-1} x}{\sin 2x} dx = \frac{\pi}{4(p-1)} \sec \frac{p\pi}{2} \quad [p^2 < 1]. \quad \text{BH [311] (4)}$$

3.756

$$1. \int_0^{\frac{\pi}{2}} \frac{\ln \sin x dx}{1-2p \cos 2x + p^2} = \frac{\pi}{2(1-p^2)} \ln \frac{1-p}{2} \quad [p^2 < 1];$$

$$= \frac{\pi}{2(p^2-1)} \ln \frac{p-1}{2p} \quad [p^2 > 1]. \quad \text{BH [321] (1)}$$

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos 2x \ln \sin x}{1-2p \cos 2x + p^2} dx = \frac{\pi}{2p(1-p^2)} \left\{ \frac{1+p^2}{2} \ln(1-p) - p^2 \ln 2 \right\} \quad [p^2 < 1];$$

$$= \frac{\pi}{2p(p^2-1)} \left\{ \frac{1+p^2}{2} \ln \frac{p-1}{p} - \ln 2 \right\} \quad [p^2 > 1].$$

BH [321] (2)

$$3. \int_0^{\frac{\pi}{2}} \frac{\ln \cos x dx}{1-2p \cos 2x + p^2} = \frac{\pi}{2(1-p^2)} \ln \frac{1+p}{2} \quad [p^2 < 1];$$

$$= \frac{\pi}{2(p^2-1)} \ln \frac{p+1}{2p} \quad [p^2 > 1]. \quad \text{BH [321] (8)}$$

$$4. \int_0^{\frac{\pi}{2}} \frac{\cos 2x \ln \cos x}{1-2p \cos 2x + p^2} dx = \frac{\pi}{2p(1-p^2)} \left[\frac{1+p^2}{2} \ln(1+p) - p^2 \ln 2 \right] \quad [p^2 < 1];$$

$$= \frac{\pi}{2p(p^2-1)} \left[\frac{1+p^2}{2} \ln \frac{1+p}{p} - \ln 2 \right] \quad [p^2 > 1].$$

BH [321] (9)

3.757

$$1. \int_0^{\pi} \ln \sin x \sin^{2a} 2x \cos 2x dx = -\frac{(2a-1)!!}{2^a \cdot a!} \frac{\pi}{4a+2}. \quad \text{BH [330] (9)}$$

$$2. \int_0^{\pi/2} \ln \sin x \cos^a 2x \sin 2x \, dx = -\frac{1}{4(a+1)} \left(\ln 2 + \sum_{k=0}^a \frac{1}{k+1} \right). \quad \text{BH [285] (2)}$$

$$3. \int_0^{\pi/2} \ln \cos x \cos^{p-1} 2x \operatorname{tg} 2x \, dx = \frac{1}{4(1-p)} \beta(p) \quad [p > 1]. \quad \text{BH [286] (2)}$$

3.758

$$1. \int_0^{\pi/2} \ln \sin x \cos qx \, dx = -\frac{\pi}{8q} \quad [q > 1]. \quad \text{BH [305] (6)}$$

$$2. \int_0^{\pi/2} \ln \sin x \cos 2ax \, dx = -\frac{1}{2a}. \quad \text{BH [330] (7)}$$

$$3. \int_0^{\pi/2} \ln \sin x \cos [2b(x-a)] \, dx = \frac{\pi \cos 2ab}{2b}. \quad \text{LI [330] (8)}$$

$$4. \int_0^{\pi/2} \ln \cos x \cos^p x \cos px \, dx = -\frac{\pi}{2^p} \ln 2 \quad [p > -1]. \quad \text{BH [337] (6)}$$

$$5. \int_0^{\pi/2} \ln \cos x \cos^{p-1} x \sin x \sin px \, dx = \frac{\pi}{2^{p+1}} \left[C + \phi(p) - \frac{1}{p} - 2 \ln 2 \right] \\ [p > 0]. \quad \text{BH [306] (12)}$$

3.759

$$1. \int_0^{\pi/2} (\ln \cos x)^2 \, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]. \quad \text{BH [306] (14)}$$

$$2. \int_0^{\pi/2} \frac{\operatorname{cosec}^q x - \sin^q x}{(\ln \operatorname{cosec} x)^p} \frac{dx}{\cos x} = \Gamma(1-p) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-q-1)^{1-p}} - \frac{1}{(2k+q-1)^{1-p}} \right\} \\ [p < 1, q^2 < 1]. \quad \text{BH [326] (12)}$$

3.761

$$1. \int_0^{\pi/2} \ln(\sin x \cos x) \frac{\sin^{2a} x}{\cos^{2a+2} x} \, dx = \\ = \frac{1}{2a+1} \left[(-1)^{a+1} \frac{\pi}{2} - \ln 2 + \frac{1}{2a+1} + 2 \sum_{k=0}^{a-1} \frac{(-1)^{k-1}}{2a-2k-1} \right]. \quad \text{BH [294] (8)}$$

$$2. \int_0^{\pi/2} \ln(\sin x \cos x) \frac{\sin^{2a-1} x}{\cos^{2a+1} x} \, dx = \\ = \frac{1}{2a} \left[(-1)^a \ln 2 - \ln 2 + \frac{1}{2a} + (-1)^a \sum_{k=1}^{a-1} \frac{(-1)^k}{k} \right]. \quad \text{BH [294] (9)}$$

3.762

1. $\int_0^{\frac{\pi}{2}} \sin x \ln \operatorname{tg} x \, dx = \ln 2.$ BH [307] (3)
2. $\int_0^{\frac{\pi}{2}} \sin x \ln \operatorname{ctg} \frac{x}{2} \, dx = \ln 2.$ Lo III 290
3. $\int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg} x \, dx}{\sqrt{1-k^2 \sin^2 x}} = -\ln k' \cdot K(k).$ BH [322] (11)
4. $\int_0^{\frac{\pi}{2}} \frac{\sin x \ln \operatorname{ctg} \frac{x}{2}}{1-\cos^2 \alpha \sin^2 x} \, dx =$
 $= \operatorname{cosec} 2\alpha \left\{ \frac{\pi}{2} \ln 2 + L(\varphi - \alpha) - L(\varphi + \alpha) - L\left(\frac{\pi}{2} - 2\alpha\right) \right\}$
 $[\operatorname{tg} \varphi = \operatorname{ctg} \alpha \cos u; 0 < u < \pi].$ Lo III 290
5. $\int_0^{\frac{\pi}{2}} \frac{\sin x \ln \operatorname{ctg} \frac{x}{2}}{1-\cos^2 \alpha \sin^2 x} \, dx = 2 \operatorname{cosec} 2\alpha \left\{ \left(\frac{\pi}{2} - \alpha\right) \ln 2 - L\left(\frac{\pi}{2} - \alpha\right) \right\}.$
Lo III 290
6. $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x \ln \operatorname{ctg} \frac{x}{2} \, dx}{(\sin^2 \alpha + \operatorname{tg}^2 \beta \sin^2 x) \sqrt{\sin^2 x - \sin^2 \alpha}} =$
 $= \frac{\pi}{2} \frac{\cos^2 \beta}{\sin \alpha \sin \beta} \ln \frac{\sin \beta + \sqrt{1 - \cos^2 \alpha \cos^2 \beta}}{\sin \alpha (1 + \sin \beta)} \quad \left[0 < \alpha < \frac{\pi}{2}, 0 < \beta < \pi \right].$
Lo III 285
7. $\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x}{1+p \sin 2x} \, dx = \frac{1}{16p} \{4(\arccos p)^2 - \pi^2\} \quad [p^2 \leq 1].$ BH [291] (2)
8. $\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x}{1-p^2 \sin^2 2x} \, dx = -\frac{\pi}{4p} \arcsin p \quad [p^2 \leq 1].$ BH [291] (9)
9. $\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x}{1+p^2 \sin^2 2x} \, dx = -\frac{\pi}{4p} \ln(p + \sqrt{1+p^2}) \quad [p^2 < 1].$
BH [291] (10)
10. $\int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \cos^{q-1} x \operatorname{ctg} x \sin [(q+1)x] \, dx = -\frac{\pi}{2} [C + \psi(q+1)]$
 $[q > -1].$ BH [307] (11)
11. $\int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \cos^{q-1} x \cos [(q+1)x] \, dx = -\frac{\pi}{2q} \quad [q > 0].$ BH [307] (10)

3.763

$$1. \int_0^{\frac{\pi}{2}} (\ln \operatorname{ctg} x)^{a-1} \operatorname{tg} qx \, dx = (a-1)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(q+2k+1)^a}. \quad \text{BH [286] (22)}$$

$$2. \int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2a-1} \frac{dx}{\cos 2x} = \frac{1-2^{2a}}{2a} \pi^{2a} |B_{2a}| \quad \text{BH [312] (6)}$$

Vezi și 3.767.

3.764

$$1. \int_0^{\frac{\pi}{2}} \ln(1+p \sin x) \frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1]. \quad \text{BH [313] (1)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln(1+p \cos x) \frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1]. \quad \text{BH [313] (8)}$$

$$3. \int_0^{\frac{\pi}{2}} \ln(1+p \cos x) \frac{dx}{\cos x} = \pi \arcsin p \quad [p^2 < 1]. \quad \text{BH [331] (1)}$$

$$4. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1+\cos \alpha \cos x)}{1-\cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2}-\alpha\right)-\alpha \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right].$$

Lo III 291

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1-\cos \alpha \cos x)}{1-\cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2}-\alpha\right)+(\pi-\alpha) \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right].$$

Lo III 291

3.765

$$1. \int_0^{\pi} \ln(1-2p \cos x + p^2) \cos ax \, dx = -\frac{\pi}{a} p^a \quad [p^2 < 1]. \quad \text{BH [330] (11)}$$

$$2. \int_0^{2\pi} \ln(1-2p \cos x + p^2) \cos ax \, dx = -\frac{2\pi}{a} p^a \quad [p^2 < 1]. \quad \text{BH [332] (5)}$$

$$3. \int_0^{\pi} \ln(1-2p \cos x + p^2) \sin ax \sin x \, dx = \frac{\pi}{2} \left(\frac{p^{a+1}}{a+1} - \frac{p^{a-1}}{a-1} \right) \quad [p^2 < 1].$$

BH [330] (10)

$$4. \int_0^{2\pi} \ln(1-2p \cos x + p^2) \sin ax \sin x \, dx = \pi \left(\frac{p^{a+1}}{a+1} - \frac{p^{a-1}}{a-1} \right) \quad [p^2 < 1].$$

BH [332] (4)

$$5. \int_0^{\pi} \ln(1-2p \cos x + p^2) \cos ax \cos x \, dx = -\frac{\pi}{2} \left(\frac{p^{a+1}}{a+1} + \frac{p^{a-1}}{a-1} \right) \quad [p^2 < 1].$$

BH [330] (12)

$$6. \int_0^{2\pi} \ln(1-2p \cos x + p^2) \cos ax \cos x dx = -\pi \left(\frac{p^{2a+1}}{a+1} + \frac{p^{a-1}}{a-1} \right) \quad [p^2 < 1].$$

BH [332] (6)

3.766

$$1. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin^2 x dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') K(k) + (2 - \ln k') E(k) \}.$$

BH [323] (3)

$$2. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\cos^2 x dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{1}{k^2} \{ (2 - k^2 - k'^2 \ln k') K(k) - (2 - \ln k') E(k) \}.$$

BH [323] (6)

$$3. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x dx}{\sqrt{(1-k^2 \sin^2 x)^{2a+1}}} = \frac{2}{(2a-1)^2 k^2} \{ [1 + (2a-1) \ln k'] (k')^{1-2a} - 1 \}$$

BH [324] (17)

$$4. \int_0^{\frac{\pi}{2}} \frac{\ln(1-\cos^2 \beta \cos^2 x)}{1-\cos^2 \alpha \cos^2 x} dx = -\frac{\pi}{\sin \alpha} \ln \frac{1+\sin \alpha}{\sin \alpha + \sin \beta}$$

Lo III 285

$\left[0 < \beta < \frac{\pi}{2}, 0 < \alpha < \frac{\pi}{2} \right].$

3.767

$$1. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8}.$$

BH [293] (1)

$$2. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\operatorname{tg} 2x} = \pm \frac{\pi^2}{16}.$$

BH [293] (2)

$$3. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2a} \frac{dx}{\sin 2x} = \pm \frac{2^{2a+2}-1}{4(a+1)(2a+1)} \pi^{2a+2} |B_{2a+2}|$$

BH [294] (24)

$$4. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2a-1} \frac{dx}{\sin 2x} = \pm \frac{1-2^{2a+1}}{2^{2a+2}a} \cdot (2a)! \zeta(2a+1).$$

BH [294] (25)

$$5. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \sin 2x)^{a-1} \frac{dx}{\operatorname{tg} 2x} = \frac{(-1)^{a-1}}{2} (a-1)! \zeta(a+1).$$

Li [294] (20)

Vezi și 3.762.

3.768

$$1. \int_0^{\pi} \ln \frac{1+\sin x}{1+\cos \lambda \sin x} \frac{dx}{\sin x} = \lambda^2 \quad [\lambda^2 < \pi^2]. \quad \text{BH [331] (2)}$$

$$2. \int_0^{\frac{\pi}{2}} \ln \frac{p+q \sin ax}{p-q \sin ax} \frac{dx}{\sin ax} = \int_0^{\frac{\pi}{2}} \ln \frac{p+q \cos ax}{p-q \cos ax} \frac{dx}{\cos ax} = \int_0^{\frac{\pi}{2}} \ln \frac{p+q \operatorname{tg} ax}{p-q \operatorname{tg} ax} \frac{dx}{\operatorname{tg} ax} = \\ = \pi \arcsin \frac{q}{p} \quad [p > q > 0].$$

F II 695m BH [315] (5) BH [315] (13) BH [315] (17)m

$$3. \int_0^{\frac{\pi}{2}} \frac{\cos x}{1-\cos^2 \alpha \cos^2 x} \ln \frac{1+\cos \beta \cos x}{1-\cos \beta \cos x} dx = \frac{2\pi}{\sin 2\alpha} \ln \frac{\cos \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\beta}{2}} \\ \left[0 < \alpha \leq \beta < \frac{\pi}{2} \right]. \quad \text{Lo III 284}$$

3.769

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln (1 + \sqrt{\sin^2 \beta - \cos^2 \beta} \operatorname{tg}^2 \alpha \sin^2 x)}{1 - \sin^2 \alpha \cos^2 x} dx = \\ = \operatorname{cosec} 2\alpha \{ (2\alpha + 2\gamma - \pi) \ln \cos \beta + 2L(\alpha) - 2L(\gamma) + \\ + L(\alpha + \gamma) - L(\alpha - \gamma) \} \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; 0 < \alpha < \beta < \frac{\pi}{2} \right]. \quad \text{Lo III 291}$$

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln (1 - \sqrt{\sin^2 \beta - \cos^2 \beta} \operatorname{tg}^2 \alpha \sin^2 x)}{1 - \sin^2 \alpha \cos^2 x} dx = \\ = \operatorname{cosec} 2\alpha \{ (\pi + 2\alpha - 2\gamma) \ln \cos \beta + 2L(\alpha) + 2L(\gamma) - \\ - L(\alpha + \gamma) + L(\alpha - \gamma) \} \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; 0 < \alpha < \beta < \frac{\pi}{2} \right]. \quad \text{Lo III 291}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\ln (\sin x + \sqrt{\sin^2 x - \sin^2 \beta})}{1 - \cos^2 \alpha \cos^2 x} dx = \\ = -\operatorname{cosec} \alpha \left\{ \operatorname{arctg} \left(\frac{\operatorname{tg} \beta}{\sin \alpha} \right) \ln \sin \beta + \frac{\pi}{2} \ln \frac{1 + \sin \alpha}{\sin \alpha - \sqrt{1 - \cos^2 \alpha \cos^2 \beta}} \right\} \\ \left[0 < \alpha < \pi, 0 < \beta < \frac{\pi}{2} \right]. \quad \text{Lo III 285}$$

$$3.771 \quad \int_0^{\frac{\pi}{4}} (\ln \cos 2x)^{a-1} \operatorname{tg} x \cos^2 2x dx = \frac{1}{2} (-1)^{a-1} \cdot (a-1)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(q+k+1)^a} \\ [q > -1]. \quad \text{BH [286] (10)}$$

$$3.772 \quad \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x (\ln \cos 2x)^{a-1} \operatorname{tg} 2x dx = (-1)^{a-1} \frac{(a-1)!}{4} \sum_{k=0}^{\infty} \frac{1}{(1+2k)^{a+1}} = \\ = (-1)^{a-1} \frac{(a-1)!}{2^{a+1}} \zeta \left(a+1, \frac{1}{2} \right) \quad \text{BH [287] (20)}$$

3.773

$$1. \int_0^{\pi} \frac{\ln \sin rx \, dx}{1-2p \cos x + p^2} = \frac{\pi}{1-p^2} \ln \frac{1-p^{2r}}{2} \quad [p^2 < 1]. \quad \text{BH [331] (10)}$$

$$2. \int_0^{\pi} \frac{\ln \cos rx \, dx}{1-2p \cos x + p^2} = \frac{\pi}{1-p^2} \ln \frac{1+p^{2r}}{2} \quad [p^2 < 1]. \quad \text{BH [331] (11)}$$

$$3.774 \quad \int_0^{\pi} \frac{\ln \operatorname{tg} rx \, dx}{1-2p \cos x + p^2} = \frac{\pi}{1-p^2} \ln \frac{1-p^{2r}}{1+p^{2r}} \quad [p^2 < 1]. \quad \text{BH [331] (12)}$$

$$3.775 \quad \int_0^{\frac{\pi}{2}} \ln(p \operatorname{tg} x) \sin^{q-1} 2x \, dx = 2^{q-2} B\left(\frac{q}{2}, \frac{q}{2}\right) \ln p \quad [p > 0, q > 0].$$

Li [307] (8)

3.78 Funcția logaritmică, funcții trigonometrice și funcția rațională

$$3.781 \quad \int_0^{\infty} \sin x \ln x \frac{dx}{x} = -\frac{\pi}{2} C \quad [z' = z \operatorname{sgn} \operatorname{Re}(z)]. \quad \text{F II 810}$$

3.782

$$1. \int_0^{\infty} \sin x \ln x \frac{x dx}{x^2 + x^2} = \frac{\pi}{2} e^{-x'} \ln z' - \frac{\pi}{4} [e^{-x'} \operatorname{Ei}(z') + e^{x'} \operatorname{Ei}(-z')] \\ [z' = z \operatorname{sgn} \operatorname{Re}(z)]. \quad \text{NI 27 (10)}$$

$$2. \int_0^{\infty} \cos x \ln x \frac{x' dx}{x^2 + x^2} = \frac{\pi}{2} e^{-x'} \ln z' - \frac{\pi}{4} [e^{-x'} \operatorname{Ei}(z') - e^{x'} \operatorname{Ei}(-z')] \\ [z' = z \operatorname{sgn} \operatorname{Re}(z)] \quad \text{NI 27 (9)}$$

$$3.783 \quad \int_0^{\infty} \sin x \ln^2 x \frac{dx}{x} = \frac{\pi}{2} C^2 + \frac{\pi^2}{24}. \quad \text{F II 810}$$

$$3.784 \quad \int_0^1 \frac{(1+x)x}{\ln x} \sin(\ln x) \, dx = \frac{\pi}{4}. \quad \text{BH [326] (2)m}$$

$$3.785 \quad \int_0^{\infty} \sin qx \ln(px) \frac{dx}{x} = \frac{\pi}{2} \left(\ln \frac{p}{q} - C \right) \quad [p > 0, q > 0]. \quad \text{BH [312] (8)m}$$

3.786

$$1. \int_0^{\infty} \sin px \ln(1+x^2) \frac{dx}{x} = \pi \operatorname{Ei}(-p). \quad \text{BH [310] (14)m}$$

$$2. \int_0^1 \ln(1-x^2) \cos(p \ln x) \frac{dx}{x} = \frac{1}{2p^2} + \frac{\pi}{2p} \operatorname{cth} \frac{p\pi}{2} \quad \text{Li [309] (1) m}$$

3.79 Funcția logaritmică, funcții trigonometrice și funcții hiperbolice

3.791

$$1. \int_0^{\infty} e^{-qx} \sin px \ln x \, dx = \frac{1}{p^2 + q^2} \left[q \arctg \frac{p}{q} - pC + \frac{p}{2} \ln(p^2 + q^2) \right] \\ [q > 0, p > 0]. \quad \text{BH [467] (1)m}$$

$$2. \int_0^{\infty} e^{-qx} \cos px \ln x \, dx = -\frac{1}{p^2 + q^2} \left[\frac{q}{2} \ln(p^2 + q^2) + p \operatorname{arctg} \frac{p}{q} + qC \right] \\ [q > 0]. \quad \text{BH [467] (2)}$$

$$3.792 \int_0^{\frac{\pi}{2}} \frac{e^{-p \operatorname{tg} x} \ln \cos x \, dx}{\sin x \cos x} = -\frac{1}{2} [\operatorname{ci}(x)]^2 + \frac{1}{2} [\operatorname{si}(x)]^2 \\ [\operatorname{Re}(p) > 0]. \quad \text{NI 32.(11)}$$

3.8 FUNCȚII TRIGONOMETRICE INVERSE

3.81 Funcții trigonometrice inverse

$$3.811 \int_0^{\infty} \operatorname{arctg} px \operatorname{arctg} qx \, dx = \\ = \frac{\pi}{2} \left\{ \frac{1}{p} \ln \left(1 + \frac{p}{q} \right) + \frac{1}{q} \ln \left(1 + \frac{q}{p} \right) \right\} [p > 0, q > 0]. \quad \text{BH [77] (8)}$$

$$3.812 \int_0^{\pi} \operatorname{arctg}(\cos x) \, dx = 0. \quad \text{BH [345] (1)}$$

3.8 3.82 Arcsinus, arccosinus și funcția putere

3.821

$$1. \int_0^1 \frac{\arcsin x}{x} \, dx = \frac{\pi}{2} \ln 2. \quad \text{F II 614 F II 623}$$

$$2. \int_0^1 \frac{\arccos x}{1 \pm x} \, dx = \mp \frac{\pi}{2} \ln 2 + 2G. \quad \text{BH [231] (7) BH [231] (8)}$$

$$3. \int_0^1 \arcsin x \frac{x}{1+qx^2} \, dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \quad [q > -1]. \quad \text{BH [231] (1)}$$

$$4. \int_0^1 \arcsin x \frac{x}{1-p^2x^2} \, dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \quad [p^2 < 1]. \quad \text{Li [231] (3)}$$

$$5. \int_0^1 \arccos x \frac{dx}{\sin^2 \lambda - x^2} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}. \quad \text{BH [231] (10)}$$

$$6. \int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \quad [q > -1]. \quad \text{BH [235] (10)}$$

$$7. \int_0^1 \arcsin x \frac{x}{(1+qx^2)^2} \, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1]. \quad \text{BH [234] (2)}$$

$$8. \int_0^1 \arccos x \frac{x}{(1+qx^2)^2} \, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{\sqrt{1+q}} \quad [q > -1]. \quad \text{BH [234] (4)}$$

3.822

$$1. \int_0^1 x \sqrt{1-k^2 x^2} \arccos x \, dx = \frac{1}{9k^3} \left[\frac{3}{2} \pi + k'^2 K(k) - 2(1+k'^2) E(k) \right]. \quad \text{BH [236] (9)}$$

$$2. \int_0^1 x \sqrt{1-k^2 x^2} \arcsin x \, dx = \frac{1}{9k^3} \left[-\frac{3}{2} \pi k'^3 - k'^2 K(k) + 2(1+k'^2) E(k) \right]. \quad \text{BH [236] (1)}$$

$$3. \int_0^1 x \sqrt{k'^2 + k^2 x^2} \arcsin x \, dx = \frac{1}{9k^3} \left[\frac{3}{2} \pi + k'^2 K(k) - 2(1+k'^2) E(k) \right]. \quad \text{BH [236] (5)}$$

$$4. \int_0^1 \frac{x \arcsin x}{\sqrt{1-k^2 x^2}} \, dx = \frac{1}{k^3} \left[-\frac{\pi}{2} k' + E(k) \right]. \quad \text{BH [237] (1)}$$

$$5. \int_0^1 \frac{x \arccos x}{\sqrt{1-k^2 x^2}} \, dx = \frac{1}{k} \left[\frac{\pi}{2} - E(k) \right]. \quad \text{BH [240] (1)}$$

$$6. \int_0^1 \frac{x \arcsin x}{\sqrt{k'^2 + k^2 x^2}} \, dx = \frac{1}{k^2} \left[\frac{\pi}{2} - E(k) \right]. \quad \text{BH [238] (1)}$$

$$7. \int_0^1 \frac{x \arccos x}{\sqrt{k'^2 + k^2 x^2}} \, dx = \frac{1}{k^2} \left[-\frac{\pi}{2} k' + E(k) \right]. \quad \text{BH [241] (1)}$$

$$8. \int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda) \sqrt{1-x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin [(2k+1)\lambda]}{(2k+1)^3}. \quad \text{BH [243] (11)}$$

$$9. \int_0^1 \frac{x \arcsin kx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \, dx = -\frac{\pi}{2k} \ln k'. \quad \text{BH [239] (1)}$$

$$10. \int_0^1 \frac{x \arccos kx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \, dx = \frac{\pi}{2k} \ln(1+k). \quad \text{BH [242] (1)}$$

3.823

$$1. \int_0^1 x^{2a} \arcsin x \, dx = \frac{1}{2a+1} \left[\frac{\pi}{2} - \frac{2^a \cdot a!}{(2a+1)!!} \right]. \quad \text{BH [229] (1)}$$

$$2. \int_0^1 x^{2a-1} \arcsin x \, dx = \frac{\pi}{4a} \left[1 - \frac{(2a-1)!!}{2^a \cdot a!} \right]. \quad \text{BH [229] (2)}$$

$$3. \int_0^1 x^{2a} \arccos x \, dx = \frac{2^a \cdot a!}{(2a+1) \cdot (2a+1)!!}. \quad \text{BH [229] (4)}$$

$$4. \int_0^1 x^{2a-1} \arccos x \, dx = \frac{\pi}{4a} \frac{(2a-1)!!}{2^a \cdot a!}. \quad \text{BH [229] (5)}$$

$$5. \int_{-1}^1 (1-x^2)^a \arccos x \, dx = \pi \frac{2^a \cdot a!}{(2a+1)!!}. \quad \text{BH [254] (2)}$$

$$6. \int_{-1}^1 (1-x^2)^{a-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2 (2a-1)!!}{2 \cdot 2^a \cdot a!}. \quad \text{BH [254] (3)}$$

3.824

$$1. \int_0^1 (\arcsin x)^2 \frac{dx}{x^2 \sqrt{1-x^2}} = \pi \ln 2. \quad \text{BH [243] (13)}$$

$$2. \int_0^1 (\arccos x)^2 \frac{dx}{(\sqrt{1-x^2})^3} = \pi \ln 2. \quad \text{BH [244] (9)}$$

3.83 Arctangenta, arccotangenta și funcția putere

3.831

$$1. \int_0^1 \frac{\operatorname{arctg} x}{x} dx = \int_1^\infty \frac{\operatorname{arctg} x}{x} dx = G. \quad \text{F II 482 BH [253] (8)}$$

$$2. \int_0^\infty \frac{\operatorname{arctg} x}{1 \pm x} dx = \pm \frac{\pi}{4} \ln 2 + G. \quad \text{BH [248] (6) BH [248] (7)}$$

$$3. \int_0^1 \frac{\operatorname{arctg} x}{x(1+x)} dx = -\frac{\pi}{8} \ln 2 + G. \quad \text{BH [235] (11)}$$

$$4. \int_0^\infty \frac{\operatorname{arctg} x}{1-x^2} dx = -G. \quad \text{BH [248] (2)}$$

$$5. \int_0^1 \operatorname{arctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)} \operatorname{arctg} q$$

[p > -1]. BH [234] (7)

$$6. \int_0^1 \operatorname{arctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{1+q^2}{(1+p)^2} +$$

+ $\frac{p}{p^2+q^2} \operatorname{arctg} q + \frac{1}{1+p} \operatorname{arctg} q \quad [p > -1]. \quad \text{BH [234] (10)}$

$$7. \int_0^1 \frac{\operatorname{arctg} x}{x(1+x^2)} dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G. \quad \text{BH [235] (12)}$$

$$8. \int_0^\infty \frac{x \operatorname{arctg} x}{1+x^4} dx = \frac{\pi^2}{16}. \quad \text{BH [248] (3)}$$

$$9. \int_0^\infty \frac{x \operatorname{arctg} x}{1-x^4} dx = -\frac{\pi}{8} \ln 2. \quad \text{BH [248] (4)}$$

$$10. \int_0^\infty \frac{x \operatorname{arctg} x}{1-x^4} dx = \frac{\pi}{8} \ln 2. \quad \text{BH [248] (12)}$$

$$11. \int_0^\infty \frac{\operatorname{arctg} x}{x \sqrt{1+x^2}} dx = \int_0^\infty \frac{\operatorname{arctg} x}{\sqrt{1+x^2}} dx = 2G. \quad \text{BH [251] (3) BH [251] (10)}$$

$$12. \int_0^1 \frac{\operatorname{arctg} x}{x \sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(1+\sqrt{2}). \quad \text{F II 694}$$

$$31. \int_0^1 \frac{x \operatorname{arctg} x dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2} \left[F\left(\frac{\pi}{4}, k\right) - \frac{\pi}{2\sqrt{2(1+k^2)}} \right]. \quad \text{BH [244] (14)}$$

3.832

1. $\int_0^1 x^p \operatorname{arctg} x \, dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} - \beta \left(\frac{p}{2} + 1 \right) \right] \quad [p > -2]. \quad \text{BH [229] (7)}$
2. $\int_0^\infty x^p \operatorname{arctg} x \, dx = \frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2} \quad [-1 > p > -2]. \quad \text{BH [246] (1)}$
3. $\int_0^1 x^p \operatorname{arccctg} x \, dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} + \beta \left(\frac{p}{2} + 1 \right) \right] \quad [p > -1]. \quad \text{BH [229] (8)}$
4. $\int_0^\infty x^p \operatorname{arccctg} x \, dx = -\frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2} \quad [-1 < p < 0]. \quad \text{BH [246] (2)}$
5. $\int_0^\infty \left(\frac{x^p}{1+x^{2p}} \right)^{q/2} \operatorname{arctg} x \, \frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2} p} \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)} \quad [q > 0]. \quad \text{BH [250] (10)}$

3.833

1. $\int_0^\infty (1 - x \operatorname{arccctg} x) \, dx = \frac{\pi}{4}. \quad \text{BH [246] (3)}$
2. $\int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x \right) \frac{dx}{1-x} = -\frac{\pi}{8} \ln 2 + G. \quad \text{BH [232] (2)}$
3. $\int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x \right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} G. \quad \text{BH [235] (25)}$
4. $\int_0^1 \left(x \operatorname{arccctg} x - \frac{1}{x} \operatorname{arctg} x \right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2. \quad \text{BH [232] (1)}$

$$3.834 \quad \int_0^\infty (\operatorname{arctg} x)^2 \frac{dx}{x^2 \sqrt{1+x^2}} = \int_0^\infty (\operatorname{arccctg} x)^2 \frac{x \, dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4G. \\ \text{BH [251] (9) BH [251] (17)}$$

3.835

1. $\int_0^1 \frac{\operatorname{arctg} px}{1+p^2 x} \, dx = \frac{1}{2p^2} \operatorname{arctg} p \ln(1+p^2). \quad \text{BH [231] (19)}$
2. $\int_0^1 \frac{\operatorname{arccctg} px}{1+p^2 x} \, dx = \frac{1}{p^2} \left\{ \frac{\pi}{4} + \frac{1}{2} \operatorname{arccctg} p \right\} \ln(1+p^2) \\ [p > 0]. \quad \text{BH [231] (24)}$
3. $\int_0^\infty \frac{\operatorname{arctg} qx}{(p+x)^2} \, dx = -\frac{q}{1+p^2 q^2} \left(\ln pq - \frac{\pi}{2} pq \right) \\ [p > 0, q > 0]. \quad \text{BH [249] (1)}$
4. $\int_0^\infty \frac{\operatorname{arccctg} qx}{(p+x)^2} \, dx = \frac{q}{1+p^2 q^2} \left(\ln pq + \frac{\pi}{2pq} \right) \quad [p > 0, q > 0]. \quad \text{BH [249] (8)}$

$$5. \int_0^{\infty} \frac{x \operatorname{arctg} px}{q^2 + x^2} dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \quad [p > 0, q > 0]. \quad \text{BH [248] (9)}$$

$$6. \int_0^{\infty} \frac{x \operatorname{arctg} px}{x^2 - q^2} dx = \frac{\pi}{4} \ln \frac{1+p^2q^2}{p^2q^2} \quad [p > 0, q > 0]. \quad \text{BH [248] (10)}$$

$$7. \int_0^{\infty} \frac{\operatorname{arctg} px}{x(1+x^2)} dx = \frac{\pi}{2} \ln(1+p) \quad [p \geq 0]. \quad \text{F II 745}$$

$$8. \int_0^{\infty} \frac{\operatorname{arctg} px}{x(1-x^2)} dx = \frac{\pi}{4} \ln(1+p^2) \quad [p \geq 0]. \quad \text{BH [250] (6)}$$

$$9. \int_0^1 \operatorname{arctg} px \frac{dx}{1+p^2x} = \frac{1}{2p^2} \operatorname{arctg} p \ln(1+p^2) \quad [p > 0]. \quad \text{BH [231] (17)}$$

$$10. \int_0^1 \operatorname{arctg} px \frac{dx}{1+p^2x} = \frac{1}{p^2} \left\{ \frac{\pi}{4} + \frac{1}{2} \operatorname{arctg} p \right\} \ln(1+p^2) \quad [p > 0] \\ \text{BH [231] (24)}$$

$$11. \int_0^{\infty} \operatorname{arctg} qx \frac{dx}{x(p^2+x^2)} = \frac{\pi}{2p^2} \ln(1+pq) \quad [p > 0, q \geq 0] \quad \text{BH [250] (3)}$$

$$12. \int_0^{\infty} \operatorname{arctg} qx \frac{dx}{x(1-p^2x^2)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \quad [q \geq 0]. \quad \text{BH [250] (6)}$$

$$13. \int_0^{\infty} \frac{x \operatorname{arctg} qx}{(p^2+x^2)^2} dx = \frac{\pi q}{4p(1+pq)} \quad [p > 0, q \geq 0]. \quad \text{BH [252] (12)m}$$

$$14. \int_0^{\infty} \frac{x \operatorname{arctg} qx}{(p^2+x^2)^2} dx = \frac{\pi q}{4p^2(1+pq)} \quad [p > 0, q \geq 0]. \quad \text{BH [252] (20)m}$$

$$15. \int_0^1 \frac{\operatorname{arctg} qx}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(q + \sqrt{1+q^2}). \quad \text{BH [244] (11)}$$

3.836

$$1. \int_0^{\infty} \operatorname{arctg} qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q\pi \ln \frac{1+\sqrt{1+q^2}}{\sqrt{1+q^2}} + \\ + \frac{\pi}{2} \ln(q + \sqrt{1+q^2}) - \frac{\pi}{2} \operatorname{arctg} q. \quad \text{BH [230] (7)}$$

$$2. \int_0^{\infty} \frac{\operatorname{arctg} px - \operatorname{arctg} qx}{x} dx = \frac{\pi}{2} \ln \frac{p}{q} \quad [p > 0, q > 0]. \quad \text{F II 635}$$

$$3. \int_0^{\infty} \frac{\operatorname{arctg} px \operatorname{arctg} qx}{x^2} dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p \cdot q^q} \quad [p > 0, q > 0]. \quad \text{F II 745}$$

3.837

$$1. \int_0^1 \operatorname{arctg}(\sqrt{1-x^2}) \frac{dx}{1-x^2 \cos^2 \lambda} = \frac{\pi}{\cos \lambda} \ln \left[\cos \left(\frac{\pi-4\lambda}{8} \right) \operatorname{cosec} \left(\frac{\pi+4\lambda}{8} \right) \right]. \\ \text{BH [245] (9)}$$

$$2. \int_0^1 \operatorname{arctg}(p\sqrt{1-x^2}) \frac{dx}{1-x^2} = \frac{1}{2} \pi \ln(p + \sqrt{1+p^2}) \quad [p > 0].$$

BH [245] (10)

$$3. \int_0^1 \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1-k^2x^2}) \sqrt{\frac{1-x^2}{1-k^2x^2}} dx =$$

$$= \frac{\pi}{2k^2} [E(\lambda, k) - k'^2 F(\lambda, k)] - \frac{\pi}{2k^2} \operatorname{ctg} \lambda (1 - \sqrt{1-k^2 \sin^2 \lambda}). \quad \text{BH [245] (12)}$$

$$4. \int_0^1 \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1-k^2x^2}) \sqrt{\frac{1-k^2x^2}{1-x^2}} dx =$$

$$= \frac{\pi}{2} E(\lambda, k) - \frac{\pi}{2} \operatorname{ctg} \lambda (1 - \sqrt{1-k^2 \sin^2 \lambda}). \quad \text{BH [245] (11)}$$

$$5. \int_0^1 \frac{\operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1-k^2x^2})}{\sqrt{(1-x^2)(1-k^2x^2)}} dx = \frac{\pi}{2} F(\lambda, k). \quad \text{BH [245] (13)}$$

3.838

$$1. \int_0^\infty \operatorname{arctg} x^3 \frac{dx}{1+x^2} = \int_0^\infty \operatorname{arctg} x^3 \frac{dx}{1+x^2}; \quad \text{BH [252] (10) BH [252] (11)}$$

$$= \int_0^\infty \operatorname{arccotg} x^3 \frac{dx}{1+x^2} = \int_0^\infty \operatorname{arccotg} x^3 \frac{dx}{1+x^2} = \frac{\pi^2}{8}. \quad \text{BH [252] (18) BH [252] (19)}$$

$$2. \int_0^1 \frac{1-x^2}{x^3} \operatorname{arctg} x^2 dx = \frac{\pi}{2} (\sqrt{2} - 1). \quad \text{BH [244] (10)m}$$

$$3.839 \quad \int_0^\infty \operatorname{arctg} \left(\frac{p \sin qx}{1+p \cos qx} \right) \frac{x dx}{1+x^2} = \frac{\pi}{2} \ln(1+pe^{-q})$$

[p > -e^q]. BH [341] (14)m

3.84 Arctangenta, arccotangenta și funcția exponențială

$$3.841 \quad \int_0^\infty \frac{\operatorname{arctg} \frac{x}{e^{2\pi x} - 1}}{e^{2\pi x} - 1} dx = \frac{1}{2} \left[\ln \Gamma(q) - \left(q - \frac{1}{2}\right) \ln q + q - \frac{1}{2} \ln 2\pi \right]$$

[q > 0]. WW II 25

$$3.842 \quad \int_0^\infty \left(\frac{2}{\pi} \operatorname{arccotg} x - e^{-px} \right) \frac{dx}{x} = C + \ln p \quad [p > 0]. \quad \text{NI 66 (12)}$$

3.85 Arctangenta și funcția hiperbolică

$$3.851 \quad \int_{-\infty}^\infty \frac{\operatorname{arctg} e^{-x}}{\operatorname{ch}^{2q} px} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\Pi(x)}{\operatorname{ch}^{2q} px} dx = \frac{\sqrt{\pi^2}}{4p} \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)} \quad [q > 0].$$

Li [282] (10)

3.86 Funcții trigonometrice directe și funcții trigonometrice inverse

$$3.861 \quad \int_0^{\frac{\pi}{2}} \arcsin(k \sin x) \frac{\sin x dx}{\sqrt{1-k^2 \sin^2 x}} = -\frac{\pi}{2k} \ln k'. \quad \text{BH [344] (2)}$$

$$3.862 \quad \int_0^{\infty} \left(\frac{2}{x} \operatorname{arctg} x - \cos px \right) dx = C + \ln p \quad [p > 0]. \quad \text{NI 66 (12)}$$

3.863

$$1. \quad \int_0^{\infty} \operatorname{arctg} qx \sin px \, dx = \frac{\pi}{2p} (1 - e^{-\frac{p}{q}}) \quad [p > 0, q > 0]. \quad \text{BH [347] (1)m}$$

$$2. \quad \int_0^{\infty} \operatorname{arctg} qx \cos px \, dx = \frac{1}{2p} \left[e^{-\frac{p}{q}} \operatorname{Ei} \left(\frac{p}{q} \right) - e^{\frac{p}{q}} \operatorname{Ei} \left(-\frac{p}{q} \right) \right] \quad [p > 0, q > 0].$$

BH [347] (2)m

$$3. \quad \int_0^{\infty} \operatorname{arctg} rx \frac{\sin px \, dx}{1 \pm 2q \cos px + q^2} =$$

$$= \pm \frac{\pi}{2pq} \ln \frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}} \quad [q^2 < 1, r > 0, p > 0];$$

$$= \pm \frac{\pi}{2pq} \ln \frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \quad [q^2 > 1, r > 0, p > 0]. \quad \text{BH [347] (10)}$$

$$4. \quad \int_0^{\infty} \operatorname{arctg} px \frac{\operatorname{tg} x \, dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln \left(1 + \frac{r}{q} \operatorname{th} \frac{1}{p} \right)$$

$$[p > 0, q > 0, r > 0]. \quad \text{BH [347] (9)}$$

3.864

$$1. \quad \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \sin ax \, dx = \frac{\pi}{2a} p^a \quad (p^2 < 1). \quad \text{BH [345] (4)}$$

$$2. \quad \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \sin ax \cos x \, dx = \frac{\pi}{4} \left(\frac{p^{a+1}}{a+1} + \frac{p^{a-1}}{a-1} \right) \quad [p > 0].$$

BH [345] (5)

$$3. \quad \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \cos ax \sin x \, dx = \frac{\pi}{4} \left(\frac{p^{a+1}}{a+1} - \frac{p^{a-1}}{a-1} \right) \quad [p > 0].$$

BH [345] (6)

3.865

$$1. \quad \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p} \quad [p^2 < 1]. \quad \text{BH [346] (1)}$$

$$2. \quad \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \frac{dx}{\operatorname{tg} x} = -\frac{\pi}{2} \ln (1 - p^2) \quad [p^2 < 1]. \quad \text{BH [346] (3)}$$

3.866

$$1. \quad \int_0^{\frac{\pi}{2}} \operatorname{arctg} (\operatorname{tg} \lambda \sqrt{1 - k^2 \sin^2 x}) \frac{\sin^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= \frac{\pi}{2k^2} [F(\lambda, k) - E(\lambda, k) + \operatorname{ctg} \lambda (1 - \sqrt{1 - k^2 \sin^2 \lambda})]. \quad \text{BH [344] (4)}$$

$$2. \int_0^{\frac{\pi}{2}} \operatorname{arctg} (\operatorname{tg} \lambda \sqrt{1-k^2 \sin^2 x}) \frac{\cos^2 x \, dx}{\sqrt{1-k^2 \sin^2 x}} =$$

$$= \frac{\pi}{2k^2} [E(\lambda, k) - k'^2 F(\lambda, k) + \operatorname{ctg} \lambda (\sqrt{1-k^2 \sin^2 \lambda} - 1)]. \quad \text{BH [344] (5)}$$

3.87 Funcții trigonometrice inverse și directe și funcția putere

$$3.871 \quad \int_0^{\infty} \operatorname{arctg} x \cos px \frac{dx}{x} = \int_0^{\infty} \operatorname{arctg} \frac{x}{p} \cos x \frac{dx}{x} =$$

$$= -\frac{\pi}{2} \operatorname{Ei}(-p) \quad [\operatorname{Re}(p) > 0]. \quad \text{F.III 654 NI 25 (13)}$$

3.88 Funcții trigonometrice inverse și funcția logaritmică

3.881

$$1. \int_0^1 \arcsin x \ln x \, dx = 2 - \ln 2 - \frac{1}{2} \pi. \quad \text{BH [339] (1)}$$

$$2. \int_0^1 \arccos x \ln x \, dx = \ln 2 - 2. \quad \text{BH [339] (2)}$$

$$3.882 \quad \int_0^1 \arccos x \frac{dx}{\ln x} = - \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k \cdot k!} \frac{\ln(2k+2)}{2k+1}. \quad \text{BH [339] (8)}$$

3.883

$$1. \int_0^1 \operatorname{arctg} x \ln x \, dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48} \pi^2. \quad \text{BH [339] (3)}$$

$$2. \int_0^1 \operatorname{arctg} x \ln x \, dx = -\frac{1}{48} \pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2. \quad \text{BH [339] (4)}$$

$$3.884 \quad \int_0^1 \operatorname{arctg} x (\ln x)^{a-1} (\ln x + a) \, dx = \frac{a!}{(-2)^{a+1}} (2^a - 1) \zeta(a+1). \quad \text{BH [339] (7)}$$

3.9 INTEGRALE MULTIPLE

3.90 Schimbarea de variabilă în integralele multiple

3.901

$$1. \iint_{(\sigma)} f(x, y) \, dx \, dy = \iint_{(\sigma')} f[\varphi(u, v), \psi(u, v)] |\Delta| \, du \, dv,$$

unde $x = \varphi(u, v)$, $y = \psi(u, v)$, iar

$$\Delta = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} = \frac{D(\varphi, \psi)}{D(u, v)}$$

este determinantul funcțional (jacobianul) funcțiilor φ și ψ .

$$2. \iiint_{(V)} f(x, y, z) dx dy dz = \iiint_{(V')} f[\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w)] |\Delta| du dv dw,$$

unde $x = \varphi(u, v, w)$, $y = \psi(u, v, w)$, $z = \chi(u, v, w)$, iar

$$\Delta = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & \frac{\partial \varphi}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \chi}{\partial u} & \frac{\partial \chi}{\partial v} & \frac{\partial \chi}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi, \psi, \chi)}{D(u, v, w)}$$

este determinantul funcțional al funcțiilor φ, ψ, χ .

Se presupune atât în 3.901 2., cât și în 3.901 1., că

a) funcțiile φ, ψ, χ , precum și derivatele lor parțiale de ordinul întâi sînt continue în domeniul de integrare;

b) determinantul funcțional nu-și schimbă semnul în acest domeniu;

c) între vechile variabile x, y, z și noile variabile u, v, w există o corespondență biunivocă în domeniul de integrare;

d) domeniul V (sau, respectiv, σ) se transformă în domeniul V' (sau, respectiv, σ'), atunci cînd trecem de la variabilele x, y, z , la variabilele u, v, w .

3.902 Transformarea în coordonate polare

$$x = r \cos \varphi, \quad y = r \sin \varphi; \quad \frac{D(x, y)}{D(r, \varphi)} = r.$$

3.903 Transformarea în coordonate sferice

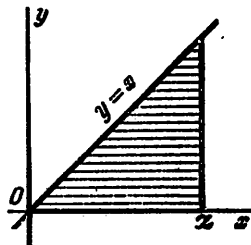
$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

$$\frac{D(x, y, z)}{D(r, \theta, \varphi)} = r^2 \sin \theta.$$

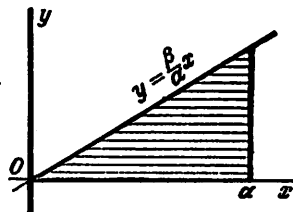
3.91 Schimbarea ordinii de integrare

3.911

$$1. \int_0^a dx \int_0^x f(x, y) dy = \int_0^a dy \int_y^a f(x, y) dx.$$

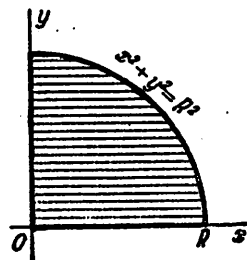


$$2. \int_0^a \int_0^{\frac{b}{a}x} f(x, y) dx dy = \int_0^{\frac{b}{a}} \int_{ay}^a f(x, y) dy dx.$$

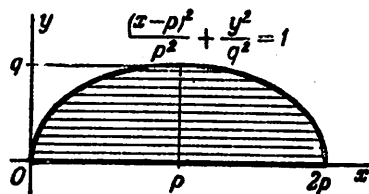


3.912

$$\begin{aligned}
 1. \quad \int_0^R \int_0^{\sqrt{R^2-x^2}} f(x, y) dx dy &= \\
 &= \int_0^R \int_0^{\sqrt{R^2-y^2}} f(x, y) dy dx.
 \end{aligned}$$

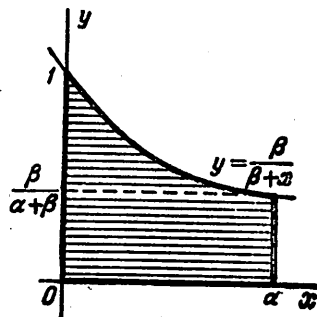


$$\begin{aligned}
 2. \quad \int_0^{2p} \int_0^{\frac{q}{p}\sqrt{2px-x^2}} f(x, y) dx dy &= \\
 &= \int_0^q \int_{p\left[1-\sqrt{1-\left(\frac{y}{q}\right)^2}\right]}^{p\left[1+\sqrt{1-\left(\frac{y}{q}\right)^2}\right]} (xy) dy dx.
 \end{aligned}$$

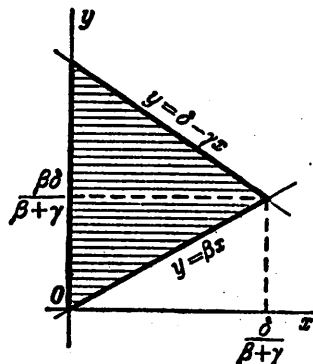


3.913

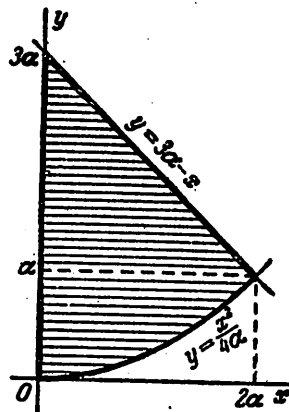
$$\begin{aligned}
 1. \quad \int_0^{\frac{\beta}{\beta+\alpha}} \int_0^{\frac{\beta}{\beta+x}} f(x, y) dx dy &= \\
 &= \int_0^{\frac{\beta}{\beta+\alpha}} \int_0^{\frac{\beta}{\beta+x}} f(x, y) dy dx + \\
 &+ \int_{\frac{\beta}{\beta+\alpha}}^1 \int_0^{\frac{\beta}{y}(1-y)} f(x, y) dy dx.
 \end{aligned}$$



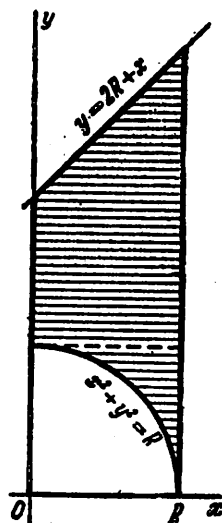
$$\begin{aligned}
 2. \quad \int_0^{\frac{\delta}{\beta+\gamma}} \int_{\beta x}^{\delta-\gamma x} f(x, y) dx dy &= \\
 &= \int_0^{\frac{\delta}{\beta+\gamma}} \int_0^{\frac{y}{\beta}} f(x, y) dy dx + \int_{\frac{\delta}{\beta+\gamma}}^{\frac{\delta}{\beta}} \int_0^{\frac{\delta-y}{\gamma}} f(x, y) dy dx \\
 &\quad \left[\alpha = \frac{\delta}{\beta+\gamma}, \alpha > 0, \beta > 0, \gamma > 0 \right].
 \end{aligned}$$



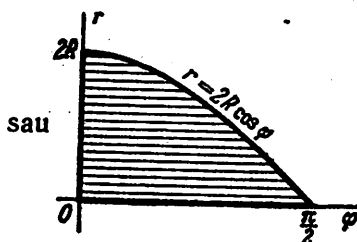
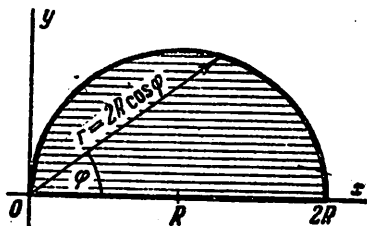
$$\begin{aligned}
 3. \quad & \int_0^{2\alpha} \int_{\frac{x^2}{4\alpha}}^{3\alpha-x} f(x, y) dx dy = \\
 & = \int_0^{\alpha} \int_0^{2\sqrt{\alpha y}} f(x, y) dy dx + \int_0^{3\alpha} \int_0^{3\alpha-y} f(x, y) dy dx.
 \end{aligned}$$



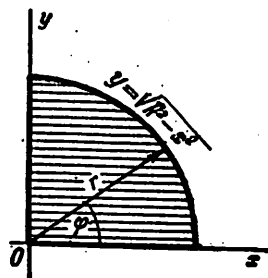
$$\begin{aligned}
 4. \quad & \int_0^R \int_{\sqrt{R^2-x^2}}^{x+2R} f(x, y) dx dy = \int_0^R \int_{\sqrt{R^2-y^2}}^R f(x, y) dy dx + \\
 & + \int_R^{2R} \int_0^R f(x, y) dy dx + \int_{2R}^{3R} \int_{y-2R}^R f(x, y) dy dx.
 \end{aligned}$$



$$3.914 \quad \int_0^{\frac{\pi}{2}} \int_0^{2R \cos \varphi} f(r, \varphi) d\varphi dr = \int_0^{2R} \int_0^{\arccos \frac{r}{2R}} f(r, \varphi) dr d\varphi.$$

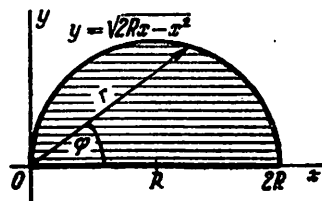


$$\begin{aligned}
 3.915 \quad & \int_0^R \int_0^{\sqrt{R^2-x^2}} f(x, y) dx dy = \\
 & = \int_0^{\frac{\pi}{2}} \int_0^R f(r \cos \varphi, r \sin \varphi) r d\varphi dr.
 \end{aligned}$$



$$3.916 \quad \int_0^{2R} \int_0^{\sqrt{2Rx-x^2}} f(x, y) dx dy =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2R \cos \varphi} f(r \cos \varphi, r \sin \varphi) r d\varphi dr.$$



$$3.917 \quad \int_a^\beta dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \int_0^\beta dx \int_0^{\varphi_2(x)} f(x, y) dy - \int_0^\beta dx \int_0^{\varphi_1(x)} f(x, y) dy -$$

$$- \int_0^a dx \int_0^{\varphi_2(x)} f(x, y) dy + \int_0^a dx \int_0^{\varphi_1(x)} f(x, y) dy$$

[$\varphi_1(x) \leq \varphi_2(x)$ pentru $a \leq x \leq \beta$].

$$3.918 \quad \int_0^\gamma dx \int_0^{\varphi(x)} f(x, y) dy = \int_0^\gamma dx \int_0^1 f[x, z\varphi(x)] \varphi(x) dz \quad [y = z\varphi(x)];$$

$$= \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y) dy \quad [x = \gamma z].$$

$$3.919 \quad \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dx dy = \int_{x_0}^{x_1} \int_0^1 (y_1 - y_0) f[x, y_0 + (y_1 - y_0)t] dx dt$$

[$y = y_0 + (y_1 - y_0)t$].

3.92 Integrale duble și triple cu limite de integrare constante

3.920 Formule cu caracter general

$$1. \quad \int_0^\pi d\omega \int_0^\infty f'(p \operatorname{ch} x + q \cos \omega \operatorname{sh} x) \operatorname{sh} x dx =$$

$$= -\frac{\pi \operatorname{sgn} p}{\sqrt{p^2 - q^2}} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2}) \quad [p^2 > q^2, \lim_{x \rightarrow +\infty} f(x) = 0]. \quad \text{Lo III 389}$$

$$2. \quad \int_0^{2\pi} d\omega \int_0^\infty f'[p \operatorname{ch} x + (q \cos \omega + r \sin \omega) \operatorname{sh} x] dx =$$

$$= -\frac{2\pi \operatorname{sgn} p}{\sqrt{p^2 - q^2 - r^2}} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2 - r^2}) \quad [p^2 > q^2 + r^2, \lim_{x \rightarrow +\infty} f(x) = 0].$$

Lo III 390

$$3. \quad \int_0^\pi \int_0^\pi \frac{dx dy}{\sin x \sin^2 y} f' \left[\frac{p - q \cos x}{\sin x \sin y} + r \operatorname{ctg} y \right] =$$

$$= -\frac{2\pi \operatorname{sgn} p}{\sqrt{p^2 - q^2 - r^2}} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2 - r^2}) \quad [p^2 > q^2 + r^2, \lim_{x \rightarrow +\infty} f(x) = 0].$$

Lo III 280

$$4. \quad \int_{-\infty}^\infty dx \int_{-\infty}^\infty f'(p \operatorname{ch} x \operatorname{ch} y + q \operatorname{sh} x \operatorname{ch} y + r \operatorname{sh} y) \operatorname{ch} y dy =$$

$$= -\frac{2\pi \operatorname{sgn} p}{\sqrt{p^2 - q^2 - r^2}} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2 - r^2}) \quad [p^2 > q^2 + r^2, \lim_{x \rightarrow +\infty} f(x) = 0]. \quad \text{Lo III 390}$$

$$5. \int_0^{\infty} dx \int_0^{\pi} f(p \operatorname{ch} x + q \cos \omega \operatorname{sh} x) \operatorname{sh}^2 x \sin \omega d\omega =$$

$$= 2 \int_0^{\infty} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2} \operatorname{ch} x) \operatorname{sh}^2 x dx [\lim_{x \rightarrow +\infty} f(x) = 0]. \quad \text{Lo III 391}$$

$$6. \int_0^{\infty} dx \int_0^{2\pi} d\omega \int_0^{\pi} f[p \operatorname{ch} x + (q \cos \omega + r \sin \omega) \sin \theta \operatorname{sh} x] \operatorname{sh}^2 x \sin \theta d\theta =$$

$$= 4 \int_0^{\infty} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2 - r^2} \operatorname{ch} x) \operatorname{sh}^2 x dx$$

$$[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0]. \quad \text{Lo III 390}$$

$$7. \int_0^{\infty} dx \int_0^{2\pi} d\omega \int_0^{\pi} f\{p \operatorname{ch} x + [(q \cos \omega + r \sin \omega) \sin \theta + s \operatorname{ch} \theta] \operatorname{sh} x\} \times$$

$$\times \operatorname{sh}^2 x \sin \theta d\theta = 4\pi \int_0^{\infty} f(\operatorname{sgn} p \cdot \sqrt{p^2 - q^2 - r^2 - s^2} \operatorname{ch} x) \operatorname{sh}^2 x dx$$

$$[p^2 > q^2 + r^2 + s^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0]. \quad \text{Lo III 391}$$

3.921

$$1. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = \frac{\pi}{2\sqrt{1 - k^2}}. \quad \text{Lo I 252}$$

$$2. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = K(k). \quad \text{Lo I 252}$$

$$3. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin \alpha \sin y dx dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2}. \quad \text{Lo I 253}$$

3.922

$$1. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi K^2 \left(\frac{\sqrt{2}}{2} \right). \quad \text{MO 137}$$

$$2. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx dy dz}{3 - \cos y \cos z - \cos x \cos z - \cos x \cos y} = \sqrt{3} \pi K^2 \left(\sin \frac{\pi}{12} \right). \quad \text{MO 137}$$

$$3. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} =$$

$$= 4\pi [18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}] K^2 [(2 - \sqrt{3})(\sqrt{3} - \sqrt{2})]. \quad \text{MO 137}$$

3.923

$$\int_0^{\infty} \int_0^{\infty} \varphi(a^2 x^2 + b^2 y^2) dx dy = \frac{\pi}{4ab} \int_0^{\infty} \varphi(x) x dx.$$

$$\begin{aligned}
 3.924 \quad & \int_0^\pi \int_0^{2\pi} f(\alpha \cos \theta + \beta \sin \theta \cos \psi + \gamma \sin \theta \sin \psi) \sin \theta \, d\theta \, d\psi = \\
 & = 2\pi \int_0^{\frac{\pi}{2}} f(R \cos p) \sin p \, dp = 2\pi \int_{-1}^1 f(Rt) \, dt \quad [R = \sqrt{\alpha^2 + \beta^2 + \gamma^2}]
 \end{aligned}$$

Vezi și 4.517, 6.835.

3.93-3.94 Integrale multiple

$$3.931 \quad \int_p^x dt_{a-1} \int_p^{t_{a-1}} dt_{a-2} \dots \int_p^{t_1} f(t) \, dt = \frac{1}{(a-1)!} \int_p^x (x-t)^{a-1} f(t) \, dt,$$

unde $f(t)$ este o funcție continuă pe segmentul $[p, q]$ și $p \leq x \leq q$. F II 692

3.932

$$1. \quad \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_a \geq 0 \\ x_1 + x_2 + \dots + x_a \leq h}} dx_1 dx_2 \dots dx_a = \frac{h^a}{a!} \quad [\text{volumul unui simplex cu } a \text{ dimensiuni}] \quad \text{F III 472}$$

$$2. \quad \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq R^2} dx_1 dx_2 \dots dx_a = \frac{V \pi^{\frac{a}{2}}}{\Gamma(\frac{a}{2} + 1)} R^a [\text{volumul unei sfere cu } a \text{ dimensiuni}]. \quad \text{F III 473}$$

$$3.933 \quad \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} \frac{dx_1 dx_2 \dots dx_a}{\sqrt{1 - x_1^2 - x_2^2 - \dots - x_a^2}} = \frac{\pi^{\frac{a+1}{2}}}{\Gamma(\frac{a+1}{2})} \quad (a > 1)$$

[jumătatea ariei sferei cu $(a+1)$ dimensiuni

$$x_1^2 + x_2^2 + \dots + x_{a+1}^2 = 1].$$

F III 474

$$3.934 \quad \int_{\substack{x_1 > 0, x_2 > 0, \dots, x_a > 0 \\ (\frac{x_1}{q_1})^{p_1} + (\frac{x_2}{q_2})^{p_2} + \dots + (\frac{x_a}{q_a})^{p_a} \leq 1}} \int \dots \int x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} dx_1 dx_2 \dots dx_a =$$

$$= \frac{q_1^{p_1} q_2^{p_2} \dots q_a^{p_a}}{a_1 a_2 \dots a_a} \frac{\Gamma(\frac{p_1}{a_1}) \Gamma(\frac{p_2}{a_2}) \dots \Gamma(\frac{p_a}{a_a})}{\Gamma(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \dots + \frac{p_a}{a_a} + 1)}$$

$$[a_i > 0, p_i > 0, q_i > 0, i = 1, 2, \dots, n].$$

F III 477

3.935

$$\begin{aligned}
 1. \quad & \int_{\substack{x_1 > 0, x_2 > 0, \dots, x_a > 0 \\ (\frac{x_1}{q_1})^{a_1} + (\frac{x_2}{q_2})^{a_2} + \dots + (\frac{x_a}{q_a})^{a_a} > 1}} \int \dots \int f\left[\left(\frac{x_1}{q_1}\right)^{a_1} + \left(\frac{x_2}{q_2}\right)^{a_2} + \dots + \left(\frac{x_a}{q_a}\right)^{a_a}\right] \times \\
 & \times x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} dx_1 dx_2 \dots dx_a = \\
 & = \frac{q_1^{p_1} q_2^{p_2} \dots q_a^{p_a}}{a_1 a_2 \dots a_a} \frac{\Gamma(\frac{p_1}{a_1}) \Gamma(\frac{p_2}{a_2}) \dots \Gamma(\frac{p_a}{a_a})}{\Gamma(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \dots + \frac{p_a}{a_a})} \int_1^\infty f(x) x^{\frac{p_1}{a_1} + \frac{p_2}{a_2} + \dots + \frac{p_a}{a_a} - 1} dx,
 \end{aligned}$$

presupunind că integrala din membrul al doilea este absolut convergentă.

F III 487

$$\begin{aligned}
 2. \quad & \int_{x_1 \geq 0} \int_{x_2 \geq 0} \dots \int_{x_a \geq 0} f \left[\left(\frac{x_1}{q_1} \right)^{\alpha_1} + \left(\frac{x_2}{q_2} \right)^{\alpha_2} + \dots + \left(\frac{x_a}{q_a} \right)^{\alpha_a} \right] \times \\
 & \left(\frac{x_1}{q_1} \right)^{\alpha_1} + \left(\frac{x_2}{q_2} \right)^{\alpha_2} + \dots + \left(\frac{x_a}{q_a} \right)^{\alpha_a} \leq 1 \\
 & \times x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} dx_1 dx_2 \dots dx_a = \\
 & = \frac{q_1^{p_1} q_2^{p_2} \dots q_a^{p_a}}{\alpha_1 \alpha_2 \dots \alpha_a} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a}\right)} \int_0^1 f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a} - 1} dx
 \end{aligned}$$

presupunind că integrala simplă din membrul al doilea este absolut convergentă; toate numerele q_i , α_i , p_i sînt pozitive. F III 479

În particular,

$$\begin{aligned}
 3. \quad & \int_{x_1 \geq 0} \int_{x_2 \geq 0} \dots \int_{x_a \geq 0} x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} e^{-q(x_1+x_2+\dots+x_a)} dx_1 dx_2 \dots dx_a = \\
 & \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a)}{\Gamma(p_1+p_2+\dots+p_a)} \int_0^1 x^{p_1+p_2+\dots+p_a-1} e^{-qx} dx \\
 & [a > 0, p_1 > 0, p_2 > 0, \dots, p_a > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_{x_1 \geq 0} \int_{x_2 \geq 0} \dots \int_{x_a \geq 0} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{(1-x_1^{\alpha_1}-x_2^{\alpha_2}-\dots-x_a^{\alpha_a})^\mu} dx_1 dx_2 \dots dx_a = \\
 & = \frac{1}{\alpha_1 \alpha_2 \dots \alpha_a} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(1-\mu+\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_a}{\alpha_a}\right)} \Gamma(1-\mu) \\
 & [p_1 > 0, p_2 > 0, \dots, p_a > 0, \mu < 1]. \quad \text{F III 480}
 \end{aligned}$$

3.936

$$\begin{aligned}
 1. \quad & \int_{x_1 \geq 0} \int_{x_2 \geq 0} \dots \int_{x_a \geq 0} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_a^{\alpha_a})^\mu} dx_1 dx_2 \dots dx_a = \\
 & = \frac{1}{\alpha_1 \alpha_2 \dots \alpha_a} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(\mu - \frac{p_1}{\alpha_1} - \frac{p_2}{\alpha_2} - \dots - \frac{p_a}{\alpha_a}\right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a}\right)} \\
 & \left[p_1 > 0, p_2 > 0, \dots, p_a > 0; \mu > \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a} \right]. \quad \text{F III 488}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_{x_1 \geq 0} \int_{x_2 \geq 0} \dots \int_{x_a \geq 0} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_a^{\alpha_a})^\mu} dx_1 dx_2 \dots dx_a = \\
 & = \frac{1}{\alpha_1 \alpha_2 \dots \alpha_a} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a} - \mu\right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_a}{\alpha_a}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a}\right)} \\
 & \left[\mu < \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_a}{\alpha_a} \right]. \quad \text{F III 480}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_{x_1 > 0, x_2 > 0, \dots, x_a > 0} \int_{x_1^{a_1} + x_2^{a_2} + \dots + x_a^{a_a} \leq 1} \dots \int x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} \sqrt{\frac{1-x_1^{a_1}-x_2^{a_2}-\dots-x_a^{a_a}}{1+x_1^{a_1}+x_2^{a_2}+\dots+x_a^{a_a}}} \times \\
 & \times dx_1 dx_2 \dots dx_a = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p_1}{a_1}\right) \Gamma\left(\frac{p_2}{a_2}\right) \dots \Gamma\left(\frac{p_a}{a_a}\right)}{a_1 a_2 \dots a_a} \frac{1}{\Gamma(m)} \times \\
 & \times \left\{ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \right\},
 \end{aligned}$$

$$\text{unde } m = \frac{p_1}{a_1} + \frac{p_2}{a_2} + \dots + \frac{p_a}{a_a}.$$

F III 480

$$\begin{aligned}
 3.937 \quad & \int_{x_1 > 0, x_2 > 0, \dots, x_a > 0} \int_{x_1 + x_2 + \dots + x_a \leq 1} \dots \int f(x_1 + x_2 + \dots + x_a) \times \\
 & \times \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} dx_1 dx_2 \dots dx_a}{(q_1 x_1 + q_2 x_2 + \dots + q_a x_a + r)^{p_1 + p_2 + \dots + p_a}} = \\
 & = \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a)}{\Gamma(p_1 + p_2 + \dots + p_a)} \int_0^1 f(x) \frac{x^{p_1 + p_2 + \dots + p_a - 1}}{(q_1 x + r)^{p_1} (q_2 x + r)^{p_2} \dots (q_a x + r)^{p_a}} dx,
 \end{aligned}$$

unde $f(x)$ este o funcție continuă pe segmentul $[0,1]$.

F III 481

$$[q_1 \geq 0, q_2 \geq 0, \dots, q_a \geq 0; b > 0]$$

3.938

$$\begin{aligned}
 1. \quad & \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1} e^{-(q_1 x_1 + q_2 x_2 + \dots + q_a x_a)}}{(r_0 + r_1 x_1 + r_2 x_2 + \dots + r_a x_a)^s} dx_1 dx_2 \dots dx_a = \\
 & = \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a)}{\Gamma(s)} \int_0^\infty \frac{e^{-r_0 x} x^{s-1} dx}{(q_1 + r_1 x)^{p_1} (q_2 + r_2 x)^{p_2} \dots (q_a + r_a x)^{p_a}}
 \end{aligned}$$

pentru p_i, q_i, r_i, s pozitivi; acest rezultat are loc și pentru $r_0 = 0$, cu condiția ca $p_1 + p_2 + \dots + p_a > s$.

F III 481

$$\begin{aligned}
 2. \quad & \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{(r_0 + r_1 x_1 + r_2 x_2 + \dots + r_a x_a)^s} dx_1 dx_2 \dots dx_a = \\
 & = \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a) \Gamma(s - p_1 - p_2 - \dots - p_a)}{r_1^{p_1} r_2^{p_2} \dots r_a^{p_a} r_0^{s - p_1 - p_2 - \dots - p_a} \Gamma(s)} \\
 & [p_i > 0, r_i > 0, s > 0].
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{[1 + (r_1 x_1)^{q_1} + (r_2 x_2)^{q_2} + \dots + (r_a x_a)^{q_a}]^s} dx_1 dx_2 \dots dx_a = \\
 & = \frac{\Gamma\left(\frac{p_1}{q_1}\right) \Gamma\left(\frac{p_2}{q_2}\right) \dots \Gamma\left(\frac{p_a}{q_a}\right) \Gamma\left(s - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \dots - \frac{p_a}{q_a}\right)}{q_1^{p_1} q_2^{p_2} \dots q_a^{p_a} r_1^{p_1 q_1} r_2^{p_2 q_2} \dots r_a^{p_a q_a} \Gamma(s)} \\
 & [p_i > 0, q_i > 0, r_i > 0, s > 0].
 \end{aligned}$$

3.939

$$1. \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} (p_1 x_1 + p_2 x_2 + \dots + p_a x_a)^{2b} dx_1 dx_2 \dots dx_a =$$

$$= \frac{(2b-1)!!}{2^b} \frac{\sqrt{\pi^a}}{\Gamma\left(\frac{a}{2} + b + 1\right)} (p_1^2 + p_2^2 + \dots + p_a^2)^b. \quad \text{F III 482}$$

$$2. \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} (p_1 x_1 + p_2 x_2 + \dots + p_a x_a)^{2b+1} dx_1 dx_2 \dots dx_a = 0. \quad \text{F III 483}$$

3.941

$$1. \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \dots + p_a x_a} dx_1 dx_2 \dots dx_a =$$

$$= \sqrt{\pi^a} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma\left(\frac{a}{2} + k + 1\right)} \left(\frac{p_1^2 + p_2^2 + \dots + p_a^2}{4}\right)^k. \quad \text{F III 483}$$

$$2. \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \dots + p_a x_a} dx_1 dx_2 \dots dx_a =$$

$$= \frac{(2\pi)^a I_a(\sqrt{p_1^2 + p_2^2 + \dots + p_a^2})}{(p_1^2 + p_2^2 + \dots + p_a^2)^a}. \quad \text{F III 483m}$$

$$3.942 \quad \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 \leq R^2} f(\sqrt{x_1^2 + x_2^2 + \dots + x_a^2}) dx_1 dx_2 \dots dx_a =$$

$$= \frac{2\sqrt{\pi^a}}{\Gamma\left(\frac{a}{2}\right)} \int_0^R x^{a-1} f(x) dx,$$

unde $f(x)$ este o funcție continuă pe segmentul $[0, R]$. F III 485

$$3.943 \quad \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 x_2 \dots x_a) (1-x_1)^{p_1-1} (1-x_2)^{p_2-1} \dots (1-x_a)^{p_a-1} \times$$

$$\times x_1^{p_1} x_2^{p_2} \dots x_a^{p_a} dx_1 dx_2 \dots dx_a =$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a)}{\Gamma(p_1 + p_2 + \dots + p_a)} \int_0^1 f(x) (1-x)^{p_1 + p_2 + \dots + p_a - 1} dx,$$

presupunind că integrala din membrul al doilea converge absolut. F III 488

$$3.944 \quad \overbrace{\iint \dots \int_{x_1^2 + x_2^2 + \dots + x_a^2 = 1}}^{a-1} f(p_1 x_1 + p_2 x_2 + \dots + p_a x_a) \frac{dx_1 dx_2 \dots dx_{a-1}}{|x_a|} =$$

$$= 2 \iint \dots \int_{x_1^2 + x_2^2 + \dots + x_{a-1}^2 \leq 1} f(p_1 x_1 + p_2 x_2 + \dots + p_a x_a) \frac{dx_1 dx_2 \dots dx_{a-1}}{\sqrt{1 - x_1^2 - x_2^2 - \dots - x_{a-1}^2}} =$$

$$= \frac{2\sqrt{\pi^{a-1}}}{\Gamma\left(\frac{a-1}{2}\right)} \int_0^{\pi} f(\sqrt{p_1^2 + p_2^2 + \dots + p_a^2} \cos x) \sin^{a-2} x dx \quad [a \geq 3],$$

unde $f(x)$ este o funcție continuă pe segmentul $[-\sqrt{p_1^2 + p_2^2 + \dots + p_a^2}, \sqrt{p_1^2 + p_2^2 + \dots + p_a^2}]$. F III 489

3.945 Fie funcțiile $f(x_1, x_2, \dots, x_a)$ și $g(x_1, x_2, \dots, x_a)$ continue în domeniul mărginit închis (D) , valoarea cea mai mare și valoarea cea mai mică a funcției g în domeniul (D) fiind respectiv M și m fie $\varphi(u)$ o funcție continuă pentru $m \leq u \leq M$. Să notăm cu $\psi(u)$ integrala

$$1.. \psi(u) = \iint \dots \int_{m \leq g(x_1, x_2, \dots, x_a) \leq u} f(x_1, x_2, \dots, x_a) dx_1 dx_2 \dots dx_a,$$

extinsă la acea parte a domeniului (D) în care se verifică inegalitatea $m \leq g(x_1, \dots, x_a) \leq u$. Atunci

$$\begin{aligned} 2. \quad & \iint \dots \int_{m \leq g(x_1, x_2, \dots, x_a) \leq M} f(x_1, x_2, \dots, x_a) \varphi[g(x_1, x_2, \dots, x_a)] dx_1 dx_2 \dots dx_a = \\ & = (S) \int_m^M \varphi(u) d\psi(u) = (R) \int_m^M \varphi(u) \frac{d\psi(u)}{du} du, \end{aligned}$$

unde integrala din mijloc trebuie considerată ca o integrală Stieltjes: dacă există derivata continuă $\frac{d\psi}{du}$, atunci integrala în sensul lui Riemann din membrul al doilea există.

În formula 3.945 2., M poate fi și $+\infty$, în care caz, prin $\int_m^{+\infty}$ trebuie să înțelegem $\lim_{M \rightarrow +\infty} \int_m^M$. F III 491

$$\begin{aligned} 3.946 \quad & \iint \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_a \geq 0 \\ x_1 + x_2 + \dots + x_a \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_a^{p_a-1}}{(q_1 x_1 + q_2 x_2 + \dots + q_a x_a)^r} dx_1 dx_2 \dots dx_a = \\ & = \frac{\Gamma(p_1) \Gamma(p_2) \dots \Gamma(p_a)}{\Gamma(p_1 + p_2 + \dots + p_a - r + 1) \Gamma(r)} \int_0^\infty \frac{x^{r-1} dx}{(1 + q_1 x)^{p_1} (1 + q_2 x)^{p_2} \dots (1 + q_a x)^{p_a}} \\ & \quad [p_1 > 0, p_2 > 0, \dots, p_a > 0, q_1 > 0, q_2 > 0, \dots, q_a > 0, \\ & \quad p_1 + p_2 + \dots + p_a > r > 0]. \end{aligned} \quad \text{F III 493}$$

$$\begin{aligned} 3.947 \quad & \iint \dots \int_{0 \leq x_1^2 + x_2^2 + \dots + x_a^2 \leq 1} \exp \left\{ \frac{q_1 x_1 + q_2 x_2 + \dots + q_a x_a}{\sqrt{x_1^2 + x_2^2 + \dots + x_a^2}} \right\} dx_1 dx_2 \dots dx_a = \\ & = \frac{2 \sqrt{\pi^a}}{a (p_1^2 + p_2^2 + \dots + p_a^2)^{\frac{a-1}{4}-\frac{1}{2}}} I_{\frac{a}{2}-1} \left(\sqrt{p_1^2 + p_2^2 + \dots + p_a^2} \right). \end{aligned} \quad \text{F III 495}$$

$$\begin{aligned} 3.948 \quad & \int_0^\infty \int_0^\infty \dots \int_0^\infty \exp \left[- \left(x_1 + x_2 + \dots + x_a + \frac{\lambda^{a+1}}{x_1 x_2 \dots x_a} \right) \right] \times \\ & \times x_1^{\frac{1}{a+1}-1} x_2^{\frac{2}{a+1}-1} \dots x_a^{\frac{a}{a+1}-1} dx_1 dx_2 \dots dx_a = \\ & = \frac{1}{\sqrt{a+1}} (2\pi)^{\frac{a}{2}} e^{-(a+1)\lambda}. \end{aligned} \quad \text{F III 496}$$

4. INTEGRALE DEFINITE ALE FUNCTIILOR SPECIALE

4.1 INTEGRALE ELIPTICE

4.11 Forme care conțin $F(x, k)$

4.111

$$\int_0^{\frac{\pi}{2}} F(x, k) \operatorname{ctg} x \, dx = \frac{\pi}{4} K(k') + \frac{1}{2} \ln k \cdot K(k). \quad \text{BH [350] (1)}$$

4.112

$$1. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1+k \sin^2 x} \, dx = \frac{1}{4k} K(k) \ln \frac{(1+k) \sqrt{k}}{2} + \frac{\pi}{16k} K(k'). \quad \text{BH [350] (6)}$$

$$2. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1-k \sin^2 x} \, dx = \frac{1}{4k} K(k) \ln \frac{2}{(1-k) \sqrt{k}} - \frac{\pi}{16k} K(k'). \quad \text{BH [350] (7)}$$

$$3. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1-k^2 \sin^2 x} \, dx = -\frac{1}{(2k)^2} \ln k' \cdot K(k). \quad \text{BH [350] (2)}$$

4.12 Forme care conțin $E(x, k)$

$$4.121 \quad \int_0^{\frac{\pi}{2}} E(x, k) \frac{\sin x \cos x}{1-k^2 \sin^2 x} \, dx = \frac{1}{2k^2} \{(1+k'^2) K(k) - (2 + \ln k') E(k)\}. \quad \text{BH [350] (4)}$$

$$4.122 \quad \int_0^{\frac{\pi}{2}} E(x, k) \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{1}{2} \{E(k) K(k) - \ln k'\}. \quad \text{BH [350] (10)}$$

$$4.123 \quad \int_0^{\frac{\pi}{2}} E(\sin x, k) \frac{\sin x}{1-k^2 \sin^2 x} \, dx = \frac{\pi}{2k'}. \quad \text{BH [350] (3)}$$

4.13 Integrale eliptice complete

1. $\int_0^1 K(k) dk = 2G.$ F II 755
2. $\int_0^1 E(k) dk = 1 + G.$ J 45 (175)
3. $\int_0^a \frac{E(k) k dk}{k'^2 \sqrt{a^2 - k^2}} = \frac{\pi a}{2 \sqrt{1 - a^2}}.$ Lo I 252
4. $\int_0^{\frac{\pi}{2}} \frac{E(p \sin x)}{1 - p^2 \sin^2 x} \sin x dx = \frac{\pi}{2 \sqrt{1 - p^2}}.$ F II 489

4.2 FUNCȚIA EXPONENȚIALĂ INTEGRALĂ ȘI FUNCȚII ÎNRUDITE

4.21 Logaritmul integral

- 4.211 $\int_0^1 \text{li}(x) dx = -\ln 2.$ BH [79] (5)
- 4.212
 1. $\int_0^1 \text{li}(x) x^{p-1} dx = -\frac{1}{p} \ln(p+1) \quad [p \geq -1].$ BH [255] (2)
 2. $\int_1^{\infty} \text{li}(x) \frac{dx}{x^{q+1}} = -\frac{1}{q} \ln(q-1) \quad [q > 1].$ BH [255] (4)
 3. $\int_0^1 \text{li}\left(\frac{1}{x}\right) x dx = 0.$ BH [255] (1)
- 4.213
 1. $\int_0^1 \text{li}\left(\frac{1}{x}\right) \left(\ln \frac{1}{x}\right)^{p-1} dx = -\pi \text{ctg } p\pi \Gamma(p) \quad [0 < p < 1].$ BH [340] (1)
 2. $\int_1^{\infty} \text{li}\left(\frac{1}{x}\right) (\ln x)^{p-1} dx = -\frac{\pi}{\sin p\pi} \Gamma(p) \quad [p > 0].$ BH [340] (9)

4.22 Funcția exponențială integrală

- 4.221
 1. $\int_0^p \text{Ei}(ax) dx = p \text{Ei}(ap) + \frac{1 - e^{ap}}{a}.$ NI 11 (7)
 2. $\int_0^{\infty} \text{Ei}(-px) \text{Ei}(-qx) dx = \left(\frac{1}{p} + \frac{1}{q}\right) \ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q}.$ F II 653 NI 53 (3)
- 4.222 $\int_0^{\infty} \text{Ei}(-px) x^{q-1} dx = -\frac{\Gamma(q)}{q p^q} [\text{Re}(q) > 0, \text{Re}(p) > 0].$ NI 55 (7)

4.223

$$1. \int_0^{\infty} \text{Ei}(-px) e^{-qx} dx = -\frac{1}{q} \ln\left(1 + \frac{q}{p}\right) [\text{Re}(p+q) \geq 0; q \neq 0, p+q \neq 0];$$

$$= 1 \quad [q=0]. \quad \text{F II 652 NI 48 (8). F II 652}$$

$$2. \int_0^{\infty} \text{Ei}(px) e^{-qx} dx = -\frac{1}{q} \ln\left(\frac{q}{p} - 1\right) \quad [\text{Re}(q-p) \geq 0, q-p \neq 0].$$

$$\text{BH [283] (3)}$$

$$3. \int_0^{\infty} \text{Ei}(-x^2) e^{-px^2} dx = -\sqrt{\frac{\pi}{p}} \ln[\sqrt{p} + \sqrt{1+p}] \quad [p > 0].$$

$$\text{BH [283] (5)}$$

$$4. \int_0^{\infty} \text{Ei}(-x^2) e^{px^2} dx = -\sqrt{\frac{\pi}{p}} \arcsin p \quad [p > 0]. \quad \text{NI 59 (9)m}$$

4.224

$$1. \int_0^{\infty} \text{Ei}(-x) \sin qx dx = -\frac{1}{2q} \ln(1+q^2). \quad \text{BH [473] (1)}$$

$$2. \int_0^{\infty} \text{Ei}(-x) \cos qx dx = -\frac{1}{q} \arctg q. \quad \text{BH [473] (2)}$$

$$3. \int_0^{\infty} \text{Ei}(-x) e^{-px} \sin qx dx = -\frac{1}{p^2+q^2} \left\{ \frac{q}{2} \ln[(1+p)^2+q^2] - p \arctg \frac{q}{1+p} \right\}.$$

$$\text{BH [473] (7)}$$

$$4. \int_0^{\infty} \text{Ei}(x) e^{-x} \sin qx dx = \frac{1}{1+q^2} \left(\frac{\pi}{2} - q \ln q \right). \quad \text{BH [473] (4)}$$

$$5. \int_0^{\infty} \text{Ei}(-x) e^{-px} \cos qx dx = -\frac{1}{p^2+q^2} \left\{ \frac{p}{2} \ln[(1+p)^2+q^2] + \right.$$

$$\left. + q \arctg \frac{q}{1+p} \right\}. \quad \text{BH [473] (8)}$$

$$6. \int_0^{\infty} \text{Ei}(x) e^{-x} \cos qx dx = -\frac{1}{1+q^2} \left(\frac{q}{2} \pi + \ln q \right). \quad \text{BH [473] (6)}$$

$$4.225 \quad \int_0^{\infty} \text{Ei}(-x) \ln x dx = C + 1. \quad \text{NI 56 (10)}$$

Vezi și 4.481 1., 4.481 4., 4.481 7.

4.23 Sinus integral și cosinus integral

4.231

$$1. \int_0^p \text{si}(ax) dx = p \text{si}(ap) + \frac{\cos ap - 1}{a}. \quad \text{NI 11 (9)}$$

$$2. \int_0^p \text{ci}(ax) dx = p \text{ci}(ap) - \frac{\sin ap}{a}. \quad \text{NI 11 (8)}$$

$$3. \int_0^{\infty} \text{si}(px) \text{si}(qx) dx = \frac{\pi}{2p} \quad [p \geq q]. \quad \text{F II 653}$$

$$4. \int_0^{\infty} \text{ci}(px) \text{ci}(qx) dx = \frac{\pi}{2p} \quad [p \geq q]. \quad \text{F II 653}$$

$$5. \int_0^{\infty} \text{si}(px) \text{ci}(qx) dx = \frac{1}{4q} \ln \left(\frac{p+q}{p-q} \right)^2 + \frac{1}{4p} \ln \frac{(p^2 - q^2)^2}{q^4} \quad [p \neq q];$$

$$= \frac{1}{q} \ln 2 \quad [p = q]. \quad \text{F II 653}$$

4.232

$$1. \int_0^{\infty} \left[\text{si}(px) + \frac{\pi}{2} \right] \frac{x dx}{q^2 + x^2} = \frac{\pi}{2} \text{Ei}(-pq). \quad \text{BH [255] (5)}$$

$$2. \int_0^{\infty} \left[\text{si}(px) + \frac{\pi}{2} \right] \frac{x dx}{q^2 - x^2} = -\frac{\pi}{2} \text{ci}(pq). \quad \text{BH [255] (6)}$$

$$3. \int_0^{\infty} \text{ci}(px) \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \text{Ei}(-pq). \quad \text{BH [255] (7)}$$

$$4. \int_0^{\infty} \text{ci}(px) \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} [-\text{si}(pq)]. \quad \text{BH [255] (8)}$$

$$5. \int_0^{\infty} \text{si}(px) x^{q-1} dx = -\frac{\Gamma(q) \sin \frac{\pi q}{2}}{qp^q} \quad [\text{Re}(q) > 0, p > 0]. \quad \text{NI 56 (9)}$$

$$6. \int_0^{\infty} \text{ci}(px) x^{q-1} dx = -\frac{\Gamma(q) \cos \frac{\pi q}{2}}{qp^q} \quad [\text{Re}(q) > 0, p > 0]. \quad \text{NI 56 (8)}$$

4.233

$$1. \int_0^{\infty} e^{-px} \text{si}(qx) dx = -\frac{1}{p} \arctg \frac{p}{q} \quad [\text{Re}(q + ip) \geq 0]. \quad \text{NI 49 (12)}$$

$$2. \int_0^{\infty} e^{-px} \text{ci}(qx) dx = -\frac{1}{p} \ln \sqrt{1 + \frac{p^2}{q^2}} \quad [\text{Re}(q + ip) \geq 0]. \quad \text{NI 49 (11)}$$

4.234

$$1. \int_0^{\infty} \sin px \text{si}(qx) dx = -\frac{\pi}{2p} \quad [p^2 > q^2];$$

$$= -\frac{\pi}{4p} \quad [p^2 = q^2];$$

$$= 0 \quad [p^2 < q^2]. \quad \text{F II 652 NI 50 (8)}$$

$$2. \int_0^{\infty} \cos px \text{si}(qx) dx = -\frac{1}{4p} \ln \left(\frac{p+q}{p-q} \right)^2 \quad [p \neq 0, p^2 \neq q^2];$$

$$= 1 \quad [p = 0], \quad \text{F II 652 NI 50 (10)}$$

$$3. \int_0^{\infty} \sin px \operatorname{ci}(qx) dx = -\frac{1}{4p} \ln\left(\frac{p^2}{q^2} - 1\right)^2 \quad [p \neq 0, p^2 \neq q^2].$$

F II 652 NI 50 (9)

$$4. \int_0^{\infty} \cos px \operatorname{ci}(qx) dx = -\frac{\pi}{2p} \quad [p^2 > q^2];$$

$$= -\frac{\pi}{4p} \quad [p^2 = q^2];$$

$$= 0 \quad [p^2 < q^2].$$

F II 651 NI 50 (7)

4.235

$$1. \int_0^{\infty} \operatorname{si}(x) \ln x dx = C + 1. \quad \text{NI 56 (10)}$$

$$2. \int_0^{\infty} \operatorname{ci}(x) \ln x dx = \frac{\pi}{2}. \quad \text{NI 56 (11)}$$

Vezi și 4.245, 4.254, 4.481 2., 4.481 3., 4.481 6., 4.481 7.

4.24 Integrala de probabilitate

$$4.241 \int_0^p \Phi(ax) dx = p \Phi(ap) + \frac{e^{-a^2 p^2} - 1}{a \sqrt{\pi}}. \quad \text{NI 12 (20)}$$

$$4.242 \int_0^{\infty} [1 - \Phi(px)] x^{2q-1} dx = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2 \sqrt{\pi} q p^{2q}} \quad [\operatorname{Re}(q) > 0, \operatorname{Re}(p^2) > 0].$$

NI 56 (12)

4.243

$$1. \int_0^{\infty} \left[1 - \Phi\left(\frac{q}{\sqrt{x}}\right)\right] e^{-px} dx = \frac{1}{p} e^{-q} \sqrt{p} \quad [\operatorname{Re}(p) > 0]$$

(vezi și 5.245, 5.251). ED 147 (235)

$$2. \int_0^{\infty} [1 - \Phi(px)] e^{-q^2 x^2} dx = \frac{1}{2q^2} \left(1 - \frac{p^2}{p^2 + q^2}\right). \quad \text{NI 49 (14)}$$

4.244

$$1. \int_0^{\infty} [1 - \Phi(x)] \sin 2px dx = \frac{2(1 - e^{-p^2})}{\pi p} \quad [p > 0]. \quad \text{NI 61 (10)}$$

$$2. \int_0^{\infty} [1 - \Phi(x)] \cos 2px dx = \frac{2e^{-p^2}}{\pi p i} \Phi(pi). \quad \text{NI 64 (6)}$$

$$4.245 \int_0^{\infty} [1 - \Phi(x)] \operatorname{si}(2px) dx = \frac{2}{\pi p} (1 - e^{-p^2}) - \frac{2}{\sqrt{\pi}} (1 - \Phi(p)) \quad [p > 0].$$

NI 61 (13)

4.25 Integrala Fresnel

4.251

$$1. \int_0^p S(ax) dx = p S(ap) + \frac{\cos(a^2 p^2) - 1}{a \sqrt{2\pi}}.$$

NI 12 (22)m

$$2. \int_0^p C(ax) dx = p C(ap) - \frac{\sin a^2 p^2}{a \sqrt{2\pi}}. \quad \text{NI 12 (21)}m$$

4.252

$$1. \int_0^\infty \left[\frac{1}{2} - S(px) \right] x^{2q-1} dx = \frac{\sqrt{2} \Gamma\left(q + \frac{1}{2}\right) \sin \frac{2q+1}{4} \pi}{\sqrt{\pi q p^{2q}}} \left[0 < \operatorname{Re}(q) < \frac{3}{2}, p > 0 \right]. \quad \text{NI 56 (14)}m$$

$$2. \int_0^\infty \left[\frac{1}{2} - C(px) \right] x^{2q-1} dx = \frac{\sqrt{2} \Gamma\left(q + \frac{1}{2}\right) \cos\left(\frac{2q+1}{4} \pi\right)}{\sqrt{\pi q p^{2q}}} \left[0 < \operatorname{Re}(q) < \frac{3}{2}, p > 0 \right]. \quad \text{NI 56 (13)}m$$

4.253

$$1. \int_0^\infty \left[\frac{1}{2} - S(x) \right] \sin 2px dx = -\frac{2\sqrt{2} \cos \frac{\pi}{8}}{\pi} \frac{\sin \frac{p^2}{2}}{p} \quad [p > 0]. \quad \text{NI 61 (12)}m$$

$$2. \int_0^\infty \left[\frac{1}{2} - C(x) \right] \sin 2px dx = -\frac{2\sqrt{2} \sin \frac{\pi}{8}}{\pi} \frac{\sin \frac{p^2}{2}}{p} \quad [p > 0]. \quad \text{NI 61 (11)}m$$

4.254

$$1. \int_0^\infty \left[\frac{1}{2} - S(x) \right] \operatorname{si}(2px) dx = \frac{\sqrt{8} \cos \frac{\pi}{8}}{\sqrt{\pi}} \left[\frac{1}{2} - S(p\sqrt{2}) \right] \quad [p > 0]. \quad \text{NI 61 (15)}m$$

$$2. \int_0^\infty \left[\frac{1}{2} - C(x) \right] \operatorname{si}(2px) dx = \frac{\sqrt{8} \sin \frac{\pi}{8}}{\sqrt{\pi}} \left[\frac{1}{2} - S(p\sqrt{2}) \right] \quad [p > 0] \quad \text{NI 61 (14)}m$$

Vezi și 5.246—5.249.

4.3 INTEGRALE EULERIENE ȘI FUNCȚII ÎNRUDITE

4.31 Funcția Γ

4.311

$$1. \left. \begin{aligned} \int_{-\infty}^{\infty} \frac{\cos rx}{\Gamma(p+x) \Gamma(q-x)} dx &= \\ &= \frac{2^{p+q-2} \cos^{p+q-2} \frac{r}{2} \cos \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad [|r| < \pi] \\ &= 0 \quad [|r| > \pi] \end{aligned} \right\} \begin{aligned} &[r \text{ real}; \\ &\operatorname{Re}(p+q) > 1]. \end{aligned} \quad \text{MO 10 } m$$

$$\left. \begin{aligned}
 2. \quad & \int_{-\infty}^{\infty} \frac{\sin rx \, dx}{\Gamma(p+x) \Gamma(q-x)} = \\
 & = \frac{2^{p+q-2} \cos^{p+q-2} \frac{r}{2} \sin \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad [|r| < \pi] \\
 & = 0 \quad [|r| > \pi]
 \end{aligned} \right\} \begin{aligned} & [r \text{ real}; \\ & \operatorname{Re}(p+q) > 1]. \end{aligned}$$

MO 10m

Vezi și 6.419, 7.213, 7.323, 7.342.

4.32 Logaritmul funcției Γ .

4.321

$$1. \quad \int_p^{p+1} \ln \Gamma(x) \, dx = \frac{1}{2} \ln 2\pi + p \ln p - p. \quad \text{F II 784}$$

$$2. \quad \int_0^1 \ln \Gamma(x) \, dx = \int_0^1 \ln \Gamma(1-x) \, dx = \frac{1}{2} \ln 2\pi. \quad \text{F II 783}$$

$$3. \quad \int_0^1 \ln \Gamma(x+q) \, dx = \frac{1}{2} \ln 2\pi + q \ln q - q. \quad \text{NH 89 (17)}$$

$$4. \quad \int_0^z \ln \Gamma(x+1) \, dx = \frac{z}{2} \ln 2\pi - \frac{z(z+1)}{2} + z \ln \Gamma(z+1) - \ln G(z+1),$$

unde

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp \left(-\frac{z(z+1)}{2} - \frac{Gz^2}{2} \right) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k} \right)^k \exp \left(-z + \frac{z^2}{2k} \right) \right\}.$$

WW II 43

4.322

$$1. \quad \int_0^1 \ln \Gamma(x) \sin 2a\pi x \, dx = \frac{1}{2a\pi} [\ln(2a\pi) + C] \quad (\text{vezi și 4.332 2}) \quad \text{NH 203 (5)}$$

$$2. \quad \int_0^1 \ln \Gamma(x) \sin (2a+1)\pi x \, dx = \frac{\ln 2\pi}{(2a+1)\pi}. \quad \text{NH 203 (5)}$$

$$3. \quad \int_0^1 \ln \Gamma(x) \cos 2a\pi x \, dx = \frac{1}{4a} \quad (\text{vezi și 4.332 3.}) \quad \text{NH 203 (6)}$$

$$4. \quad \int_0^1 \ln \Gamma(x) \cos (2a+1)\pi x \, dx = 0 \quad (\text{vezi și 4.332 4.}) \quad \text{NH 203 (6)}$$

4.33 Funcția $\psi(x)$.

$$4.331 \quad \int_1^x \psi(x) \, dx = \ln \Gamma(x).$$

4.332

$$1. \int_0^1 \phi(x) \sin \pi x \, dx = -\pi \int_0^1 \ln \Gamma(x) \cos \pi x \, dx = 0. \quad \text{NH 204}$$

$$2. \int_0^1 \phi(x) \sin^2 \pi x \, dx = -\pi \int_0^1 \ln \Gamma(x) \sin 2\pi x \, dx = \\ = -\frac{1}{2} [C + \ln(2\pi)]. \quad \text{NH 204}$$

$$3. \int_0^1 \phi(x) \sin \pi x \cos \pi x \, dx = -\pi \int_0^1 \ln \Gamma(x) \cos 2\pi x \, dx = -\frac{\pi}{4}. \quad \text{NH 204}$$

$$4. \int_0^1 \phi(x) \sin \pi x \sin 2\pi x \, dx = 0. \quad \text{NH 204 (8)}$$

$$5. \int_0^1 \phi(x) \sin \pi x \sin (2a+1)\pi x \, dx = \frac{1}{2} \ln \frac{a}{a+1}. \quad \text{NH 204 (8)}$$

4.4 FUNCȚII CILINDRICE

4.40 Funcții cilindrice

4.401

$$1. \int_0^x J_{\frac{1}{2}}(t) \, dt = 2S(\sqrt{x}) \quad \text{W 599 (4)}$$

$$2. \int_0^x J_{-\frac{1}{2}}(t) \, dt = 2C(\sqrt{x}) \quad \text{W 599 (3)}$$

$$3. \int_0^x J_p(x) \, dx = \frac{1}{\pi} \int_0^\pi \frac{\sin(x \sin \varphi)}{\sin \varphi} \cos p\varphi \, d\varphi. \quad \text{MO 42}$$

$$4.402 \quad \int_0^\infty N_{2p}(ax) \, dx = -\frac{\operatorname{tg} p\pi}{a} \quad \left[\operatorname{Re}(p) < \frac{1}{2} \right]. \quad \text{W 432 (7)}$$

4.403

$$1. \int_0^\infty J_{p+a}(at) J_{p-a-1}(\beta t) \, dt = \frac{\beta^{p-a-1} \Gamma(p)}{a^{p-a} \Gamma(p-a)} F\left(p, -a; p-a; \frac{\beta^2}{a^2}\right) \\ [0 < \beta < a];$$

$$= (-1)^a \frac{1}{2a} \quad [0 < \beta = a]; \\ = 0 \quad [0 < a < \beta] \quad [\operatorname{Re}(p) > 0].$$

MO 50

$$2. \int_0^\infty J_p(ax) J_{p-1}(\beta x) \, dx = \frac{\beta^{p-1}}{a^p} \quad \left. \begin{array}{l} [\beta < a] \\ = \frac{1}{2\beta} \quad [\beta = a] \\ = 0 \quad [\beta > a] \end{array} \right\} \begin{array}{l} [\operatorname{Re}(p) > 0]. \\ \text{W 444 (8), C 153 (40)m} \end{array}$$

$$3. \int_0^z J_p(x) J_q(z-x) dx = 2 \sum_{k=0}^{\infty} (-1)^k J_{p+q+2k+1}(z) \\ [\operatorname{Re}(p) > -1, \operatorname{Re}(q) > -1] \quad (\text{vezi și 4.456}). \quad W 414 (2)$$

$$4. \int_0^z J_p(x) J_{-p}(z-x) dx = \sin z \quad [-1 < \operatorname{Re}(p) < 1]. \quad W 415 (4)$$

$$5. \int_0^z J_p(x) J_{1-p}(z-x) dx = J_0(z) - \cos(z) \quad [-1 < \operatorname{Re}(p) < 2]. \quad W 415 (4)$$

$$4.404 \quad \int_0^z J_0(t) J_0\left(\frac{z}{t}\right) dt = J_0(2\sqrt{z}) \quad [z > 0]. \quad MO 46$$

$$4.405 \quad \int_0^z J_0(\sqrt{z^2 - x^2}) dx = \sin z. \quad MO 48$$

$$4.406 \quad \int_0^z K_{2p}(2x \operatorname{sh} x) dx = \frac{\pi^2}{8 \cos p\pi} (J_p^2(z) + N_p^2(z)) \\ \left[\operatorname{Re}(z) > 0, -\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2} \right]. \quad MO 45$$

4.407

$$1. \int_0^{\operatorname{Re} z} J_{2p}(2x \cos x) dx = \frac{\pi}{2} J_p^2(z) \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right]. \quad WW II 198$$

$$2. \int_0^{\operatorname{Re} z} J_{2a}(2x \sin x) dx = \frac{\pi}{2} J_a^2(z). \quad W 42 (1)$$

4.41 Funcții cilindrice și funcția putere

4.411

$$1. \int_0^{\infty} \frac{J_0(xt) t^2}{t^4 + 1} dt = \operatorname{ker} x. \quad MO 46$$

$$2. \int_0^{\infty} \frac{J_0(xt) t}{t^4 + 1} dt = -\operatorname{kei} x. \quad MO 46$$

$$3. \int_0^{\infty} \frac{N_0(ax)}{z^2 + x^2} dx = -\frac{K_0(az)}{z}. \quad W 466 (6)$$

$$4. \int_0^{\infty} \frac{J_p(ax)}{\sqrt{z^2 + x^2}} dx = I_p\left(\frac{az}{2}\right) K_{\frac{p}{2}}\left(\frac{az}{2}\right) \\ \left[\operatorname{Re}(p) > -1, |\arg z| < \frac{\pi}{2}, a > 0 \right]. \quad W 477 (3) \quad MO 44$$

4.412

$$1. \int_0^{\infty} \frac{J_p(ax) dx}{x^{p-q}} = \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)}{2^{p-q} a^{q-p+1} \Gamma\left(p - \frac{1}{2}q + \frac{1}{2}\right)} \left[-1 < \operatorname{Re}(q) < \operatorname{Re}(p) + \frac{1}{2} \right]$$

C f44 (5) W 428 (1)

$$2. \int_0^{\infty} \frac{J_p(at) t^{p+1}}{(t^2+x^2)^{q+1}} dt = \frac{a^q x^{p-q}}{2^q \Gamma(q+1)} K_{p-q}(ax) \\ \left[-1 < \operatorname{Re} p < \operatorname{Re} \left(2q + \frac{3}{2} \right), a > 0, x > 0 \right]. \quad \text{MO 43}$$

$$3. \int_0^{\infty} \frac{J_p(ax) x^{p+1}}{(x^4+4x^2)^{p+\frac{1}{2}}} dx = \frac{a^p \sqrt{\pi}}{2^{2p+2} \Gamma(p+\frac{1}{2})} J_p(ax) K_p(ax) \\ \left[\operatorname{Re}(p) > -\frac{1}{2}, |\arg z| < \frac{\pi}{4}, a > 0 \right]. \quad \text{W 478 (5)}$$

$$4. \int_0^{\infty} \frac{J_{p+1}(ax) x^{p+4}}{(x^4+4x^2)^{p+\frac{3}{2}}} dx = \frac{a^{p+1} \sqrt{\pi}}{2^{2p+2} \Gamma(p+\frac{3}{2})} J_p(ax) K_p(ax) \\ \left[\operatorname{Re}(p) > -\frac{5}{6}, |\arg z| < \frac{\pi}{4}, a > 0 \right]. \quad \text{W 478 (6)}$$

$$5. \int_0^{\infty} K_p(x) x^{q-1} dx = 2^{q-3} \Gamma\left(\frac{q-p}{2}\right) \Gamma\left(\frac{q+p}{2}\right) [p^2 < q^2]. \quad \text{W 424 (8)}$$

4.413 Dacă $J_p(\alpha) = J_p(\beta) = 0$ pentru $\alpha \neq \beta$ și $p > -1$, atunci

$$\int_0^1 x J_p(\alpha x) J_p(\beta x) dx = 0 \quad [\alpha \neq \beta]; \\ = \frac{1}{2} \{J_{p+1}(\alpha)\}^2 \quad [\alpha = \beta]. \quad \text{WW II 198}$$

$$4.414 \int_0^{\infty} \frac{J_0^2(ax)}{x^2+x^4} x dx = I_0(ax) K_0(ax). \quad \text{MO 46}$$

4.415

$$1. \int_0^{\infty} J_p(at) J_q(\beta t) t^{-s} dt = \frac{a^p \Gamma\left(\frac{p+q-s+1}{2}\right)}{2^s \beta^{p-s+1} \Gamma\left(\frac{-p+q+s+1}{2}\right) \Gamma(p+1)} \times \\ \times F\left(\frac{p+q-s+1}{2}, \frac{p-q-s+1}{2}; p+1; \frac{a^2}{\beta^2}\right) \\ [\operatorname{Re}(p+q-s+1) > 0, \operatorname{Re}(s) > -1, 0 < \alpha < \beta]. \quad \text{W 439 (2)m MO 49}$$

Dacă înlocuim, în același timp, p și q , precum și α și β , unul prin altul, atunci funcția din membrul doi al acestei egalități se schimbă. Așadar, ea este funcție de α/β , neanalitică pentru $\frac{\alpha}{\beta} = 1$. În cazul cînd $\alpha = \beta$, are loc egalitatea

$$2. \int_0^{\infty} J_p(at) J_q(at) t^{-s} dt = \frac{a^{s-1} \Gamma(s) \Gamma\left(\frac{p+q-s+1}{2}\right)}{2^s \Gamma\left(\frac{-p+q+s+1}{2}\right) \Gamma\left(\frac{p+q+s+1}{2}\right) \Gamma\left(\frac{p-q+s+1}{2}\right)} \\ [\operatorname{Re}(p+q+1) > \operatorname{Re}(s) > 0; a > 0]. \quad \text{MO 49}$$

$$3. \int_0^{\infty} J_p(at) J_q(\beta t) t^{-s} dt = \frac{\beta^s \Gamma\left(\frac{p+q-s+1}{2}\right)}{2^s a^{q-s+1} \Gamma\left(\frac{p-q+s+1}{2}\right) \Gamma(q+1)} \times \\ \times F\left(\frac{p+q-s+1}{2}, \frac{-p+q-s+1}{2}; q+1; \frac{\beta^2}{a^2}\right) \\ [\operatorname{Re}(p+q-s+1) > 0, \operatorname{Re}(s) > -1, 0 < \beta < \alpha].$$

Dacă $q-p+s+1$ (sau $p-q+s+1$) este un număr întreg și negativ, atunci membrul al doilea al egalității 4.415 1. (sau 4.415 3.) se anulează. Deosebit de importante sînt cazurile cînd funcția hipergeometrică F din 4.415 3. (sau din 4.415 1.) se reduce la o funcție elementară. MO 50

4.416 Cazuri particulare

$$1. \int_0^{\infty} J_p(ax) J_q(ax) \frac{dx}{x} = \frac{2}{\pi} \frac{\sin\left(\frac{p-q}{2}\pi\right)}{p^2 - q^2} \quad [\operatorname{Re}(p+q) > 0; \alpha > 0]. \text{ MO 50}$$

$$2. \int_0^{\infty} J_p(ax) J_p(\beta x) \frac{dx}{x} = \frac{1}{2p} \left(\frac{\beta}{\alpha}\right)^p \quad [\alpha > \beta];$$

$$= \frac{1}{2p} \left(\frac{\alpha}{\beta}\right)^p \quad [\alpha < \beta]$$

$$[\operatorname{Re}(p) > 0, \alpha > 0, \beta > 0]. \quad \text{C 152 (37)m W 443 (1)}$$

$$3. \int_0^{\infty} J_{p+1}(\alpha t) J_q(\beta t) t^{q-p} dt = 0 \quad [\alpha < \beta];$$

$$= \frac{(\alpha^2 - \beta^2)^{p-q} \beta^q}{2^{p-q} \alpha^{p+1} \Gamma(p-q+1)} \quad [\alpha > \beta]$$

$$[\operatorname{Re}(q) > -1; \operatorname{Re}(p-q+1) > 0, \alpha > 0, \beta > 0]. \text{ MO 51}$$

$$4. \int_0^{\infty} \frac{J_p(x) J_q(x)}{x^{p+q}} dx = \frac{\sqrt{\pi} \Gamma(p+q)}{2^{p+q} \Gamma\left(p+q+\frac{1}{2}\right) \Gamma\left(p+\frac{1}{2}\right) \Gamma\left(q+\frac{1}{2}\right)}$$

$$[\operatorname{Re}(p+q) > 0]. \quad \text{C 147 (17) W 434 (1)}$$

Vezi și 4.403.

4.417

$$1. \int_0^{\infty} J_p(ux) J_q(vx) J_q(wx) x^{1-p} dx =$$

$$= \frac{2^{1-p} (vw)^q}{u^p \Gamma(p-q) \Gamma\left(q+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^A (u^2 - v^2 - w^2 + 2vw \cos \varphi)^{p-q-1} \sin^{2q} \varphi d\varphi,$$

unde $A=0$ pentru $u^2 < (v-w)^2$; $A = \arccos \frac{v^2 + w^2 - u^2}{2vw}$ pentru

$(v-w)^2 < u^2 < (v+w)^2$; $A = \pi$ pentru $(v+w)^2 < u^2$ MO 52

$$\left[u > 0, v > 0, w > 0, \operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(q) > -\frac{1}{2} \right].$$

În particular, pentru $p=q$,

$$2. \int_0^{\infty} J_q(ux) J_q(vx) J_q(wx) x^{1-q} dx = \frac{2^{q-1} \Delta^{2q-1}}{\sqrt{\pi} (uvw)^q \Gamma\left(q+\frac{1}{2}\right)},$$

unde Δ este aria unui triunghi avînd laturile egale cu u, v, w ; în cazul cînd segmentele de lungime u, v, w nu pot forma un triunghi, valoarea integralei este egală cu zero $[\operatorname{Re}(q) > -\frac{1}{2}]$. MO 52

$$3. \int_0^{\infty} \prod_{k=1}^n J_p(u_k x) \frac{dx}{x^{ap-2p-1}} = 0$$

$[u_1, v, u_2, \dots > \dots > u_n > 0; u_1 > u_2 + u_3 + \dots + u_n, \operatorname{Re}(p) > -1]. \text{ MO 52}$

4.418

$$1. \int_0^z J_p(x) J_q(z-x) \frac{dx}{x} = \frac{J_{p+q}(z)}{p} \quad [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > -1]. \quad W 415 (3)$$

$$2. \int_0^z \frac{J_p(x)}{x} \frac{J_q(z-x)}{z-x} dx = \left(\frac{1}{p} + \frac{1}{q} \right) \frac{J_{p+q}(z)}{z} \\ [\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0]. \quad W 415 (5)$$

4.419

$$1. \int_0^\infty J_p(\alpha \sqrt{x^2+z^2}) \frac{x^{2q+1}}{\sqrt{(x^2+z^2)^p}} dx = \frac{2^q \Gamma(q+1)}{\alpha^{q+1} z^{p-q-1}} J_{p-q-1}(\alpha z) \\ [\alpha > 0, \operatorname{Re}\left(\frac{1}{2}p - \frac{1}{4}\right) > \operatorname{Re}(q) > -1]. \quad W 457 (5)$$

$$2. \int_0^\infty \frac{J_p(\alpha \sqrt{t^2+1})}{\sqrt{t^2+1}} dt = -\frac{\pi}{2} J_{\frac{p}{2}}\left(\frac{\alpha}{2}\right) N_{\frac{p}{2}}\left(\frac{\alpha}{2}\right) \quad [\operatorname{Re}(p) > -1, x > 0]. \\ MO 46$$

$$3. \int_0^\infty K_p(\alpha \sqrt{x^2+z^2}) \frac{x^{2q+1}}{\sqrt{(x^2+z^2)^p}} dx = \frac{2^q \Gamma(q+1)}{\alpha^{q+1} z^{p-q-1}} K_{p-q-1}(\alpha z) \\ [\alpha > 0, \operatorname{Re}(q) > -1]. \quad W 457 (6)$$

$$4. \int_0^\infty J_p(\beta x) \frac{J_{q-1}(\alpha \sqrt{x^2+z^2})}{(x^2+z^2)\sqrt{(x^2+z^2)^q}} x^{p+1} dx = \frac{\alpha^{q-1} z^p}{2^{q-1} \Gamma(q)} K_p(\beta z) \\ [\alpha < 1, R(q+2) > R(p) > -1]. \quad W 459 (11)$$

$$5. \int_0^\infty J_p(\beta x) \frac{J_q(\alpha \sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^q}} x^{p-1} dx = \frac{2^{p-1} \Gamma(p)}{\beta^p} \frac{J_q(\alpha z)}{z^q} \\ [\operatorname{Re}(q+2) > \operatorname{Re}(p) > 0, \beta > \alpha > 0]. \quad W 459 (12)$$

$$6. \int_0^\infty J_p(\beta x) \frac{J_q(\alpha \sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^q}} x^{p+1} dx = 0 \quad [0 < \alpha < \beta]; \\ = \frac{\beta^p}{\alpha^q} \left\{ \frac{\sqrt{\alpha^2 - \beta^2}}{z} \right\}^{q-p-1} J_{q-p-1}\{z \sqrt{\alpha^2 - \beta^2}\} \quad [\alpha > \beta > 0]; \\ [\operatorname{Re}(q) > \operatorname{Re}(p) > -1]. \quad W 455 (1)$$

$$7. \int_0^\infty J_p(\beta x) \frac{K_q(\alpha \sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^q}} x^{p+1} dx = \frac{\beta^p}{\alpha^q} \left(\frac{\sqrt{\alpha^2 + \beta^2}}{z} \right)^{q-p-1} K_{q-p-1}(z \sqrt{\alpha^2 + \beta^2}) \\ [\alpha > 0, \beta > 0, \operatorname{Re}(p) > -1, |\arg z| < \frac{\pi}{2}]. \quad C 151 (31), W 456 (2)$$

$$8. \int_0^\infty J_p(\beta t) \frac{K_q(\alpha \sqrt{t^2-y^2})}{\sqrt{(t^2-y^2)^q}} t^{p+1} dt = \frac{\pi}{2} \frac{\beta^p}{\alpha^q} \left\{ \frac{\sqrt{\alpha^2 + \beta^2}}{y} \right\}^{q-p-1} \times \\ \times \exp\left[-\frac{\pi}{2}\left(q-p-\frac{1}{2}\right)\right] \{J_{q-p-1}[y \sqrt{\alpha^2 + \beta^2}] - i N_{q-p-1}[y \sqrt{\alpha^2 + \beta^2}]\}$$

[$\operatorname{Re}(q) < 1$; aici se presupune că drumul de integrare nu conține singularitatea $t=y$, care este înlăturată prin alegerea convenabilă a drumului (singularitatea $t=y$ este lăsată sub acest drum), și că semnul lui $\sqrt{t^2-y^2}$ se alege în așa fel, încît expresia considerată să fie pozitivă pentru $t>y$; $\alpha>0$; $\beta>0$, $y>0$]. W 456 (8)

$$9. \int_0^\infty J_p(ux) K_q(v \sqrt{x^2-y^2}) (x^2-y^2)^{-\frac{q}{2}} x^{p+1} dx = \\ = \frac{\pi}{2} \exp \left[-i\pi \left(q-p-\frac{1}{2} \right) \right] \cdot \frac{u^p}{v^q} \cdot \left[\frac{\sqrt{u^2+v^2}}{y} \right]^{q-p-1} H_{q-p-1}^{(2)}(y \sqrt{u^2+v^2})$$

[$\operatorname{Re}(q) < 1$, $\operatorname{Re}(p) > -1$, $u > 0$, $v > 0$, $y > 0$; $\arg \sqrt{x^2-y^2} = 0$ pentru $x > y$; dacă $x < y$, atunci $\arg(x^2-y^2)^\sigma = \pi\sigma$, unde $\sigma = \frac{1}{2}$ sau $\sigma = -\frac{q}{2}$]. MO 43

$$10. \int_0^\infty J_p(ux) H_q^{(2)}(v \sqrt{x^2+y^2}) (x^2+y^2)^{-\frac{q}{2}} x^{p+1} dx = \\ = \frac{u^p}{v^q} \left[\frac{\sqrt{v^2-u^2}}{y} \right]^{q-p-1} H_{q-p-1}^{(2)}(y \sqrt{v^2-u^2}) \quad [u < v]$$

[$\operatorname{Re}(q) < \operatorname{Re}(p)$, $\operatorname{Re}(p) > -1$, $u > 0$, $v > 0$, $y > 0$; $\arg \sqrt{v^2-u^2} = 0$ pentru $v > u$, $\arg(v^2-u^2)^\sigma = -\pi\sigma$ pentru $v < u$, unde $\sigma = \frac{1}{2}$ sau $\sigma = \frac{q-p-1}{2}$]. MO 43

$$11. \int_0^\infty J_p(\beta x) J_q(\alpha \sqrt{x^2+z^2}) J_q(\gamma \sqrt{x^2+z^2}) \frac{x^{p-1}}{(x^2+z^2)^q} dx = \\ = \frac{2^{p-1} \Gamma(p)}{\beta^p} \frac{J_q(\alpha z)}{z^q} \frac{J_q(\gamma z)}{z^q} \\ \left[\alpha > 0; \beta > \alpha + \gamma; \gamma > 0, \operatorname{Re} \left(2q + \frac{5}{2} \right) > \operatorname{Re}(p) > 0 \right]. \quad \text{W 459 (14)}$$

$$12. \int_0^\infty J_p(\beta t) \prod_{k=1}^a J_q(\alpha_k \sqrt{t^2+x^2}) \sqrt{(t^2+x^2)^{-aq}} t^{p-1} dt = \\ = 2^{p-1} \beta^{-p} \Gamma(p) \prod_{k=1}^a [x^{-q} J_q(\alpha_k x)] \\ \left[x > 0, \alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_a > 0, \beta > \sum_{k=1}^a \alpha_k; \right. \\ \left. \operatorname{Re} \left(aq + \frac{1}{2} a + \frac{1}{2} \right) > \operatorname{Re}(p) > 0 \right] \quad \text{MO 43} \\ \text{Vezi și } 5.311, 5.313, 5.314 \text{ 3.}$$

4.42 Funcții cilindrice și funcția exponențială

4.421

$$1. \int_0^\infty e^{-\alpha x} J_p(\beta x) dx = \frac{[\sqrt{\alpha^2 + \beta^2} - \alpha]^p}{\beta^p \sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re}(p) > -1, \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|].$$

W 422 (8)

$$2. \int_0^\infty e^{-\alpha x} N_0(\beta x) dx = \frac{-2}{\pi \sqrt{\alpha^2 + \beta^2}} \ln \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{\beta} \quad [\alpha \geq 0, \beta > 0]. \quad \text{MO 47}$$

$$3. \int_0^{\infty} e^{-x \operatorname{ch} u} K_p(x) dx = \frac{\pi}{\sin p\pi} \frac{\operatorname{sh} pu}{\operatorname{sh} u} \\ | -1 < \operatorname{Re}(p) < 1; \operatorname{Re}(\operatorname{ch} u) > -1]. \quad \text{W 424 (9)}$$

$$4. \int_0^{\infty} e^{-\alpha x} K_0(\beta x) dx = \frac{\arccos \frac{\alpha}{\beta}}{\sqrt{\beta^2 - \alpha^2}} \quad [0 < \alpha < \beta]; \quad \text{W 424} \\ = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \ln \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right) \quad [0 \leq \beta < \alpha]. \quad \text{MO 48}$$

4.422

$$1. \int_0^{\infty} e^{-q^2 x^2} J_{2p}(\alpha x) dx = \frac{\sqrt{\pi}}{2q} \exp \left(-\frac{\alpha^2}{8q^2} \right) I_p \left(\frac{\alpha^2}{8q^2} \right) \\ \left[|\arg q| < \frac{\pi}{4}, \operatorname{Re}(p) > -\frac{1}{2} \right]. \quad \text{C 146 (14) W 432 (5)}$$

$$2. \int_0^{\infty} e^{-q^2 x^2} N_{2p}(\alpha x) dx = -\frac{\sqrt{\pi}}{2q} \exp \left(-\frac{\alpha^2}{8q^2} \right) \left\{ I_p \left(\frac{\alpha^2}{8q^2} \right) \operatorname{tg} p\pi + \right. \\ \left. + \frac{1}{\pi} K_p \left(\frac{\alpha^2}{8q^2} \right) \sec p\pi \right\} \quad \left[-\frac{1}{2} < \operatorname{Re}(q) < \frac{1}{2} \right]. \quad \text{C 146 (15) W 432 (6)}$$

$$4.423 \int_0^{\infty} e^{-xz} J_{p+\frac{1}{2}} \left(\frac{x^2}{2} \right) dx = \frac{\Gamma(p+1)}{\sqrt{\pi}} D_{-p-1} \left(ze^{\frac{\pi}{4}i} \right) D_{-p-1} \left(ze^{-\frac{\pi}{4}i} \right) [\operatorname{Re}(p) > -1].$$

MO 122

4.424

$$1. \int_{-\infty}^{\infty} e^{itx} H_0^{(1)}(r \sqrt{\alpha^2 - t^2}) dt = -2i \frac{e^{ia \sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}} \\ [0 \leq \arg \sqrt{\alpha^2 - t^2} < \pi, 0 \leq \arg \alpha < \pi; r, x \text{ reali}] \quad \text{MO 49}$$

$$2. \int_{-\infty}^{\infty} e^{-itx} H_0^{(2)}(r \sqrt{\alpha^2 - t^2}) dt = 2i \frac{e^{-ia \sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}} \\ [-\pi < \arg \sqrt{\alpha^2 - t^2} \leq 0, -\pi < \arg \alpha \leq 0, r, x \text{ reali}] \quad \text{MO 49}$$

4.425

$$1. \int_0^{\infty} K_{q-p}(2z \operatorname{sh} x) e^{(p+q)x} dx = \frac{\pi^2}{4 \sin[(p-q)\pi]} [J_p(z) N_q(z) - J_q(z) N_p(z)] \\ [\operatorname{Re}(z) > 0, -1 < \operatorname{Re}(p-q) < 1]. \quad \text{MO 44}$$

$$2. \int_0^{\infty} K_0(2z \operatorname{sh} x) e^{-2px} dx = -\frac{\pi}{4} \left\{ J_p(z) \frac{\partial N_p(z)}{\partial p} - N_p(z) \frac{\partial J_p(z)}{\partial p} \right\}$$

Vezi și 6.232–6.234 $[\operatorname{Re}(z) > 0]$.

MO 44

4.43 Funcții cilindrice, funcția exponențială și funcția putere

4.431

$$1. \int_0^{\infty} (J_0(x) - e^{-ax}) \frac{dx}{x} = \ln 2a \quad [a > 0]. \quad \text{NI 66 (13)}$$

$$2. \int_0^{\infty} \frac{e^{i(u+x)}}{u+x} J_0(x) dx = \frac{\pi}{2} i H_0^{(1)}(u). \quad \text{MO 44}$$

$$3. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} I_p(x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} Q_{p-\frac{1}{2}}(\operatorname{ch} \alpha). \quad \text{W 424 (5)}$$

$$4.432 \quad \int_0^{\infty} e^{-\alpha x} J_p(\beta x) x^{q-1} dx =$$

$$= \frac{\left(\frac{\beta}{2\alpha}\right)^p \Gamma(p+q)}{\alpha^q \Gamma(p+1)} F\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p+1; -\frac{\beta^2}{\alpha^2}\right); \quad \text{W 421 (2)}$$

$$= \frac{\left(\frac{\beta}{2\alpha}\right)^p \Gamma(p+q)}{\alpha^q \Gamma(p+1)} \left(1 + \frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-q} F\left(\frac{p-q+1}{2}, \frac{p-q}{2}+1; p+1; -\frac{\beta^2}{\alpha^2}\right);$$

$$\quad \text{W 421 (3)}$$

$$= \frac{\left(\frac{\beta}{2}\right)^p \Gamma(p+q)}{\sqrt{(\alpha^2 + \beta^2)^{p+q}} \Gamma(p+1)} F\left(\frac{p+q}{2}, \frac{\alpha-q+p}{2}; p+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$[\operatorname{Re}(p+q) > 0, \operatorname{Re}(\alpha + i\beta) > 0, \operatorname{Re}(\alpha - i\beta) > 0]. \quad \text{W 421 (3)}$$

Cazuri particulare

$$1. \int_0^{\infty} e^{-\alpha x} J_p(\beta x) x^p dx = \frac{(2\beta)^p \Gamma\left(p + \frac{1}{2}\right)}{\sqrt{\pi} (\alpha^2 + \beta^2)^{p+\frac{1}{2}}} [\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|].$$

$$\quad \text{W 422 (5)}$$

$$2. \int_0^{\infty} e^{-\alpha x} J_p(\beta x) x^{p+1} dx = \frac{2\alpha (2\beta)^p \Gamma\left(p + \frac{3}{2}\right)}{\sqrt{\pi} (\alpha^2 + \beta^2)^{p+\frac{3}{2}}}$$

$$[\operatorname{Re}(p) > -1, \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|]. \quad \text{W 422 (6)}$$

$$3. \int_0^{\infty} e^{-\alpha x} J_p(\beta x) \frac{dx}{x} = \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^{\frac{p}{2}}}{p \beta^p}$$

$$[\operatorname{Re}(p) > 0; \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|] \quad (\text{cf. 4.421 1.}). \quad \text{W 422 (7)}$$

4.433

$$1. \int_0^{\infty} e^{-\alpha x} N_p(\beta x) x^{q-1} dx =$$

$$= \operatorname{ctg} p\pi \frac{\left(\frac{\beta}{2}\right)^p \Gamma(p+q)}{\sqrt{(\alpha^2 + \beta^2)^{p+q}} \Gamma(p+1)} F\left(\frac{p+q}{2}, \frac{p-q+1}{2}; p+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right) -$$

$$- \operatorname{cosec} p\pi \frac{\left(\frac{\beta}{2}\right)^{-p} \Gamma(q-p)}{\sqrt{(\alpha^2 + \beta^2)^{q-p}} \Gamma(1-p)} F\left(\frac{q-p}{2}, \frac{1-p-q}{2}; 1-p; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$[\operatorname{Re}(q) > |\operatorname{Re}(p)|, \operatorname{Re}(\alpha \pm i\beta) > 0]. \quad \text{W 421 (4)}$$

$$2. \int_0^{\infty} e^{-x \cos \beta} J_{-p}(x \sin \beta) x^q dx = \Gamma(q-p+1) P_q^p(\cos \beta) \\ \left[0 < \beta < \frac{\pi}{2}, \operatorname{Re}(q-p) > -1 \right]. \quad \text{W424(3) WW II 175m}$$

$$3. \int_0^1 e^{\frac{xu}{2}} (1-x)^{2p-1} x^{q-p} J_{q-p}\left(\frac{ixu}{2}\right) dx = \\ = 2^{2(p-q)} e^{\frac{\pi}{2}(q-p)} \frac{B(2p, 2q-2p+1)}{\Gamma(q-p+1)} \frac{e^{\frac{u}{2}}}{u^{\frac{p+1}{2}}} M_{p,q}(u). \quad \text{MO 118m}$$

$$4. \int_0^{\infty} e^{-xz} K_{p+\frac{1}{2}}(x) x^{q-\frac{1}{2}} dx = \sqrt{\frac{\pi}{2}} \Gamma(p+q+1) \Gamma(q-p) (z^2-1)^{-\frac{q}{2}} \mathfrak{P}_p^{-q}(z) \\ [\operatorname{Re}(z) > 0, \operatorname{Re}(p+q+1) > 0, \operatorname{Re}(q-p) > 0]. \quad \text{MO 90}$$

$$5. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} I_p(x \operatorname{sh} \alpha) x^q dx = \Gamma(p+q+1) \mathfrak{P}_q^{-p}(\operatorname{ch} \alpha) \\ [\operatorname{Re}(q) > -2]. \quad \text{W 423 (1)}$$

$$6. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} K_p(x \operatorname{sh} \alpha) x^q dx = \frac{\sin q\pi}{\sin(p+q)\pi} \Gamma(q-p+1) \Omega_q^p(\operatorname{ch} \alpha) \\ [\operatorname{Re}(q+1) > |\operatorname{Re}(q)|]. \quad \text{W 423 (2)}$$

1.434

$$1. \int_0^{\infty} e^{-x^2} x^{2\alpha+\beta+1} J_{\beta}(2x \sqrt{z}) dx = \frac{1}{2} e^{-z} z^{\frac{\beta}{2}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} {}_1F_1(-\alpha; \beta+1; z) \\ \left[\operatorname{Re}(\alpha+\beta) > -1, |\arg z| < \frac{\pi}{2} \right]. \quad \text{MO 115m}$$

$$2. \int_0^{\infty} J_p(\alpha x) e^{-\beta^2 x^2} x^{q-1} dx = \\ = \frac{\Gamma\left(\frac{p+q}{2}\right) \left(\frac{\alpha}{2\beta}\right)^p}{2\beta^q \Gamma(p+1)} \exp\left(-\frac{\alpha^2}{4\beta^2}\right) {}_1F_1\left(\frac{p+q}{2}+1; p+1; \frac{\alpha^2}{4\beta^2}\right) \\ \left[\operatorname{Re}(p+q) > 0, |\arg \beta| < \frac{\pi}{4}, \alpha > 0 \right]. \quad \text{W 431(3)m}$$

În particular,

$$3. \int_0^{\infty} J_p(\alpha x) e^{-\beta^2 x^2} x^{p+1} dx = \frac{\alpha^p}{(2\beta^2)^{p+1}} \exp\left(-\frac{\alpha^2}{4\beta^2}\right) \\ \left[\operatorname{Re}(p) > -1; |\arg \beta| < \frac{\pi}{4} \right]. \quad \text{W 431 (4)}$$

$$4. \int_0^{\infty} J_0(\beta t) \frac{\exp(-\alpha \sqrt{t^2-y^2})}{\sqrt{t^2-y^2}} t dt = \frac{\exp(\mp ty \sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}},$$

unde semnul se alege după cum punctul singular a fost ocolit deasupra axei sau dedesubtul ei; $\alpha > 0, \beta > 0, y > 0$.

W 457 (4)

$$4.435 \quad \int_0^{\infty} e^{-2\alpha x} J_p(\beta x) J_q(\beta x) x^{p+q} dx =$$

$$= \frac{\Gamma\left(p+q+\frac{1}{2}\right) \beta^{p+q}}{\sqrt{\pi^3}} \int_0^{\frac{\pi}{2}} \frac{\cos^{p+q} \varphi \cos(p-q)\varphi}{(\alpha^2 + \beta^2 \cos^2 \varphi)^{p+q} \sqrt{\alpha^2 + \beta^2 \cos^2 \varphi}} d\varphi$$

$$\left[\operatorname{Re}(\alpha) > |\operatorname{Im} \beta|, \operatorname{Re}(p+q) > -\frac{1}{2} \right]. \text{W 427 (1)}$$

Cazuri particulare

$$1. \quad \int_0^{\infty} e^{-2\alpha x} J_0(\beta x) J_1(\beta x) x dx = \frac{K\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right) - E\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)}{2\pi\beta \sqrt{\alpha^2 + \beta^2}}. \text{W 427 (2)}$$

$$2. \quad \int_0^{\infty} e^{-2\alpha x} J_1^2(\beta x) dx = \frac{(2\alpha^2 + \beta^2) K\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right) - 2(\alpha^2 + \beta^2) E\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)}{\pi\beta^2 \sqrt{\alpha^2 + \beta^2}}.$$

W 428 (3)

$$3. \quad \int_0^{\infty} e^{-2\alpha x} J_0^2(\beta x) dx = \frac{K\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)}{\pi \sqrt{\alpha^2 + \beta^2}}.$$

W 428 (4)

$$4. \quad \int_0^{\infty} e^{-2\alpha x} I_0(\beta x) I_1(\beta x) x dx = \frac{1}{2\pi\beta} \left\{ \frac{\alpha}{\alpha^2 - \beta^2} E\left(\frac{\beta}{\alpha}\right) - \frac{1}{\alpha} K\left(\frac{\beta}{\alpha}\right) \right\}$$

[Re(α) > Re(β)]. W 428 (5).

$$4.436 \quad \int_0^{\infty} e^{-\frac{1}{2}x^2} J_p(\alpha x) J_p(\beta x) x dx = \frac{1}{2\rho^2} \exp\left(-\frac{\alpha^2 + \beta^2}{4\rho^2}\right) I_p\left(\frac{\alpha\beta}{2\rho^2}\right)$$

[Re(p) > -1, |arg p| < $\frac{\pi}{4}$, α > 0, β > 0]. C 146 (16)m W 433 (1)

$$4.437 \quad \int_0^{\infty} \exp\left\{-\frac{1}{2}\left[x + \frac{z^2 + \zeta^2}{x}\right]\right\} K_p\left(\frac{z\zeta}{x}\right) \frac{dx}{x} = 2K_p(z) K_p(\zeta)$$

[|arg z| < π, |arg ζ| < π, |arg(z+ζ)| < $\frac{\pi}{4}$] (vezi și 6.439). MO 45

(vezi și 5.256–5.259, 5.312, 5.314).

4.44 Funcții cilindrice și funcții hiperbolice

4.441

$$1. \quad \int_0^{\infty} K_0(2z \operatorname{sh} t) \operatorname{ch} 2pt dt = \frac{\pi^2}{8} \{J_p^2(z) + N_p^2(z)\} \quad [\operatorname{Re}(z) > 0]. \quad \text{MO 44}$$

$$2. \quad \int_0^{\infty} K_{2q}(z \operatorname{sh} 2t) \operatorname{cth}^{2p} t dt =$$

$$= \frac{1}{4z} \Gamma\left(\frac{1}{2} + q - p\right) \Gamma\left(\frac{1}{2} - q - p\right) W_{p,q}(iz) W_{p,q}(-iz)$$

[|arg z| ≤ $\frac{\pi}{2}$, |Re(q)| + Re(p) < $\frac{1}{2}$]. MO 119

$$3. \int_0^{\infty} t h^{2p} t \{J_{2q}(u \operatorname{sh} 2t) \sin [(q-p)\pi] + N_{2q}(u \operatorname{sh} 2t) \cos [(q-p)\pi]\} dt = \\ = -\frac{W_{p,q}(u) W_{-p,q}(u)}{2u} \left[|\operatorname{Re}(q)| - \operatorname{Re}(p) < \frac{1}{2}, u > 0 \right]. \quad \text{MO 119}$$

4.45 Funcții cilindrice și funcții trigonometrice

4.451

$$1. \int_0^z \sin(z-x) J_0(x) dx = z J_1(z); \quad \text{W 415 (2)}$$

$$2. \int_0^z \cos(z-x) J_0(x) dx = z J_0(z). \quad \text{W 415 (1)}$$

$$3. \int_0^z \sin(z-x) J_1(x) dx = \sin z - z J_0(z). \quad \text{W 416 (3)}$$

$$4. \int_0^{\infty} J_p(ax) \sin \beta x dx = \begin{cases} \frac{\sin\left(p \arcsin \frac{\beta}{a}\right)}{\sqrt{a^2 - \beta^2}} & [\beta < a]; \\ \infty \text{ sau } 0 & [\beta = a]; \\ \frac{a^p \cos \frac{p\pi}{2}}{\sqrt{\beta^2 - a^2} (\beta + \sqrt{\beta^2 - a^2})^p} & [\beta > a]. \end{cases} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ [\operatorname{Re}(p) > -2]. \end{array} \quad \text{W 444 (4)}$$

$$5. \int_0^{\infty} J_p(ax) \cos \beta x dx = \begin{cases} \frac{\cos\left(p \arcsin \frac{\beta}{a}\right)}{\sqrt{a^2 - \beta^2}} & [\beta < a]; \\ \infty \text{ sau } 0 & [\beta = a]; \\ \frac{a^p \sin \frac{p\pi}{2}}{\sqrt{\beta^2 - a^2} (\beta + \sqrt{\beta^2 - a^2})^p} & [\beta > a]. \end{cases} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ [\operatorname{Re}(p) > -1]. \end{array} \quad \text{W 444 (5)}$$

$$6. \int_0^{\infty} K_0(\beta x) \sin ax dx = \frac{1}{\sqrt{a^2 + \beta^2}} \ln \left(\frac{\beta}{a} + \sqrt{\frac{\beta^2}{a^2} + 1} \right) \\ [a > 0; \beta > 0]. \quad \text{W 425 (11) m MO 48}$$

$$7. \int_0^{\infty} K_0(\beta x) \cos ax dx = \frac{\pi}{2\sqrt{a^2 + \beta^2}} \\ [\alpha \text{ și } \beta \text{ reali}; \beta > 0], \quad \text{W 425 (10) m MO 48}$$

4.452

$$1. \int_0^{\infty} J_0(a\sqrt{x^2 + z^2}) \cos \beta x dx = \frac{\cos z \sqrt{a^2 - \beta^2}}{\sqrt{a^2 - \beta^2}} \quad [0 < \beta < a, z > 0]; \\ = 0 \quad [0 < a < \beta, z > 0]. \quad \text{MO 47m}$$

$$2. \int_0^{\infty} N_0(\alpha \sqrt{x^2 + z^2}) \cos \beta x \, dx = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \sin(z \sqrt{\alpha^2 - \beta^2}) \quad [0 < \beta < \alpha, z > 0];$$

$$= -\frac{1}{\sqrt{\beta^2 - \alpha^2}} \exp(-z \sqrt{\alpha^2 - \beta^2}) \quad [0 < \alpha < \beta, z > 0]. \quad \text{MO 47m}$$

$$3. \int_0^z J_0(\sqrt{z^2 - x^2}) \cos x \, dx = \frac{1}{\sqrt{2}} \sin(z \sqrt{2}). \quad \text{MO 48}$$

4.453

$$1. \int_0^{\frac{\pi}{2}} J_{p+q}(2x \cos x) \cos[(p-q)x] \, dx = \frac{\pi}{2} J_p(z) J_q(z)$$

$$[\operatorname{Re}(p+q) > -1]. \quad \text{MO 42}$$

$$2. \int_0^{\pi} J_0(2x \cos x) \cos 2ax \, dx = (-1)^a \pi J_a^2(z). \quad \text{MO 45}$$

$$3. \int_0^{\pi} J_0(2x \sin x) \cos 2ax \, dx = \pi J_a^2(z). \quad \text{MO 45}$$

$$4. \int_0^{\frac{\pi}{2}} J_p(qx \sin t) \cos(qx \cos t) \, dt =$$

$$= \frac{\pi}{2} J_{\frac{p}{2}}\left(q \frac{\sqrt{x^2 + z^2} + x}{2}\right) J_{\frac{p}{2}}\left(q \frac{\sqrt{x^2 + z^2} - x}{2}\right)$$

$$[\operatorname{Re}(p) > -1, \operatorname{Re}(z) > 0]. \quad \text{MO 46}$$

$$5. \int_0^{\frac{\pi}{2}} J_{p-\frac{1}{2}}(x \sin t) \sin^{p+\frac{1}{2}} t \, dt = \sqrt{\frac{\pi}{2x}} J_p(x)$$

[p poate fi zero, un număr natural, $1/2$, un număr natural plus $1/2$; $x > 0$]
MO 45

$$6. \int_0^{\frac{\pi}{2}} J_p(z \sin x) \sin^{p+1} x \cos^{2q+1} x \, dx = \frac{2^q \Gamma(q+1)}{z^{q+1}} J_{p+q+1}(z)$$

$$[\operatorname{Re}(p) > -1, \operatorname{Re}(q) > -1]. \quad \text{C 148 (23) W 406 (1)}$$

$$7. \int_0^{\frac{\pi}{2}} J_p(z \sin x) \sin^p x \cos^{2p} x \, dx = 2^{p-1} \sqrt{\pi} \Gamma\left(p + \frac{1}{2}\right) z^{-p} J_p^2\left(\frac{z}{2}\right)$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}\right]. \quad \text{MO 42m}$$

$$4.454 \int_0^{\frac{\pi}{2}} J_p(z \sin x) I_q(z \cos x) t g^{p+1} x \, dx = \frac{\left(\frac{z}{2}\right)^p \Gamma\left(\frac{q-p}{2}\right)}{\Gamma\left(\frac{q+p}{2} + 1\right)} J_q(z)$$

$$[\operatorname{Re}(p) > \operatorname{Re}(q) > -1]. \quad \text{W 407 (4)}$$

$$4.455 \quad \int_0^{\frac{\pi}{2}} J_p(z_1 \sin x) J_q(z_2 \cos x) \sin^{p+1} x \cos^{q+1} x \, dx = \frac{z_1^p z_2^q J_{p+q+1}(\sqrt{z_1^2 + z_2^2})}{\sqrt{z_1^2 + z_2^2}^{p+q+1}} \\ [\operatorname{Re}(p) > -1, \operatorname{Re}(q) > -1]. \quad \text{W 410 (1)}$$

$$4.456 \quad \int_0^{\frac{\pi}{2}} J_p(z \cos^2 x) J_q(z \sin^2 x) \sin x \cos x \, dx = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k J_{p+q+2k+1}(z) \\ [\operatorname{Re}(p) > -1, \operatorname{Re}(q) > -1] \quad (\text{vezi și 4.403 3.}). \text{W 414 (1)}$$

4.46 Funcții cilindrice, funcții trigonometrice și funcția putere

$$4.461 \quad \int_0^{\infty} [J_0(x) - \cos ax] \frac{dx}{x} = \ln 2a. \quad \text{NI 66 (13)}$$

4.462

$$1. \quad \left. \begin{aligned} \int_0^{\infty} J_p(ax) \sin \beta x \frac{dx}{x} &= \frac{1}{p} \sin \left(p \arcsin \frac{\beta}{a} \right) & [\beta < a] \\ &= \frac{a^p \sin \frac{p\pi}{2}}{p(\beta + \sqrt{\beta^2 - a^2})^p} & [\beta > a] \end{aligned} \right\} \quad \begin{aligned} &[\operatorname{Re}(p) > -1]. \\ &\text{W 443 (2)} \end{aligned}$$

$$2. \quad \int_0^{\infty} J_0(ax) \sin x \frac{dx}{x} = \frac{\pi}{2} \quad [0 < a < 1]; \\ = \operatorname{arccosec} a \quad [a > 1]. \quad \text{WW II 200}$$

$$3. \quad \int_0^{\infty} J_0(x) \sin \beta x \frac{dx}{x} = \frac{\pi}{2} \quad [\beta > 1]; \\ = \arcsin \beta \quad [\beta^2 < 1]; \\ = -\frac{\pi}{2} \quad [\beta < -1].$$

$$4. \quad \left. \begin{aligned} \int_0^{\infty} J_p(ax) \cos \beta x \frac{dx}{x} &= \frac{1}{p} \cos \left(p \arcsin \frac{\beta}{a} \right) & [\beta < a] \\ &= \frac{a^p \cos \frac{p\pi}{2}}{p(\beta + \sqrt{\beta^2 - a^2})^p} & [\beta > a] \end{aligned} \right\} \quad \begin{aligned} &[\operatorname{Re}(p) > 0]. \\ &\text{W 443 (3)} \end{aligned}$$

4.463

$$1. \quad \int_0^{\infty} \frac{\sin ax}{\beta^2 + x^2} J_0(ux) \, dx = \frac{\operatorname{sh} a\beta}{\beta} K_0(\beta u) \quad [a > 0, \beta > 0, u > a]. \quad \text{MO 46}$$

$$2. \quad \int_0^{\infty} \frac{\cos ax}{\beta^2 + x^2} J_0(ux) \, dx = \frac{\pi}{2} \frac{e^{-a\beta}}{\beta} I_0(\beta u) \quad [a > 0, \beta > 0, -a < u < a]. \text{MO 46}$$

4.464

$$1. \quad \int_0^{\infty} \frac{\sin(x+t)}{x+t} J_0(t) \, dt = \frac{\pi}{2} J_0(x) \quad [x > 0]. \quad \text{W 475 (4)}$$

$$2. \int_0^{\infty} \frac{\cos(x+t)}{x+t} J_0(t) dt = -\frac{\pi}{2} N_0(x) \quad [x > 0]. \quad \text{W 475 (5)}$$

4.465

$$1. \int_0^z J_p(x) \sin(z-x) \frac{dx}{x} = \frac{2}{p} \sum_{k=0}^{\infty} (-1)^k J_{p+2k+1}(z) \quad [\operatorname{Re}(p) > 0]. \quad \text{W 416 (4)}$$

$$2. \int_0^z J_p(x) \cos(z-x) \frac{dx}{x} = \\ = \frac{1}{p} J_p(z) + \frac{2}{p} \sum_{k=1}^{\infty} (1-)^k J_{p+2k}(z) \quad [\operatorname{Re}(p) > 0]. \quad \text{W 416 (5)}$$

$$4.466 \quad \int_0^{\infty} \sin \left[x \cos \varphi + \frac{\pi}{2} (q-p-1) \right] K_q(x \cos \varphi) x^p dx = \\ = -\frac{\pi}{2} \Gamma(p+q+1) P_p^{-q}(\cos \varphi) \quad [0 < \varphi < \pi; \operatorname{Re}(p+q+1) > 0]. \quad \text{MO 89}$$

4.467

$$1. \int_0^{\infty} J_0(\beta x) \sin(\alpha x^2) x dx = \frac{1}{2\alpha} \cos \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \beta > 0]. \quad \text{MO 47}$$

$$2. \int_0^{\infty} J_0(\beta x) \cos(\alpha x^2) x dx = \frac{1}{2\alpha} \sin \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \beta > 0]. \quad \text{MO 47}$$

Funcții cilindrice, funcții trigonometrice și funcția exponențială

$$4.468 \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(p-q)x} \left(\frac{2 \cos x}{R^2 e^{ix} + r^2 e^{-ix}} \right)^{\frac{p+q}{2}} J_{p+q}(z \sqrt{2 \cos x [R^2 e^{ix} + r^2 e^{-ix}]}) dx = \\ = \pi R^{-p} r^{-q} J_p(Rz) J_q(rz) \\ [\operatorname{Re}(p+q) > -1; R > 0, r > 0] \quad (\text{vezi și 6.541}). \quad \text{MO 42}$$

4.47 Funcții cilindrice, logaritmul și arctangenta

4.471

$$1. \int_0^{\infty} J_0(x) \ln x dx = -C - \ln 2. \quad \text{W 430 (4)}$$

$$2. \int_0^{\infty} J_1(tx) \ln \sqrt{1+t^2} dt = \frac{2}{x} \operatorname{ker} x. \quad \text{MO 46}$$

4.472

$$\int_0^{\infty} J_1(tx) \arctg t^2 dt = -\frac{2}{x} \operatorname{kei} x. \quad \text{MO 46}$$

4.48 Funcții cilindrice, sinusul integral etc.

4.481

$$1. \int_0^{\infty} \text{Ei}(-x) J_0(2\sqrt{zx}) dx = \frac{e^z - 1}{z}. \quad \text{NI 60 (4)}$$

$$2. \int_0^{\infty} \text{si}(x) J_0(2\sqrt{zx}) dx = -\frac{\sin z}{z}. \quad \text{NI 60 (6)}$$

$$3. \int_0^{\infty} \text{ci}(x) J_0(2\sqrt{zx}) dx = \frac{\cos z - 1}{z}. \quad \text{NI 60 (5)}$$

$$4. \int_0^{\infty} \text{Ei}(-x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\text{Ei}(-z) - \mathcal{O} - \ln z}{\sqrt{z}}. \quad \text{NI 60 (7)}$$

$$5. \int_0^{\infty} \text{si}(x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \text{si}(z)}{\sqrt{z}}. \quad \text{NI 60 (9)}$$

$$6. \int_0^{\infty} \text{ci}(x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\text{ci}(z) - \mathcal{O} - \ln z}{\sqrt{z}}. \quad \text{NI 60 (8)}$$

$$7. \int_0^{\infty} \text{Ei}(-x) N_0(2\sqrt{zx}) dx = \frac{\mathcal{O} + \ln z - e^z \text{Ei}(-z)}{\pi z}. \quad \text{NI 63 (5)}$$

Integralele produselor dintre funcțiile cilindrice și polinoamele $C_n^p(x)$, vezi 4.62.

Integralele produselor dintre funcțiile cilindrice și alte funcții speciale, vezi 5.315 — 5.317.

4.49 Integrarea funcțiilor cilindrice după indice

Pentru $-\pi < t < \pi$ $\text{Re}(p+q) > 1$ și $x > 0$, $y > 0$,

$$1. \int_{-\infty}^{\infty} \frac{J_{p+t}(x) J_{q-t}(y)}{x^{p+t} y^{q-t}} e^{it\xi} dt =$$

$$= \left\{ \frac{2 \cos \frac{1}{2} \xi}{x^{\frac{p}{2}} e^{-\frac{1}{2} t \xi} + y^{\frac{q}{2}} e^{\frac{1}{2} t \xi}} \right\}^{\frac{p+q}{2}} e^{\frac{q-p}{2} t \xi} J_{p+q} \left[\sqrt{2 \left(x^2 e^{-\frac{1}{2} t \xi} + y^2 e^{\frac{1}{2} t \xi} \right) \cos \frac{1}{2} \xi} \right].$$

Pentru toate celelalte valori (reale) ale lui t , această integrală se anulează. W 495 (1)

$$2. \int_{-\infty}^{\infty} J_{p+t}(x) J_{q-t}(x) dt = J_{p+q}(2x) \quad [\text{Re}(p+q) > 1, x > 0]. \quad \text{W 495 (2)}$$

4.5 FUNCȚII SFERICE

4.51 Funcții sferice

$$4.510 \quad \int_{\cos \varphi}^1 P_p(x) dx = \sin \varphi P_p^{-1}(\cos \varphi). \quad \text{MO 90}$$

4.511

$$1. \quad \int_{\cos \varphi}^1 x P_p(x) dx = \frac{\sin \varphi}{(p-1)(p-2)} [\sin \varphi P_p(\cos \varphi) + \cos \varphi P_p^1(\cos \varphi)]. \quad \text{MO 90}$$

$$2. \quad \int_u^\infty \Omega_p(x) (x-u)^{q-1} dx = \Gamma(q) e^{q\pi i} (u^2-1)^{\frac{q}{2}} \Omega_p^{-q}(u) \\ [u > 1, \operatorname{Re}(q) > 0, \operatorname{Re}(p-q+1) > 0]. \quad \text{MO 90}$$

$$4.512 \quad \int_0^\infty \frac{e^{-2x} x^{-p-\frac{1}{2}} \left[\frac{2(t+x)(u+x)}{tu} - 1 \right]}{[(t+x)(u+x)]^{\frac{1}{2}p+\frac{1}{4}}} dx = \frac{(tu)^{\frac{1}{4}-\frac{1}{2}p}}{\pi e^{s+u}} K_p(t) K_p(u) \\ [t > 0, u > 0]. \quad \text{MO 90}$$

$$4.513 \quad \left. \begin{aligned} \int_{-1}^1 P_a^c(x) P_b^c(x) dx &= 0 & [a \neq b]; \\ &= \frac{2}{2a+1} \frac{(a+c)!}{(a-c)!} & [a=b]. \end{aligned} \right\} \quad [c < a, c < b]. \\ \text{Sm III 185 WW II 120}$$

4.514

$$1. \quad \int_0^1 [P_a^b(x)]^2 \frac{dx}{1-x^2} = \frac{(a+b)!}{2b \cdot (a-b)!} \quad [0 < b \leq a]. \quad \text{MO 74}$$

$$2. \quad \int_{-1}^1 P_a^b(x) P_a^c(x) \frac{dx}{1-x^2} = 0 \quad [0 \leq b \leq a, 0 \leq c \leq a, b \neq c]. \quad \text{MO 74}$$

4.515

$$1. \quad \int_1^\infty e^{-xu} (x^2-1)^{\frac{q}{2}} \mathfrak{P}_p^{-q}(x) dx = \sqrt{\frac{2}{\pi}} z^{-q-\frac{1}{2}} K_{p+\frac{1}{2}}(u) \\ [\operatorname{Re}(u) > 0, \operatorname{Re}(q) > -1]. \quad \text{MO 90}$$

$$2. \quad \int_1^\infty e^{-\frac{1}{2}zx} \left(\frac{x+1}{x-1} \right)^{\frac{q}{2}} \mathfrak{P}_{p-\frac{1}{2}}^q(x) dx = \frac{2}{z} W_{q,p}(z) \\ \left[\operatorname{Re}(q) < 1, p - \frac{1}{2} \neq 0, \pm 1, \pm 2, \dots \right]. \quad \text{MO 118}$$

$$4.516 \quad \int_0^\pi P_p^{-q}(\cos \varphi) \sin^{r-1} \varphi d\varphi = \\ = \frac{\pi \Gamma\left(\frac{r+q}{2}\right) \Gamma\left(\frac{r-q}{2}\right)}{2^q \Gamma\left(\frac{r+p+1}{2}\right) \Gamma\left(\frac{r-p}{2}\right) \Gamma\left(\frac{p+q}{2}+1\right) \Gamma\left(\frac{q-p+1}{2}\right)} [\operatorname{Re}(r \pm q) > 0]. \quad \text{MO 90}$$

4.517

$$1. \int_0^{2\pi} d\varphi \int_0^{\pi} \exp[\pm (b_1 - b_2) \varphi i] P_{a_1}^{b_1}(\cos \varphi) P_{a_2}^{b_2}(\cos \varphi) \sin \varphi d\varphi = 0$$

[$a_1 \neq a_2$ $b_1 \neq b_2$]. MO 74

$$2. \int_0^{2\pi} d\varphi \int_0^{\pi} [P_a^b(\cos \varphi)]^2 \sin \varphi d\varphi = \frac{4\pi}{2a+1} \frac{(a+b)!}{(a-b)!}.$$

MO 74

Vezi și 6.835.

4.52 Polinoame Legendre

4.520

$$1. \int_{-1}^1 P_a(x) P_b(x) dx = 0 \quad [a \neq b];$$

$$= \frac{2}{2a+1} \quad [a = b] \quad \left. \vphantom{\int_{-1}^1} \right\} \text{[ortogonalitate].}$$

Sm III 382 WW II 94

$$2. \int_0^1 P_a(x) P_b(x) dx = \frac{1}{2a+1} \quad [a = b];$$

$$= 0 \quad [a - b \text{ par}, a \neq b];$$

$$= \frac{(-1)^{\frac{1}{2}(a+b+1)} a! b!}{2^{a+b-1} (a-b) (a+b+1) \left[\left(\frac{a}{2} \right)! \left(\frac{b-1}{2} \right)! \right]^2}$$

[a par, b impar]. WW II 96

$$3. \int_0^{2\pi} P_{2a}(\cos \varphi) d\varphi = 2\pi \left[\left(\frac{2a}{a} \right) 2^{-2a} \right]^2.$$

MO 70

4.521

$$1. \int_{-1}^1 x^b P_a(x) dx = 0 \quad [b < a].$$

$$2. \int_0^1 x^p P_{2a}(x) dx = \frac{p(p-2) \dots (p-2a+2)}{(p+2a+1)(p+2a-1) \dots (p+1)} \quad [\operatorname{Re}(p) > -1].$$

H 39 MO 71

$$3. \int_0^1 x^p P_{2a+1}(x) dx = \frac{(p-1)(p-3) \dots (p-2a+1)}{(p+2a+2)(p+2a) \dots (p+2)} \quad [\operatorname{Re}(p) > -1].$$

H 40 MO 71

$$4. \int_{-1}^1 (1-x^2)^b P_{2a}(x) dx = \frac{2b^a}{(b-a)(2b+2a+1)} \int_{-1}^1 (1-x^2)^{b-1} P_{2a}(x) dx$$

[$a < b$]. WW II 102

$$5. \int_{-1}^1 (1+px^2)^{-a-\frac{3}{2}} P_{2a}(x) dx = \frac{2}{2a+1} (-p)^a (1+p)^{-a-\frac{1}{2}} \quad [|p| < 1].$$

MO 74

4.522

$$1. \int_0^1 x^2 P_{a+1}(x) P_{a-1}(x) dx = \frac{a(a+1)}{(2a-1)(2a+1)(2a+3)}. \quad \text{WW II 129}$$

$$2. \int_{-1}^1 \frac{1}{1-x} \{P_a(x) P_{a-1}(z) - P_{a-1}(x) P_a(z)\} dx = -\frac{2}{a}. \quad \text{WW II 131}$$

$$4.523 \quad \int_{-1}^1 (\operatorname{ch} 2p - x)^{-\frac{1}{2}} P_a(x) dx = \frac{2\sqrt{2}}{2a+1} \exp[-(2a+1)p] \quad [p > 0]. \quad \text{WW II 96}$$

4.524

$$1. \int_0^{2\pi} P_{2a+1}(\cos \varphi) \cos \varphi d\varphi = \frac{\pi}{2^{4a+1}} \binom{2a}{a} \binom{2a+2}{a+1}. \quad \text{MO 70}$$

$$2. \int_0^{\pi} P_a(\cos \varphi) \sin b\varphi d\varphi = \frac{2(b-a+1)(b-a+3)\dots(b+a-1)}{(b-a)(b-a+2)\dots(b+a)}$$

$$= 0 \quad [b \leq a \text{ sau } b+a \text{ par}]; \quad [b > a \text{ și } b+a \text{ impar}]; \quad \text{MO 71}$$

$$3. \int_0^{\pi} P_a(1-2\sin^2 x \sin^2 \varphi) \sin x dx = 2 \frac{\sin(2a+1)\varphi}{(2a+1)\sin \varphi}. \quad \text{MO 71}$$

$$4.525 \quad \int_{-1}^1 P_a(x) \arcsin x dx = 0 \quad [a \text{ par}];$$

$$= \pi \frac{(a-2)!}{\sqrt{2^{a+1}} \cdot \left(\frac{a+1}{2}\right)!} [a \text{ impar}]. \quad \text{WW II 129}$$

4.526

$$1. \int_{-1}^1 \frac{dP_a(x)}{dx} \frac{dP_b(x)}{dx} dx = a(a+1) [b \geq a, a \text{ și } b \text{ fiind ori amândoi pari, ori amândoi impari}]. \quad \text{WW II 100}$$

$$2. \int_{-1}^1 \frac{d^2 P_a(x)}{dx^2} \frac{d^2 P_b(x)}{dx^2} dx = [1 + (-1)^{a+b}] \times$$

$$\times \frac{(b-1)b(b+1)(b+2)}{48} [3a(a+1) - b(b+1) + 6] \quad [b \geq a].$$

WW II 100

4.527

$$1. \int_{-1}^1 (1-x^2) \left[\frac{dP_a(x)}{dx} \right]^2 dx = \frac{2a(a+1)}{2a+1}. \quad \text{WW II 102}$$

$$2. \int_{-1}^1 (1-x^2)^2 \frac{dP_a(x)}{dx} \frac{d^2 P_b(x)}{dx^2} dx = 8a(a+1) [b-a \text{ par și pozitiv}];$$

$$= -\frac{2a(a^2-1)(a-2)}{2a+1} \quad [a=b];$$

$$= 0 \text{ [pentru toate celelalte valori ale lui } a \text{ și } b].$$

WW II 129

4.6 POLINOAME $C_n^p(t)$

$$4.60 \quad \int_0^\pi \sin^{2p}\varphi C_a^p(\cos\varphi) C_b^p(\cos\varphi) d\varphi = \frac{\pi \Gamma(2p+a)}{2^{2p-1} (p+a) \cdot a! [\Gamma(p)]^2} \quad [a=b];$$

$$= 0 \quad [a \neq b]. \text{ MO 98}$$

$$4.61 \quad \int_0^\pi e^{iz \cos \varphi} C_a^p(\cos \varphi) \sin^{2p} \varphi d\varphi = \frac{2^p \Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma(2p+a)}{a! \Gamma(2p)} \cdot i^a \cdot \frac{J_{p+a}(z)}{z^p}$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2} \right]. \text{ MO 99}$$

4.62

$$1. \quad \int_0^\pi e^{iz \cos \varphi \cos \varphi} J_{p-\frac{1}{2}}(z \sin \varphi \sin \varphi) C_a^p(\cos \varphi) \sin^{p+\frac{1}{2}} \varphi d\varphi =$$

$$= \sqrt{\frac{2\pi}{z}} i^a \sin^{p-\frac{1}{2}} \varphi C_a^p(\cos \varphi) J_{p+a}(z) \quad \left[\operatorname{Re}(p) > -\frac{1}{2}, |\arg z| < \pi \right]. \text{ MO 99}$$

$$2. \quad \int_0^\pi J_{p-\frac{1}{2}}(z \sin x \sin \varphi) \cos(z \cos x \cos \varphi) C_a^p(\cos x) \sin^{p+\frac{1}{2}} x dx =$$

$$= (-1)^{\frac{a}{2}} \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \sin^{p-\frac{1}{2}} \varphi C_a^p(\cos \varphi) J_{p+a}(z) \quad [a \text{ par}];$$

$$= 0 \quad [a \text{ impar}]. \text{ W 414 (2)}$$

$$3. \quad \int_0^\pi J_{p-\frac{1}{2}}(z \sin x \sin \varphi) \sin(z \cos x \cos \varphi) C_a^p(\cos x) \sin^{p+\frac{1}{2}} x dx =$$

$$= 0 \quad [a \text{ par}];$$

$$= (-1)^{\frac{a-1}{2}} \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \sin^{p-\frac{1}{2}} \varphi C_a^p(\cos \varphi) J_{p+a}(z) \quad [a \text{ impar}]. \text{ W 414 (3)}$$

$$4.63 \quad \int_{-1}^1 (1-x^2)^{\frac{p-1}{2}} F(x) C_a^{\frac{p}{2}}(x) dx =$$

$$= \frac{1}{a!} \frac{p(p+1) \dots (p+a-1)}{(p+1)(p+3) \dots (p+2a-1)} \int_{-1}^1 (1-x^2)^{a+\frac{p-1}{2}} \frac{d^a F}{dx^a} dx,$$

unde $F(x)$ este o funcție oarecare avînd derivatele, pînă la cea de ordinul $(n+1)$ inclusiv, continue. MO 100

4.7 POLINOAME ORTOGONALE

4.71 Polinoame Cebîșev

$$1. \quad \int_{-1}^1 \frac{T_a(x) T_b(x)}{\sqrt{1-x^2}} dx = 0 \quad [a \neq b];$$

$$= \frac{\pi}{2} \quad [a = b \neq 0];$$

$$= \pi \quad [a = b = 0].$$

MO 104

$$2. \int_{-1}^1 \frac{U_a(x) U_b(x)}{\sqrt{1-x^2}} dx = 0 \quad [a \neq b \text{ sau } a=b=0];$$

MO 105

$$= \frac{\pi}{2} \quad [a=b \neq 0].$$

4.72 Polinoame Hermite

$$1. \int_{-\infty}^{\infty} e^{-x^2} H_a(x) H_b(x) dx = 0 \quad [a \neq b];$$

Sm III 567

$$= 2^a a! \sqrt{\pi} \quad [a=b].$$

Sm III 568

Vezi și 5.261.

4.73 Polinoame Jacobi

$$1. \int_0^1 x^{q-1} (1-x)^{p-q} G_a(p, q; x) G_b(p, q; x) dx = 0 \quad [a \neq b];$$

$$= \frac{a!}{p+2aq(q+1) \dots (q+a-1)\Gamma(p+a)} \frac{\Gamma(q)\Gamma(p-q+a+1)}{\Gamma(p+q+1)}$$

$$[a=b, \operatorname{Re}(q) > 0, \operatorname{Re}(p-q) > -1].$$

MO 108

4.74 Polinoame Laguerre

$$4.741 \quad \int_0^{\infty} e^{-x} x^p L_a^{(p)}(x) L_b^{(p)}(x) dx = 0 \quad [a \neq b];$$

$$= a! \Gamma(p+a+1) \quad [a=b].$$

Sm III 573

4.742

$$1. \int_0^t L_n(x) dx = L_n(t) - L_{n+1}(t).$$

MO 110

$$2. \sum_{k=0}^{\infty} \left[\int_0^t L_k(x) dx \right]^2 = e^t - 1 \quad [t > 0].$$

MO 110

Vezi și 5.262, 5.315.

4.8 FUNCȚII HIPERGEOMETRICE DEGENERATE

4.81 Funcții Whittaker

$$1. \int_0^t e^{-\frac{1}{2}(t-x)} (t-x)^{2\lambda-1} x^{\mu-\lambda-\frac{1}{2}} M_{0, \mu-\lambda}(x) dx =$$

$$= t^{\mu-\frac{1}{2}} B(2\lambda, 2\mu-2\lambda+1) M_{\lambda, \mu}(t)$$

$$\left[t > 0, \operatorname{Re}(\mu-\lambda) > -\frac{1}{2}, \operatorname{Re}(\lambda) > 0 \right].$$

MO 118

Vezi și 5.264, 5.265.

4.82 Funcțiile cilindrului parabolic

$$4.821. \int_{-\infty}^{\infty} D_a(x) D_b(x) dx = 0 \quad (a \neq b); \quad \text{WW II 158}$$

$$= a! (2\pi)^{\frac{1}{2}} \quad [a = b]. \quad \text{WW II 158}$$

$$4.822. \int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2} (x-z)^{-1} D_a(x) dx = \pm i e^{\mp \alpha \pi} \sqrt{2\pi} \cdot a! e^{-\frac{1}{4}z^2} D_{-a-1}(\mp iz)$$

[semnul se alege după cum partea imaginară a lui z este pozitivă sau negativă].
WW II 162

$$4.823. \int_0^{\infty} e^{-\frac{3}{4}x^2} x^q D_{q+1}(z) dz = (\sqrt{2})^{-1-q} \Gamma(q+1) \sin\left(\frac{1}{4} - \frac{1}{4}q\right) \pi$$

[Re(q) > -1].

WW II 161

$$4.824. \int_{-\infty}^{(0+)} e^{\left(\frac{1}{4}-\alpha\right)x^2} x^q D_p(z) dz = \frac{\pi^{\frac{3}{2}} 2^{\frac{1}{2}p-q} e^{\pi i\left(q-\frac{1}{2}\right)}}{\Gamma(-q) \Gamma\left(\frac{1}{2}q - \frac{1}{2}p + 1\right) \alpha^{\frac{1}{2}(q+1)}} \times$$

$$\times F\left(-\frac{1}{2}p, \frac{1}{2}q + \frac{1}{2}; \frac{1}{2}q - \frac{1}{2}p + 1; 1 - \frac{1}{2}\alpha^{-1}\right) \left[|\arg \alpha| < \frac{\pi}{2}\right]. \quad \text{WW II 161}$$

Vezi și 5.266, 5.317.

4.9 INTEGRAREA COSINUȘILOR POLINOAMELOR $T_a(t, x)$

4.900 Polinoamele $T_a(t, x)$ se definesc prin egalitatea

$$T_a(t, x) = t^a F\left(-\frac{a}{2}, \frac{1-a}{2}; 1-a; -\frac{4x}{t^2}\right).$$

Astfel

1. $T_2(t, x) = t^2 + 2x.$
2. $T_3(t, x) = t^3 + 3tx.$
3. $T_4(t, x) = t^4 + 4t^2x + 2x^2.$
4. $T_5(t, x) = t^5 + 5t^3x + 5tx^2.$

Pentru $x > 0$ au loc egalitățile

$$4.911. \int_0^{\infty} \cos T_a(t, -x) dt = \frac{\pi \sqrt{x}}{2a \sin \frac{\pi}{2a}} \left[J_{\frac{1}{a}}\left(2x^{\frac{a}{2}}\right) + J_{-\frac{1}{a}}\left(2x^{\frac{a}{2}}\right) \right]. \quad \text{MO 41}$$

$$4.912. \int_0^{\infty} \cos T_{2b}(t, x) dt = \frac{\pi \sqrt{x}}{4b \sin \frac{\pi}{4b}} \left[J_{-\frac{1}{2b}}\left(2x^b\right) - J_{\frac{1}{2b}}\left(2x^b\right) \right]. \quad \text{MO 41}$$

$$4.913. \int_0^{\infty} \cos T_{2b+1}(t, x) dt = \frac{2 \sqrt{x} \cos \frac{\pi}{4b+2}}{2b+1} K_{\frac{1}{2b+1}}\left(2x^{b+\frac{1}{2}}\right). \quad \text{MO 41}$$

5. TRANSFORMĂRI INTEGRALE ȘI IMAGINILE LOR

5.0 INTRODUCERE

5.00 Funcțiile de variabilă reală $f(t)$, cu care vom avea de a face în această secțiune, trebuie să satisfacă următoarele condiții:

1. Pe orice segment finit care face parte din domeniul de definiție al funcției $f(t)$, această funcție trebuie să satisfacă condițiile lui Dirichlet (vezi 0.322).

2. Pentru orice valoare t_0 din domeniul de definiție al funcției, trebuie să fie satisfăcută relația

$$f(t_0) = \frac{1}{2} \left[\lim_{t \rightarrow t_0 + 0} f(t) + \lim_{t \rightarrow t_0 - 0} f(t) \right].$$

3. În prima dintre aceste condiții, cuvintele „să satisfacă condițiile lui Dirichlet” se pot înlocui cu cuvintele „să aibă o variație mărginită” (vezi 0.323).

5.01 Unele relații între diferitele transformări integrale.

Dacă

$$g(p) = \int_0^{\infty} e^{-pt} f(t) dt; \quad F(u) = \int_0^{\infty} J_0(2\sqrt{tu}) f(t) dt;$$

$$v(z) = \int_0^{\infty} e^{-pz} g(p) dp,$$

atunci

$$g(p) = \frac{1}{p} \int_0^{\infty} e^{-\frac{u}{p}} F(u) du;$$

$$v(z) = \int_0^{\infty} \frac{f(t)}{z+t} dt = 2 \int_0^{\infty} K_0(2\sqrt{zu}) F(u) du.$$

MO 160

5.1 TRANSFORMATĂ FOURIER

5.10 Fie $F(x) = F_1(x) + iF_2(x)$, unde $F_1(x)$ și $F_2(x)$ sînt funcții reale de variabilă reală x , definite pentru toate valorile lui x ($-\infty < x < +\infty$) și satisfăcînd condițiile menționate la 5.00. Să presupunem, totodată, că

$\int_{-\infty}^{\infty} [F(x)] dx$ converge. Atunci, pentru toate valorile reale ale lui y , există funcția

$$1. f(y) = \int_{-\infty}^{\infty} e^{ixy} F(x) dx$$

și pentru fiecare punct în care funcția $F(x)$ este finită,

$$2. F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} f(y) dy,$$

integrala $\int_{-\infty}^{\infty}$ urmînd a fi înțeleasă ca valoare principală.

Funcția $f(y)$ se numește *spectrul* funcției $F(x)$.

MO 160

În exemplele care urmează, s-au notat cu literele p, q parametrii reali și pozitivi.

$F(x)$		$f(y)$	
5.111			
1.	$\frac{1}{x^2 + q^2}$	$\frac{\pi}{q} e^{-q y }$	MO 162
2.	$\frac{1}{x^2 + q^2} \operatorname{sgn} x$	$\frac{i}{q} [e^{-qy} \operatorname{Ei}(qy) - e^{qy} \operatorname{Ei}(-qy)]$	MO 162
5.112	$\frac{x}{x^2 + q^2} \operatorname{sgn} x$	$-\frac{1}{q} [e^{-qy} \operatorname{Ei}(qy) + e^{qy} \operatorname{Ei}(-qy)]$	MO 162
5.113			
1.	$\frac{1}{\sqrt{q^2 - x^2}}$	$\pi J_0(qy)$	MO 163
	0		
2.	0	$\pi N_0(q y)$	MO 163
	$-\frac{1}{\sqrt{x^2 - q^2}}$		
3.	0	$\pi i \operatorname{sgn} y \cdot J_0(qy)$	MO 163
	$\frac{\operatorname{sgn} x}{\sqrt{x^2 - q^2}}$		
5.114	$\frac{1}{\sqrt{x^2 + q^2}}$	$2K_0(q y)$	MO 161
5.115			
1.	$\frac{1}{\sqrt{ x }}$	$\sqrt{\frac{2\pi}{ y }}$	MO 161
2.	$\frac{\operatorname{sgn} x}{\sqrt{ x }}$	$i \operatorname{sgn} y \cdot \sqrt{\frac{2\pi}{ y }}$	MO 161

$F(x)$	$f(y)$
5.116	
1. $\frac{1}{2} \frac{1}{(x^2 + g^2)^{r + \frac{1}{2}}}$	$\left(\frac{ y }{2g}\right)^r \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(r + \frac{1}{2}\right)} K_r(g y)$ MO 163
	$\left[\operatorname{Re}(r) > -\frac{1}{2}\right]$
2. $\frac{1}{ x ^r}$	$2\Gamma(1-r) \sin\left(\frac{r\pi}{2}\right) \frac{1}{ y ^{1-r}}$ MO 161
	$[0 < \operatorname{Re}(r) < 1]$
3. $\frac{\operatorname{sgn} x}{ x ^r}$	$2i \operatorname{sgn} y \cdot \Gamma(1-r) \cos\left(\frac{r\pi}{2}\right) \frac{1}{ y ^{1-r}}$ MO 161
	$[0 < \operatorname{Re}(r) < 2]$
5.117 $\begin{cases} e^{i\omega x} & [g < x < h] \\ 0 & [x < g \text{ sau } x > h] \end{cases}$	$i \frac{\exp[ig(\omega + y)] - \exp[ih(\omega + y)]}{\omega + y}$ MO 164
5.118 $\begin{cases} e^{-\beta x} e^{i\omega x} & [x > 0] \\ 0 & [x < 0] \end{cases}$	$\frac{i}{\omega + y + i\beta}$ MO 161
5.119 $e^{-\lambda x^2}$	$\sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{y^2}{4\lambda}\right)$ MO 161
	$[\operatorname{Re}(\lambda) > 0, \operatorname{Re}\sqrt{\lambda} > 0]$
5.121 $e^{i q^2 x^2}$	$\begin{cases} \frac{2}{3q} \sqrt{\frac{y}{q}} K_{\frac{1}{3}}\left(\frac{2y}{3q} \sqrt{\frac{y}{3q}}\right) & [y > 0] \\ \frac{2\pi}{3q} \sqrt{\frac{ y }{3q}} \left[J_{\frac{1}{3}}\left(\frac{2 y }{3q} \sqrt{\frac{ y }{3q}}\right) + \right. \\ \quad \left. + J_{-\frac{1}{3}}\left(\frac{2 y }{3q} \sqrt{\frac{ y }{3q}}\right) \right] & [y < 0]. \end{cases}$ MO 164
5.122	
1. $\frac{\exp(-p\sqrt{q^2 + x^2})}{\sqrt{q^2 + x^2}}$	$2K_0(q\sqrt{p^2 + y^2})$ MO 164
2. $\frac{\exp(ip\sqrt{q^2 + x^2})}{\sqrt{q^2 + x^2}}$	$\pi i H_0^{(1)}(q\sqrt{p^2 - y^2})$ MO 164
5.123	
1. $\frac{e^{-q x }}{\sqrt{ x }}$	$\sqrt{2\pi} \frac{\sqrt{\sqrt{q^2 + y^2} + q}}{\sqrt{q^2 + y^2}}$ MO 162
2. $\frac{e^{-q x }}{\sqrt{ x }} \operatorname{sgn} x$	$i \sqrt{2\pi} \operatorname{sgn} y \frac{\sqrt{\sqrt{q^2 + y^2} - q}}{\sqrt{q^2 + y^2}}$ MO 162

$F(x)$	$f(y)$
5.124	
1. $\frac{\exp\left(-\frac{q}{\sqrt{ x }}\right)}{\sqrt{ x }}$	$\sqrt{\frac{2\pi}{ y }} (\cos \sqrt{2q y } - \sin \sqrt{2q y })$ MO 163
2. $\frac{\exp\left(-\frac{q}{\sqrt{ x }}\right)}{\sqrt{ x }} \operatorname{sgn} x$	$i \operatorname{sgn} y \cdot \sqrt{\frac{2\pi}{ y }} (\cos \sqrt{2q y } + \sin \sqrt{2q y })$ MO 163
5.125	
1. $\frac{\operatorname{sh} qx}{\operatorname{sh} \pi x}$	$\frac{\sin q}{\operatorname{ch} y + \cos q}$ MO 163 [$-\pi < q < \pi$]
2. $\frac{\operatorname{sh} qx}{\operatorname{ch} \pi x}$	$2i \frac{\sin \frac{q}{2} \operatorname{sh} \frac{y}{2}}{\operatorname{ch} y + \cos q}$ MO 163 [$-\pi < q < \pi$]
3. $\frac{\operatorname{ch} qx}{\operatorname{ch} \pi x}$	$2 \frac{\cos \frac{q}{2} \operatorname{ch} \frac{y}{2}}{\operatorname{ch} y + \cos q}$ MO 163 [$-\pi < q < \pi$]
5.126 $\frac{1}{\operatorname{ch} qx}$	$\frac{\pi}{q} \frac{1}{\operatorname{ch} \frac{\pi y}{2q}}$ MO 161
5.127 $\frac{\operatorname{ch}(p\sqrt{q^2-x^2})}{\sqrt{q^2-x^2}}$ 0	$\left. \begin{array}{l} \{ x < q \} \\ \{ x > q \} \end{array} \right\} \pi J_0(q\sqrt{y^2+p^2})$ MO 164
5.128	
1. $\sin qx^2$	$\sqrt{\frac{\pi}{q}} \cos\left(\frac{y^2}{4q} + \frac{\pi}{4}\right)$ MO 161
2. $\cos qx^2$	$\sqrt{\frac{\pi}{q}} \cos\left(\frac{y^2}{4q} - \frac{\pi}{4}\right)$ MO 161
5.129	
1. $\operatorname{sgn} x \cdot \sin qx^2$	$i \sqrt{\frac{2\pi}{q}} \left\{ \cos \frac{y^2}{4q} C\left(\frac{y}{2\sqrt{q}}\right) + \sin \frac{y^2}{4q} S\left(\frac{y}{2\sqrt{q}}\right) \right\}$ MO 162 m
2. $\operatorname{sgn} x \cdot \cos qx^2$	$i \sqrt{\frac{2\pi}{q}} \left\{ \sin \frac{y^2}{4q} C\left(\frac{y}{2\sqrt{q}}\right) - \cos \frac{y^2}{4q} S\left(\frac{y}{2\sqrt{q}}\right) \right\}$ MO 162 m

$F(x)$	$f(y)$
5.131 $\frac{\sin qx}{x}$	$\begin{cases} \pi & y < q \\ 0 & y > q \end{cases}$ MO 161
5.132	
1. $\frac{\sin qx}{\sqrt{ x }}$	$i \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{ y-q }} - \frac{1}{\sqrt{ y+q }} \right)$ MO 162
2. $\frac{\cos qx}{\sqrt{ x }}$	$\sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{ y-q }} + \frac{1}{\sqrt{ y+q }} \right)$ MO 162
5.133	
1. $\frac{\sin^2 qx}{x}$	$\begin{cases} \operatorname{sgn} y \cdot i \frac{\pi}{2} & y < 2q \\ 0 & y > 2q \end{cases}$ MO 162
2. $\frac{\sin^2 qx}{x^2}$	$\begin{cases} \pi \left(a - \frac{1}{2} y \right) & y < 2q \\ 0 & y > 2q \end{cases}$ MO 161
5.134	
1. $\frac{\sin qx^2}{x}$	$-i\pi \left[S\left(\frac{y}{2\sqrt{q}}\right) - C\left(\frac{y}{2\sqrt{q}}\right) \right]$ MO 162m
2. $\frac{\sin qx^2}{x^2}$	$\pi y \left[S\left(\frac{y}{2\sqrt{q}}\right) - C\left(\frac{y}{2\sqrt{q}}\right) \right] +$ $+ 2\sqrt{\pi q} \sin\left(\frac{y^2}{4q} + \frac{\pi}{4}\right)$ MO 162m
5.135	
1. $\frac{\sin p \sqrt{q^2 + x^2}}{\sqrt{q^2 + x^2}}$	$\begin{cases} 0 & y > p \\ \pi J_0(q \sqrt{p^2 - y^2}) & y < p \end{cases}$ MO 164
2. $\frac{\cos(p \sqrt{q^2 + x^2})}{\sqrt{q^2 + x^2}}$	$\begin{cases} 2K_0(q \sqrt{y^2 - p^2}) & y > p \\ -\pi N_0(q \sqrt{p^2 - y^2}) & y < p \end{cases}$ MO 164
5.136	
1. $\frac{\cos(p \sqrt{q^2 - x^2})}{\sqrt{q^2 - x^2}}$ $\begin{cases} x < q \\ 0 & x > q \end{cases}$	$\pi J_0(q \sqrt{p^2 + y^2})$ MO 164
2. $\operatorname{sgn} x \cdot \frac{\cos(p \sqrt{x^2 - q^2})}{\sqrt{x^2 - q^2}}$ $\begin{cases} x > q \\ 0 & x < q \end{cases}$	$\begin{cases} \pi i \operatorname{sgn} y \cdot J_0(q \sqrt{y^2 - p^2}) & y > p \\ 0 & y < p \end{cases}$ MO 164
5.137	
1. $-\frac{\sin(p \sqrt{x^2 - q^2})}{\sqrt{x^2 - q^2}}$ $\begin{cases} x > q \\ \frac{\exp(-p \sqrt{q^2 - x^2})}{\sqrt{q^2 - x^2}} & x < q \end{cases}$	$\begin{cases} \pi J_0(q \sqrt{y^2 - p^2}) & y > p \\ 0 & y < p \end{cases}$ MO 164

$F(x)$	$f(y)$	
$2. \left. \begin{aligned} &\frac{\sin(p\sqrt{q^2-x^2})}{\sqrt{q^2-x^2}} & \ x < q \\ &-\frac{\exp(-p\sqrt{x^2-q^2})}{\sqrt{x^2-q^2}} & \ x > q \end{aligned} \right\}$	$\pi N_0(q\sqrt{p^2+y^2})$	MO 163
$5.138 \frac{\operatorname{Arsh}\left(\frac{x}{q}\right)}{\sqrt{x^2+q^2}}$	$\pi i \operatorname{sgn} y \cdot K_0(q y)$	MO 164
$5.139 \left. \begin{aligned} &\frac{\arcsin\left(\frac{x}{q}\right)}{\sqrt{q^2-x^2}} & \ x < q \\ &\frac{\ln\left[\frac{x}{q}-\sqrt{\left(\frac{x}{q}\right)^2-1}\right]}{\sqrt{x^2-q^2}} & \ x > q \end{aligned} \right\}$	$i \frac{\pi^2}{2} \operatorname{sgn} y \cdot N_0(q y)$	MO 163
5.141		
$1. \frac{\sin(p\sqrt{q^2+x^2})}{\sqrt{q^2+x^2}} \times$ $\times [\operatorname{ci}(p\sqrt{q^2+x^2}+px) +$ $+ \operatorname{ci}(p\sqrt{q^2+x^2}-px)] -$ $-\frac{\cos(p\sqrt{q^2+x^2})}{\sqrt{q^2+x^2}} \times$ $\times [\operatorname{si}(p\sqrt{q^2+x^2}+px) +$ $+ \operatorname{si}(p\sqrt{q^2+x^2}-px) + \pi]$	$\left. \begin{aligned} &\pi^2 N_0(q\sqrt{p^2-y^2}) & \ y < p \\ &0 & \ y > p \end{aligned} \right\}$	MO 165
$2. -\frac{\cos(p\sqrt{q^2+x^2})}{\sqrt{q^2+x^2}} \times$ $\times [\operatorname{si}(p\sqrt{q^2+x^2}+px) +$ $+ \operatorname{si}(p\sqrt{q^2+x^2}-px)] +$ $+\frac{\sin(p\sqrt{q^2+x^2})}{\sqrt{q^2+x^2}} \times$ $\times [\operatorname{ci}(p\sqrt{q^2+x^2}+px) +$ $+ \operatorname{ci}(p\sqrt{q^2+x^2}-px)]$	$\left. \begin{aligned} &2\pi K_0(q\sqrt{y^2-p^2}) & \ y > p \\ &0 & \ y < p \end{aligned} \right\}$	MO 165
$5.142 \frac{J_r(p\sqrt{q^2-x^2})}{\sqrt{q^2-x^2}} \quad \begin{aligned} &\ x < q \\ &0 & \ x > q \end{aligned}$	$\pi J_{\frac{r}{2}}\left[\frac{q}{2}(\sqrt{p^2+y^2}-y)\right] \times$ $\times J_{\frac{r}{2}}\left[\frac{q}{2}(\sqrt{p^2+y^2}+y)\right]$	MO 165

$F(x)$	$f(y)$
5.143 $\frac{J_r(p\sqrt{q^2+x^2})}{(q^2+x^2)^{\frac{r}{2}}}$	$\begin{cases} \frac{\sqrt{2\pi q}}{p^r q^r} (p^2-y^2)^{\frac{r-1}{2}} J_{r-\frac{1}{2}}(q\sqrt{p^2-y^2}) & y < p \\ 0 & y > p \end{cases}$ MO 165
5.144 $(q^2-x^2)^{\frac{r}{2}} J_r(p\sqrt{q^2-x^2})$ 0 $\left. \begin{matrix} x < q \\ x > q \end{matrix} \right\}$	$\sqrt{2\pi q} p^r q^r \frac{J_{r+\frac{1}{2}}(q\sqrt{p^2+y^2})}{(p^2+y^2)^{\frac{r+1}{2}}}$ MO 165
5.145 $(q^2+x^2)^{\frac{r}{2}} K_r(p\sqrt{q^2+x^2})$	$\sqrt{2\pi q} p^r q^r \frac{K_{r+\frac{1}{2}}(q\sqrt{p^2+y^2})}{(p^2+y^2)^{\frac{r+1}{2}}}$ MO 165
5.146 $\frac{H_r^{(2)}(p\sqrt{q^2+x^2})}{(q^2+x^2)^{\frac{r}{2}}}$	$i \sqrt{\frac{8q}{\pi}} \frac{(y^2-p^2)^{\frac{r-1}{2}}}{p^r q^r} K_{r-\frac{1}{2}}(q\sqrt{y^2-p^2})$ MO 165
5.147 $P_\alpha(x)$ 0 $\left. \begin{matrix} x < 1 \\ x > 1 \end{matrix} \right\}$	$i^\alpha \sqrt{\frac{2\pi}{y}} J_{\alpha+\frac{1}{2}}(y)$ MO 165
5.148 $\exp\left(-\frac{x^2}{2}\right) H_\alpha(x)$	$\sqrt{2\pi} i^\alpha \exp\left(-\frac{y^2}{2}\right) H_\alpha(y)$ MO 165m
5.149 $\exp\left(-\frac{x^2}{2}\right) H_\alpha\left(\frac{x}{\sqrt{2}}\right) H_b\left(\frac{x}{\sqrt{2}}\right)$	$\sqrt{2\pi} 2^\alpha (iy)^{b-\alpha} \exp\left(-\frac{y^2}{2}\right) L_\alpha^{(b-\alpha)}(y^2)$ MO 165m

5.2 TRANSFORMATĂ LAPLACE

5.200 Transformata Laplace transformă funcția $f(t)$, care satisface condițiile de la 5.00, în funcția

$$1. \quad g(p) = \int_0^\infty e^{-pt} f(t) dt.$$

Transformarea inversă, care definește funcția $f(t)$ ca soluție a ecuației integrale 5.200 1., dacă se cunoaște funcția $g(p)$, este dată de formula

$$2. \quad f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} g(p) dp.$$

Dacă există, totodată, o constantă reală c_0 , în așa fel încît integrala

$$\int_0^{\infty} e^{-c_0 t} |f(t)| dt$$

să fie convergentă, atunci

a) funcția $g(p)$ este o funcție analitică de variabilă complexă $p = \sigma + i\tau$, uniformă și regulată în tot semiplanul $\sigma \geq c_0$.

Pentru ca formula 5.200 2. să dea în adevăr soluția ecuației 5.200 1., este suficient ca funcția $g(p)$ să satisfacă condiția a) și, în afară de aceasta, ca

b) pentru $\sigma > c_0$, funcția $g(p)$ să posedă o majorantă,

$$G(\tau) \geq |g(\sigma + i\tau)|,$$

asa încît integrala

$$\int_{-\infty}^{\infty} G(\tau) d\tau$$

să aibă o valoare principală; pentru aceasta, este suficient ca această condiție să fie satisfăcută, nu pentru însăși funcția $g(p)$, ci pentru o funcție oarecare

$$g_0(p) = g(p) - \frac{q_1}{p^{r_1}} - \frac{q_2}{p^{r_2}} - \dots - \frac{q_a}{p^{r_a}},$$

unde $q_1, \dots, q_a$; $r_1, \dots, r_a$ sînt anumite constante, $\text{Re}(r_1) > 0, \dots, \text{Re}(r_a) > 0$; c_0 trebuie luat, în orice caz, pozitiv.

Constanta c , care intră în semnul de limită în integrala 5.200 2., trebuie să satisfacă inegalitatea $c \geq c_0$.

Dacă sînt satisfăcute condițiile a) și b), rezultă, din formula 5.200 2., că $f(t) = 0$, pentru $t < 0$.

În egalitățile 5.200 1. și 5.200 2., funcția $f(t)$ se numește *originalul*, iar funcția $g(p)$, *reprezentarea* sau *imaginea*.

M O 166

Relația dintre original și reprezentare în transformarea Laplace se notează simbolic

$$g(p) = \mathcal{L}f(t), \quad f(t) = \mathcal{L}^{-1}g(p).$$

În calculul operațional, se folosește frecvent transformarea

$$h(p) = p \int_0^{\infty} e^{-pt} f(t) dt = pg(p).$$

În acest caz, relația dintre funcțiile $f(t)$ și $h(p) \equiv pg(p)$ („originalul“ și „reprezentarea“) se notează simbolic

$$f(t) \doteq pg(p).$$

În formulele care urmează, s-au notat cu literele q și s parametrii reali și pozitivi.

Originalul	Imaginea	
5.201 $f(qt)$	$\frac{1}{q} g\left(\frac{p}{q}\right)$	ED 90m
	$[q > 0]$	
5.202		
1. $\frac{df}{dt}$	$pg(p) - f(0)$	ED 43 (7)m
2. $\frac{d^2 f}{dt^2}$	$p^2 g(p) - \sum_{k=0}^{n-1} p^{n-k-1} f^{(k)}(0)$	ED 49m
3. $\int_0^t f(\tau) d\tau$	$\frac{g(p)}{p}$	ED 43 (6)m
4. $\int_t^\infty \frac{f(\tau)}{\tau} d\tau$	$\frac{1}{p} \int_0^p g(s) ds$	ED 93m
5. $t^\alpha f(t)$	$(-1)^\alpha \frac{d^\alpha g}{dp^\alpha}$	ED 49m
6. $\frac{f(t)}{t}$	$\int_p^\infty g(s) ds$	MO 168
5.203		
1. $\begin{cases} f(t-q) & [t \geq q] \\ 0 & [0 \leq t < q] \end{cases}$	$e^{-qp} g(p)$	ED 94
	$[q > 0]$	
2. $f(t+q)$	$e^{qp} \left[g(p) - \int_0^q e^{-p\tau} f(\tau) d\tau \right]$	MO 168
	$[q > 0]$	
3. $e^{at} f(t)$	$g(p-a)$	ED 93m MO 169
5.204 $f(t^2)$	$\frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\frac{p^2 s^2}{4}\right) g\left(\frac{1}{s^2}\right) \frac{ds}{s^3}$	MO 169
5.205 $\frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{s^2}{4t}\right) f(s) ds$	$\frac{1}{\sqrt{p}} g(\sqrt{p})$	MO 169
5.206		
1. $t^{\frac{r}{2}} \int_0^\infty s^{-\frac{r}{2}} J_r(2\sqrt{st}) f(s) ds$	$p^{-r-1} g\left(\frac{1}{p}\right)$	MO 169
	$[Re(r) > -\frac{1}{2}]$	
2. $\int_0^t J_0[2\sqrt{s(t-s)}] f(s) ds$	$\frac{1}{p} g\left(p + \frac{1}{p}\right)$	MO 169
3. $f(t) - \int_0^t f(\sqrt{t^2-s^2}) J_1(s) ds$	$g(\sqrt{p^2+1})$	MO 169

Originalul	Imaginea
5.207 1. $\int_0^{\infty} \frac{t^s f(s)}{\Gamma(s+1)} ds$ 2. $t^{-\frac{\alpha+1}{2}} \int_0^{\infty} \exp\left(-\frac{s^2}{4t}\right) \times$ $\times H_{\alpha}\left(\frac{s}{2\sqrt{t}}\right) f(s) ds$	$\frac{1}{p} g(\ln p)$ MO 169 $2^{\alpha} \sqrt{\pi} p^{\frac{\alpha-1}{2}} g(p)$ MO 169
5.208 Dacă $\mathfrak{L} f_1(t) = g_1(p)$ $\mathfrak{L} f_2(t) = g_2(p)$, to $\int_0^t f_1(\tau) f_2(t-\tau) d\tau$	$g_1(p) g_2(p)$ ED 95 (15)
5.209 Transformata Laplace pentru funcții periodice. Fie $f(t)$ o funcție periodică pentru $t > 0$, avind perioada egală cu q ($q > 0$), adică fie $f(t+q) = f(t)$. Să punem	
$\int_0^q f(t) e^{-pt} dt = g_0(p).$	
Atunci	
$g(p) = \int_0^{\infty} f(t) e^{-pt} dt = \frac{g_0(p)}{1 - e^{-pq}}.$	
5.211 1. $\frac{1}{1+q^t}$ 2. $\frac{1}{1+q^{t^2}}$ 3. $\frac{t}{1+q^{t^2}}$	$-\frac{1}{q} e^{\frac{p}{q}} \text{Ei}\left(-\frac{p}{q}\right)$ (vezi și 3.222 2.) MO 170 $\frac{1}{q} \left[\sin \frac{p}{q} \text{ci}\left(\frac{p}{q}\right) - \cos \frac{p}{q} \text{si}\left(\frac{p}{q}\right) \right]$ (vezi și 3.223 1.) MO 170 $-\frac{1}{q^2} \left[\sin \frac{p}{q} \text{si}\left(\frac{p}{q}\right) + \cos \frac{p}{q} \text{ci}\left(\frac{p}{q}\right) \right]$ (vezi și 3.223 2.) MO 170
5.212 $\frac{1}{\sqrt{1+q^t}}$	$\sqrt{\frac{\pi}{qp}} e^{\frac{p}{q}} \left[1 - \Phi\left(\sqrt{\frac{p}{q}}\right) \right]$ MO 170
5.213 t^r	$\frac{\Gamma(r+1)}{p^{r+1}}$ (vezi și 3.225 3.) ED 107 (70)
5.214	$[\text{Re}(r) > -1]$
1. $(t^2 - q^2)^r$ 0	$\frac{2^{r+\frac{1}{2}} \Gamma(r+1) q^{r+\frac{1}{2}} K_{r+\frac{1}{2}}(qp)}{\sqrt{\pi} p^{r+\frac{1}{2}}}$ MO 170
$[t > q]$ $[t < q]$	$[\text{Re}(r) > -1]$

Originalul	Imaginea
2. $\frac{(2qt - t^2)^{a-\frac{1}{2}}}{0} \quad \left\{ \begin{array}{l} [0 < t < 2q] \\ [t > 2q] \end{array} \right\}$	$\sqrt{\pi} \Gamma\left(a + \frac{1}{2}\right) 2^a q^a p^{-a} e^{-pq} I_a(pq)$ MO 170
3. $(2qt + t^2)^r$	$\frac{\Gamma(r+1)}{\sqrt{\pi}} 2^{r+\frac{1}{2}} q^{r+\frac{1}{2}} p^{-r-\frac{1}{2}} e^{qp} K_{r+\frac{1}{2}}(qp)$ MO 170
4. $\left(t + \frac{i}{2} t^2\right)^{r-\frac{1}{2}}$	$\frac{i \sqrt{\pi} \Gamma\left(r + \frac{1}{2}\right)}{2p^r} e^{-\frac{ip}{2}} H_r^{(1)}\left(\frac{p}{2}\right)$ MO 170
	$[\operatorname{Re}(r) > -1]$
	$\left[\operatorname{Re}(r) > -\frac{1}{2}\right]$
5.215 $\frac{1}{(1+qt)^r}$	$\frac{p^{\frac{r}{2}-1}}{q^{\frac{r}{2}}} e^{\frac{p}{2q}} W_{-\frac{r}{2}, \frac{1-r}{2}}\left(\frac{p}{q}\right)$ (vezi și 3.226 2) MO 170
5.216	
1. $\frac{t^{\frac{r}{2}-1}}{(1+t)^{\frac{r+1}{2}}}$	$2^{\frac{r}{2}} e^{\frac{p}{2}} \Gamma\left(\frac{r}{2}\right) D_{-r}(\sqrt{2p})$ MO 171
	$[\operatorname{Re}(r) > 0]$
2. $\frac{t^{\mu-\lambda-\frac{1}{2}}}{(1+t)^{-\mu-\lambda+\frac{1}{2}}}$	$e^{\frac{p}{2}} p^{-\frac{1}{2}-\mu} \Gamma\left(\mu + \frac{1}{2} - \lambda\right) W_{\lambda, \mu}(p)$ MO 170
5.217	
1. $\frac{(qt + \sqrt{q^2 t^2 - 1})^r + (qt - \sqrt{q^2 t^2 - 1})^r}{\sqrt{q^2 t^2 - 1}} \quad \left\{ \begin{array}{l} 0 \\ [z < \frac{1}{q}] \end{array} \right\}$	$\frac{2}{q} K_r\left(\frac{p}{q}\right)$ MO 171
2. $\frac{(qt + \sqrt{q^2 t^2 - 1})^r - (qt - \sqrt{q^2 t^2 - 1})^r}{0} \quad \left\{ \begin{array}{l} 0 \\ [t < \frac{1}{q}] \end{array} \right\}$	$\frac{2r}{p} K_r\left(\frac{p}{q}\right)$ MO 171
5.218	
1. e^{qt}	$\frac{1}{p-q}$ ED 67 (21)m
2. $\frac{e^t}{e^x + 1}$	$\beta(p)$ MO 171
3. $\left(1 - e^{-\frac{t}{q}}\right)^{r-1}$	$qB(pq, r)$ (vezi și 3.212) MO 171
	$[\operatorname{Re}(r) > 0]$

Originalul	Imaginea
5.219	
1. $e^{-q^2 t^2}$	$\frac{\sqrt{\pi}}{2q} \exp\left(\frac{p^2}{4q^2}\right) \left[1 - \Phi\left(\frac{p}{2q}\right)\right]$ MO 174
2. $\frac{t}{1 - e^{-qt}}$	$\frac{d^2}{dp^2} \ln \Gamma(qp) = q \frac{d}{dp} \phi(qp)$ MO 171
3. $\frac{e^{qt}}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{p-q}}$ MO 174
5.221	
1. $\frac{\exp\left(-\frac{q}{t}\right)}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} e^{-2\sqrt{qp}}$ ED 70 (23)m
2. $q^r \frac{\exp\left(-\frac{q^2}{4t}\right)}{(2t)^{r+1}}$	$p^{\frac{r}{2}} K_r(q\sqrt{p})$ MO 174
3. $t^{r-1} \exp(-q^2 t^2)$	$2^{-\frac{r}{2}} q^{-r} \Gamma(r) \exp\left(\frac{p^2}{8q^2}\right) D_{-r}\left(\frac{p}{q\sqrt{2}}\right)$ MO 174
	[Re(r) > 0]
4. $\frac{\exp(-q\sqrt{t})}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} \left[1 - \Phi\left(\frac{q}{2\sqrt{p}}\right)\right] \exp\left(\frac{q^2}{4p}\right)$ MO 174
5.222	
1. $\text{sh } qt$	$\frac{q}{p^2 - q^2}$ MO 173
2. $\text{ch } qt$	$\frac{p}{p^2 - q^2}$ MO 173
3. $\frac{1}{\text{ch } t}$	$\beta\left(\frac{p+1}{2}\right)$ MO 8 m
4. $\text{sh}(2q\sqrt{t})$	$\sqrt{\pi} qp^{-\frac{3}{2}} \exp \frac{q^2}{p}$ MO 173 m
5.223	
1. $\frac{\text{sh } qt}{t}$	$\frac{1}{2} \ln \frac{p+q}{p-q}$ MO 174
	[Re(p) > Re(q)]
2. $\frac{\text{sh}^2 qt}{t}$	$-\frac{1}{2} \ln \sqrt{1 - \frac{4q^2}{p^2}}$ MO 174
5.224 $\frac{\text{ch}(2q\sqrt{t})}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} \exp \frac{q^2}{p}$ MO 174 m
5.225	
1. $\sin qt$	$\frac{q}{p^2 + q^2}$ ED 67 (20)m
2. $\cos qt$	$\frac{p}{p^2 + q^2}$ ED 67 (20)m

Originalul	Imaginea
3. $\sin^{2a} qt$	$\frac{(2a)! q^{2a}}{p [(2q)^2 + p^2] [(4q)^2 + p^2] \dots [(2aq)^2 + p^2]}$ (vezi și 3.562 6) MO 172
4. $\sin^{2a+1} qt$	$\frac{(2a+1)! q^{2a+1}}{[q^2 + p^2] [(3q)^2 + p^2] \dots [(2a+1)^2 q^2 + p^2]}$ (vezi și 3.562 8) MO 172
5.226	
1. $\sin qt^2$	$\sqrt{\frac{\pi}{2q}} \left\{ \cos \frac{p^2}{4q} \left[\frac{1}{2} - C \left(\frac{p}{2q} \right) \right] + \right. \\ \left. + \sin \frac{p^2}{4q} \left[\frac{1}{2} - S \left(\frac{p}{2q} \right) \right] \right\} \text{ MO 172}$
2. $\cos qt^2$	$\sqrt{\frac{\pi}{2q}} \left\{ \cos \frac{p^2}{4q} \left[\frac{1}{2} - S \left(\frac{p}{2q} \right) \right] - \right. \\ \left. - \sin \frac{p^2}{4q} \left[\frac{1}{2} - C \left(\frac{p}{2q} \right) \right] \right\} \text{ MO 172}$
5.227 $\sin(q\sqrt{t})$	$\frac{q}{2p} \sqrt{\frac{\pi}{p}} \exp\left(-\frac{q^2}{4p}\right) \text{ MO 172}$
5.228 $\sin(q\sqrt{t^2-s^2})$ 0	$\left. \begin{matrix} [t > s] \\ [t < s] \end{matrix} \right\} \frac{qs}{\sqrt{p^2+q^2}} K_1(s\sqrt{p^2+q^2}) \text{ MO 173}$
5.229	
1. $\frac{\sin qt}{t}$	$\arctg \frac{q}{p} \text{ MO 173}$
2. $\frac{\sin qt}{\sqrt{t}}$	$\sqrt{\frac{\pi}{2}} \frac{\sqrt{\sqrt{p^2+q^2}-p}}{\sqrt{p^2+q^2}} \text{ MO 173}$
3. $\frac{\cos qt}{\sqrt{t}}$	$\sqrt{\frac{\pi}{2}} \frac{\sqrt{\sqrt{p^2+q^2}+p}}{\sqrt{p^2+q^2}} \text{ MO 173}$
4. $t^r \sin qt$	$\frac{1}{2i} \frac{\Gamma(r+1)}{(p^2+q^2)^{r+1}} [(p+iq)^{r+1} - (p-iq)^{r+1}]$ (vezi și 3.573) MO 172
5. $t^r \cos qt$	$\frac{1}{2} \frac{\Gamma(r+1)}{(p^2+q^2)^{r+1}} [(p+iq)^{r+1} + (p-iq)^{r+1}]$ (vezi și 3.575 1–3.575 3) MO 172
5.231	
1. $\frac{\sin qt \sin st}{t}$	$\frac{1}{4} \ln \frac{p^2+(q+s)^2}{p^2+(q-s)^2} \text{ BH [365] (5)m}$
	$[q > 0, s > 0]$
2. $\frac{\sin st \cos qt}{t}$	$\frac{1}{2} \arctg \frac{2ps}{p^2+q^2-s^2} \text{ F II 744m}$
	$[s > 0]$
3. $\frac{\sin qt \sin st}{t^2}$	$\frac{q+s}{2} \arctg \frac{q+s}{p} - \frac{q-s}{2} \arctg \frac{q-s}{p} +$ $+ \frac{p}{4} \ln \frac{p^2+(q-s)^2}{p^2+(q+s)^2} \text{ F II 744m}$
	$[q > 0, s > 0]$

Originalul	Imaginea
4. $\frac{\sin^2 qt}{t}$	$\frac{1}{2} \ln \sqrt{1 + \frac{4q^2}{p^2}}$ MO 172
5.232	
1. $\frac{\sin\left(\frac{q}{t}\right)}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} \exp(-\sqrt{2pq}) \sin \sqrt{2pq}$ MO 173
2. $\frac{\cos\left(\frac{q}{t}\right)}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} \exp(-\sqrt{2pq}) \cos \sqrt{2pq}$ MO 173
5.233	
1. $\frac{\sin(q\sqrt{t})}{t}$	$\pi \Phi\left(\frac{q}{2\sqrt{p}}\right)$ MO 172
2. $\sqrt{t} \cos(q\sqrt{t})$	$\frac{1}{2p} \sqrt{\frac{\pi}{p}} \left(1 - \frac{q^2}{2p}\right) \exp\left(-\frac{q^2}{4p}\right)$ MO 172
3. $t^{-\frac{3}{4}} \sin(q\sqrt{t})$	$\frac{\pi}{4} \sqrt{\frac{8q}{p}} \exp\left(-\frac{q^2}{8p}\right) I_{\frac{1}{4}}\left(\frac{q^2}{8p}\right)$ MO 173
4. $t^{-\frac{3}{4}} \cos(q\sqrt{t})$	$\frac{\pi}{4} \sqrt{\frac{8q}{p}} \exp\left(-\frac{q^2}{8p}\right) I_{-\frac{1}{4}}\left(\frac{q^2}{8p}\right)$ MO 173
5.234	
1. $e^{-st} \sin(qt \pm \varphi)$	$\frac{q \cos \varphi \pm (p+s) \sin \varphi}{(p+s)^2 + q^2}$ MO 171
2. $e^{-st} \cos(qt \pm \varphi)$	$\frac{(p+s) \cos \varphi \mp q \sin \varphi}{(p+s)^2 + q^2}$ MO 171
5.235 $\ln \sqrt{1+t^2}$	$-\frac{1}{p} [\cos p \operatorname{ci}(p) + \sin p \operatorname{si}(p)]$ MO 174
5.236 $\ln t + \sqrt{t^2 - 1} $	$\frac{1}{p} K_0(p)$ MO 175
5.237 $t^r \ln t$	$\frac{\Gamma(r+1)}{p^{r+1}} [\psi(r+1) - \ln p]$
	$[\operatorname{Re}(r) > -1] \quad (\text{vezi și } 3.722, 3.723) \text{ MO 170}$
5.238 $\frac{\ln(1+t^2)}{t}$	$[\operatorname{ci}(p)]^2 + [\operatorname{si}(p)]^2$ NI 32 (11)m
5.239	
1. $\sin qt \ln t$	$\frac{1}{p^2 + q^2} \left[p \arctg \frac{q}{p} - qC - \frac{q}{2} \ln(p^2 + q^2) \right]$ BH [467] (1)m
2. $\cos qt \ln t$	$-\frac{1}{p^2 + q^2} \left[\frac{p}{2} \ln(p^2 + q^2) + q \arctg \frac{q}{p} + pC \right]$ BH [467] 2m
5.241	
1. $\delta_1(\pi v, i\pi t)$	$-\frac{\operatorname{sh}(2v\sqrt{p})}{\sqrt{p} \operatorname{ch} \sqrt{p}}$ MO 182m
	$\left[-\frac{1}{2} \leq v \leq \frac{1}{2} \right]$

Originalul	Imaginea	
2. $\theta_2(\pi v, i\pi t)$	$-\frac{\text{sh}[(2v-1)\sqrt{p}]}{\sqrt{p} \text{ch} \sqrt{p}}$	MO 182m
	$[0 < v < 1]$	
3. $\theta_3(\pi v, i\pi t)$	$\frac{\text{ch}[(2v-1)\sqrt{p}]}{\sqrt{p} \text{sh} \sqrt{p}}$	MO 182m
	$[0 < v < 1]$	
4. $\theta_4(\pi v, i\pi t)$	$\frac{\text{ch}(2v\sqrt{p})}{\sqrt{p} \text{sh} \sqrt{p}}$	MO 182m
	$[-\frac{1}{2} < v < \frac{1}{2}]$	
5.242 $\text{Ei}(qt)$	$-\frac{1}{p} \ln\left(\frac{p}{q} - 1\right)$	MO 174
5.243 $\text{si}(qt)$	$-\frac{1}{p} \text{arctg}\left(\frac{p}{q}\right)$	ED 109 (73)m
5.244 $\text{ci}(qt)$	$-\frac{1}{p} \ln \sqrt{\left(\frac{p}{q}\right)^2 + 1}$	ED 109 (74)m
5.245 $\Phi(qt)$	$\frac{1}{p} \left[1 - \Phi\left(\frac{p}{2q}\right)\right] \exp\left(\frac{p^2}{4q^2}\right)$	MO 175
5.246 $S(t)$	$\frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right)\right] + \right.$ $\left. + \sin \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right)\right] \right\}$	MO 173m
5.247 $C(t)$	$\frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right)\right] - \right.$ $\left. - \sin \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right)\right] \right\}$	MO 172m
5.248 $S(\sqrt{t})$	$\frac{(\sqrt{p^2+1}-p)^{\frac{1}{2}}}{2p \sqrt{p^2+1}}$	ED 122 (58)m
5.249 $C(\sqrt{t})$	$\frac{(\sqrt{p^2+1}+p)^{\frac{1}{2}}}{2p \sqrt{p^2+1}}$	ED 122 (58)m
5.251		
1. $\Phi(q\sqrt{t})$	$\frac{q}{p} \frac{1}{\sqrt{p^2+q^2}}$	MO 175
2. $\Phi\left(\frac{q}{\sqrt{t}}\right)$	$\frac{1}{p} (1 - \exp(-2q\sqrt{p}))$ (vezi și 4.243 1)	ED 147 (235)m MO 175
5.252		
1. $J_r(qt)$	$\frac{1}{\sqrt{p^2+q^2}} \frac{q^r}{(p+\sqrt{p^2+q^2})^r}$	ED 83 (43) MO 177

[Re(r) > -1]

Originalul	Imagina
2. $J_{\frac{1}{2}}(qt)$	$\frac{1}{\sqrt{q}} \frac{\sqrt{p^2 + q^2 - p}}{\sqrt{p^2 + q^2}}$ MO 177
3. $J_{-\frac{1}{2}}(qt)$	$\frac{1}{\sqrt{q}} \frac{\sqrt{p^2 + q^2 + p}}{\sqrt{p^2 + q^2}}$ MO 177
4. $J_0^2(qt)$	$\frac{2}{\pi \sqrt{p^2 + 4q^2}} K\left(\frac{2q}{\sqrt{p^2 + 4q^2}}\right)$ MO 178
5.253	
1. $J_{r+\frac{1}{2}}\left(\frac{t^2}{2}\right)$	$\frac{\Gamma(r+1)}{\sqrt{\pi}} D_{-r-1}\left(pe^{\frac{i\pi}{4}}\right) D_{-r-1}\left(pe^{-\frac{i\pi}{4}}\right)$ [Re(r) > -1] MO 178
2. $J_0(2\sqrt{qt})$	$\frac{1}{p} \exp\left(-\frac{q}{p}\right)$ ED 145m
3. $J_r(2q_1\sqrt{t}) J_r(2q_2\sqrt{t})$	$\frac{1}{p} I_r\left(\frac{2q_1 q_2}{p}\right) \exp\left(-\frac{q_1 + q_2}{p}\right)$ MO 178 [Re(r) > -1]
4. $J_r(q\sqrt{t})$	$\frac{q}{4} \sqrt{\frac{\pi}{p^3}} \exp\left(-\frac{q^2}{8p}\right) \times$ $\times \left[I_{r-\frac{1}{2}}\left(\frac{q^2}{8p}\right) - I_{r+\frac{1}{2}}\left(\frac{q^2}{8p}\right) \right]$ MO 178
5. $J_0(q\sqrt{t^2 + 2st})$	$\frac{1}{\sqrt{p^2 + q^2}} \exp[s(p - \sqrt{p^2 + q^2})]$ MO 179
6. $J_0(q\sqrt{t^2 - s^2})$ 0	$\left. \begin{matrix} [t > s] \\ [t < s] \end{matrix} \right\} \frac{1}{\sqrt{p^2 + q^2}} \exp(-s\sqrt{p^2 + q^2}) =$ $= \sqrt{\frac{2s}{\pi}} \frac{K_{\frac{1}{2}}(s\sqrt{p^2 + q^2})}{\sqrt{p^2 + q^2}}$ MO 179
5.254	
1. $N_0(qt)$	$-\frac{2}{\pi \sqrt{p^2 + q^2}} \ln \left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q}\right)^2} \right]$ MO 179
2. $N_r(qt)$	$\frac{\left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q}\right)^2} \right]^r}{\sqrt{p^2 + q^2}} \operatorname{ctg} r\pi -$ $-\frac{\left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q}\right)^2} \right]^r}{\sqrt{p^2 + q^2}} \operatorname{cosec} r\pi.$ MO 179 [-1 < Re(r) < 1]

Originalul	Imaginea
3. $N_{\frac{1}{2}}(qt)$	$-\frac{(p + \sqrt{p^2 + q^2})^{\frac{1}{2}}}{\sqrt{q} \sqrt{p^2 + q^2}} \quad \text{MO 179}$
4. $N_{-\frac{1}{2}}(qt)$	$\frac{(\sqrt{p^2 + q^2} - p)^{\frac{1}{2}}}{\sqrt{q} \sqrt{p^2 + q^2}} \quad \text{MO 180}$
5. $H_0^{(1)}(qt)$	$\frac{1}{\sqrt{p^2 + q^2}} \left\{ 1 - \frac{2t}{\pi} \ln \left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q} \right)^2} \right] \right\} \quad \text{MO 180}$
6. $H_0^{(2)}(qt)$	$\frac{1}{\sqrt{p^2 + q^2}} \left\{ 1 + \frac{2t}{\pi} \ln \left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q} \right)^2} \right] \right\} \quad \text{MO 180}$
7. $H_r^{(1,2)}(qt)$	$\frac{(\sqrt{p^2 + q^2} - p)^r}{q^r \sqrt{p^2 + q^2}} \left\{ 1 \pm \frac{t}{\sin r\pi} \times \right. \\ \left. \times \left[\cos r\pi - \frac{(p + \sqrt{p^2 + q^2})^{2r}}{q^{2r}} \right] \right\}$
	$[-1 < \text{Re}(r) < 1] \quad \text{MO 180}$
8. $H_{\frac{1}{2}}^{(1)}(qt)$	$\frac{1}{\sqrt{p^2 + q^2}} \sqrt{\frac{\sqrt{p^2 + q^2} - p}{q}} \times \\ \times \left\{ 1 - i \left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q} \right)^2} \right] \right\} \quad \text{MO 180}$
9. $H_{-\frac{1}{2}}^{(2)}(qt)$	$\frac{1}{\sqrt{p^2 + q^2}} \sqrt{\frac{\sqrt{p^2 + q^2} - p}{q}} \times \\ \times \left\{ 1 + i \left[\frac{p}{q} + \sqrt{1 + \left(\frac{p}{q} \right)^2} \right] \right\} \quad \text{MO 180}$
10. $I_r(qt)$	$\frac{q^r}{\sqrt{p^2 - q^2} (p + \sqrt{p^2 - q^2})^r} \quad \text{MO 180}$
	$[\text{Re}(r) > -1]$
11. $K_r(qt)$	$\frac{\pi \left[\frac{p}{q} + \sqrt{\left(\frac{p}{q} \right)^2 - 1} \right]^r}{2 \sin r\pi \sqrt{p^2 - q^2}} \times \\ \times \left\{ 1 - \frac{q^{2r}}{(p + \sqrt{p^2 - q^2})^{2r}} \right\} \quad \text{MO 180}$
	$[-1 < \text{Re}(r) < 1]$
12. $K_{\pm \frac{1}{2}}(qt)$	$\frac{\pi}{\sqrt{2q}} \frac{1}{\sqrt{p + q}} \quad \text{MO 180}$
13. $K_1(q\sqrt{t})$	$\frac{q}{8} \sqrt{\frac{\pi}{p^3}} \left[K_1\left(\frac{q^2}{8p}\right) - K_0\left(\frac{q^2}{8p}\right) \right] \times \\ \times \exp\left(\frac{q^2}{8p}\right) \quad \text{MO 181}$

Originalul	Imagina
5.255	
1. $\text{ber } t$	$\frac{\sqrt{V p^4 + 1 + p^2}}{\sqrt{2(p^4 + 1)}} \quad \text{MO 181}$
2. $\text{bei } t$	$\frac{V \sqrt{V p^4 + 1 - p^2}}{\sqrt{2(p^4 + 1)}} \quad \text{MO 181}$
3. $\text{ber}_r t + i \text{bei}_r t$	$\frac{i^{\frac{3}{2}r}}{\sqrt{p^2 - i(p + \sqrt{p^2 - i})^r}} \quad \text{MO 181}$
4. $\text{ker}_r t + i \text{kei}_r t$	$\frac{i^{-\frac{3}{2}r} \pi (p + \sqrt{p^2 - i})^r}{2 \sin r\pi \sqrt{p^2 - i}} \times$ $\times [1 - i^r (p + \sqrt{p^2 - i})^{-2r}] \quad \text{MO 181}$
	$[-1 < \text{Re}(r) < 1]$
5.256	
1. $t J_r(qt)$	$\frac{q^r (p + r \sqrt{p^2 + q^2})}{V(p^2 + q^2)^2 (p + \sqrt{p^2 + q^2})^r} \quad \text{MO 178}$
	$[\text{Re}(r) > -2]$
2. $\frac{1}{t} J_r(qt)$	$\frac{q^r}{r(p + \sqrt{p^2 + q^2})^r} \quad \text{MO 178}$
	$[\text{Re}(r) > 0]$
3. $t^r J_r(qt)$	$2^r q^r \frac{\Gamma(r + \frac{1}{2})}{\sqrt{\pi}} (p^2 + q^2)^{-r - \frac{1}{2}}$ ED 84 (44)m MO 177
	$[\text{Re}(r) > -\frac{1}{2}]$
4. $t^{r+1} J_r(qt)$	$2^{r+1} q^r p \frac{\Gamma(r + \frac{3}{2})}{\sqrt{\pi}} (p^2 + q^2)^{-r - \frac{3}{2}}$ ED 84 (44)m MO 177
	$[\text{Re}(r) > -1]$
5. $t^\alpha J_r(qt)$	$p^{-r-\alpha-1} \frac{\Gamma(r+\alpha+1)}{2^r \Gamma(r+1)} \times$ $\times q^r F\left(\frac{r+\alpha+1}{2}, \frac{r+\alpha+2}{2}; r+1; -\frac{q^2}{p^2}\right) \quad \text{MO 178}$
6. $t^{r_1+r_2} J_{r_1}(qt) J_{r_2}(qt)$	$\frac{2}{V \pi^3} q^{r_1+r_2} \Gamma\left(r_1+r_2+\frac{1}{2}\right) \times$ $\times \int_0^{\frac{\pi}{2}} \frac{\cos^{r_1+r_2} \varphi \cos(r_1-r_2) \varphi}{(p^2 + 4q^2 \cos^2 \varphi)^{r_1+r_2+\frac{1}{2}}} d\varphi$ $[\text{Re}(r_1+r_2) > -\frac{1}{2}] \quad \text{MO 178m}$

Originalul	Imaginea
5.257	
1. $\frac{J_{2r}(\sqrt{qt})}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}} J_r\left(\frac{q}{8p}\right) \exp\left(-\frac{q}{8p}\right)$ $\left[\operatorname{Re}(r) > -\frac{1}{2}\right]$ MO 178
2. $t^{\lambda-\frac{1}{2}} J_{2\mu}(2\sqrt{t})$	$\frac{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)}{\Gamma(2\mu+1)} \frac{\exp\left(-\frac{1}{2p}\right)}{p^{-1}} M_{\lambda,\mu}\left(\frac{1}{p}\right)$ $\left[\operatorname{Re}(\lambda+\mu) > -\frac{1}{2}\right]$ MO 177
3. $t^{\frac{r}{2}} J_r(2\sqrt{qt})$	$\frac{q^{\frac{r}{2}}}{p^{r+1}} \exp\left(-\frac{q}{p}\right)$ ED 76m $[\operatorname{Re}(r) > -1]$
4. $t^a J_r(2\sqrt{qt})$	$\frac{\Gamma\left(\frac{r}{2}+a+1\right) q^{\frac{r}{2}} p^{-\frac{r}{2}-a-1}}{\Gamma(r+1)} \times$ $\times {}_1F_1\left(\frac{r}{2}+a+1; r+1; -\frac{q}{p}\right)$ MO 178
5. $t^{a+\frac{r}{2}} J_r(2\sqrt{qt})$	$\frac{q^{\frac{r}{2}} \exp\left(-\frac{q}{p}\right)}{p^{a+r+1}} L_a^{(r)}\left(\frac{q}{p}\right)$ MO 178 $[r+a > -1]$
5.258	
1. $t J_0(q\sqrt{t^2-r^2})$ 0	$\left. \begin{array}{l} [t > r] \\ [t < r] \end{array} \right\} \frac{p}{p^2+q^2} \left(r + \frac{1}{\sqrt{p^2+q^2}}\right) \times$ $\times \exp\left(-r\sqrt{p^2+q^2}\right)$ MO 179
2. $\frac{J_1(q\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}}$ 0	$\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\} \frac{1}{qs} (e^{-sp} - e^{-s\sqrt{p^2+q^2}})$ MO 179
3. $\frac{J_r(q\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}}$ 0	$\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\} I_{\frac{r}{2}} \left[\frac{s}{2} (\sqrt{p^2+q^2} - p) \right] \times$ $\times K_{\frac{r}{2}} \left[\frac{s}{2} (\sqrt{p^2+q^2} + p) \right]$ MO 179
4. $t \frac{J_1(q\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}}$ 0	$\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\} \frac{1}{q} \left[\exp(-sp) - \frac{p}{\sqrt{p^2+q^2}} \times \right.$ $\left. \times \exp(-s\sqrt{p^2+q^2}) \right]$ MO 179
5. $(t^2-s^2)^{\frac{r}{2}} J_r(q\sqrt{t^2-s^2})$ 0	$\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\} \sqrt{\frac{2s}{\pi}} q^r s^r (p^2+q^2)^{-\frac{r}{2}-\frac{1}{4}} K_{r+\frac{1}{2}}(s\sqrt{p^2+q^2})$ MO 179

Originalul	Imaginea
6. $\left(\frac{t}{t+2s}\right)^{\frac{r}{2}} J_r(q\sqrt{t^2+2st})$	$\frac{\exp[s(p-\sqrt{p^2+q^2})]}{\sqrt{p^2+q^2}} \left(\frac{q}{p+\sqrt{p^2+q^2}}\right)^r$ MO 179
[Re(r) > -1]	
7. $\left(\frac{t-s}{t+s}\right)^{\frac{r}{2}} J_r(q\sqrt{t^2-s^2})$ 0 $\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\}$	$\frac{\exp(-s\sqrt{p^2+q^2})}{\sqrt{p^2+q^2}} \left(\frac{q}{p+\sqrt{p^2+q^2}}\right)^r$ ED 89 (52)
[Re(r) > -1]	
5.259	
1. $\left(\frac{t-s}{t+s}\right)^{\frac{r}{2}} I_r(q\sqrt{t^2-s^2})$ 0 $\left. \begin{array}{l} [t > s] \\ [t < s] \end{array} \right\}$	$\frac{\exp(-s\sqrt{p^2-q^2})}{\sqrt{p^2-q^2}} \left(\frac{q}{p+\sqrt{p^2-q^2}}\right)^r$ MO 180
[Re(r) > -1]	
2. $tK_0(qt)$	$\frac{1}{(p^2-q^2)} \left\{ \frac{p}{\sqrt{p^2-q^2}} \times \right.$ $\left. \times \ln \left[\frac{p}{q} + \sqrt{\left(\frac{p}{q}\right)^2 - 1} \right] - 1 \right\}$ MO 181
3. $t^{\frac{1}{2}} K_{\pm \frac{1}{2}}(qt)$	$\sqrt{\frac{\pi}{2q}} \frac{1}{p+q}$ MO 181
5.261	
1. $H_{2a+1}(\sqrt{t})$	$(-1)^a 2^a \cdot (2a-1)!! \sqrt{\pi} (p-1)^a p^{-a-\frac{3}{2}}$ ED 151 (261)m
2. $t^{-\frac{1}{2}} H_{2a}(\sqrt{t})$	$(-1)^a 2^a \cdot (2a-1)!! \sqrt{\pi} (p-1)^a p^{-a-\frac{1}{2}}$ MO 177m
3. $t^{-\frac{a+1}{2}} \exp\left(-\frac{q^2}{4t}\right) H_a\left(\frac{q}{2\sqrt{t}}\right)$	$2^a \sqrt{\pi} p^{\frac{a-1}{2}} \exp(-q\sqrt{p})$ ED 129 (117)
5.262	
1. $t^a L_a(t)$	$\frac{(a!)^2}{p^{a+1}} P_a\left(1-\frac{2}{p}\right)$ MO 176
2. $t^r L_a^{(r)}(t)$	$\Gamma(r+a+1) \frac{(p-1)^a}{p^{r+a+1}}$ MO 176
[Re(r) > -1]	

Originalul	Imaginea
5.263	
1. ${}_aF_b(a_1, \dots, a_a; \gamma_1, \dots, \gamma_b; t)$	$\frac{1}{p} \cdot (a+1)F_b\left(a_1, \dots, a_a, 1; \gamma_1, \dots, \gamma_b; \frac{1}{p}\right)$ MO 174
	$[b \geq a]$
2. ${}_aF_b(a_1, \dots, a_a; \gamma_1, \dots, \gamma_{b-1}; t)$	$\frac{1}{p} \cdot {}_aF_{b-1}\left(a_1, \dots, a_a; \gamma_1, \dots, \gamma_{b-1}; \frac{1}{p}\right)$ MO 175
	$[b \geq a]$
3. ${}_aF_b(a_1, \dots, a_a; \gamma_1, \dots, \gamma_b; t^2)$	$\frac{1}{p} \times$ $\times (a+2)F_b\left(a_1, \dots, a_a, 1, \frac{1}{2}; \gamma_1, \dots, \gamma_b; \frac{4}{p^2}\right)$ MO 176
	$[b > a]$
4. ${}_aF_a\left(\frac{1}{a}, \frac{2}{a}, \dots, \frac{a-1}{a}, 1; \frac{t^a}{a^a}\right)$	$\frac{1}{p} \exp(p^{-a})$ MO 176
5.264	
1. $t M_{\mu, \nu}(t)$	$\frac{\Gamma\left(\alpha + \nu + \frac{3}{2}\right)}{\left(p + \frac{1}{2}\right)^{\alpha + \nu + \frac{3}{2}}} \times$ $\times F\left(\alpha + \nu + \frac{3}{2}, -\mu + \nu + \frac{1}{2}; 2\nu + 1; \frac{2}{2p+1}\right)$ MO 176m
	$\left[\operatorname{Re}(\alpha + \nu) > -\frac{3}{2}\right]$
2. $t^{\mu - \frac{1}{2}} M_{\lambda, \mu}(qt)$	$q^{\mu + \frac{1}{2}} \Gamma(2\mu + 1) \frac{\left(p - \frac{1}{2}q\right)^{1 - \mu - \frac{1}{2}}}{\left(p + \frac{1}{2}q\right)^{1 + \mu + \frac{1}{2}}}$ MO 176m
	$\left[\operatorname{Re}(\mu) > -\frac{1}{2}\right]$
5.265 $t^a W_{\lambda, \mu}(qt)$	
	$\frac{\Gamma\left(\alpha + \mu + \frac{3}{2}\right) \Gamma\left(\alpha - \mu + \frac{3}{2}\right) q^{\mu + \frac{1}{2}}}{\Gamma(\alpha - \lambda + 2) \left(p + \frac{q}{2}\right)^{\alpha + \mu + \frac{3}{2}}} \times$ $\times F\left(\alpha + \mu + \frac{3}{2}, \mu - \lambda + \frac{1}{2}; \alpha - \lambda + 2; \frac{2p - q}{2p + q}\right)$ MO 176
	$\left[\operatorname{Re}\left(\alpha \pm \mu + \frac{3}{2}\right) > 0, \operatorname{Re}(p) > \frac{q}{2}, q > 0\right]$

Originalul	Imaginea
5.266	
1. $D_{2a+1}(\sqrt{2t})$	$\sqrt{2}(-2)^a \Gamma\left(a + \frac{3}{2}\right) \frac{\left(p - \frac{1}{2}\right)^a}{\left(p + \frac{1}{2}\right)^{a+\frac{3}{2}}}$ MO 177
2. $t^{-\frac{1}{2}} D_{2a}(\sqrt{2t})$	$(-2)^a \Gamma\left(a + \frac{1}{2}\right) \frac{\left(p - \frac{1}{2}\right)^a}{\left(p + \frac{1}{2}\right)^{a+\frac{1}{2}}}$ MO 177
3. $(2t)^{\frac{r-1}{2}} \exp\left(-\frac{t}{2}\right) D_{-r}(\sqrt{2t})$	$\sqrt{\frac{\pi}{2}} \frac{(\sqrt{p+1}-1)^r}{p^r \sqrt{p+1}}$ [Re(r) > -1] MO 175
4. $(2t)^{\frac{r-1}{2}} \exp\left(-\frac{t}{2}\right) D_{-r-1}(\sqrt{2t})$	$\sqrt{\frac{\pi}{2}} \frac{(\sqrt{p+1}-1)^{r+1}}{(r+1)p^{r+1}}$ [Re(r) > -1] MO 175
5. $(2t)^{-\frac{r}{2}} \exp\left(-\frac{q^2}{8t}\right) D_{r-1}\left(\frac{q}{\sqrt{2t}}\right)$	$\sqrt{\frac{\pi}{2}} p^{\frac{r}{2}-1} e^{-\frac{q}{2}\sqrt{p}}$ MO 175

5.3 TRANSFORMATA HANKEL

5.30 Fie funcția $F(x)$, definită pentru $0 \leq x < +\infty$, și să presupunem că partea ei reală și partea ei imaginară satisfac condițiile 5.00. Să presupunem mai departe, că integrala

$$\int_0^{\infty} |F(x)| dx$$

este convergentă. Atunci, pentru valorile reale ale lui $y \geq 0$ și pentru valorile reale ale lui $r \geq -\frac{1}{2}$, există integrala

$$1. f(y) = \int_0^{\infty} J_r(xy) F(x) \sqrt{xy} dx,$$

iar pentru $x > 0$, avem

$$2. F(x) = \int_0^{\infty} J_r(xy) f(y) \sqrt{xy} dy \quad [x > 0].$$

Se spune despre funcția $f(y)$ că este obținută din funcția $F(x)$ printr-o transformare Hankel de ordinul r . MO 182

În formulele care urmează, s-a notat cu litera q un parametru a cărui parte reală este pozitivă.

$F(x)$	r	$f(y)$
5.311 $x^{-\frac{1}{2}}$	$r > -1$	$y^{-\frac{1}{2}}$ MO 183
5.312 $\frac{e^{-qx}}{\sqrt{x}}$	$r > -1$	$\frac{y^{-r-\frac{1}{2}} (\sqrt{q^2+y^2}-q)^r}{\sqrt{q^2+y^2}}$ MO 183
5.313 $\sqrt{x} J_a\left(\frac{x^2}{2}\right)$	$r=2a$	$\sqrt{y} J_a\left(\frac{y^2}{2}\right)$ MO 183
5.314		
1. $\sqrt{x} \exp\left(-\frac{x^2+1}{q}\right) I_a\left(\frac{2x}{q}\right)$	$r=a$	$\frac{q}{2} \exp\left(-\frac{qy^2}{4}\right) \sqrt{y} J_a(y)$ MO 183
2. $x^{\frac{r+1}{3}+\frac{1}{6}} e^{-\frac{x^2}{4}} I_{\frac{r}{3}-\frac{1}{6}}\left(\frac{x^2}{4}\right)$	$r > -1$	$y^{\frac{r+1}{3}+\frac{1}{6}} e^{-\frac{y^2}{4}} I_{\frac{r}{3}-\frac{1}{6}}\left(\frac{y^2}{4}\right)$ MO 183
3. $\sqrt{x} K_{\frac{r}{4}}\left(\frac{x^2}{4}\right) I_{\frac{r}{4}}\left(\frac{x^2}{4}\right)$	$r > -1$	$\sqrt{y} K_{\frac{r}{4}}\left(\frac{y^2}{4}\right) I_{\frac{r}{4}}\left(\frac{y^2}{4}\right)$ MO 183
5.315		
1. $e^{-\frac{x^2}{2}} x^{2a+r+\frac{1}{2}} L_a^{(a+r)}\left(\frac{x^2}{2}\right)$	$r > -1$	$e^{-\frac{y^2}{2}} y^{2a+r+\frac{1}{2}} L_a^{(a+r)}\left(\frac{y^2}{2}\right)$ MO 183
2. $(-1)^a e^{-\frac{x^2}{2}} x^{r+\frac{1}{2}} L_a^{(r)}(x^2)$	$r > -1$	$e^{-\frac{y^2}{2}} y^{r+\frac{1}{2}} L_a^{(r)}(y^2)$ MO 183
3. $\frac{(q-1)^a}{q^{a+r+1}} e^{-\frac{x^2}{2q}} x^{r+\frac{1}{2}} \times$ $\times L_a^{(r)}\left(\frac{x^2}{2q(1-q)}\right)$	$r > 0$	$e^{-\frac{qy^2}{2}} y^{r+\frac{1}{2}} L_a^{(r)}\left(\frac{y^2}{2}\right)$ MO 183
5.316 $x^{4s-r+\frac{1}{2}} e^{-\frac{x^2}{2}} \times$ $\times {}_1F_1\left(r-2s; 2s+1; \frac{x^2}{2}\right)$	$r \geq 0$	$y^{4s-r+\frac{1}{2}} e^{-\frac{y^2}{2}} \times$ $\times {}_1F_1\left(r-2s; 2s+1; \frac{y^2}{2}\right)$ MO 183
$\left[\operatorname{Re}\left(4s-r+\frac{3}{2}\right) > 0, \operatorname{Re}(2s+1) > 0\right]$		
5.317		
1. $x^{a+\frac{1}{2}} e^{-\frac{x^2}{4}} D_{2a+2}(x)$	$r=a$	$(-1)^{a+1} y^{a+\frac{1}{2}} e^{-\frac{y^2}{4}} D_{2a+2}(y)$ MO 183
2. $x^{a+\frac{1}{2}} e^{-\frac{x^2}{4}} D_{2a+1}(x)$	$r=a+1$	$(-1)^a y^{a+\frac{1}{2}} e^{-\frac{y^2}{4}} D_{2a+1}(y)$ MO 183

6.-7. FUNCȚII SPECIALE.

6.1 INTEGRALE ȘI FUNCȚII ELIPTICE

6.11 Integrale eliptice

6.110

1. Orice integrală de forma $\int R(x, P(x))dx$ unde $P(x)$ este un polinom de gradul trei sau patru, poate fi redusă la o funcție rațională de integrale care duc la funcții elementare și la următoarele trei integrale:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx, \quad \int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

care se numesc respectiv *integrale eliptice de speța întâi, a doua și a treia, în forma normală Legendre*. Rezultatele acestei reduceri pentru integralele care se întâlnesc des sînt date în formulele 3.13 – 3.14. Numărul k se numește *modulul* acestor integrale, numărul $k' = \sqrt{1-k^2}$, *modulul complementar*, iar numărul n , *parametrul* integralei de speța a treia. F II 97–106

2. Prin substituția $x = \sin \varphi$, integralele eliptice se reduc la *forma trigonometrică normală*

$$\int \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}, \quad \int \sqrt{1-k^2 \sin^2 \varphi} d\varphi, \quad \int \frac{d\varphi}{(1+h \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}. \quad \text{F II 106}$$

Reducerea integralelor de funcții trigonometrice la forma normală, vezi 2.47 – 2.48.

3. Integralele eliptice, luate între limitele 0 și $\frac{\pi}{2}$, se numesc *integrale eliptice complete*.

6.111 Notății.

1. $\Delta\varphi = \sqrt{1-k^2 \sin^2 \varphi}; \quad k' = \sqrt{1-k^2}; \quad k^2 < 1.$

2. Integrala eliptică de speța întâi

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

3. Integrala eliptică de speța a doua

$$E(\varphi, k) = \int_0^\varphi \sqrt{1-k^2 \sin^2 \alpha} d\alpha = \int_0^{\sin \varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx. \quad \text{F II 135}$$

4. Integrala eliptică de speța a treia

$$E(\varphi, n, k) = \int_0^{\varphi} \frac{da}{(1+n \sin^2 a) \sqrt{1-k^2 \sin^2 a}} = \int_0^{\sin \varphi} \frac{dx}{(1+nx^2) \sqrt{(1-x^2)(1-k^2 x^2)}}. \quad \text{Si 13}$$

$$5. \quad D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2} = \int_0^{\varphi} \frac{\sin^2 a \, da}{\sqrt{1-k^2 \sin^2 a}} = \int_0^{\sin \varphi} \frac{x^2 dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}.$$

6.112 Integrale eliptice complete

$$1. \quad K(k) = F\left(\frac{\pi}{2}, k\right) = K'(k'). \quad 2. \quad E(k) = E\left(\frac{\pi}{2}, k\right) = E'(k').$$

$$3. \quad K'(k) = F\left(\frac{\pi}{2}, k'\right) = K(k). \quad 4. \quad E'(k) = E\left(\frac{\pi}{2}, k'\right) = E(k').$$

$$5. \quad D = D\left(\frac{\pi}{2}, k\right) = \frac{K-E}{k^2}.$$

La scrierea integralelor eliptice complete, modulul k , care servește ca variabilă independentă, se omite adeseori și, deci, se scrie

$$K(=K(k)), \quad K'(=K'(k)), \quad E(=E(k)), \quad E'(=E'(k)).$$

Dezvoltarea în serie

6.113

$$1. \quad K = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots + \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 k^{2n} + \dots \right\} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \quad \text{F II 487 WW II 342}$$

$$2. \quad K = \frac{\pi}{1+k'} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \dots \right\}. \quad \text{D (773.2)}$$

$$3. \quad K = \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2}\right) k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4}\right) k'^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) k'^6 + \dots \quad \text{D (773.3)}$$

Vezi și 6.197 1, 6.197 2.

6.114

$$1. \quad E = \frac{\pi}{2} \left\{ 1 - \frac{1}{2^2} k^2 - \frac{1^2 \cdot 3}{2^3 \cdot 4^2} k^4 - \dots - \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 \frac{k^{2n}}{2n-1} - \dots \right\} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \quad \text{F II 487}$$

$$2. \quad E = \frac{(1+k')\pi}{4} \left\{ 1 + \frac{1}{2^2} \left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^3 \cdot 4^2} \left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left[\frac{(2n-3)!!}{2^n \cdot n!}\right]^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \dots \right\}. \quad \text{D (774.2)}$$

$$3. \quad E = 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{1 \cdot 2}\right) k'^2 + \frac{1^2 \cdot 3}{2^3 \cdot 4^2} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4}\right) k'^4 + \frac{1^2 \cdot 3^2 \cdot 5}{2^3 \cdot 4^2 \cdot 6^2} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6}\right) k'^6 + \dots \quad \text{D (774.3)}$$

$$6.115 \quad D = \pi \left\{ \frac{1}{1} \left(\frac{1}{2}\right)^2 + \frac{2}{3} \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^2 + \dots + \frac{n}{2n-1} \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 k^{2(n-1)} + \dots \right\} \quad \text{J 43 (158)}$$

$$6.116 \quad \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2 \sin^2 \varphi}}{1-n^2 \sin^2 \varphi} d\varphi = \sqrt{n'^2 - k'^2} \left(\frac{\arccos \frac{1}{n'}}{n' \sqrt{n'^2 - 1}} + R \right),$$

$$\begin{aligned} \text{unde } R = & \frac{k'^2}{2} \left(p + \frac{1}{2} \right) \frac{1}{n'^2} + \frac{k'^4}{16} \left[-1 + \left(p + \frac{1}{4} \right) \frac{1}{n'^2} \left(1 + \frac{6}{n'^2} \right) \right] + \\ & + \frac{k'^6}{16} \left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p + \frac{1}{6} \right) \frac{1}{n'^2} \left(\frac{3}{8} + \frac{1}{n'^2} + \frac{5}{n'^4} \right) \right] + \\ & + \frac{15k'^8}{256} \left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p + \frac{1}{8} \right) \frac{1}{n'^2} \left(\frac{5}{24} + \frac{9}{20n'^2} + \frac{1}{n'^4} + \frac{14}{3n'^6} \right) \right] + \dots, \\ & p = \ln \frac{4}{k'}, \quad k' = 4e^{-p}, \quad k'^2 = 1 - k^2, \quad n'^2 = 1 - n^2. \quad] 44(163') \end{aligned}$$

Serii trigonometrice

6.117 Pentru valori mici ale lui k și φ , putem folosi seriile

$$1. \quad F(\varphi, k) = \frac{2}{\pi} K \varphi - \sin \varphi \cos \varphi \left(a_0 + \frac{2}{3} a_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a_2 \sin^4 \varphi + \dots \right),$$

unde

$$a_0 = \frac{2}{\pi} K - 1; \quad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n \cdot n!} \right]^2 k^{2n}. \quad] 10(19)$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} E \varphi - \sin \varphi \cos \varphi \left(b_0 + \frac{2}{3} b_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b_2 \sin^4 \varphi + \dots \right),$$

unde

$$b_0 = 1 - \frac{2}{\pi} E; \quad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{2^n \cdot n!} \right]^2 \frac{k^{2n}}{2n-1}. \quad] 27(86)$$

6.118 Pentru k apropiat de 1, putem folosi seriile

$$1. \quad F(\varphi, k) = \frac{2}{\pi} K' \ln \operatorname{tg} \left(\frac{\varphi}{2} + \frac{\pi}{4} \right) - \frac{\operatorname{tg} \varphi}{\cos \varphi} \left(a'_0 - \frac{2}{3} a'_1 \operatorname{tg}^2 \varphi + \right. \\ \left. + \frac{2 \cdot 4}{3 \cdot 5} a'_2 \operatorname{tg}^4 \varphi - \dots \right),$$

unde

$$a'_0 = \frac{2}{\pi} K' - 1; \quad a'_n = a'_{n-1} - \left[\frac{(2n-1)!!}{2^n \cdot n!} \right]^2 k'^{2n}. \quad] 10(23)$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} E' \ln \operatorname{tg} \left(\frac{\varphi}{2} + \frac{\pi}{4} \right) + \frac{\operatorname{tg} \varphi}{\cos \varphi} \left(b'_0 - \frac{2}{3} b'_1 \operatorname{tg}^2 \varphi + \right. \\ \left. + \frac{2 \cdot 4}{3 \cdot 5} b'_2 \operatorname{tg}^4 \varphi - \dots \right),$$

unde

$$b'_0 = \frac{2}{\pi} E' - 1; \quad b'_n = b'_{n-1} - \left[\frac{(2n-1)!!}{2^n \cdot n!} \right]^2 \frac{k'^{2n}}{2n-1}. \quad] 27(90)$$

Descompunerea integralelor eliptice complete în polinoame Legendre, vezi 6.828.

6.119 Dezvoltarea sub formă de produs infinit

$$1. \quad K(k) = \frac{\pi}{2} \prod_{n=1}^{\infty} (1 + k_n),$$

unde

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}; \quad k_0 = k.$$

Vezi și 6.197.

6.12 6.12 Relații funcționale între integralele eliptice

6.121

$$1. F(-\varphi, k) = -F(\varphi, k); \quad 2. E(-\varphi, k) = -E(\varphi, k); \quad \text{JE 151}$$

$$3. F(n\pi \pm \varphi, k) = 2nK(k) \pm F(\varphi, k);$$

$$4. E(n\pi \pm \varphi) = 2nE(k) \pm E(\varphi, k). \quad \text{JE 151}$$

$$6.122 \quad E(k) K'(k) + E'(k) K(k) + K(k) K'(k) = \frac{\pi}{2}. \quad \text{F II 691 F II 791}$$

6.123

$$1. \frac{\partial F}{\partial k} = \frac{1}{k'^2} \left(\frac{E - k'^2 F}{k} - \frac{\sin \varphi \cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right). \quad \text{MO 138}$$

$$2. \frac{dK(k)}{dk} = \frac{E(k)}{kk'^2} - \frac{K(k)}{k}. \quad \text{F II 691}$$

$$3. \frac{\partial E}{\partial k} = \frac{E - F}{k}. \quad \text{MO 138}$$

$$4. \frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}. \quad \text{F II 690}$$

6.124

1. Funcțiile K și K' satisfac ecuația

$$\frac{d}{dk} \left\{ kk'^2 \frac{du}{dk} \right\} - ku = 0. \quad \text{WW II 371}$$

2. Funcțiile E și $E' - K'$ satisfac ecuația

$$k'^2 \frac{d}{dk} \left(k \frac{du}{dk} \right) + ku = 0. \quad \text{WW II 371}$$

Formule de transformare

6.125

$$\left. \begin{aligned} 1. F\left(\psi, \frac{1-k'}{1+k'}\right) &= (1+k') F(\varphi, k) \\ 2. E\left(\psi, \frac{1-k'}{1+k'}\right) &= \frac{2}{1+k'} [E(\varphi, k) + k' F(\varphi, k)] - \frac{1-k'}{1+k'} \sin \psi \end{aligned} \right\} \begin{aligned} & \\ & \left[\begin{aligned} \operatorname{tg}(\psi - \varphi) &= \\ &= k' \operatorname{tg} \varphi. \end{aligned} \right] \end{aligned}$$

MO 131

$$\left. \begin{aligned} 3. F\left(\psi, \frac{2\sqrt{k}}{1+k}\right) &= (1+k) F(\varphi, k). \\ 4. E\left(\psi, \frac{2\sqrt{k}}{1+k}\right) &= \frac{1}{1+k} \left[2E(\varphi, k) - k'^2 F(\varphi, k) + \right. \\ &\quad \left. + 2k \frac{\sin \varphi \cos \varphi}{1+k \sin^2 \varphi} \sqrt{1 - k^2 \sin^2 \varphi} \right] \end{aligned} \right\} \left[\sin \psi = \frac{(1+k) \sin \varphi}{1+k \sin^2 \varphi} \right].$$

MO 131

6.126 În particular,

$$1. K\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2} K(k). \quad \text{MO 130}$$

$$2. E\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k} [E(k) + k' K(k)]. \quad \text{MO 130}$$

$$3. K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k) K(k). \quad \text{MO 130}$$

$$4. E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k} [2E(k) - k'^2 K(k)]. \quad \text{MO 130}$$

6.127

k_1	$\sin \varphi_1$	$\cos \varphi_1$	$F(\varphi_1, k_1)$	$E(\varphi_1, k_1)$
$i \frac{k}{k'}$	$k' \frac{\sin \varphi}{\Delta \varphi}$	$\frac{\cos \varphi}{\Delta \varphi}$	$k' F(\varphi, k)$	$\frac{1}{k'} \left[E(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$
k'	$-i \operatorname{tg} \varphi$	$\sec \varphi$	$-i F(\varphi, k)$	$i [E(\varphi, k) - F(\varphi, k) - \Delta \varphi \operatorname{tg} \varphi]$
$\frac{1}{k}$	$k \sin \varphi$	$\Delta \varphi$	$k F(\varphi, k)$	$\frac{1}{k} \left[E(\varphi, k) - k'^2 F(\varphi, k) \right]$
$\frac{1}{k'}$	$-ik' \operatorname{tg} \varphi$	$\frac{\Delta \varphi}{\cos \varphi}$	$-ik' F(\varphi, k)$	$\frac{i}{k'} \left[E(\varphi, k) - k'^2 F(\varphi, k) - \Delta \varphi \operatorname{tg} \varphi \right]$
$\frac{k'}{ik}$	$-\frac{ik \sin \varphi}{\Delta \varphi}$	$\frac{1}{\Delta \varphi}$	$-ik F(\varphi, k)$	$\frac{i}{k} \left[E(\varphi, k) - F(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$

(vezi 6.111 1).

MO 131

6.128 În particular,

$$1. \quad K\left(i \frac{k}{k'}\right) = k' K(k). \quad \text{MO 130}$$

$$2. \quad K'\left(i \frac{k}{k'}\right) = k' [K(k') - i K(k)]. \quad \text{MO 130}$$

$$3. \quad K\left(\frac{1}{k}\right) = k K(k) + i K'(k). \quad \text{MO 130}$$

Integralele integralelor eliptice, vezi 4.11–4.13; integralele nedefinite ale integralelor eliptice complete, vezi 2.82.

6.129 Valori particulare

$$1. \quad K\left(\sin \frac{\pi}{4}\right) = K\left(\frac{\sqrt{2}}{2}\right) = K'\left(\frac{\sqrt{2}}{2}\right) = \sqrt[4]{2} \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right) \right]^2. \quad \text{MO 130}$$

$$2. \quad K'(\sqrt{2}-1) = \sqrt{2} K(\sqrt{2}-1). \quad \text{MO 130}$$

$$3. \quad K'\left(\sin \frac{\pi}{18}\right) = \sqrt{3} K\left(\sin \frac{\pi}{18}\right). \quad \text{MO 130}$$

$$4. \quad K'\left(\operatorname{tg}^2 \frac{\pi}{8}\right) = K'\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2K\left(\operatorname{tg}^2 \frac{\pi}{8}\right). \quad \text{MO 130}$$

6.13 Funcții eliptice

6.130 Definiție și proprietăți generale.

1. Funcția rațională $f(z)$ de variabilă complexă se numește *eliptică*, dacă admite două perioade (dacă este *biperiodică*) $2\omega_1$ și $2\omega_2$, adică dacă

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \quad [m, n \text{ întregi}].$$

Raportul perioadelor unei funcții eliptice nu poate fi un număr real. Pentru funcția eliptică $f(z)$, planul z poate fi reprezentat ca împărțit în paralelograme — *paralelogramele perioadelor* — ale căror vîrfuri sînt punctele $z_0 + 2m\omega_1 + 2n\omega_2$. În punctele corespunzătoare ale acestor paralelograme, funcția $f(z)$ are aceleași valori,

J117 Si 299.

2. Fie α unghiul dintre laturile a și b ale paralelogramului perioadelor. Atunci

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b} e^{i\alpha},$$

$$q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi \sin \alpha} \left[\cos \left(\frac{a}{b} \pi \cos \alpha \right) + i \sin \left(\frac{a}{b} \pi \cos \alpha \right) \right]$$

3. Derivata unei funcții eliptice este tot o funcție eliptică (avînd aceleași perioade). Sm III 598

4. O funcție eliptică diferită de o constantă are, în paralelogramul perioadelor, un număr finit de poluri: cel puțin două poluri ordinare sau un pol de ordinul al doilea. Fie aceste poluri în punctele $a_1, a_2, \dots, a_n$ și avînd respectiv ordinele $\alpha_1, \alpha_2, \dots, \alpha_n$. Fie zerourile $b_1, b_2, \dots, b_n$ ale funcției eliptice situate într-un singur paralelogram al perioadelor și fie ordinul acestor zerouri egal respectiv cu $\beta_1, \beta_2, \dots, \beta_n$. Atunci

$$\gamma = \alpha_1 + \alpha_2 + \dots + \alpha_n = \beta_1 + \beta_2 + \dots + \beta_n.$$

Numărul γ , egal cu această sumă, se numește *ordinul funcției eliptice*.

Sm III 599 J118 Si 300–301

5. Suma reziduurilor unei funcții eliptice în raport cu toate polurile care aparțin paralelogramului perioadelor este egală cu zero. J118

6. Diferența dintre suma tuturor zerourilor și suma tuturor polurilor unei funcții eliptice așezate în paralelogramul perioadelor este egală cu una dintre perioade. Si 302

7. Între fiecare pereche de funcții eliptice cu perioade identice, există o relație algebrică. Goll, 151

8. O funcție uniformă nu poate avea mai mult de două perioade. Goll, 147

9. Funcția eliptică de ordinul γ ia orice valoare în paralelogramul perioadelor, de γ ori. Sm III 601 Si 301

6.14 Funcții eliptice Jacobi

6.141 Considerînd limita superioară φ a integralei

$$u = \int_0^{\varphi} \frac{da}{\sqrt{1 - k^2 \sin^2 a}}$$

ca o funcție de u , folosim notația

$$\varphi = \operatorname{am} u$$

și numim această limită superioară *amplitudine*. Mărimea u se numește *argument* și dependența ei de φ se scrie

$$u = \arg \varphi.$$

6.142 Amplitudinea este funcție infinit multiformă de u , avînd perioada egală cu $4K$. Punctele de ramificație ale amplitudinii corespund valorilor

$$u = 2mK + (2n+1)K'i,$$

J67–69

ale argumentului, m și n fiind numere întregi arbitrare (vezi și 6.151).

6.143 Funcțiile

$$\operatorname{sn} u = \sin \varphi = \sin \operatorname{am} u,$$

$$\operatorname{cn} u = \cos \varphi = \cos \operatorname{am} u,$$

$$\operatorname{dn} u = \Delta \varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du}$$

se numesc respectiv *sinusul amplitudinii* sau *sinusul eliptic*, *cosinusul amplitudinii* sau *cosinusul eliptic* și *delta amplitudinii*. Toate aceste funcții eliptice au fost introduse de Jacobi și îi poartă numele. Si 16

Funcțiile eliptice ale lui Jacobi sînt funcții biperiodice, avînd în paralelogramul perioadelor două poluri ordinare. J69

6.144

$$\left. \begin{aligned} 1. \quad u &= \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} \\ 2. \quad u &= \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2 + k^2 t^2)}} \\ 3. \quad u &= \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2 - k'^2)}} \end{aligned} \right\} \quad \text{Si 21 (23)}$$

6.145 Dezvoltarea în serie de puteri

$$1. \quad \operatorname{sn} u = u - \frac{1+k^2}{8!} u^3 + \frac{1+14k^2+k^4}{5!} u^5 - \frac{1+135k^2+135k^4+k^6}{7!} u^7 + \\ + \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!} u^9 - \dots \quad [|u| < |K'|]. \quad \text{J 81 (97)}$$

$$2. \quad \operatorname{cn} u = 1 - \frac{1}{2!} u^2 + \frac{1+4k^2}{4!} u^4 - \frac{1+44k^2+16k^4}{6!} u^6 + \\ + \frac{1+408k^2+912k^4+64k^6}{8!} u^8 - \dots \quad [|u| < |K'|]. \quad \text{J 81 (98)}$$

$$3. \quad \operatorname{dn} u = 1 - \frac{k^2}{2!} u^2 + \frac{k^2(4+k^2)}{4!} u^4 - \frac{k^2(16+44k^2+k^4)}{6!} u^6 + \\ + \frac{k^2(64+912k^2+408k^4+k^6)}{8!} u^8 - \dots \quad [|u| < |K'|]. \quad \text{J 81 (99)}$$

$$4. \quad \operatorname{am} u = u - \frac{k^2}{8!} u^3 + \frac{k^2(4+k^2)}{5!} u^5 - \frac{k^2(16+44k^2+k^4)}{7!} u^7 + \\ + \frac{k^2(64+912k^2+408k^4+k^6)}{9!} u^9 - \dots \quad [|u| < |K'|]. \quad \text{La 380 (4)}$$

6.146 Dezvoltarea în serie trigonometrică sau reprezentarea sub formă de produs

$$(q = e^{-\frac{\pi K'}{K}})^1$$

$$1. \quad \operatorname{sn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}. \quad \text{J 84 (108)}$$

¹ Dezvoltările 1-22 sînt valabile pe toată banda $\left| \operatorname{Im} \left(\frac{\pi u}{2K} \right) \right| < \frac{1}{2} \pi \operatorname{Im}(\tau)$, unde

$\operatorname{Im}(\tau) = \int \operatorname{sn} \left(\frac{2Kx}{\pi} \right) e^{i\pi x} dx$, luată de-a lungul paralelogramului, ale cărui vîrfuri sînt $-\pi, \pi, \pi\tau, -2\pi + \pi\tau$ [$\tau = \ln q$; $\operatorname{Im}(\tau) > 0$]; dezvoltările 23-25 sînt valabile pentru orice valoare finită a lui u .

$$2. \quad \operatorname{cn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}. \quad \text{J 84 (109)}$$

$$3. \quad \operatorname{dn} u = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K}. \quad \text{J 84 (110)}$$

$$4. \quad \operatorname{am} u = \frac{\pi u}{2K} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin \frac{n\pi u}{K}. \quad \text{WW II 358}$$

$$5. \quad \frac{1}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right]. \quad \text{La 369 (3)}$$

$$6. \quad \frac{1}{\operatorname{cn} u} = \frac{\pi}{2k'K} \left[\frac{1}{\cos \frac{\pi u}{2K}} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right]. \quad \text{La 369 (3)}$$

$$7. \quad \frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'K} \left[1 + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K} \right]. \quad \text{La 369 (3)}$$

$$8. \quad \frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'K} \left[\operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{K} \right]. \quad \text{La 369 (4)}$$

$$9. \quad \frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'K} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}. \quad \text{La 369 (4)}$$

$$10. \quad \frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{K} \right]. \quad \text{La 369 (5)}$$

$$11. \quad \frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{kK} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}. \quad \text{La 369 (5)}$$

$$12. \quad \frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right]. \quad \text{La 369 (6)}$$

$$13. \quad \frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left[\frac{1}{\cos \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right]. \quad \text{La 369 (6)}$$

$$14. \quad \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^n} \sin \frac{n\pi u}{K} \right]. \quad \text{La 369 (7)}$$

$$15. \quad \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left\{ \operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right\}. \quad \text{La 369 (7)}$$

$$16. \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2 K} \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K}. \quad \text{La 369 (7)}$$

$$17. \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2(1-k^2)K} \left[\operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^n} \sin \frac{n\pi u}{K} \right]. \quad \text{La 369 (8)}$$

$$18. \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right]. \quad \text{La 369 (8)}$$

$$19. \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{K} \left[\frac{1}{\sin \frac{\pi u}{K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2(2n-1)}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K} \right]. \quad \text{La 369 (8)}$$

$$20. \ln \operatorname{sn} u = \ln \frac{2K}{\pi} + \ln \sin \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^n} \sin^2 \frac{n\pi u}{2K}. \quad \text{La 369 (2)}$$

$$21. \ln \operatorname{cn} u = \ln \cos \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+(-1)^n q^n} \sin^2 \frac{n\pi u}{2K}. \quad \text{La 369 (2)}$$

$$22. \ln \operatorname{dn} u = -8 \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin^2(2n-1) \frac{\pi u}{2K}. \quad \text{La 369 (2)}$$

$$23. \operatorname{sn} u = \frac{2\sqrt{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1-2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}. \quad \text{J 86 (145)}$$

$$24. \operatorname{cn} u = \frac{2\sqrt{K}\sqrt{q}}{\sqrt{k}} \cos \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1+2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}. \quad \text{J 86 (146)}$$

$$25. \operatorname{dn} u = \sqrt{k'} \prod_{n=1}^{\infty} \frac{1+2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}. \quad \text{J 86 (147)}$$

$$26. \operatorname{sn}^2 u = \sum_{n=0}^{\infty} \left[\frac{1+k^2}{2k^2} - \frac{(2n+1)^2}{2k^2} \frac{\pi^2}{4K^2} \right] \frac{2\pi q^{n+\frac{1}{2}} \sin(2n+1) \frac{\pi u}{2K}}{K(1-q^{2n+1})} \\ \left[\left| \operatorname{Im} \left(\frac{u}{2K} \right) \right| < \operatorname{Im}(\tau) \right]. \quad \text{MO 147}$$

$$27. \frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4K^2} \operatorname{cosec}^2 \frac{\pi u}{2K} + \frac{K-E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^{2n} \cos \frac{n\pi u}{K}}{1-q^{2n}} \\ \left[\left| \operatorname{Im} \left(\frac{u}{2K} \right) \right| < \frac{1}{2} \operatorname{Im}(\tau) \right]. \quad \text{MO 148}$$

6. 147

$$1. \operatorname{sn} u = \frac{\pi}{2kK} \sum_{n=-\infty}^{\infty} \frac{1}{\sin \frac{\pi}{2K} [u - (2n-1) iK']}. \quad \text{MO 149}$$

$$2. \quad \operatorname{cn} u = \frac{\pi i}{2kK} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin \frac{\pi}{2K} [u - (2n-1) iK']}. \quad \text{MO 150}$$

$$3. \quad \operatorname{dn} u = \frac{\pi i}{2K} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\operatorname{tg} \frac{\pi}{2K} [u - (2n-1) iK']}. \quad \text{MO 150}$$

6.148 Dezvoltările lui Weierstrass pentru funcțiile $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

$$\operatorname{sn} u = \frac{B}{A}, \quad \operatorname{cn} u = \frac{C}{A}, \quad \operatorname{dn} u = \frac{D}{A},$$

unde

$$A = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!}$$

$$[a_2 = 2k^2, \quad a_3 = 8(k^2 + k^4), \quad a_4 = 32(k^2 + k^6) + 68k^4,$$

$$a_5 = 128(k^2 + k^6) + 480(k^4 + k^8),$$

$$a_6 = 512(k^2 + k^{10}) + 3008(k^4 + k^8) + 5400k^6, \dots].$$

$$B = \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!}$$

$$[b_0 = 1, \quad b_1 = 1 + k^2, \quad b_2 = 1 + k^4 + 4k^2,$$

$$b_3 = 1 + k^6 + 9(k^2 + k^4), \quad b_4 = 1 + k^8 + 16(k^2 + k^6) - 6k^4,$$

$$b_5 = 1 + k^{10} + 25(k^2 + k^6) - 49k^4(k^4 + k^8),$$

$$b_6 = 1 + k^{12} + 36(k^2 + k^{10}) - 5781(k^4 + k^8) - 12184k^6, \dots].$$

$$C = \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!}$$

$$[c_0 = 1, \quad c_1 = 1, \quad c_2 = 1 + 2k^2, \quad c_3 = 1 + 6k^2 + 8k^4,$$

$$c_4 = 1 + 12k^2 + 60k^4 + 32k^6,$$

$$c_5 = 1 + 20k^2 + 348k^4 + 448k^6 + 128k^8,$$

$$c_6 = 1 + 30k^2 + 2372k^4 + 4600k^6 + 2880k^8 + 512k^{10}, \dots].$$

$$D = \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!}$$

$$[d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2 + k^4, \quad d_3 = 8k^2 + 6k^4 + k^6,$$

$$d_4 = 32k^2 + 60k^4 + 12k^6 + k^8,$$

$$d_5 = 128k^2 + 448k^4 + 348k^6 + 20k^8 + k^{10},$$

$$d_6 = 512k^2 + 2880k^4 + 4600k^6 + 2372k^8 + 30k^{10} + k^{12}, \dots].$$

6.15 Proprietățile funcțiilor eliptice Jacobi și relații funcționale dintre ele

6.151 Perioadele, zerourile, polurile și reziduurile funcțiilor eliptice Jacobi

1.	Perioade	Zerouri	Poluri	Reziduuri
$\operatorname{sn} u$	$4mK + 2nK'$	$2mK + 2nK'$	$2mK + (2n+1)K'$	$(-1)^m \frac{1}{k}$
$\operatorname{cn} u$	$4mK + 2n(K+K')$	$(2m+1)K + 2nK'$	$2mK + (2n+1)K'$	$(-1)^{m-1} \frac{1}{k}$
$\operatorname{dn} u$	$2mK + 4nK'$	$(2m+1)K + (2n+1)K'$	$2mK + (2n+1)K'$	$(-1)^{n-1} i$

Sm III 630, J 69-72

2.	$u^*=u+K$	$u+iK'$	$u+K+iK'$	$u+2K$	$u+2iK'$	$u+2K+2iK'$
	$\operatorname{sn} u^*=\frac{\operatorname{cn} u}{\operatorname{dn} u}$	$\frac{1}{k \operatorname{sn} u}$	$\frac{1}{k} \frac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\operatorname{sn} u$	$\operatorname{sn} u$	$-\operatorname{sn} u$
	$\operatorname{cn} u^*=-k' \frac{\operatorname{sn} u}{\operatorname{dn} u}$	$-\frac{i}{k} \frac{\operatorname{dn} u}{\operatorname{sn} u}$	$-\frac{ik'}{k \operatorname{cn} u}$	$-\operatorname{cn} u$	$-\operatorname{cn} u$	$\operatorname{cn} u$
	$\operatorname{dn} u^*=k' \frac{1}{\operatorname{dn} u}$	$-i \frac{\operatorname{cn} u}{\operatorname{sn} u}$	$ik' \frac{\operatorname{sn} u}{\operatorname{cn} u}$	$\operatorname{dn} u$	$-\operatorname{dn} u$	$-\operatorname{dn} u$

Sm III 630

3.	$u^*=0$	$-u$	$\frac{1}{2} K$	$\frac{1}{2} (K+iK')$	$\frac{1}{2} iK'$	$u+2mK+2nK'$
	$\operatorname{sn} u^*=0$	$-\operatorname{sn} u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k}+i\sqrt{1-k}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m \operatorname{sn} u$
	$\operatorname{cn} u^*=1$	$\operatorname{cn} u$	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n} \operatorname{cn} u$
	$\operatorname{dn} u^*=1$	$\operatorname{dn} u$	$\sqrt{k'}$	$\frac{\sqrt{k'}(\sqrt{1+k}-i\sqrt{1-k})}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n \operatorname{dn} u$

Si 19 Si 18 (13) WW II 344

WW II 352

WW II 352 WW II 348

6.152 Formule de transformare

u_1	k_1	$\operatorname{sn}(u_1, k_1)$	$\operatorname{cn}(u_1, k_1)$	$\operatorname{dn}(u_1, k_1)$
ku	$\frac{1}{k}$	$k \operatorname{sn}(u, k)$	$\operatorname{dn}(u, k)$	$\operatorname{cn}(u, k)$
iu	k'	$i \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$
$k'u$	$i \frac{k'}{k}$	$k \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$
iku	$i \frac{k'}{k}$	$i k \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$
$ik'u$	$\frac{1}{k'}$	$i k' \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k) \operatorname{sn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{1-k \operatorname{sn}^2(u, k)}{1+k \operatorname{sn}^2(u, k)}$
$(1+k')u$	$\frac{1-k'}{1+k'}$	$(1+k') \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1-(1+k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$	$1-(1-k') \operatorname{sn}^2(u, k)$
$\frac{(1+k')^2}{2} u$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \right)^2$	$\frac{k^2 \operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\sqrt{k_1} [1+\operatorname{dn}(u, k)] [k' + \operatorname{dn}(u, k)]}$	$\frac{\operatorname{dn}(u, k) - \sqrt{k'}}{1-\sqrt{k'}} \times \sqrt{\frac{2(1+k')}{[1+\operatorname{dn}(u, k)] [k' + \operatorname{dn}(u, k)]}}$	$\frac{\sqrt{1+k_1} (\operatorname{dn}(u, k) + \sqrt{k'})}{\sqrt{[1+\operatorname{dn}(u, k)] [k' + \operatorname{dn}(u, k)]}}$

JE 191

6.153

$$1. \operatorname{sn}(iu; k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}. \quad \text{Si 50 (64)}$$

$$2. \operatorname{cn}(iu; k) = \frac{1}{\operatorname{cn}(u, k')}. \quad \text{Si 50 (65)}$$

$$3. \operatorname{dn}(iu; k) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}.$$

$$4. \operatorname{sn}(u, k) = k^{-1} \operatorname{sn}(ku, k^{-1}).$$

$$5. \operatorname{cn}(u, k) = \operatorname{dn}(ku, k^{-1}).$$

$$6. \operatorname{dn}(u, k) = \operatorname{cn}(ku, k^{-1}).$$

$$7. \operatorname{sn}(u, ik) = \frac{1}{\sqrt{1+k^2}} \frac{\operatorname{sn}(u\sqrt{1+k^2}, k(1+k^2)^{-1/2})}{\operatorname{dn}(u\sqrt{1+k^2}, k(1+k^2)^{-1/2})}.$$

$$8. \operatorname{cn}(u, ik) = \frac{\operatorname{cn}(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2})}{\operatorname{dn}(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2})}.$$

$$9. \operatorname{dn}(u, ik) = \frac{1}{\operatorname{dn}(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2})}.$$

Relații funcționale

6.154

$$1. \operatorname{sn}^2 u = \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}. \quad \text{MO 146}$$

$$2. \operatorname{cn}^2 u = \frac{\operatorname{cn} 2u + \operatorname{dn} 2u}{1 + \operatorname{dn} 2u}. \quad \text{MO 146}$$

$$3. \operatorname{dn}^2 u = \frac{\operatorname{dn} 2u + k^2 \operatorname{cn} 2u + k'^2}{1 + \operatorname{dn} 2u}. \quad \text{MO 146}$$

$$4. \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1.$$

$$5. \operatorname{dn}^2 u + k^2 \operatorname{cn}^2 u = 1.$$

6.155

$$1. \frac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} = k^2 \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn}^2 u}. \quad \text{MO 146}$$

$$2. \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u} = \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{\operatorname{cn}^2 u}. \quad \text{MO 146}$$

6.156

$$1. \operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \quad \text{Si 46 (56)}$$

$$2. \operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \quad \text{Si 46 (57)}$$

$$3. \operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \quad \text{Si 46 (58)}$$

6.157

$$1. \operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}. \quad \text{Si 47 (61) Si 67 (15)}$$

$$2. \operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}}. \quad \text{Si 48 (62) Si 67 (16)}$$

$$3. \operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u - \operatorname{cn} u}}. \quad \text{Si 48 (63) Si 67 (17)}$$

6.158

$$\left. \begin{aligned} 1. \quad \frac{d}{du} \operatorname{sn} u &= \operatorname{cn} u \operatorname{dn} u. \\ 2. \quad \frac{d}{du} \operatorname{cn} u &= -\operatorname{sn} u \operatorname{dn} u. \\ 3. \quad \frac{d}{du} \operatorname{dn} u &= -k^2 \operatorname{sn} u \operatorname{cn} u. \end{aligned} \right\} \quad \text{Si 21 (21))}$$

6.159 Funcțiile eliptice Jacobi sînt soluțiile următoarelor ecuații diferențiale:

$$\left. \begin{aligned} 1. \quad \frac{d}{du} \operatorname{sn} u &= \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)}. \\ 2. \quad \frac{d}{du} \operatorname{cn} u &= -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)}. \\ 3. \quad \frac{d}{du} \operatorname{dn} u &= -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}. \end{aligned} \right\} \quad \text{Si 21 (22)}$$

Integralele funcțiilor eliptice Jacobi (integralele nedefinite), vezi 2.81.

6.16 Funcția $\wp(u)$ a lui Weierstrass

6.160 Funcția eliptică Weierstrass $\wp(u)$ este definită de egalitatea

$$1. \quad \wp(u) = \frac{1}{u^2} + \sum'_{m,n} \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3} - \frac{1}{(2m\omega_1 + 2n\omega_2)^3} \right\}, \quad \text{Si 307 (6)}$$

unde semnul $\sum'$ arată că sumarea se extinde la toate combinațiile de valori întregi ale lui m și n , cu excepția combinației $m=n=0$; $2\omega_1$ și $2\omega_2$ sînt perioadele funcției $\wp(u)$. Evident,

$$2. \quad \wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \quad \text{și} \quad \operatorname{Im} \left(\frac{\omega_1}{\omega_2} \right) \neq 0,$$

$$3. \quad \frac{d}{du} \wp(u) = -2 \sum'_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2},$$

unde sumarea se extinde la toate valorile întregi ale lui m și n .

Seriile 5.150 1. și 5.150 3. converg pretutindeni, cu excepția polurilor, adică a punctelor $2m\omega_1 + 2n\omega_2$ (m și n întregi).

4. Funcția $\wp(u)$ este o funcție periodică de ordinul doi, avînd în paralelogramul perioadelor un pol de ordinul doi.

Si 306

6.161 Funcția $\wp(u)$ satisface ecuația diferențială

$$1. \quad \left[\frac{d}{du} \wp(u) \right]^2 = 4\wp^3(u) - g_2\wp(u) - g_3, \quad \text{Si 142 Si 310}$$

unde

$$2. \quad g_2 = 60 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-4}; \quad g_3 = 140 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-6}. \quad \text{WW II 267}$$

Numerele g_2 și g_3 se numesc *invariantii* funcției $\wp(u)$.

$$6.162 \quad u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}},$$

unde e_1, e_2, e_3 sînt rădăcinile ecuației $4z^3 - g_2z - g_3 = 0$, adică

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \quad \text{Si 142 Si 143}$$

6.163 $\wp(\omega_1) = e_1$, $\wp(\omega_1 + \omega_2) = e_2$, $\wp(\omega_2) = e_3$; în acest caz se presupune că, dacă punctele e_1, e_2, e_3 sînt situate pe aceeași dreaptă în planul complex, e_2 este situat între e_1 și e_3 .

6.164 Numărul $\Delta = g_2^3 - 27g_3^2$ se numește *discriminantul funcției $\wp(u)$* . Dacă $\Delta > 0$, atunci toate rădăcinile e_1, e_2, e_3 ale ecuației $4z^3 - g_2z - g_3 = 0$ (g_2 și g_3 reali) sînt reale. În acest caz, numerotarea numerelor e_1, e_2, e_3 se face în așa fel, încît $e_1 > e_2 > e_3$.

1. Dacă $\Delta > 0$, atunci

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}; \quad \omega_2 = i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2z - 4z^3}},$$

unde ω_1 este un număr real, iar ω_2 un număr pur imaginar; valorile radicalului de sub semnul integrală se aleg în așa fel, încît ω_1 și $\frac{\omega_2}{i}$ să fie pozitivi.

Si 150 (15) Si 150 (16) WW II 276m

2. Dacă $\Delta < 0$, atunci rădăcina e_2 a ecuației $4z^3 - g_2z - g_3 = 0$ este reală, iar celelalte două, e_1 și e_3 , sînt complex conjugate. Fie $e_1 = \alpha + i\beta$, $e_3 = \alpha - i\beta$. În acest caz, este comod să alegem ca semiperioade principale

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} \text{ și } \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}.$$

Integrarea în prima integrală se efectuează pe un drum situat în întregime în semiplanul superior, iar în a doua, pe un drum situat în întregime în semiplanul inferior.

Si 151 (21) Si 151 (22)

6.165 Dezvoltarea în serie

$$1. \quad \wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4 \cdot 5} + \frac{g_3 u^4}{4 \cdot 7} + \frac{g_2^2 u^6}{2^4 \cdot 8 \cdot 5^3} + \frac{3g_2 g_3 u^8}{2^4 \cdot 5 \cdot 7 \cdot 11} + \dots$$

WW II 268

6.166 Relații funcționale

$$1. \quad \wp(u) = \wp(-u); \quad \wp'(u) = -\wp'(-u).$$

$$2. \quad \wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2. \quad \text{Si 163 (32)}$$

$$6.167 \quad \wp(u; g_2, g_3) = \mu^2 \wp\left(\mu u; \frac{g_2}{\mu^4}, \frac{g_3}{\mu^6}\right) \text{ [formula de omogenitate].} \quad \text{Si 149 (13)}$$

Caz particular: $\mu = i$

$$1. \quad \wp(u; g_2, g_3) = -\wp(iu; g_2, -g_3).$$

6.168 Orice funcție eliptică poate fi exprimată prin funcția eliptică $\wp(u)$ avînd aceleași perioade ca și funcția dată, și prin derivata ei $\wp'(u)$; această expresie este rațională în raport cu $\wp(u)$ și liniară în raport cu $\wp'(u)$.

6.169 Legătura cu funcțiile eliptice Jacobi.

Pentru $\Delta > 0$, (vezi 6.164 1).

$$\begin{aligned} 1. \quad \wp\left(\frac{u}{\sqrt{e_1 - e_3}}\right) &= e_1 + (e_1 - e_3) \frac{\text{cn}^2(u; k)}{\text{sn}^2(u; k)}; \\ &= e_2 + (e_1 - e_3) \frac{\text{dn}^2(u; k)}{\text{sn}^2(u; k)}; \\ &= e_3 + (e_1 - e_3) \frac{1}{\text{sn}^2(u, k)}; \end{aligned}$$

Si 145 (5) J 120 (197—199) m

$$2. \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

Si 154 (29)

unde

$$3. \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}. \quad \text{Si 146 (7)}$$

Pentru $\Delta < 0$ (vezi 6.164 2.).

$$4. \quad \wp\left(\frac{u}{\sqrt{9\alpha^2 + \beta^2}}\right) = e_2 + \sqrt{9\alpha^2 + \beta^2} \frac{1 + \operatorname{cn}(2u; k)}{1 - \operatorname{cn}(2u; k)}; \quad \text{Si 147 (12)}$$

$$5. \quad \omega' = \frac{K - iK'}{2\sqrt{9\alpha^2 + \beta^2}}, \quad \omega'' = \frac{K + iK'}{\sqrt{9\alpha^2 + \beta^2}}, \quad \text{Si 153 (28)}$$

unde

$$6. \quad k = \sqrt{\frac{1}{2} - \frac{3e_3}{\sqrt{9\alpha^2 + \beta^2}}}; \quad k' = \sqrt{\frac{1}{2} + \frac{3e_3}{\sqrt{9\alpha^2 + \beta^2}}}. \quad \text{Si 147}$$

Pentru $\Delta = 0$, toate rădăcinile e_1, e_2, e_3 sînt reale și două dintre ele egale (dacă $g_2 g_3 \neq 0$).Dacă $e_1 - e_2 \neq e_3$, atunci

$$7. \quad \wp(u) = \frac{3g_2}{g_3} - \frac{9g_2}{2g_3} \operatorname{ch}^2\left(u \sqrt{-\frac{9g_2}{2g_3}}\right). \quad \text{Si 148}$$

Dacă $e_1 \neq e_2 = e_3$, atunci

$$8. \quad \wp(u) = -\frac{3g_2}{2g_3} + \frac{9g_2}{2g_3} \frac{1}{\sin^2\left(u \sqrt{\frac{9g_2}{2g_3}}\right)}. \quad \text{Si 149}$$

Dacă $g_2 = g_3$, atunci $e_1 - e_2 - e_3 = 0$ și

$$9. \quad \wp(u) = \frac{1}{u^2}. \quad \text{Si 149}$$

6.17 Funcțiile $\zeta(u)$ și $\sigma(u)$

6.171 Definiție

$$1. \quad \zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2} \right) dz. \quad \text{Si 181 (45)}$$

$$2. \quad \sigma(u) = u \exp \left\{ \int_0^u \left(\zeta(z) - \frac{1}{z} \right) dz \right\}. \quad \text{Si 181 (46)}$$

6.172 Dezvoltarea în serie și în produse infinite

$$1. \quad \zeta(u) = \frac{1}{u} + \sum' \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 + 2n\omega_2)^2} \right). \quad \text{Si 307 (8)}$$

$$2. \quad \sigma(u) = u \prod' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2} \right) \exp \left\{ \frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2} \right\}. \quad \text{Si 309 (9)}$$

6.173

$$1. \quad \zeta(u) = u - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^3 \cdot 5 \cdot 7} - \frac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \frac{3g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \dots \quad \text{Si 181 (49)}$$

$$2. \quad \sigma(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^5 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^6 \cdot 3^2 \cdot 5 \cdot 7} - \frac{g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \dots \quad \text{Si 181 (50)}$$

$$6.174 \quad \zeta(u) = \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \left\{ \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} + m\pi \frac{\omega_2}{\omega_1} \right) + \right. \\ \left. + \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} - m\pi \frac{\omega_2}{\omega_1} \right) \right\}; \quad \text{MO 154}$$

$$= \frac{\zeta(\omega_1)}{\omega_1} + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin \frac{\pi n u}{\omega_1}. \quad \text{MO 155}$$

Relații funcționale și proprietăți :

6.175 $\zeta(u) = -\zeta(-u); \sigma(u) = -\sigma(-u).$

Si 181

6.176

1. $\zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1);$ 2. $\zeta(u + 2\omega_2) = \zeta(u) + 2\zeta(\omega_2).$ Si 184 (57)

3. $\sigma(u + 2\omega_1) = -\sigma(u) \exp \{2(u + \omega_1)\zeta(\omega_1)\};$

4. $\sigma(u + 2\omega_2) = -\sigma(u) \exp \{2(u + \omega_2)\zeta(\omega_2)\}.$ Si 185 (60)

5. $\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{\pi}{2}i.$ Si 186 (62)

6.177

1. $\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}.$ Si 182 (53)

2. $\wp(u) - \wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$ Si 183 (54)

3. $\zeta(u-v) + \zeta(u+v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}.$ Si 182 (51)

6.178

1. $\zeta(u; \omega_1, \omega_2) = t\zeta(tu; t\omega_1, t\omega_2).$ MO 154

2. $\sigma(u; \omega_1, \omega_2) = t^{-1}\sigma(tu; t\omega_1, t\omega_2).$ MO 156

6.178 Integralele funcțiilor eliptice Weierstrass (integralele nedefinite, vezi 2.83.

6.18-6.19 Funcții $\wp$ 6.180 Funcțiile $\wp$ se definesc ca sumele (pentru $|q| < 1$) următoarelor serii

1. $\wp_2(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nu} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu.$ WW II 300

2. $\wp_1(u) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} =$
 $= 2 \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u.$ WW II 300

3. $\wp_2(u) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} q^{(n-\frac{1}{2})^2} \cos(2n-1)u.$

WW II 300

4. $\wp_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nu} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu.$ WW II 300

6.181 Produse infinite pentru funcțiile $\wp$

1. $\wp_4(u) = \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2u + q^{2(2n-1)})(1 - q^{2n}).$ Si 200 (9) J 90 (9)

2. $\wp_3(u) = \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2u + q^{2(2n-1)})(1 - q^{2n}).$ Si 200 (9) J 90 (9)

$$3'. \quad \vartheta_1(u) = 2\sqrt{q} \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}).$$

Si 200 (9) J 90 (9)

$$4. \quad \vartheta_2(u) = 2\sqrt{q} \cos u \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}).$$

Si 200 (9) J 90 (9)

Relații funcționale și proprietăți

6.182 Cvasiperiodicitatea. Fie $q = e^{\pi\tau i}$ ($\text{Im}(\tau) > 0$); atunci funcțiile, care sînt funcții periodice de u , apar ca funcții cvasiperiodice de τ și u . Această proprietate rezultă din următoarele egalități:

$$1. \quad \vartheta_4(u + \pi) = \vartheta_4(u); \quad 2. \quad \vartheta_4(u + \tau\pi) = -\frac{1}{q} e^{-2iu} \vartheta_4(u). \quad \text{Si 200 (10)}$$

$$3. \quad \vartheta_1(u + \pi) = -\vartheta_1(u); \quad 4. \quad \vartheta_1(u + \tau\pi) = -\frac{1}{q} e^{-2iu} \vartheta_1(u). \quad \text{Si 200 (10)}$$

$$5. \quad \vartheta_2(u + \pi) = -\vartheta_2(u); \quad 6. \quad \vartheta_2(u + \tau\pi) = \frac{1}{q} e^{-2iu} \vartheta_2(u). \quad \text{Si 200 (10)}$$

$$7. \quad \vartheta_3(u + \pi) = \vartheta_3(u); \quad 8. \quad \vartheta_3(u + \tau\pi) = \frac{1}{q} e^{-2iu} \vartheta_3(u). \quad \text{Si 200 (10)}$$

6.183

$$1. \quad \vartheta_4\left(u + \frac{1}{2}\pi\right) = \vartheta_3(u); \quad 2. \quad \vartheta_1\left(u + \frac{1}{2}\pi\right) = \vartheta_2(u).$$

$$3. \quad \vartheta_2\left(u + \frac{1}{2}\pi\right) = -\vartheta_1(u); \quad 4. \quad \vartheta_3\left(u + \frac{1}{2}\pi\right) = \vartheta_4(u). \quad \text{Si 199 (5) WW II 301}$$

$$5. \quad \vartheta_4\left(u + \frac{1}{2}\pi\tau\right) = iq^{\frac{1}{4}} e^{-iu} \vartheta_1(u);$$

$$6. \quad \vartheta_1\left(u + \frac{1}{2}\pi\tau\right) = iq^{\frac{1}{4}} e^{-iu} \vartheta_4(u).$$

$$7. \quad \vartheta_2\left(u + \frac{1}{2}\pi\tau\right) = q^{\frac{1}{4}} e^{-iu} \vartheta_3(u);$$

$$8. \quad \vartheta_3\left(u + \frac{1}{2}\pi\tau\right) = q^{\frac{1}{4}} e^{-iu} \vartheta_2(u).$$

WW II 301

6.184 Paritate și imparitate

$$1. \quad \vartheta_1(-u) = -\vartheta_1(u); \quad 2. \quad \vartheta_2(-u) = \vartheta_2(u);$$

$$3. \quad \vartheta_3(-u) = \vartheta_3(u); \quad 4. \quad \vartheta_4(-u) = \vartheta_4(u).$$

WW II 301

$$\mathbf{6.185} \quad \vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u).$$

WW II 306

6.186 Considerind funcțiile ϑ drept funcții de două variabile independente u și τ , vom avea

$$\pi i \frac{\partial^2 \vartheta_k(u, \tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u, \tau)}{\partial \tau} = 0 \quad [k = 1, 2, 3, 4]. \quad \text{WW II 308}$$

6.187 Derivatele parțiale în raport cu u ale funcțiilor ϑ le vom nota cu accente și le vom considera drept funcții de un singur argument u ; atunci

$$1. \quad \vartheta_1'(0) = \vartheta_2(0) \vartheta_3(0) \vartheta_4(0).$$

WW II 308

$$2. \quad \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}.$$

WW II 332

$$6.188 \quad \vartheta_1(u) \vartheta_2(u) \vartheta_3(u) \vartheta_4(u) = \frac{1}{2} \vartheta_1(2u) \vartheta_2(0) \vartheta_3(0) \vartheta_4(0).$$

WW II 332

6.189 Zerourile funcțiilor ϑ

$$1. \quad \vartheta_4(u) = 0 \text{ pentru } u = 2m \frac{\pi}{2} + (2n-1) \frac{\pi\tau}{2}.$$

Si 201

$$2. \quad \vartheta_1(u) = 0 \text{ pentru } u = 2m \frac{\pi}{2} + 2n \frac{\pi\tau}{2}.$$

Si 201

$$3. \quad \vartheta_2(u) = 0 \text{ pentru } u = (2m-1) \frac{\pi}{2} + 2n \frac{\pi\tau}{2}.$$

Si 201

$$4. \quad \vartheta_3(u) = 0 \text{ pentru } u = (2m-1) \frac{\pi}{2} + (2n-1) \frac{\pi\tau}{2}$$

Si 201

[m și n întregi].Integralele funcțiilor ϑ , vezi 5.241.

6.191 Legătura cu funcțiile eliptice Jacobi

Pentru $\tau = i \frac{K'}{K}$, adică pentru $q = \exp\left(-\pi \frac{K'}{K}\right)$,

$$1. \quad \operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}.$$

Si 206 (22) Si 209 (35)

$$2. \quad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}.$$

Si 207 (23) Si 209 (35)

$$3. \quad \operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}.$$

Si 207 (24) Si 209 (35)

6.192 Dezvoltarea în serie a funcțiilor H, H_1, Θ, Θ_1 .

$$1. \quad \Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos \frac{n\pi u}{K}.$$

Si 212 (42)

$$2. \quad H(u) = \vartheta_1\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{q^{(2n-1)^2}} \sin(2n-1) \frac{\pi u}{2K}.$$

Si 212 (43)

$$3. \quad \Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos \frac{n\pi u}{K}.$$

Si 212 (45)

$$4. \quad H_1(u) = \vartheta_2\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} \sqrt{q^{(2n-1)^2}} \cos(2n-1) \frac{\pi u}{2K}.$$

Si 212 (44)

În formulele 6.192., $q = \exp\left(-\pi \frac{K'}{K}\right)$.

6.193 Legătura cu funcțiile eliptice Weierstrass

$$1. \quad \wp(u) = e_1 + \left[\frac{H_1(u \sqrt{\lambda}) H'(0)}{H_1(0) H(u \sqrt{\lambda})} \right]^2 \lambda = e_2 + \left[\frac{\Theta_1(u \sqrt{\lambda}) H'(0)}{\Theta_1(0) H(u \sqrt{\lambda})} \right]^2 \lambda = \\ = e_3 + \left[\frac{\Theta(u \sqrt{\lambda}) H'(0)}{\Theta(0) H(u \sqrt{\lambda})} \right]^2 \lambda.$$

Si 235 (77) Si 235 (78)

$$2. \quad \zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda} \frac{H'(u\sqrt{\lambda})}{H(u\sqrt{\lambda})}. \quad \text{Si 234 (73)}$$

$$3. \quad \sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \cdot \frac{H(u\sqrt{\lambda})}{H'(0)}, \quad \text{Si 234 (72)}$$

unde

$$\lambda = e_1 - e_3; \quad \eta_1 = \zeta(\omega_2) - \frac{\omega_1 \lambda}{3} \frac{H'''(0)}{H'(0)}. \quad \text{Si 236}$$

6.194 Legătura cu integralele eliptice

$$1. \quad E(u, k) = u - u \frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)}. \quad \text{Si 228 (65)}$$

$$2. \quad \Pi(u, -k^2 \sin^2 a, k) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2 a \sin^2 \varphi} =$$

$$= u + \frac{\sin a}{\operatorname{cn} a \operatorname{dn} a} \left[\frac{\Theta'(a)}{\Theta(a)} u + \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} \right]. \quad \text{Si 232 (69)}$$

$$\text{Serii și produse } q \left[q = \exp\left(-\pi \frac{K'}{K}\right) \right]$$

$$6.195 \quad \frac{\pi}{2} \left[1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right]^2 = K = \frac{\pi}{2} \Theta^2(K) \quad (\text{cf. 6.197 1.}). \quad \text{Si 219}$$

$$6.196 \quad E = K - K \frac{\Theta''(0)}{\Theta(0)} = K - \frac{2\pi^2}{K} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}}. \quad \text{Si 230 (67)}$$

6.197

$$1. \quad 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} = \vartheta_3(0) \quad (\text{cf. 6.195}). \quad \text{WW II 319}$$

$$2. \quad \sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{K}{2\pi}} = \frac{1}{2} \vartheta_2(0). \quad \text{WW II 319}$$

$$3. \quad 4 \sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^4 = k. \quad \text{Si 206 (17) Si 206 (18)}$$

$$4. \quad \prod_{n=1}^{\infty} \left(\frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^4 = k'. \quad \text{Si 206 (19) Si 206 (20)}$$

$$5. \quad 2 \sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right)^2 = 2 \sqrt{k} \frac{K}{\pi}. \quad \text{WW II 330}$$

$$6. \quad \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1+q^{2n}} \right)^2 = 2 \sqrt{k'} \frac{K}{\pi}. \quad \text{WW II 330m}$$

6.198

$$1. \quad \lambda = \frac{1}{2} \frac{1 - \sqrt{K}}{1 + \sqrt{K}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{4n^2}} \text{ pentru } 0 < k < 1 \text{ avem } 0 < \lambda < \frac{1}{2} \quad \text{WW II 327}$$

La definirea lui q după modulul k dat, servește seria

$$2. \quad q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \dots$$

WW II 327

6.2 FUNCȚIA EXPONENȚIALĂ INTEGRALĂ ȘI FUNCȚII ÎNRUDITE

6.21 Logarițmul integral $\text{li}(x)$ și funcția exponențială integrală $\text{Ei}(x)$

6.210 Definiții

$$1. \quad \text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Ei}(\ln x) \quad [x < 1].$$

$$2. \quad \text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt = \text{li}(e^x) \quad [x < 0].$$

$$3. \quad \text{li}(x) = \lim_{\epsilon \rightarrow +0} \left[\int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^x \frac{dt}{\ln t} \right] = \text{Ei}(\ln x) \quad [x > 1]. \quad \text{JE 97}$$

$$4. \quad \text{Ei}(x) = - \lim_{\epsilon \rightarrow +0} \left[\int_{-x}^{-\epsilon} \frac{e^{-t}}{t} dt + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t} dt \right] \quad [x > 0].$$

$$5. \quad \text{li}\{\exp(-xe^{\pm i\pi})\} = \text{Ei}(-xe^{\pm i\pi}) = \text{Ei}(x \mp i0) = \text{Ei}(x) \pm i\pi = \\ = \text{li}(e^x) \pm i\pi \quad [x > 0]. \quad \text{JE 97 NI 2(6)}$$

Reprezentări integrale

6.211

$$1. \quad \text{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t} dt = x \ln \ln \frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t dt \quad [x > 1]. \quad \text{La 281 (33)}$$

$$2. \quad \text{li}(x) = x \int_0^1 \frac{dt}{\ln x + \ln t}; \quad \text{La 280 (22)}$$

$$= \frac{x}{\ln x} + x \int_0^1 \frac{dt}{(\ln x + \ln t)^2}; \quad \text{La 280 (29)}$$

$$= x \int_1^{\infty} \frac{1}{\ln x - \ln t} \frac{dt}{t^2} \quad [x < 1]. \quad \text{La 280 (30)}$$

$$3. \quad \text{li}(a^x) = \frac{1}{\ln a} \int_{-\infty}^x \frac{a^t}{t} dt \quad [x > 0].$$

Integralele logaritmului integral, vezi 4.21.

6.212

$$1. \quad \text{Ei}(-x) = C + \ln x + \int_0^x \frac{e^{-t}-1}{t} dt \quad [x > 0]; \quad \text{I NI 11 (1)}$$

$$= C + e^{-x} \ln x + \int_0^x e^{-t} \ln t dt \quad [x > 0]. \quad \text{NI 11 (10)}$$

$$2. \operatorname{Ei}(x) = e^x \left[\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x-t)^2} \right] \quad [x > 0] \quad (\text{cf. 6.211 2.}).$$

$$3. \operatorname{Ei}(-x) = e^{-x} \left[-\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x+t)^2} \right] \quad [x > 0] \quad (\text{cf. 6.211 2.}).$$

La 281 (28)

$$4. \operatorname{Ei}(\pm x) = \pm e^{\pm x} \int_0^1 \frac{dt}{(x \pm \ln t)} \quad [x > 0] \quad (\text{cf. 6.211 2.}).$$

$$5. \operatorname{Ei}(\pm xy) = \pm e^{\pm xy} \int_0^{\infty} \frac{e^{-xt}}{y \mp t} dt \quad [\operatorname{Re}(y) > 0, \quad x > 0]. \quad \text{NI 19 (11)}$$

$$6. \operatorname{Ei}(\pm x) = -e^{\pm x} \int_0^{\infty} \frac{e^{-t}}{t \pm ix} dt \quad [x > 0]. \quad \text{NI 23 (2) NI 23 (3)}$$

$$7. \operatorname{Ei}(xy) = e^{xy} \int_0^1 \frac{t^{y-1}}{x + \ln t} dt = \quad \text{La 282 (44)}_m$$

$$= x^{-1} e^{xy} \left[\int_0^1 \frac{t^{x-1}}{(y + \ln t)^2} dt + y^{-1} \right] \quad [x > 0, \quad y > 0]. \quad \text{La 283 (46)}_m$$

$$8. \operatorname{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t} dt = \quad \text{La 282 (45)}_m$$

$$= x^{-1} e^{-xy} \left[\int_0^1 \frac{t^{x-1}}{(y - \ln t)^2} dt - y^{-1} \right] \quad [x > 0, \quad y > 0]. \quad \text{La 283 (47)}_m$$

$$9. \operatorname{Ei}(x) = e^x \int_1^{\infty} \frac{1}{x - \ln t} \frac{dt}{t^2} \quad [x > 0]. \quad \text{La 283 (48)}$$

$$10. \operatorname{Ei}(-x) = -e^{-x} \int_1^{\infty} \frac{1}{x + \ln t} \frac{dt}{t^2} \quad [x > 0]. \quad \text{La 283 (48)}$$

$$11. \operatorname{Ei}(-x) = -e^{-x} \int_0^{\infty} \frac{t \cos t + x \sin t}{t^2 + x^2} dt \quad [x > 0]. \quad \text{NI 23 (6)}$$

$$12. \operatorname{Ei}(-x) = -e^{-x} \int_0^{\infty} \frac{t \cos t - x \sin t}{t^2 + x^2} dt \quad [x < 0]. \quad \text{NI 23 (6)}$$

$$13. \operatorname{Ei}(-x) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos t}{t} \arctg \frac{t}{x} dt \quad [\operatorname{Re}(x) > 0]. \quad \text{NI 25 (13)}$$

$$14. \operatorname{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^{\infty} \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t dt \quad [x > 0]. \quad \text{NI 26 (7)}$$

$$15. \operatorname{Ei}(x) = 2 \ln x - \frac{2e^x}{\pi} \int_0^{\infty} \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t \, dt \quad [x > 0]. \quad \text{NI 27 (8)}$$

$$16. \operatorname{Ei}(-x) = -x \int_1^{\infty} e^{-tx} \ln t \, dt \quad [x > 0]. \quad \text{NI 32 (12)}$$

Vezi și 3.215 2., 3.555 2., 3.563 1., 3.563 2., 3.691, 3.692, 3.711 1., 3.711 3., 3.721, 3.786 1., 3.871.

Integralele funcției exponențiale integrale,
vezi 4.22, 4.481 1., 4.481 4., 4.481 7.

Serii și reprezentare asimptotică

6.213

$$1. \operatorname{li}(x) = C + \ln(-\ln x) + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \quad [0 < x < 1]. \quad \text{NI 3 (9)}$$

$$2. \operatorname{li}(x) = C + \ln \ln x + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \quad [x > 1]. \quad \text{NI 3 (10)}$$

6.214

$$1. \operatorname{Ei}(x) = C + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} \quad [x < 0].$$

$$2. \operatorname{Ei}(x) = C + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} \quad [x > 0].$$

$$3. \operatorname{Ei}(x) - \operatorname{Ei}(-x) = 2x \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1) \cdot (2k+1)!} \quad [x > 0]. \quad \text{NI 39 (13)}$$

$$6.215 \quad \operatorname{Ei}(-x) = e^{-x} \sum_{k=1}^n (-1)^k \frac{(k-1)!}{x^k} + R_n,$$

unde

$$|R_n| < \frac{n!}{|x|^{n+1} \cos \frac{\varphi}{2}}, \quad x = |x| e^{i\varphi}, \quad \varphi^2 < \pi^2. \quad \text{NI 37 (9)}$$

$$6.216 \quad \operatorname{Ei}(nx) - \operatorname{Ei}(-nx) = e^{nx'} \left(\frac{1}{nx} + \frac{1}{n^2 x^2} + \frac{k_n}{n^3 x^3} \right),$$

unde $x' = x \operatorname{sgn} \operatorname{Re}(x)$, $k_n = O(n^0)$, iar n este mare

NI 39 (15).

6.217 Relații funcționale

$$\begin{aligned} 1. \quad e^{x'} \operatorname{Ei}(-x') - e^{-x'} \operatorname{Ei}(x') &= -2 \int_0^{\infty} \frac{x' \sin t}{t^2 + x^2} dt = \\ &= \frac{4}{\pi} \int_0^{\infty} \frac{x' \cos t}{t^2 + x^2} \lg t \, dt - 2e^{-x'} \ln x' \quad [x' = x \operatorname{sgn} \operatorname{Re}(x)]. \end{aligned} \quad \text{NI 24 (11) NI 27 (9)}$$

$$\begin{aligned} 2. \quad e^{x'} \operatorname{Ei}(-x') + e^{-x'} \operatorname{Ei}(x') &= -2 \int_0^{\infty} \frac{t \cos t}{t^2 + x^2} dt = \\ &= 2e^{-x'} \ln x' - \frac{4}{\pi} \int_0^{\infty} \frac{t \sin t}{t^2 + x^2} \ln t \, dt \quad [x' = x \operatorname{sgn} \operatorname{Re}(x)]. \end{aligned} \quad \text{NI 24 (10) NI 27 (10)}$$

$$3. \operatorname{Ei}(-x) - \operatorname{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos t}{t} \operatorname{arctg} \frac{t\left(x - \frac{1}{x}\right)}{1+t^2} dt \quad [\operatorname{Re}(x) > 0].$$

NI 25 (14)

$$4. \operatorname{Ei}(-\alpha x) \operatorname{Ei}(-\beta x) - \ln(\alpha\beta) \operatorname{Ei}[-(\alpha + \beta)x] = \\ = e^{-(\alpha + \beta)x} \int_0^{\infty} \frac{e^{-tx} \ln[(\alpha + t)(\beta + t)]}{t + \alpha + \beta} dt. \quad \text{NI 32 (9)}$$

Vezi și 3.518 3., 3.523 1., 3.527 2., 3.542 2., 3.542 5., 3.863 2.

Legătura cu funcția hipergeometrică degenerată, vezi 7.337.

Integralele logaritmului integral, vezi 4.21.

Integralele funcției exponențiale integrale, vezi 4.22, 4.481 1., 4.481 4., 4.481 7.

6.218 Cîteva valori numerice

1. $\operatorname{li}(1,451\,369\,234\,6\dots) = 0.$ NI 88
2. $\operatorname{Ei}(-1) = -0,219\,383\,934\,395\,520\,273\,665\dots$ NI 89
3. $\operatorname{Ei}(1) = 1,895\,117\,816\,355\,936\,755\,478\dots$ NI 89

6.22 Sinusul integral $\operatorname{si}(x)$ și cosinusul integral $\operatorname{ci}(x)$

6.220

$$1. \operatorname{si}(x) = - \int_x^{\infty} \frac{\sin t}{t} dt = -\frac{\pi}{2} + \int_0^x \frac{\sin t}{t} dt. \quad \text{NI 11 (3)}$$

$$2. \operatorname{ci}(x) = - \int_x^{\infty} \frac{\cos t}{t} dt = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt \quad \text{NI 11 (2)}$$

6.221

$$1. \operatorname{si}(xy) = - \int_x^{\infty} \frac{\sin ty}{t} dt. \quad \text{NI 18 (7)}$$

$$2. \operatorname{ci}(xy) = - \int_x^{\infty} \frac{\cos ty}{t} dt. \quad \text{NI 18 (6)}$$

$$3. \operatorname{si}(x) = - \int_0^{\frac{\pi}{2}} e^{-x \cos t} \cos(x \sin t) dt. \quad \text{NI 13 (26)}$$

6.222

$$1. \operatorname{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1) \cdot (2k-1)!}. \quad \text{NI 7 (4)}$$

$$2. \operatorname{ci}(x) = C - \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k \cdot (2k)!}. \quad \text{NI 7 (3)}$$

6.223

1. $\operatorname{ci}(x) \pm i \operatorname{si}(x) = \operatorname{Ei}(\pm ix)$ NI 6m.
2. $\operatorname{ci}(x) - \operatorname{ci}(xe^{\pm \pi i}) = \mp \pi i.$ NI 7 (5)
3. $\operatorname{si}(x) + \operatorname{si}(-x) = -\pi.$ NI 7 (7)

6.224

$$1. \operatorname{Ei}(-x) - \operatorname{ci}(x) = \int_0^{\frac{\pi}{2}} e^{-x \cos \varphi} \sin(x \sin \varphi) d\varphi. \quad \text{NI 13 (27)}$$

$$2. [\operatorname{ci}(x)]^2 + [\operatorname{si}(x)]^2 = -2 \int_0^{\frac{\pi}{2}} \frac{\exp(-x \operatorname{tg} \varphi) \ln \cos \varphi}{\sin \varphi \cos \varphi} d\varphi$$

$$[\operatorname{Re}(x) > 0] \quad (\text{vezi și 3.724}). \quad \text{NI 32 (11)}$$

Vezi și 3.217 2., 3.223, 3.522, 3.523 3., 3.526, 3.527 4., 5.211, 5.235.

6.225

$$1. \lim_{x \rightarrow +\infty} (x^\rho \operatorname{si}(x)) = 0, \quad \lim_{x \rightarrow +\infty} (x^\rho \operatorname{ci}(x)) = 0 \quad [\rho < 1]. \quad \text{NI 38 (5)}$$

$$2. \lim_{x \rightarrow -\infty} \operatorname{si}(x) = -\pi, \quad \lim_{x \rightarrow -\infty} \operatorname{ci}(x) = \pm \pi i. \quad \text{NI 38 (6)}$$

Integralele sinusului integral și ale cosinusului integral, vezi 4.23, 4.245, 4.254, 4.481.

Integralele nedefinite ale sinusului integral și ale cosinusului integral, vezi 2.84.

6.23 Integrala de probabilitate $\Phi(x)$ și integralele Fresnel $S(x)$ și $C(x)$

6.230 Definiții

$$1. \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

$$2. S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt.$$

$$3. C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt.$$

Reprezentări integrale

6.231

$$1. \Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt \quad (\text{vezi și 3.224 1.}).$$

$$2. S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt.$$

$$3. C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt.$$

6.232

$$1. \Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2 y^2} dt.$$

$$2. S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(t^2 y^2) dt.$$

$$3. C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(t^2 y^2) dt.$$

$$4. \quad \Phi(xy) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} ty dt}{\sqrt{t^2 + x^2}}; \quad \text{NI 19 (11)m}$$

$$= 1 - \frac{2x}{\pi} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} dt}{t^2 + x^2} \quad [\operatorname{Re}(y^2) > 0]. \quad \text{NI 19 (13)m}$$

$$5. \quad \Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xi e^{4x^2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2 x^2} \sin(ty) dt \quad [\operatorname{Re}(x^2) > 0]. \quad \text{NI 28 (3)m}$$

$$6. \quad \Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} x e^{-\frac{y^2}{4x^2}} \int_0^\infty e^{-t^2 x^2 - ty} dt. \quad [\operatorname{Re}(x^2) > 0]. \quad \text{NI 27. (1)m}$$

Vezi și 3.275, 3.276, 3.564 3., 4.401, 5.212, 5.219 1., 5.221 4., 5.245.
Integralele integralei de probabilitate, vezi 4.24.

Integralele integralei-sinus și ale integralei-cosinus ale lui Fresnel, vezi 4.25.

6.233 Dezvoltarea în serie

$$1. \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1) \cdot (k-1)!}; \quad \text{NI 7 (9)m}$$

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!}. \quad \text{NI 10 (11)m}$$

$$2. \quad S(x) = \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)! (4k+3)!}; \quad \text{NI 8 (14)m}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!} - \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!} \right\}. \quad \text{NI 10 (13)m}$$

$$3. \quad C(x) = \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)! (4k+1)!}; \quad \text{NI 8 (13)m}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!} + \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!} \right\}. \quad \text{NI 10 (12)m}$$

Descompunerea cu ajutorul funcțiilor Bessel, vezi 6.526.

Reprezentări asimptotice

$$6.234 \quad \Phi(\sqrt{x}) = 1 - \frac{1}{\pi} e^{-x} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(k + \frac{1}{2}\right)}{x^{k+\frac{1}{2}}} + \frac{e^{-x}}{\pi} R_n,$$

$$\text{unde } |R_n| < \frac{\Gamma\left(n + \frac{1}{2}\right)}{|x|^{n+\frac{1}{2}} \cos \frac{\varphi}{2}}, \quad x = |x| e^{i\varphi} \text{ și } \varphi^2 < \pi^2.$$

NI 37 (10)m

6.235

$$1. S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}x} \cos x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty]. \quad \text{MO 127m}$$

$$2. C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}x} \sin^2 x + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty]. \quad \text{MO 127m}$$

6.236 Relații funcționale

$$1. C(z) + iS(z) = \sqrt{\frac{i}{2}} \Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-t^2} dt.$$

$$2. C(z) - iS(z) = \frac{1}{\sqrt{2i}} \Phi(z\sqrt{i}) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-t^2} dt.$$

$$3. [\cos u^2 C(u) + \sin u^2 S(u)] =$$

$$= \frac{1}{2} [\cos u^2 + \sin u^2] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2 dt \quad [\operatorname{Re}(u) \geq 0]. \quad \text{NI 28 (6)m}$$

$$4. [\cos u^2 S(u) - \sin u^2 C(u)] =$$

$$= \frac{1}{2} [\cos u^2 - \sin u^2] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2 dt \quad [\operatorname{Re}(u) \geq 0]. \quad \text{NI 28 (5)m}$$

$$5. \left[\left[C(x) \right] - \frac{1}{2} \right]^2 + \left[\left[S(x) \right] - \frac{1}{2} \right]^2 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\exp(-x^2 \operatorname{tg} \varphi) \cdot \sin \frac{\varphi}{2} \sqrt{\cos \varphi}}{\sin 2\varphi} d\varphi.$$

NI 33 (18)m

Vezi și 5.226, 5.246, 5.247.

Legătura cu funcția hipergeometrică degenerată, vezi 7.336.

Legătura cu funcția cilindrului parabolic, vezi 7.354.

6.237

$$1. \lim_{x \rightarrow +\infty} \left(x^p \left[S(x) - \frac{1}{2} \right] \right) = 0. \quad 2. \lim_{x \rightarrow +\infty} \left(x^p \left[C(x) - \frac{1}{2} \right] \right) = 0 \quad [p < 1].$$

NI 38 (11)

$$3. \lim_{x \rightarrow +\infty} S(x) = \frac{1}{2}. \quad 4. \lim_{x \rightarrow +\infty} C(x) = \frac{1}{2}.$$

NI 38 (12)m

Integralele integralei de probabilitate, vezi 4.24.

Integralele integralei-sinus și ale integralei-cosinus ale lui Fresnel, vezi 4.25.

6.24 Funcția $L(x)$ a lui Lobachevski

6.240 Definiție

$$L(x) = - \int_0^x \ln \cos t \, dt. \quad \text{Lo III 184 (10)}$$

Reprezentarea integrală a funcției $L(x)$, vezi și 3.325 5., 3.615 1.

6.241 Dezvoltarea în serie

$$L(x) = x \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin 2kx}{k^2}. \quad \text{Lo III 185 (11)}$$

6.242 Relații funcționale

$$1. L(-x) = -L(x) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right]. \quad \text{Lo III 185 (13)}$$

$$2. L(\pi - x) = \pi \ln 2 - L(x). \quad \text{Lo III 286}$$

$$3. L(\pi + x) = \pi \ln 2 + L(x). \quad \text{Lo III 286}$$

$$4. L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right) \ln 2 - \frac{1}{2} L\left(\frac{\pi}{2} - 2x\right) \quad \left[0 \leq x < \frac{\pi}{4}\right].$$

Lo III 186 (14)

Tabela valorilor funcției $L(x)$ este dată la 8.1.6.3 INTEGRALE EULERIENE DE SPEȚA ÎNTÎI ȘI A DOUA
ȘI FUNCȚII ÎNRUDITE6.31 Funcția $\Gamma(x)$ (integrala euleriană de speța a doua)

6.310 Definiție

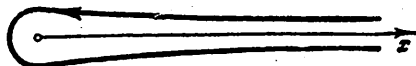
$$1. \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad [\operatorname{Re}(z) > 0]. \quad (\text{Euler}) \quad \text{F II 777 (7)}$$

Generalizare

$$2. \Gamma(z) = -\frac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t} dt$$

pentru z diferit de un număr întreg.
Conturul C este reprezentat în figură.

WW II 18



$\Gamma(z)$ este o funcție analitică rațională de z , cu poluri ordinare în punctele $z = -l$ ($l = 0, 1, 2, \dots$), cărora le corespund reziduurile $\frac{(-1)^l}{l!}$; $\Gamma(z)$ satisface relația $\Gamma(1) = 1$.

WW II 18 MO 1

Reprezentări integrale

$$6.311 \quad \Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_{-\infty}^{(0+)} e^{-t} t^{z-1} dt. \quad \text{MO 2}$$

6.312

$$1. \Gamma(z) = \int_0^1 \left(\ln \frac{1}{t}\right)^{z-1} dt \quad [\operatorname{Re}(z) > 0]. \quad \text{F II 778}$$

$$2. \Gamma(z) = x^z \int_0^{\infty} e^{-xt} t^{z-1} dt \quad [\operatorname{Re}(z) > 0, \operatorname{Re}(x) > 0]. \quad \text{F II 779 (8)}$$

$$3. \Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^{\infty} e^{-at^2} (1+t^2)^{z-\frac{1}{2}} \cos[2at + (2z-1) \operatorname{arctg} t] dt$$

[$a > 0$]. WW II 19

$$4. \quad \Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^{\infty} e^{-t^2} t^{z-1} (1 + t^2)^{\frac{z}{2}} \{3 \sin [t + z \operatorname{arctg}(-t)] + \\ + \sin [t + (z-2) \operatorname{arctg}(-t)]\} dt \\ [\operatorname{arctg} \text{ indică un unghi obtuz}] \quad \text{WW II 37}$$

$$5. \quad \Gamma(y) = x^y - i\beta y \int_0^{\infty} t^{y-1} \exp(-xte^{-i\beta}) dt \\ \left[x, y, \beta \text{ reali, } x > 0, y > 0, |\beta| < \frac{\pi}{2} \right]. \quad \text{MO 8}$$

$$6. \quad \Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^{\infty} e^{bt} (it)^{z-1} dt \quad [b > 0, 0 < \operatorname{Re}(z) < 1]. \quad \text{NH 154 (3)}$$

$$7. \quad \Gamma(z) = \frac{(\sqrt{a^2 + b^2})^z}{\cos\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^{\infty} e^{-at} \cos bt \, t^{z-1} dt; \quad \left\{ \begin{array}{l} [a > 0, \quad \text{NH 152 (1)}_m \\ b \geq 0, \\ \operatorname{Re}(z) > 0]. \quad \text{NH 152 (2)} \end{array} \right.$$

$$= \frac{(\sqrt{a^2 + b^2})^z}{\sin\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^{\infty} e^{-at} \sin bt \, t^{z-1} dt$$

$$8. \quad \Gamma(z) = \frac{b^z}{\cos \frac{\pi z}{2}} \int_0^{\infty} \cos bt \, t^{z-1} dt; \quad \left\{ \begin{array}{l} \text{NH 152 (4)} \\ [b > 0, 0 < \operatorname{Re}(z) < 1]. \end{array} \right.$$

$$= \frac{b^z}{\sin \frac{\pi z}{2}} \int_0^{\infty} \sin bt \, t^{z-1} dt \quad \text{NH 152 (5)}$$

$$9. \quad \Gamma(z) = \int_0^{\infty} e^{-t} (t-z)^{z-1} \ln t \, dt; \quad \left\{ \begin{array}{l} \text{NH 173 (7)} \\ [\operatorname{Re}(z) > 0]. \end{array} \right.$$

$$10. \quad \Gamma(z) = \int_{-\infty}^{\infty} \exp(zt - e^t) dt \quad \text{NH 145 (14)}$$

$$11. \quad \Gamma(x) \cos \alpha x = \lambda^x \int_0^{\infty} t^{x-1} e^{-\lambda t} \cos \alpha \cos(\lambda t \sin \alpha) dt;$$

$$12. \quad \Gamma(x) \sin \alpha x = \lambda^x \int_0^{\infty} t^{x-1} e^{-\lambda t} \sin \alpha \sin(\lambda t \sin \alpha) dt \\ \left[\lambda > 0, x > 0, -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right]. \quad \text{WW II 36}$$

$$13. \quad \Gamma(-z) = \int_0^{\infty} \left[\frac{e^{-t} - \sum_{k=0}^n (-1)^k \frac{t^k}{k!}}{t^{z+1}} \right] dt \quad [n = E(\operatorname{Re}(z))]. \quad \text{MO 2}$$

$$6.313 \quad \Gamma\left(\frac{z+1}{v}\right) = v u^{\frac{z+1}{v}} \int_0^{\infty} \exp(-ut^v) t^z dt$$

$$[\operatorname{Re}(u) > 0, \operatorname{Re}(x) > 0, \operatorname{Re}(z) > -1]. \quad \text{JE 110m MO 7m}$$

$$6.314 \quad \Gamma(z) = \int_1^{\infty} e^{-t} t^{z-1} dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (z+k)}.$$

MO 2

6.315

$$1. \quad \frac{1}{\Gamma(z)} = \frac{1}{2\pi} \int_C (-t)^{-z} e^{-t} dt$$

pentru z diferit de un număr întreg. Conturul este reprezentat în figura la formula 6.310 2.

WW II 18

$$2. \quad \frac{e^{ab} b^{1-s}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{at}}{(a+it)^s} dt = \frac{1}{\Gamma(z)} \text{ pentru } b > 0;$$

$$= 0 \text{ pentru } b < 0$$

$$\left[a > 0, \operatorname{Re}(z) > 0, -\frac{\pi}{2} < \arg(a+it) < \frac{\pi}{2} \right] \text{ NH 155 (8) MO 7}$$

$$3. \quad \frac{1}{\Gamma(z)} = a^{1-z} \frac{e^a}{\pi} \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} \theta - z\theta) \cos^{z-2} \theta d\theta \quad [\operatorname{Re}(z) > 1]. \text{ NH 157 (14)}$$

Vezi și 3.214 3., 3.215 1., 3.225 3., 3.287, 3.635, 3.653 1., 3.653 2., 3.723 2., 4.213, 4.222, 4.232 5., 4.232 6., 4.242.

6.32 Reprezentarea funcțiilor Γ sub formă de serii și produse

6.321 Dezvoltarea în serie

$$1. \quad \Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$

$$\left[c_0 = 1, c_{n+1} = \frac{\sum_{k=0}^n (-1)^{k+1} s_{k+1} c_{n-k}}{n+1}; s_1 = C, s_n = \zeta(n) \text{ pentru } n \geq 2, \operatorname{Re}(z) > 0 \right].$$

NH 40 (1) NH 40 (3)

$$2. \quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$

$$\left[d_0 = 1, d_{n+1} = \frac{\sum_{k=0}^n (-1)^k s_{k+1} d_{n-k}}{n+1}; s_1 = C, s_n = \zeta(n) \text{ pentru } n \geq 2 \right].$$

NH 41 (4) NH 41 (6)

Dezvoltarea sub formă de produs infinit

$$6.322 \quad \Gamma'(z) = e^{-Cz} \frac{1}{z} \prod_{k=1}^{\infty} \frac{e^{\frac{z}{k}}}{1 + \frac{z}{k}} \quad [\operatorname{Re}(z) > 0];$$

Sm III 269

$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}} \quad [\operatorname{Re}(z) > 0];$$

WW II 8

$$= \lim_{n \rightarrow \infty} \frac{n^z}{z} \prod_{k=1}^n \frac{k}{z+k} \quad [\operatorname{Re}(z) > 0].$$

Sm III 267 (130)

$$6.323 \quad \Gamma(z) = 2z^z e^{-z} \prod_{k=1}^{\infty} \sqrt[2k]{B\left(2^{k-1}z, \frac{1}{2}\right)}. \quad \text{NH 98 (12)}$$

$$6.324 \quad \Gamma(1+z) = 4^z \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right)}{\Gamma\left(\frac{1}{2}\right)}. \quad \text{MO-3}$$

6.325

$$1. \quad \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\gamma) \Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right) \left(1 - \frac{\gamma}{\beta+k}\right) \right]. \quad \text{NH 62 (2)}$$

$$2. \quad \frac{e^{Cx} \Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty} \left[\left(1 - \frac{x}{z+k}\right) e^{\frac{x}{k}} \right] \quad [\operatorname{Re}(z) > 0, \operatorname{Re}(z-x) > 0].$$

$$3. \quad \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{z}{2}\right) \Gamma\left(\frac{1}{2} - \frac{z}{2}\right)} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2k-1}\right) \left(1 + \frac{z}{2k}\right). \quad \text{MO 2}$$

6.326

$$1. \quad \frac{\Gamma(x)}{B(x+iy, x-iy)} = \left| \frac{\Gamma(x)}{\Gamma(x+iy)} \right|^2 = \prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2}\right) \quad [x, y \text{ reali } x > 0]. \quad \text{Lo V NH 63 (4)}$$

$$2. \quad \frac{\Gamma(x+iy)}{\Gamma(y)} = \frac{xe^{-iCy}}{x+iy} \prod_{n=1}^{\infty} \frac{\exp\left(\frac{iy}{n}\right)}{1 + \frac{iy}{x+n}} \quad [x, y \text{ reali, } x > 0]$$

(vezi și 3.562 1.). MO 2

6.327 Dezvoltarea asimptotică pentru valori mari ale lui $|z|$

$$\Gamma(z) = z^{-\frac{1}{2}} e^{-z} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^3} - \frac{139}{51840z^5} - \frac{571}{2488320z^7} + Q(z^{-5}) \right\} \quad [|\arg z| < \pi]. \quad \text{WW II 28}$$

Pentru valorile reale și pozitive ale lui z , restul seriei este mai mic decât ultimul termen însumat.

6.328

$$1. \quad \lim_{|y| \rightarrow \infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi} \quad [x \text{ și } y \text{ reali}] \quad \text{MO 6}$$

$$2. \quad \lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1. \quad \text{MO 6}$$

6.33 Relații funcționale pentru funcțiile Γ

$$6.331 \quad \Gamma(x+1) = x \Gamma(x).$$

6.332

$$\left. \begin{aligned} 1. \quad |\Gamma(iy)|^2 &= \frac{\pi}{y \operatorname{sh} \pi y} \\ 2. \quad \left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 &= \frac{\pi}{\operatorname{ch} \pi y} \end{aligned} \right\} [y \text{ real}] \quad \begin{array}{l} \text{MO 3} \\ \text{MO 3} \end{array}$$

$$3. \Gamma(1+ix) \Gamma(1-ix) = \frac{\pi x}{\operatorname{sh} \pi x} \quad [x \text{ și } y \text{ reali}] \quad \text{Lo V}$$

$$4. \Gamma(1+x+iy) \Gamma(1-x+iy) \Gamma(1+x-iy) \Gamma(1-x-iy) = \\ = \frac{2\pi^2 (x^2 + y^2)}{\operatorname{sh} 2y\pi - \cos 2\pi x} \quad [x \text{ și } y \text{ reali}] \quad \text{Lo V}$$

$$6.333 \quad [\Gamma(n+1)]^n = G(n+1) \prod_{k=1}^n k^k,$$

unde n este un număr natural și

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp \left[-\frac{z(z+1)}{2} - \frac{C}{2} z^2 \right] \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n \exp \left(-z + \frac{z^2}{2n}\right) \right\}.$$

WW II 43

6.334

$$1. \prod_{k=1}^n \frac{1}{\Gamma \left(-z \exp \frac{2\pi k i}{n} \right)} = -z^n \prod_{k=1}^{\infty} \left[1 - \left(\frac{z}{k} \right)^n \right] \quad [n=2, 3, \dots]. \quad \text{MO 2}$$

$$2. \Gamma \left(\frac{1}{2} + x \right) \Gamma \left(\frac{1}{2} - x \right) = \frac{\pi}{\cos \pi x}.$$

$$3. \Gamma(1-x) \Gamma(x) = \frac{\pi}{\sin \pi x}. \quad \text{F II 430}$$

$$6.335 \quad \Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{\frac{nx-1}{2}} \prod_{k=0}^{n-1} \Gamma \left(x + \frac{k}{n} \right) \quad [\text{teorema de multiplicare}]$$

F II 782m WW II 12

Cazuri particulare

$$1. \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma \left(x + \frac{1}{2} \right) \quad [\text{formula de dublare}]$$

$$2. \Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x) \Gamma \left(x + \frac{1}{3} \right) \Gamma \left(x + \frac{2}{3} \right).$$

$$3. \prod_{k=1}^{n-1} \Gamma \left(\frac{k}{n} \right) \Gamma \left(1 - \frac{k}{n} \right) = \frac{(2\pi)^{n-1}}{n}.$$

WW II 12

$$6.336 \quad \Gamma \left(-\frac{yz+xt}{2y} \right) \Gamma(1+z) = (2i)^{z+1} y \Gamma \left(1 + \frac{yz-xt}{2y} \right) \int_0^{\infty} e^{-tz} \sin^2(ty) dt$$

$$[\operatorname{Re}(yi) > 0, \operatorname{Re}(x-yzi) > 0].$$

NH 133 (10)

Legătura cu funcția ψ , vezi 6.351 1.

Legătura cu funcția B , vezi 6.381.

Integralele funcției Γ , vezi 6.417 4., 6.419, 7.323, 7.342 3., 7.342 4.

6.337

$$1. [\Gamma'(x)]^2 < \Gamma(x) \Gamma''(x) \quad [x > 0].$$

MO 1

$$2. \text{Pentru } x > 0, \min \Gamma(1+x) = 0,88560 \dots \text{ se obține când } x = 0,46163 \dots$$

JE 107

Valori particulare

6.338

1. $\Gamma(1) = \Gamma(2) = 1.$

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$

4. $\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2 [(4k+1)^2 - 1]}{[(4k-1)^2 - 1] (4k+1)^2}.$

MO 1

5. $\prod_{k=1}^{\infty} \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^3} \left(\frac{\pi}{\sqrt{3}}\right)^3.$

WW II 36

6.339 Pentru n natural,

1. $\Gamma(n+1) = n!$

2. $\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \cdot (2n-1)!!$

3. $\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}.$

4. $\frac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)} = \frac{(4p^2-1^2)(4p^2-3^2)\dots[4p^2-(2n-1)^2]}{2^{2n}}.$

W 221

6.34 Logaritmul funcției Γ , $\ln \Gamma(x)$

6.341 Reprezentare integrală

1. $\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi +$
 $+ \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t} dt \quad [\operatorname{Re}(z) > 0].$

WW II 23

2. $\ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + 2 \int_0^{\infty} \frac{\arctg \frac{t}{z}}{e^{2\pi t} - 1} dt$

$$[\operatorname{Re} z > 0 \text{ și } \arctg w = \int_0^w \frac{du}{1+u^2} \text{ se ia pe un drum rectiliniu în planul } w].$$

WW II 25

3. $\ln \Gamma(z) = \int_0^{\infty} \left\{ \frac{e^{-st} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t} \right\} \frac{dt}{t} \quad [\operatorname{Re}(z) > 0].$

WW II 24

4. $\ln \Gamma(z) = \int_0^{\infty} \left\{ (z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)} \right\} \frac{dt}{t} \quad [\operatorname{Re}(z) > 0].$

WW II 24

5. $\ln \Gamma(x) = \frac{\ln \pi - \ln \sin \pi x}{2} +$

$$+ \frac{1}{2} \int_0^{\infty} \left\{ \frac{\operatorname{sh}\left(\frac{1}{2} - x\right)t}{\operatorname{sh} \frac{t}{2}} - (1-2x)e^{-t} \right\} \frac{dt}{t} \quad [0 < x < 1].$$

WW II 24

$$6. \quad \ln \Gamma(z) = \int_0^1 \left\{ \frac{t^z - t}{t-1} - t(z-1) \right\} \frac{dt}{t \ln t} \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 38}$$

$$7. \quad \ln \Gamma(z) = \int_0^\infty \left[(z-1) \bar{e}^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t} \quad [\operatorname{Re}(z) > 0]. \quad \text{NH 187 (7)}$$

Vezi și 3.244, 3.332, 3.716.

6.342 Dezvoltarea în serie

$$\begin{aligned} 1. \quad \ln \Gamma(z+1) &= \frac{1}{2} \left[\ln \left(\frac{\pi z}{\sin \pi z} \right) - \ln \frac{1+z}{1-z} \right] + \\ &\quad + (1-C)z + \sum_{k=1}^{\infty} \frac{1-\zeta(2k+1)}{2k+1} z^{2k+1} = \\ &= -Cz + \sum_{k=2}^{\infty} (-1)^k \frac{z^k}{k} \zeta(k) \quad [|z| < 1]. \quad \text{NH 38 (16) NH 38 (12)} \end{aligned}$$

$$2. \quad \ln \Gamma(1+x) = \frac{1}{2} \ln \frac{\pi x}{\sin \pi x} - Cx - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} \{1 - \zeta(2n+1)\} \quad [|x| < 1].$$

NH 38 (14)

6.343

$$1. \quad \ln \Gamma(x) = \ln \sqrt{2\pi} + \sum_{n=1}^{\infty} \left\{ \frac{1}{2n} \cos 2n\pi x + \frac{1}{n\pi} (C + \ln 2n\pi) \sin 2n\pi x \right\}$$

[0 < x < 1]. F III 558

$$\begin{aligned} 2. \quad \ln \Gamma(z) &= z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \\ &\quad + \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{(m+1)(m+2)} \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}} \quad [|\arg z| < \pi]. \quad \text{MO 9} \end{aligned}$$

6.344 Dezvoltarea asimptotică pentru valori mari ale lui |z|

$$\ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)} z^{2k-1} + R_n(z),$$

unde

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1} \cos^{2n-1} \left(\frac{1}{2} \arg z \right)}. \quad \text{MO 5}$$

Integralele lui $\ln \Gamma(k)$, vezi 4.32.

6.35 Funcția $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$

6.351 Reprezentări integrale

$$1. \quad \psi(z) = \frac{d \ln \Gamma(z)}{dz} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt \quad [\operatorname{Re}(z) > 0]. \quad \text{NH 183 (1)}$$

$$2. \quad \psi(z) = \int_0^\infty \left\{ e^{-t} - \frac{1}{(1+t)^z} \right\} \frac{dt}{t} \quad [\operatorname{Re}(z) > 0]. \quad \text{NH 184 (7)}$$

$$3. \quad \psi(z) = \ln z - \frac{1}{2z} - 2 \int_0^{\infty} \frac{t \, dt}{(t^2 + z^2)(e^{2\pi t} - 1)} \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 26}$$

$$4. \quad \psi(z) = \int_0^1 \left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t} \right) dt \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 21}$$

$$5. \quad \psi(z) = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt - C \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \text{WW II 37}$$

$$6. \quad \psi(x) = \int_0^{\infty} \left\{ (1+t)^{-1} - (1+t)^{-x} \right\} \frac{dt}{t} - C \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 37}$$

$$7. \quad \psi(x) = \int_0^1 \frac{t^{x-1} - 1}{t-1} dt - C \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \text{F II 799}$$

$$8. \quad \psi(x) = \ln z + \int_0^{\infty} e^{-tz} \left[\frac{1}{t} - \frac{1}{1-e^{-t}} \right] dt \quad \left[|\arg(z)| < \frac{\pi}{2} \right]. \quad \text{MO 4}$$

Vezi și 3.1973., 3.2115., 3.262, 3.263, 3.281,
3.6252., 3.634, 3.645, 3.658.

Integralele funcției ψ , vezi 4.33.

Dezvoltarea în serie

6.352

$$1. \quad \psi(x) = -C - \sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k+1} \right); \quad \text{F II 799 (26) C 26 (1)}$$

$$= -C - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}. \quad \text{F II 495}$$

$$2. \quad \psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln \left(1 + \frac{1}{x+k} \right) \right]. \quad \text{MO 4}$$

$$3. \quad \psi(x) = -C + \frac{\pi^2}{6}(x-1) - (x-1) \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}. \quad \text{NH 54 (12)}$$

6.353

$$1. \quad \psi(x+1) = -C + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1} \quad \text{NH 37 (5)}$$

$$2. \quad \psi(x+1) = \frac{1}{2x} - \frac{\pi}{2} \operatorname{ctg} \pi x - \frac{x^2}{1-x^2} - C + \sum_{k=1}^{\infty} [1 - \zeta(2k+1)] x^{2k}. \quad \text{NH 38 (10)}$$

6.354

$$1. \quad \psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left(\frac{1}{y+k} - \frac{1}{x+k} \right) \quad (\text{vezi și } 3.167, 3.179, 3.2114., \\ 3.4383., 3.6231., 3.665.7). \quad \text{NH 99 (3)}$$

$$2. \quad \phi(x+iy) - \phi(x-iy) = \sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x+k)^2}.$$

6.355

$$1. \quad \phi\left(\frac{p}{q}\right) = -C + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{q}{p+kq} \right) \quad (\text{vezi și } 3.1973.) \text{NH } 20(1)$$

$$2. \quad \phi\left(\frac{p}{q}\right) = -C - \frac{\pi}{2} \operatorname{ctg} \frac{p\pi}{q} + 2 \sum_{k=1}^{E\left(\frac{q+1}{2}\right)-1} \left[\cos \frac{2kp\pi}{q} \ln \sin \frac{k\pi}{q} \right] - \ln 2q$$

$[q=2, 3, \dots, p=1, 2, \dots, q-1].$ MO 4

$$3. \quad \phi\left(\frac{p}{q}\right) - \phi\left(\frac{p-1}{q}\right) = q \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(p+kq)^{n-1}}. \quad \text{NH } 59(3)$$

$$6.356 \quad \phi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}. \quad \text{NH } 37(1)$$

Dezvoltarea sub formă de produs infinit

6.357

$$1. \quad e^{\psi(x)} = x \prod_{k=0}^{\infty} \left(1 + \frac{1}{x+k} \right) e^{-\frac{1}{x+k}}. \quad \text{NH } 65(12)$$

$$2. \quad e^{\psi y(x)} = \frac{\Gamma(x+y)}{\Gamma(x)} \prod_{k=0}^{\infty} \left(1 + \frac{y}{x+k} \right) e^{-\frac{y}{x+k}}. \quad \text{NH } 65(11)$$

Vezi și 6.39.

Legătura cu funcția ζ a lui Riemann, vezi 7.433 2.Legătura cu funcția Γ , vezi 3.619 2., 3.723 1.

Legătura cu funcția B, vezi 3.628 1.

Seriile funcției ψ , vezi 6.403 2., 6.456, 6.457 3. (funcții cilindrice), 6.761 (derivatele după indice ale funcțiilor sferice), 7.253, 7.253 (funcția hipergeometrică), 7.338 (funcția hipergeometrică degenerată).

Integralele funcției ψ , vezi 4.33.

6.358 Relații funcționale

$$1. \quad \phi(x+1) = \phi(x) + \frac{1}{x}. \quad \text{JE } 109m$$

$$2. \quad \phi\left(\frac{x+1}{2}\right) - \phi\left(\frac{x}{2}\right) = 2\beta(x) \quad (\text{cf. } 6.390).$$

$$3. \quad \phi(x+n) = \phi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}. \quad \text{Ga } 154(64)m$$

$$4. \quad \phi(n+1) = -C + \sum_{k=1}^n \frac{1}{k}. \quad \text{MO } 4$$

$$5. \quad \lim_{n \rightarrow \infty} [\phi(x+n) - \ln n] = 0. \quad \text{MO } 3$$

$$6. \quad \psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n \quad [n=2, 3, 4, \dots]. \quad \text{MO 3}$$

$$7. \quad \psi(x-n) = \psi(x) - \sum_{k=1}^n \frac{1}{x-k}.$$

$$8. \quad \psi(1-z) = \psi(z) + \pi \operatorname{ctg} \pi z.$$

Ga 155 (68)m

$$9. \quad \psi\left(\frac{1}{2} + z\right) = \psi\left(\frac{1}{2} - z\right) + \pi \operatorname{tg} \pi z.$$

JE 109m

$$10. \quad \psi\left(\frac{3}{4} - n\right) = \psi\left(\frac{1}{4} + n\right) + \pi \quad [n \text{ natural}]$$

6.359 Valori particulare

$$1. \quad \psi(1) = -C \quad (\text{cf. 6.360 1.}).$$

$$2. \quad \psi\left(\frac{1}{2}\right) = -C - 2 \ln 2 = -1,963\,510\,026 \dots$$

Ga 155m

$$3. \quad \psi\left(\frac{1}{2} \pm n\right) = -C + 2 \left[\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right].$$

JE 109m

$$4. \quad \psi\left(\frac{1}{4}\right) = -C - \frac{\pi}{2} - 3 \ln 2.$$

Ga 157m

$$5. \quad \psi\left(\frac{3}{4}\right) = -C + \frac{\pi}{2} - 3 \ln 2.$$

Ga 157m

$$6. \quad \psi\left(\frac{1}{3}\right) = -C - \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$$

Ga 157m

$$7. \quad \psi\left(\frac{2}{3}\right) = -C + \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$$

Ga 157m

$$8. \quad \psi'(1) = \frac{\pi^2}{6} = 1,644\,934\,067 \dots$$

JE 109m

$$9. \quad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4,934\,802\,201 \dots$$

JE 109m

$$10. \quad \psi'(-n) = \infty$$

JE 108m

$$11. \quad \psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$$

JE 109m

$$12. \quad \psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$$

JE 109m

$$13. \quad \psi'\left(\frac{1}{2} - n\right) = \frac{\pi^2}{2} + 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$$

JE 109m

[n natural].

6.36 Constanta C a lui Euler

6.360 Definiție

$$1. \quad C = -\psi(1) = 0,577\,215\,664\,90 \dots$$

F II 795 F II 319

$$2. \quad C = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n \right].$$

F II 801

$$3. \quad C = \lim_{x \rightarrow 1+0} \left[\zeta(x) - \frac{1}{x-1} \right].$$

F II 804

6.361 Reprezentări integrale

$$1. C = - \int_0^{\infty} e^{-t} \ln t \, dt. \quad \text{F II 807} \quad 2. C = - \int_0^1 \ln \left(\ln \frac{1}{t} \right) dt. \quad \text{F II 807}$$

$$3. C = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t} \right] dt. \quad \text{D (852.3)}$$

$$4. C = - \int_0^{\infty} \left[\cos t - \frac{1}{1+t} \right] \frac{dt}{t}. \quad \text{MO 10}$$

$$5. C = 1 - \int_0^{\infty} \left[\frac{\sin t}{t} - \frac{1}{1+t} \right] \frac{dt}{t}. \quad \text{MO 10}$$

$$6. C = - \int_0^{\infty} \left[e^{-t} - \frac{1}{1+t} \right] \frac{dt}{t}. \quad \text{F II 795 F II 802}$$

$$7. C = - \int_0^{\infty} \left[e^{-t} - \frac{1}{1+t^2} \right] \frac{dt}{t}. \quad \text{D (852.4) MO 10}$$

$$8. C = \int_0^{\infty} \left[\frac{1}{e^x - 1} - \frac{1}{te^x} \right] dt. \quad \text{D (852.5)}$$

$$9. C = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^{\infty} \frac{e^{-t}}{t} dt. \quad \text{F II 802}$$

Vezi și 6.3515. — 6.3517., 3.2115., 3.2282., 3.289, 3.291, 3.292, 3.572, 3.643, 3.7112., 3.781, 3.783, 3.785, 3.842, 3.862, 4.225, 4.2351., 4.3322.

6.362 Dezvoltări asimptotice

$$C = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^3} - \frac{1}{120n^5} + \frac{1}{252n^7} - \frac{1}{240n^9} + \dots +$$

$$+ \frac{B_{2n}}{2n} \frac{1}{k^{2n}} + \frac{B_{2n+2}}{2(n+1)} \frac{\theta}{k^{2n+2}} \quad [0 < \theta < 1]. \quad \text{F II 827}$$

6.37 Funcția $B(x, y)$ (integrala euleriană de speța întâi)

Reprezentare sub formă de integrală

6.370

$$1. B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt^1; \quad \text{F II 774 (1)}$$

$$= 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1} dt \quad \left. \vphantom{\int_0^1} \right\} [\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0].$$

6.371

$$1. B(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} \varphi \cos^{2y-1} \varphi \, d\varphi \quad \text{C 10}$$

¹ Această egalitate servește ca definiție pentru funcția $B(x, y)$.

$$2. \quad B(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x} \varphi \cos^{2y} \varphi \, d\varphi \quad \left[\operatorname{Re}(x) > -\frac{1}{2}, \operatorname{Re}(y) > -\frac{1}{2} \right]$$

MO 7

$$3. \quad B(x, y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = 2 \int_0^{\infty} \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt$$

[Re(x) > 0; Re(y) > 0]. F II 775

$$4. \quad B(x, y) = 2^{2-y-x} \int_{-1}^1 \frac{(1+t)^{2x-1} (1-t)^{2y-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0].$$

MO 7

$$5. \quad B(x, y) = \int_0^1 \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt = \int_1^{\infty} \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt.$$

F II 775

$$6. \quad B(x, y) = \frac{1}{2^{x+y-1}} \int_0^1 [(1+t)^{x-1} (1-t^{y-1}) + (1+t)^{y-1} (1-t)^{x-1}] dt \quad \left. \begin{array}{l} [\operatorname{Re}(x) > 0, \\ \operatorname{Re}(y) > 0]. \end{array} \right\}$$

BH [1] (15)

$$7. \quad B(x, y) = z^y (1+z)^x \int_0^1 \frac{t^{x-1} (1-t)^{y-1}}{(t+z)^{x+y}} dt$$

[Re(x) > 0,
Re(y) > 0.

WW II 39

$$8. \quad B(x, y) = z^y (1+z)^x \int_0^{\frac{\pi}{2}} \frac{\cos^{2x-1} \varphi \sin^{2y-1} \varphi}{(z + \cos^2 \varphi)^{x+y}} d\varphi \quad \left. \begin{array}{l} 0 > z > -1, \\ \operatorname{Re}(x+y) < 1. \end{array} \right\}$$

NH 163-(8)

Vezi și 3.171, 3.184, 3.185, 3.192, 3.195 2.,
3.212, 3.427 1., 3.455, 5.218 3.

$$6.372 \quad B(x, x) = \frac{1}{2^{2x-2}} \int_0^1 (1-t^2)^{x-1} dt = \frac{1}{2^{2x-1}} \int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}} dt$$

(vezi și 6.384, 6.378 3., precum și 3.421, 3.428, 3.518 1., 3.519 1.). F II 784

$$6.373 \quad B(x+y, x-y) = 4^{1-x} \int_0^{\infty} \frac{\operatorname{ch} 2yt}{\operatorname{ch}^2 x t} dt \quad [\operatorname{Re}(x) > |\operatorname{Re}(y)|, \operatorname{Re}(x) > 0].$$

MO 9

$$6.374 \quad B\left(x, \frac{y}{x}\right) = z \int_0^1 (1-t^x)^{x-1} t^{y-1} dt \quad [\operatorname{Re}(z) > 0, \operatorname{Re}\left(\frac{y}{x}\right) > 0, \operatorname{Re}(x) > 0]$$

F II 787m

6.375

$$\left. \begin{array}{l} 1. \quad \int_{-\infty}^{\infty} \frac{dt}{(a+it)^x (b-it)^y} = \frac{2\pi(a+b)^{1-x-y}}{(x+y-1) B(x, y)} \\ 2. \quad \int_{-\infty}^{\infty} \frac{dt}{(a-it)^x (b-it)^y} = 0 \end{array} \right\}$$

[a > 0, b > 0; x și y reali
x+y > 1.

MO 7

$$6.376 \quad B(x+iy, x-iy) = 2^{1-2x} a e^{-2iy} \int_{-\infty}^{\infty} \frac{e^{2i\alpha y t} dt}{\operatorname{ch}^{2x}(\alpha t + \gamma)}$$

[y, α, γ reali, $\alpha > 0$; $\operatorname{Re}(x) > 0$]. MO 8m

Reprezentarea integrală a lui $\ln B(x, y)$, vezi 3.245.

$$6.377 \quad \frac{1}{B(x, y)} = \frac{2^{x+y-1} (x+y-1)}{\pi} \int_0^{\frac{\pi}{2}} \cos(x-y)t \cos^{x+y-2} t dt;$$

NH 158 (5)m

$$= \frac{2^{x+y-2} (x+y-1)}{\pi \cos \left[(x-y) \frac{\pi}{2} \right]} \int_0^{\frac{\pi}{2}} \cos(x-y)t \sin^{x+y-2} t dt;$$

NH 159 (8)m

$$= \frac{2^{x+y-2} (x+y-1)}{\pi \sin \left[(x-y) \frac{\pi}{2} \right]} \int_0^{\frac{\pi}{2}} \sin(x-y)t \sin^{x+y-2} t dt$$

NH 159 (9)m

Vezi și 3.454.

6.378 Dezvoltarea în serie

$$1. \quad B(x, y) = \frac{1}{y} \sum_{n=0}^{\infty} (-1)^n \frac{y(y-1) \dots (y-n)}{n! (x+n)} \quad [y > 0].$$

WW II 36

$$2. \quad \ln B\left(\frac{1+x}{2}, \frac{1}{2}\right) = \ln \sqrt{2\pi} + \frac{1}{2} \left[\ln \left(\frac{\operatorname{tg} \frac{\pi x}{2}}{x} \right) - \ln \left(\frac{1+x}{1-x} \right) \right] +$$

$$+ \sum_{k=0}^{\infty} \frac{1 - (1-2^{-2k}) \zeta(2k+1)}{2k+1} x^{2k+1} \quad [|x| < 2].$$

NH 39 17

$$3. \quad B\left(z, \frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{(2k+1)!!}{2^k \cdot k!} \frac{1}{z+k} \quad (\text{vezi și 6.384 și 6.372}).$$

WW II 36

6.379 Dezvoltarea sub formă de produs infinit

$$B(x, y) = \prod_{k=0}^{\infty} \frac{k(x+y+k)}{(x+k)(y+k)} \quad [x, y \neq 0, -1, -2, \dots].$$

MO 2

6.38 Relații funcționale pentru funcția $B(x, y)$

$$6.381 \quad B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = B(y, x).$$

F II 779

$$6.382 \quad B(x, y) B(x+y, z) = B(y, z) B(y+z, x).$$

MO 6

$$6.383 \quad \sum_{k=-\infty}^{\infty} B(x, y+k) = B(x-1, y).$$

WW II 39

$$6.384 \quad B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right) \quad (\text{vezi și 6.372 și 6.378 3}).$$

F II 784

$$6.385 \quad B(x, x) B\left(x + \frac{1}{2}, x + \frac{1}{2}\right) = \frac{\pi}{2^{2x-1} x}.$$

WW II 38

$$6.386 \quad \frac{1}{B(n, m)} = m \binom{n+m-1}{n-1} = n \binom{n+m-1}{m-1} \quad [m \text{ și } n \text{ naturali}].$$

Legătura cu funcția ψ , vezi 3.628 1.

6.39 Funcția $\beta(x)$

6.390 Definiție

$$1. \beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right]. \quad \text{NH 16 (13)}$$

6.391 Reprezentări integrale

$$1. \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt \quad [\operatorname{Re}(x) > 0]. \quad \text{WW II 39}$$

$$2. \beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt \quad [\operatorname{Re}(x) > 0]. \quad \text{MO 4}$$

$$3. \beta\left(\frac{x+1}{2}\right) = \int_0^\infty \frac{e^{-xt}}{\cosh t} dt \quad [\operatorname{Re}(x) > 0] \quad (\text{cf. 6.391 1.}).$$

Vezi și 3.195 3., 3.322 2., 3.422, 3.644, 3.757, 3.832.

Dezvoltarea în serie

6.392

$$1. \beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}. \quad \text{NH 37 NH 101 (1)}$$

$$2. \beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)}. \quad \text{NH 101 (2)}$$

$$3. \beta(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k}. \quad \text{NH 246 (7)}$$

6.393

$$1. \beta(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} (1-2^{1-k}) \zeta(k) x^{k-1}. \quad \text{NH 37 (5)}$$

$$2. \beta(x+1) = \frac{1}{2x} - \frac{1}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=0}^{\infty} [1 - (1-2^{-2k}) \zeta(2k+1)] x^{2k}. \quad \text{NH 38 (11)}$$

$$6.394 \quad \beta^{(n)}(x) = (-1)^n \cdot n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}}. \quad \text{NH 37 (2)}$$

6.395 Dezvoltarea sub formă de sumă cu număr finit de termeni.

$$1. \beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{E\left(\frac{q-1}{2}\right)} \cos \frac{p(2k+1)\pi}{q} \ln \left(2 - 2 \cos \frac{(2k+1)\pi}{q}\right) \\ [q=2, 3, \dots, p=1, 2, 3, \dots]. \quad (\text{vezi și 6.355}). \quad \text{NH 23 (9)}$$

$$2. \beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}.$$

Relații funcționale

$$6.396 \quad \sum_{k=0}^{2n} (-1)^k \beta \left(\frac{x+k}{2n+1} \right) = (2n+1) \beta(x). \quad \text{NH 19}$$

$$6.397 \quad \sum_{k=1}^n \beta(2^k x) = \phi(2^n x) - \phi(x) - n \ln 2. \quad \text{NH 20 (10)}$$

6.4-6.5 FUNCȚII CILINDRICE

6.40 Definiții

6.401 Se numesc funcții *cilindrice* $Z_p(z)$ soluțiile ecuației diferențiale

$$\frac{d^2 Z_p}{dz^2} + \frac{1}{z} \frac{dZ_p}{dz} + \left(1 - \frac{p^2}{z^2}\right) Z_p = 0. \quad \text{C 37 (1)}$$

Formele particulare ale funcțiilor cilindrice sînt *funcțiile Bessel* (sau *funcțiile cilindrice de speța întâi*) $J_p(z)$, *funcțiile Neumann* (sau *funcțiile cilindrice de speța a doua*) $N_p(z)$ și *funcțiile Hankel* (sau *funcțiile cilindrice de speța a treia*) $H_p^{(1)}(z)$ și $H_p^{(2)}(z)$. Aceste funcții se definesc în modul următor:

$$6.402 \quad J_p(z) = \frac{z^p}{2^p} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(p+k+1)} \quad [|\arg z| < \pi]. \quad \text{C 55 (1)}$$

6.403

$$1. \quad N_p(z) = -\frac{1}{\sin p\pi} [\cos p\pi J_p(z) - J_{-p}(z)] \quad [p \text{ neîntreg}, |\arg z| < \pi]. \quad \text{C 41 (3)}$$

$$2. \quad \pi N_n(z) = 2J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - \\ - \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! (k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)]; \quad \text{C 43 (10)} \\ = 2J_n(z) \left(\ln \frac{z}{2} + C\right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - \\ - \left(\frac{z}{2}\right)^{\frac{n}{2}} \frac{1}{n!} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k! (k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^k \frac{1}{m} \right] \quad \text{C 44} \\ [n+1 \text{ natural}, |\arg z| < \pi].$$

6.404

$$1. \quad N_{-n}(z) = (-1)^n N_n(z). \quad 2. \quad J_{-n}(z) = (-1)^n J_n(z) \quad [n \text{ natural}]. \quad \text{C 41 (2)}$$

6.405

$$1. \quad H_p^{(1)}(z) = J_p(z) + iN_p(z). \quad 2. \quad H_p^{(2)}(z) = J_p(z) - iN_p(z). \quad \text{C 44 (1)}$$

În toate relațiile valabile pentru orice funcție cilindrică $Z_p(z)$, adică pentru funcțiile $J_p(z)$, $N_p(z)$ și pentru combinațiile lor liniare, de exemplu $H_p^{(1)}(z)$, $H_p^{(2)}(z)$, vom scrie, în locul literelor J , N , $H^{(1)}$, $H^{(2)}$, litera Z .

Funcții cilindrice $I_p(z)$ și $K_p(z)$ de argument imaginar

6.406

$$1. I_p(z) = e^{-\frac{\pi}{2} p t} J_p(e^{\frac{\pi}{2} t} z) \quad \left[-\pi < \arg z \leq \frac{\pi}{2} \right]. \quad \text{W 92}$$

$$2. I_p(z) = e^{\frac{3}{2} \pi p t} J_p(e^{-\frac{3}{2} \pi t} z) \quad \left[\frac{\pi}{2} < \arg z \leq \pi \right]. \quad \text{W 92}$$

Pentru n întreg,

$$3. I_n(z) = i^{-n} J_n(iz). \quad \text{C 46 (1)}$$

6.407

$$1. K_p(z) = \frac{\pi i}{2} e^{\frac{\pi}{2} p t} H_p^{(1)}(iz).$$

$$2. K_p(z) = \frac{\pi i}{2} e^{-\frac{\pi}{2} p t} H_{-p}^{(1)}(iz). \quad \text{W 92 (8)}$$

Ecuția diferențială care definește aceste funcții, vezi 6.514 1.

6.41 Reprezentări integrale ale funcțiilor $J_p(z)$

6.411

$$1. J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-n i \theta + i z \sin \theta} d\theta;$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta \quad [n \text{ natural}] \quad \text{WW II 172}$$

6.412

$$1. J_{2n}(z) = \frac{1}{\pi} \int_0^{\pi} \cos 2n\theta \cos(z \sin \theta) d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos 2n\theta \cos(z \sin \theta) d\theta$$

[n întreg] W 30 (7)

$$2. J_{2n+1}(z) = \frac{1}{\pi} \int_0^{\pi} \sin(2n+1)\theta \sin(z \sin \theta) d\theta =$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(2n+1)\theta \sin(z \sin \theta) d\theta$$

[n întreg] W 30 (6)

$$3. J_p(z) = 2 \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin^{2p} \theta \cos(z \cos \theta) d\theta \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right].$$

WW II 179m

$$4. J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} \cos(z \cos \varphi) \sin^{2p} \varphi d\varphi$$

[$\operatorname{Re}(p) > 0$] WW II 179

$$5. J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(z \sin \theta) \cos^{2p} \theta \, d\theta \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right].$$

C 65 (5) W 35 (4)m

$$6. J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi e^{\pm iz \cos \varphi} \sin^{2p} \varphi \, d\varphi$$

$$\left[\operatorname{Re}\left(p + \frac{1}{2}\right) > 0 \right]. \quad \text{WW II 178}$$

6.413

$$1. J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{p-\frac{1}{2}} \cos zt \, dt \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right].$$

C 65 (6) WW II 178

$$2. J_p(x) = 2 \frac{\left(\frac{x}{2}\right)^{-p}}{\Gamma\left(\frac{1}{2}-p\right) \Gamma\left(\frac{1}{2}\right)} \int_1^\infty \frac{\sin xt}{(t^2-1)^{p+\frac{1}{2}}} dt$$

$$\left[-\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2}, x > 0 \right]. \quad \text{MO 37}$$

$$3. J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{itz} (1-t^2)^{p-\frac{1}{2}} dt \quad \left[\operatorname{Re}(p) > -\frac{1}{2} \right].$$

W 34 (3)

$$6.414 \quad J_p(x) = \frac{2}{\pi} \int_0^\infty \sin\left(x \operatorname{ch} t - \frac{p\pi}{2}\right) \operatorname{ch} pt \, dt. \quad \text{W 199 (12)}$$

$$6.415 \quad J_p(z) = \frac{2^{p+1} z^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{p-\frac{1}{2}} \theta \sin\left(z - p\theta + \frac{1}{2}\theta\right)}{\sin^{2p+1} \theta} e^{-2z \operatorname{ctg} \theta} d\theta$$

$$\left[|\arg z| < \frac{\pi}{2}, \operatorname{Re}\left(p + \frac{1}{2}\right) > 0 \right]. \quad \text{WW II 183}$$

6.416

$$1. J_p(z) = \frac{1}{\pi} \int_0^\pi \cos(p\theta - z \sin \theta) d\theta - \frac{\sin p\pi}{\pi} \int_0^\infty e^{-p\theta - z \operatorname{sh} \theta} d\theta$$

[p oarecare $\operatorname{Re}(z) > 0$]. W 195 (4)

$$2. J_p(z) = \frac{e^{\pm p\pi i}}{\pi} \left[\int_0^\pi \cos(p\theta + z \sin \theta) d\theta - \sin p\pi \int_0^\infty e^{-p\theta + z \operatorname{sh} \theta} d\theta \right]$$

[pentru $\frac{\pi}{2} < |\arg z| < \pi$; semnul + (plus) luindu-se pentru $\arg z > \frac{\pi}{2}$, iar - (minus), pentru $\arg z < -\frac{\pi}{2}$].

WW II 174

6.417

$$1. J_p(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-p-1} \exp \left[\frac{z}{2} \left(t - \frac{1}{t} \right) \right] dt \quad \left[|\arg z| < \frac{\pi}{2} \right].$$

WW II 164 · W 195 (2)

$$2. J_p(z) = \frac{z^p}{2^{p+1} \pi i} \int_{-\infty}^{(0+)} t^{-p-1} \exp \left(t - \frac{z^2}{4t} \right) dt;$$

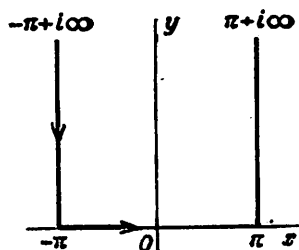
$$3. = \frac{z^p}{2^{p+1} \pi i} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!} \int_{-\infty}^{(0+)} e^t t^{-p-k-1} dt.$$

W 195 (1)

$$4. J_p(x) = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{\Gamma(-t)}{\Gamma(p+t+1)} \left(\frac{x}{2} \right)^{p+2t} dt \quad [\operatorname{Re} p \geq 0, x > 0]. \quad W 214 (7)$$

$$5. J_p(z) = \frac{\Gamma\left(\frac{1}{2}-p\right) \left(\frac{z}{2}\right)^p (1+, -1-)}{2\pi i \Gamma\left(\frac{1}{2}\right)} \int_A (t^2-1)^{p-\frac{1}{2}} \cos(zt) dt$$

$\left[\left\{ \Gamma\left(\frac{1}{2}-p\right) \right\}^{-1} \neq 0; \text{ punctul } A \text{ se află la dreapta punctului } t=1, \text{ și}$



$\arg(t-1) = \arg(t+1) = 0$ în punctul A].

WW II 177—178

$$6. J_p(z) = \frac{1}{2\pi} \int_{-\pi+i\infty}^{\pi+i\infty} e^{-iz \sin \theta + ip\theta} d\theta \quad [\operatorname{Re}(z) > 0].$$

CH 401

Drumul de integrare este arătat în figură.

$$6.418 \quad \frac{J_p(\sqrt{x^2-\zeta^2})}{(x^2-\zeta^2)^{\frac{p}{2}}} = \frac{1}{\pi(x+\zeta)^p} \left\{ \int_0^{\infty} e^{\zeta \cos t} \cos(z \sin t - pt) dt - \right.$$

$$\left. - \sin p\pi \int_0^{\infty} \exp(-z \operatorname{sh} t - \zeta \operatorname{ch} t - pt) dt \right\} \quad [\operatorname{Re}(z+\zeta) > 0]. \quad MO 40$$

$$6.419 \quad \int_{2x}^{\infty} \frac{J_0(t)}{t} dt = \frac{1}{4\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-t)}{i\Gamma(1+t)} x^{2t} dt \quad [x > 0].$$

MO 41

Vezi și 3.481—3.483, 3.585, 4.911, 4.912.

Reprezentarea integrală a lui $J_0(z)$, vezi 3.524 4., 3.528 3., 3.593.Reprezentarea integrală a lui $J_1(z)$, vezi 3.524 5., 3.528 2.6.42 Reprezentări integrale ale funcției $N_p(z)$

$$6.421 \quad N_0(x) = \frac{4}{\pi^2} \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} \sin(xt) dt - \frac{4}{\pi^2} \int_1^{\infty} \frac{\ln(t + \sqrt{t^2-1})}{\sqrt{t^2-1}} \operatorname{Ei}_{-1}(xt) dt$$

 $[x > 0].$

MO 37

$$6.422 \quad N_p(x) = -2 \frac{\left(\frac{x}{2}\right)^{-p}}{\Gamma\left(\frac{1}{2}-p\right) \Gamma\left(\frac{1}{2}\right)} \int_1^{\infty} \frac{\cos xt}{(t^2-1)^{p+\frac{1}{2}}} dt$$

$$\left[-\frac{1}{2} < \operatorname{Re} p < \frac{1}{2}, x > 0\right]. \quad \text{C (89) 28m MO 38}$$

$$6.423 \quad N_p(x) = -\frac{2}{\pi} \int_0^{\infty} \cos\left(x \operatorname{ch} t - \frac{p\pi}{2}\right) \operatorname{ch} pt \, dt \quad [-1 < \operatorname{Re}(p) < 1, x > 0].$$

W 199 (13)

$$6.424 \quad N_p(z) = \frac{1}{\pi} \int_0^{\pi} \sin(z \sin \theta - p\theta) d\theta -$$

$$-\frac{1}{\pi} \int_0^{\infty} (e^{pt} + e^{-pt} \cos p\pi) e^{-z \operatorname{sh} t} dt \quad [\operatorname{Re}(z) > 0]. \quad \text{W 197 (1)}$$

$$6.425 \quad N_p(z) = \frac{2\left(\frac{z}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \left[\int_0^{\frac{\pi}{2}} \sin(z \sin \theta) \cos^{2p} \theta d\theta - \right.$$

$$\left. - \int_0^{\infty} e^{-z \operatorname{sh} \theta} \operatorname{ch}^{2p} \theta d\theta \right] \quad \left[\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(z) > 0 \right]. \quad \text{W 181 (5)m}$$

$$6.426 \quad N_p(z) = -\frac{2^{p+\frac{1}{2}} z^p}{\Gamma\left(p+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \times$$

$$\times \int_0^{\frac{\pi}{2}} \frac{\cos^{p-\frac{1}{2}} \theta \cos\left(z - p\theta + \frac{1}{2}\theta\right)}{\sin^{2p+\frac{1}{2}} \theta} e^{-2x \operatorname{ctg} \theta} d\theta \quad \text{W 186 (8)}$$

$$\left[|\arg z| < \frac{\pi}{2}, \operatorname{Re}\left(p+\frac{1}{2}\right) > 0 \right].$$

Reprezentarea integrală a lui $N_0(z)$, vezi 3.528 4., 3.552.

6.43 Reprezentări integrale ale funcțiilor $H_p^{(1)}(z)$ și $H_p^{(2)}(z)$

6.431

$$1. \quad H_p^{(1)}(x) = \frac{e^{-\frac{p\pi i}{2}}}{\pi i} \int_{-\infty}^{\infty} e^{ix \operatorname{ch} t - p t} dt =$$

$$= \frac{2e^{-\frac{p\pi i}{2}}}{\pi i} \int_0^{\infty} e^{tx \operatorname{ch} t} \operatorname{ch} pt \, dt \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \text{W 199 (10)}$$

$$2. \quad H_p^{(2)}(x) = -\frac{e^{\frac{p\pi i}{2}}}{\pi i} \int_{-\infty}^{\infty} e^{-ix \operatorname{ch} t - p t} dt =$$

$$= -\frac{2e^{\frac{p\pi i}{2}}}{\pi i} \int_0^{\infty} e^{-tx \operatorname{ch} t} \operatorname{ch} t \, dt \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad [-1 < \operatorname{Re}(p) < 1, x > 0].$$

W 199 (11)

6.432

$$1. H_p^{(1)}(z) = -\frac{2^{p+1} t z^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{p-\frac{1}{2}} t e^{i(z-pt+\frac{1}{2})}}{\sin^{2p+1} t} \exp(-2z \operatorname{ctg} t) dt$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(z) > 0 \right]. \quad \text{W 186 (5)}$$

$$2. H_p^{(2)}(z) = \frac{2^{p+1} t z^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{p-\frac{1}{2}} t e^{-i(z-pt+\frac{1}{2})}}{\sin^{2p+1} t} \exp(-2z \operatorname{ctg} t) dt$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(z) > 0 \right]. \quad \text{W 186 (6)}$$

6.433

$$1. H_p^{(1)}(x) = -\frac{2t \left(\frac{x}{2}\right)^{-p}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - p\right)} \int_1^{\infty} \frac{e^{tx}}{(t^2-1)^{p+\frac{1}{2}}} dt$$

$$\left[-\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2}, x > 0 \right]. \quad \text{W 187 (1)}$$

$$2. H_p^{(2)}(x) = \frac{2t \left(\frac{x}{2}\right)^{-p}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - p\right)} \int_1^{\infty} \frac{e^{-tx}}{(t^2-1)^{p+\frac{1}{2}}} dt$$

$$\left[-\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2}, x > 0 \right]. \quad \text{W 187 (2)}$$

6.434

$$1. H_p^{(1)}(z) = -\frac{t}{\pi} e^{-\frac{1}{2} i p \pi} \int_0^{\infty} \exp \left[\frac{1}{2} t z \left(t + \frac{1}{t} \right) \right] t^{-p-1} dt$$

$$[0 < \arg z < \pi; \text{ sau } \arg z = 0 \text{ și } -1 < \operatorname{Re}(p) < 1]. \quad \text{MO 38}$$

$$2. H_p^{(1)}(xz) = -\frac{t}{\pi} e^{-\frac{1}{2} i p \pi} z^p \int_0^{\infty} \exp \left[\frac{1}{2} t x \left(t + \frac{z^2}{t} \right) \right] t^{-p-1} dt$$

$$\left[0 < \arg z < \frac{\pi}{2}, x > 0, \operatorname{Re}(p) > -1; \right.$$

$$\left. \text{ sau } \arg z = \frac{\pi}{2}, x > 0 \text{ și } -1 < \operatorname{Re}(p) < 1 \right]. \quad \text{MO 38}$$

$$3. H_p^{(1)}(xz) = \sqrt{\frac{2}{\pi z}} \frac{x^p \exp \left[i \left(xz - \frac{\pi}{2} p - \frac{\pi}{4} \right) \right]}{\Gamma\left(p + \frac{1}{2}\right)} \int_0^{\infty} \left(1 + \frac{u}{2z} \right)^{p-\frac{1}{2}} t^{p-\frac{1}{2}} e^{-xz} dt$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}, -\frac{\pi}{2} < \arg z < \frac{3}{2} \pi, x > 0 \right]. \quad \text{MO 39}$$

$$6.435 \quad H_p^{(1)}(z) = \frac{-2te^{-ip\pi} \left(\frac{z}{2}\right)^p}{\sqrt{\pi} \Gamma\left(p + \frac{1}{2}\right)} \int_0^{\infty} e^{tx \operatorname{ch} t} \operatorname{sh} 2p t dt$$

$$\left[0 < \arg x < \pi, \operatorname{Re}(p) > -\frac{1}{2} \text{ sau } \arg z = 0 \text{ și } -\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2} \right]. \quad \text{MO 38}$$

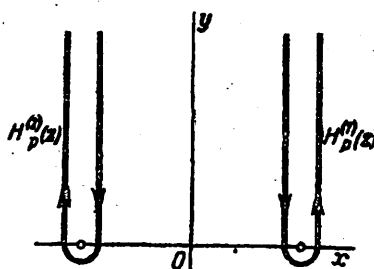
$$6.436 \quad H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\exp(i\sqrt{x^2+t^2})}{\sqrt{x^2+t^2}} dt \quad [x > 0]. \quad \text{MO 38}$$

6.437

$$1. \quad H_p^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2}-p\right)\left(\frac{z}{2}\right)^p (1+)}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{1+\infty i}^{(1+)} e^{izt} (t^2-1)^{p-\frac{1}{2}} dt \quad [-\pi < \arg z < 2\pi].$$

W 183 (4)

$$2. \quad H_p^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2}-p\right)\left(\frac{z}{2}\right)^p (-1-)}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{-1+\infty i}^{(-1-)} e^{izt} (t^2-1)^{p-\frac{1}{2}} dt \quad [-2\pi < \arg z < \pi] \quad \text{W 183 (5)}$$

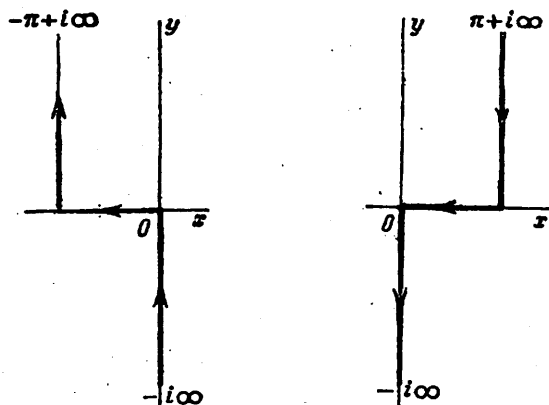


drumurile de integrare sînt arătate în figură].

6.438

$$1. \quad H_p^{(1)}(z) = -\frac{1}{\pi} \int_{-\infty i}^{-\pi+i\infty} e^{-iz \sin \theta + ip\theta} d\theta \quad [\operatorname{Re}(z) > 0]. \quad \text{W 197 (2)m}$$

$$2. \quad H_p^{(2)}(z) = -\frac{1}{\pi} \int_{\pi+i\infty}^{\infty} e^{-iz \sin \theta + ip\theta} d\theta \quad [\operatorname{Re}(z) > 0]. \quad \text{W 197 (3)m}$$



Drumurile de integrare pentru formula 1 sînt arătate în figura din stînga, iar pentru formula 2, în figura din dreapta.

6.439

$$\left. \begin{aligned} 1. \quad H_p^{(1)}(z) J_p(\zeta) &= \frac{1}{\pi i} \int_0^{\gamma+i\infty} \exp \left[\frac{1}{2} \left(t - \frac{z^2 + \zeta^2}{t} \right) \right] I_p \left(\frac{z\zeta}{t} \right) \frac{dt}{t} \\ 2. \quad H_p^{(2)}(z) J_p(\zeta) &= \frac{i}{\pi} \int_0^{\gamma-i\infty} \exp \left[\frac{1}{2} \left(t - \frac{z^2 + \zeta^2}{t} \right) \right] I_p \left(\frac{z\zeta}{t} \right) \frac{dt}{t} \end{aligned} \right\}$$

$[\gamma > 0, \operatorname{Re}(p) > -1, |\zeta| < |z|]$ (vezi și 4.437). MO 45
Vezi și 5.214 4.

6.44 Reprezentări integrale ale funcțiilor $I_p(z)$ și $K_p(z)$ Funcția $I_p(z)$

6.441

$$\left. \begin{aligned} 1. \quad I_p(z) &= \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{p-\frac{1}{2}} e^{\pm izt} dt \\ 2. \quad I_p(z) &= \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{p-\frac{1}{2}} \operatorname{ch} zt dt \end{aligned} \right\} \left[\operatorname{Re}\left(p + \frac{1}{2}\right) > 0 \right].$$

6.442

$$\left. \begin{aligned} 1. \quad I_p(z) &= \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi e^{\pm z \cos \theta} \sin^{2p} \theta d\theta \\ 2. \quad I_p(z) &= \frac{\left(\frac{z}{2}\right)^p}{\Gamma\left(p + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi \operatorname{ch}(z \cos \theta) \sin^{2p} \theta d\theta \end{aligned} \right\} \left[\operatorname{Re}\left(p + \frac{1}{2}\right) > 0 \right].$$

W 94 (9)

$$6.443 \quad I_p(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos p\theta d\theta - \frac{\sin p\pi}{\pi} \int_0^\infty e^{-z \operatorname{ch} t - pt} dt$$

$$\left[|\arg z| \leq \frac{\pi}{2}, \operatorname{Re}(p) > 0 \right].$$

W 201 (4)

Vezi și 3.227. 3., 3.473, 5.238.

Reprezentarea integrală a lui $I_0(z)$ și $I_1(z)$, vezi 3.321. 21.Funcția $K_p(z)$

6.444

$$1. \quad K_p(z) = \int_0^\infty e^{-z \operatorname{ch} t} \operatorname{ch} pt dt$$

$$\left[|\arg z| < \frac{\pi}{2} \text{ sau } \operatorname{Re}(z) = 0 \text{ și } p = 0 \right].$$

MO 39

$$2. \quad K_p(z) = \frac{\left(\frac{z}{2}\right)^p \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p + \frac{1}{2}\right)} \int_0^\infty e^{-z \operatorname{ch} t} \operatorname{sh}^{2p} t dt$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(z) > 0; \text{ sau } \operatorname{Re}(z) = 0 \text{ și } -\frac{1}{2} < \operatorname{Re}(p) < \frac{1}{2} \right].$$

W 190 (5) WW II 203m

$$6.445 \quad K_p(z) = \frac{\left(\frac{z}{2}\right)^p \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p + \frac{1}{2}\right)} \int_1^{\infty} e^{-zt} (t^2 - 1)^{p-\frac{1}{2}} dt$$

$$\left[\operatorname{Re}\left(p + \frac{1}{2}\right) > 0, |\arg z| < \frac{\pi}{2}; \text{ sau } \operatorname{Re}(z) = 0 \text{ și } p = 0 \right]. \quad \text{W 190 (4)}$$

6.446

$$1. \quad K_p(x) = \frac{1}{\cos \frac{p\pi}{2}} \int_0^{\infty} \cos(x \operatorname{sh} t) \operatorname{ch} pt \, dt$$

$$[x > 0, -1 < \operatorname{Re}(p) < 1].$$

W 202 (13)

$$2. \quad K_p(xz) = \frac{\Gamma\left(p + \frac{1}{2}\right) (2z)^p}{x^p \Gamma\left(\frac{1}{2}\right)} \int_0^{\infty} \frac{\cos xt \, dt}{(t^2 + z^2)^{p+\frac{1}{2}}}$$

$$\left[\operatorname{Re}\left(p + \frac{1}{2}\right) \geq 0, x > 0, |\arg z| < \frac{\pi}{2} \right].$$

W 191 (1)

6.447

$$1. \quad K_p(z) = \frac{1}{2} \left(\frac{z}{2}\right)^p \int_{-\infty}^{\infty} \frac{e^{-t-\frac{z^2}{4t}} dt}{t^{p+1}} \left[|\arg z| < \frac{\pi}{2}, \operatorname{Re}(z^2) > 0 \right]. \quad \text{W 203 (15)}$$

$$2. \quad K_p(xz) = \frac{z^p}{2} \int_0^{\infty} \exp\left[-\frac{x}{2}\left(t + \frac{z^2}{t}\right)\right] t^{-p-1} dt$$

$$\left[|\arg z| < \frac{\pi}{4} \text{ sau } |\arg z| = \frac{\pi}{4} \text{ și } \operatorname{Re}(p) < 1 \right].$$

MO 39

6.448

$$1. \quad K_p(xz) = \sqrt{\frac{\pi}{2x}} \frac{x^p e^{-xz}}{\Gamma\left(p + \frac{1}{2}\right)} \int_0^{\infty} e^{-xt} t^{p-\frac{1}{2}} \left(1 + \frac{t}{2x}\right)^{p-\frac{1}{2}} dt$$

$$\left[|\arg z| < \pi, \operatorname{Re}(p) > -\frac{1}{2}, x > 0 \right].$$

MO 39

$$2. \quad K_p(xz) = \frac{\sqrt{\pi}}{\Gamma\left(p + \frac{1}{2}\right)} \left(\frac{x}{2z}\right)^p \int_0^{\infty} \frac{\exp(-x \sqrt{t^2 + z^2})}{\sqrt{t^2 + z^2}} t^{2p} dt$$

$$\left[\operatorname{Re}(p) > -\frac{1}{2}, \operatorname{Re}(z) > 0, \operatorname{Re} \sqrt{t^2 + z^2} > 0, x > 0 \right].$$

MO 39

Vezi și 3.227 4., 4.512, 4.515, 5.214, 5.217, 5.221 2., 5.228.

$$6.449 \quad K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}} \int_0^{\infty} \cos(t^3 + xt) dt \quad (\text{vezi și 4.913}).$$

C 98 (31) W 211 (2)

Reprezentarea integrală a lui $K_0(z)$, vezi 3.528, 3.552, 3.717. 5.236.

6.45 Dezvoltarea în serie

Funcția $J_p(z)$

$$6.450 \quad J_n(z) = \left(\frac{z}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k} \quad [|\arg z| < \pi] \quad (\text{cf. 6.402}).$$

6.451 Cazuri particulare

$$1. \quad J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} (k!)^2}.$$

$$2. \quad J_1(z) = -J'_0(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} \cdot k!(k+1)!}.$$

$$3. \quad J_{\frac{1}{3}}(z) = \frac{\sqrt[3]{\frac{z}{2}}}{\Gamma\left(\frac{4}{3}\right)} \sum_{k=0}^{\infty} (-1)^k \frac{(z\sqrt[3]{3})^{2k}}{2^{2k} \cdot k! \cdot 1 \cdot 4 \cdot 7 \dots (3k+1)} \quad (\text{cf. 4.9}).$$

$$4. \quad J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)} \sqrt[3]{\frac{z}{2}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(z\sqrt[3]{3})^{2k}}{2^{2k} \cdot k! \cdot 2 \cdot 5 \cdot 8 \dots (3k-1)} \right\} \quad (\text{cf. 4.9}).$$

Dezvoltarea lui $J_p(z)$ în serie de polinoame Laguerre, vezi 7.145.

6.452

$$1. \quad J_p(z) J_q(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{p+q+2k} \Gamma(p+q+2k+1)}{k! \Gamma(p+q+k+1) \Gamma(p+k+1) \Gamma(q+k+1)}.$$

[Dacă $p^2 \neq q^2$, atunci $2p, 2q, 2(p+q)$ din această formulă nu pot fi numere negative întregi; dacă $p=q$, atunci $2p$ nu poate fi un număr negativ întreg; dacă $p=-q$, atunci p nu poate fi un număr negativ întreg]. W 161 (5)

$$2. \quad J_p(az) J_q(bz) = \frac{\left(\frac{az}{2}\right)^p \left(\frac{bz}{2}\right)^q}{\Gamma(q+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -p-k, q+1; \frac{b^2}{a^2}\right)}{k! \Gamma(p+k+1)}.$$

MO 28

Funcția $N_p(z)$

$$6.453 \quad N_p(z) = \frac{1}{\sin p\pi} \left\{ \cos p\pi \left(\frac{z}{2}\right)^p \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} \cdot k! \Gamma(p+k+1)} - \left(\frac{z}{2}\right)^{-p} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} \cdot k! \Gamma(k-p+1)} \right\} \quad [p \text{ neîntreg}]$$

(cf. 6.403 1.).

Pentru $p+1$ natural, vezi 6.403 2.; pentru p întreg și negativ, vezi 6.404 1.

6.454 Cazuri particulare

$$1. \quad \pi N_0(z) = 2J_0(z) \left(\ln \frac{z}{2} + C \right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^k \frac{1}{m}. \quad C 44$$

$$2. \pi N_1(z) = 2J_1(z) \left(\ln \frac{z}{2} + C \right) - \frac{2}{z} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2} \right)^{2k-1}}{k! (k-1)!} \left\{ 2 \sum_{m=1}^{k-1} \frac{1}{m} + \frac{1}{k} \right\}. \quad D (811.2)$$

Funcțiile $I_p(z)$ și $K_p(z)$

$$6.455 \quad I_p(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p+k+1)} \left(\frac{z}{2} \right)^{p+2k} \quad \text{WW II 187}$$

$$6.456 \quad K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k! \left(\frac{z}{2} \right)^{n-2k}} +$$

$$+ (-1)^{n+1} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{n+2k}}{k! (n+k)!} \left[\ln \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right]; \quad W 95 (15)$$

$$= (-1)^{n+1} I_n(z) \ln \frac{Cz}{2} + \frac{1}{2} (-1)^n \sum_{l=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{n+2l}}{l! (n+l)!} \left(\sum_{k=1}^l \frac{1}{k} + \sum_{k=1}^{n+l} \frac{1}{k} \right) +$$

$$\frac{1}{2} \sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!} \left(\frac{z}{2} \right)^{2l-n} \quad [n+1 \text{ natural}], \quad MO 29$$

6.457 Cazuri particulare

$$1. \quad I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2k}}{(k!)^2}.$$

$$2. \quad I_1(z) = I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2k+1}}{k! (k+1)!}.$$

$$3. \quad K_0(z) = -\ln \frac{z}{2} I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} (k!)^2} \psi(k+1). \quad W 95 (14)$$

6.46 Dezvoltări asimptotice ale funcțiilor cilindrice

6.461 Pentru valori mari ale lui $|z|$,

$$1. \quad J_{\pm p}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos \left(z \mp \frac{\pi}{2} p - \frac{\pi}{4} \right) \times \right.$$

$$\times \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma \left(p+2k+\frac{1}{2} \right)}{(2k)! \Gamma \left(p-2k+\frac{1}{2} \right)} + R_1 \right] -$$

$$\left. - \sin \left(z \mp \frac{\pi}{2} p - \frac{\pi}{4} \right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma \left(p+2k+\frac{3}{2} \right)}{(2k+1)! \Gamma \left(p-2k-\frac{1}{2} \right)} + R_2 \right] \right\}$$

$$[|\arg z| < \pi] \text{ (vezi 6.339 4.)} \quad W 222 (1) \quad W 222 (3)$$

$$\begin{aligned}
2. \quad N_{\pm p}(z) = & \sqrt{\frac{2}{\pi z}} \left\{ \sin \left(z \mp \frac{\pi}{2} p - \frac{\pi}{4} \right) \times \right. \\
& + \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma \left(p + 2k + \frac{1}{2} \right)}{(2k)! \Gamma \left(p - 2k + \frac{1}{2} \right)} + R_1 \right] + \\
& \left. + \cos \left(z \mp \frac{\pi}{2} p - \frac{\pi}{4} \right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma \left(p + 2k + \frac{3}{2} \right)}{(2k+1)! \Gamma \left(p - 2k - \frac{1}{2} \right)} + R_2 \right] \right\} \\
& [|\arg z| < \pi] \text{ (vezi 6.339 4.).} \quad W 222 (2) \quad W 222 (4) \quad W 222 (5)
\end{aligned}$$

$$\begin{aligned}
3. \quad H_p^{(1)}(z) = & \sqrt{\frac{2}{\pi z}} e^{i \left(z - \frac{\pi}{2} p - \frac{\pi}{4} \right)} \times \\
& \times \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2iz)^k} \frac{\Gamma \left(p + k + \frac{1}{2} \right)}{k! \Gamma \left(p - k + \frac{1}{2} \right)} + \theta_1 \frac{(-1)^n}{(2iz)^n} \frac{\Gamma \left(p + n + \frac{1}{2} \right)}{n! \Gamma \left(p - n + \frac{1}{2} \right)} \right] \\
& \left[\operatorname{Re}(p) > -\frac{1}{2}, |\arg z| < \pi \right] \text{ (vezi 6.339 4.).} \quad W 221 (5)
\end{aligned}$$

$$\begin{aligned}
4. \quad H_p^{(2)}(z) = & \sqrt{\frac{2}{\pi z}} e^{-i \left(z - \frac{\pi}{2} p - \frac{\pi}{4} \right)} \times \\
& \times \left[\sum_{k=0}^{n-1} \frac{1}{(2iz)^k} \frac{\Gamma \left(p + k + \frac{1}{2} \right)}{k! \Gamma \left(p - k + \frac{1}{2} \right)} + \theta_2 \frac{1}{(2iz)^n} \frac{\Gamma \left(p + n + \frac{1}{2} \right)}{n! \Gamma \left(p - n + \frac{1}{2} \right)} \right] \\
& \left[\operatorname{Re}(p) > -\frac{1}{2}, |\arg z| < \pi \right] \text{ (vezi 6.339 4.).} \quad W 221 (6)
\end{aligned}$$

Pentru indicii $p = \frac{2n-1}{2}$ (n natural), seriile 6.461 se întrerup. În acest caz, pentru toate valorile lui n , avem formulele închise 6.47.

$$\begin{aligned}
5. \quad I_p(z) \sim & \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2z)^k} \frac{\Gamma \left(p + k + \frac{1}{2} \right)}{k! \Gamma \left(p - k + \frac{1}{2} \right)} + \\
& + \frac{\exp \left[-z \pm \left(p + \frac{1}{2} \right) \pi i \right]}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{1}{(2z)^k} \frac{\Gamma \left(p + k + \frac{1}{2} \right)}{k! \Gamma \left(p - k + \frac{1}{2} \right)}.
\end{aligned}$$

[Semnul $+$ se ia pentru $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, iar semnul $-$, pentru $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$] (vezi 6.339 4.). $W 226 (2), W 226 (3)$

¹ Contradicția pe care, la prima vedere, o conține această afirmație se explică prin așa-numitul fenomen Stokes (vezi W 224–226).

$$6. \quad K_p(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[\sum_{k=0}^{n-1} \frac{1}{(2z)^k} \frac{\Gamma\left(p+k+\frac{1}{2}\right)}{k! \Gamma\left(p-k+\frac{1}{2}\right)} + \right. \\ \left. + \theta_3 \frac{\Gamma\left(p+n+\frac{1}{2}\right)}{(2z)^n \cdot n! \Gamma\left(p-n+\frac{1}{2}\right)} \right] \\ \text{(vezi 6.339 4.).} \quad W 231 \quad W 245 (9)$$

Evaluarea resturilor serilor asimptotice din formulele 6.461.61

$$7. \quad |R_1| < \left| \frac{\Gamma\left(p+2n+\frac{1}{2}\right)}{(2z)^{2n} \cdot (2n)! \Gamma\left(p-2n+\frac{1}{2}\right)} \right| \quad \left[n > \frac{p}{2} - \frac{1}{4} \right]. \quad W 231$$

$$8. \quad |R_2| < \left| \frac{\Gamma\left(p+2n+\frac{3}{2}\right)}{(2z)^{2n+1} \cdot (2n+1)! \Gamma\left(p-2n-\frac{1}{2}\right)} \right| \quad \left[n > \frac{p}{2} - \frac{3}{4} \right] \quad W 231$$

Pentru $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, p real și $n+\frac{1}{2} > |p|$,

$|\theta_1| < 1$, dacă $\operatorname{Im}(z) > 0$; $|\theta_1| < |\sec(\arg z)|$, dacă $\operatorname{Im}(z) \leq 0$. W 245

Pentru $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$, p real și $n+\frac{1}{2} > |p|$,

$|\theta_2| < 1$, dacă $\operatorname{Im}(z) \leq 0$; $|\theta_2| < |\sec(\arg z)|$, dacă $\operatorname{Im}(z) > 0$. W 246

Pentru p real,

$|\theta_3| < 1$ și $\operatorname{Re}(\theta_3) > 0$, dacă $\operatorname{Re}(z) > 0$;

$|\theta_3| < |\operatorname{cosec}(\arg z)|$, dacă $\operatorname{Re}(z) < 0$.

W 245

Pentru p și z reali și $n \geq p - \frac{1}{2}$,

$$0 \leq |\theta_3| \leq 1.$$

W 231

Din 6.461 7. și 6.461 8., rezultă, între altele, că, pentru valori reale și pozitive ale lui z și p , erorile $|R_1|$ și $|R_2|$ sînt mai mici decît modulul primului termen neglijat. Pentru valorile lui $|\arg z|$ apropiate de π , seriile 6.461 1. și 6.461 2. pot să fie inaplicabile în calcule; în particular, pentru $|\arg z| > 0$, eroarea poate fi mai mare decît primul termen neglijat (în modul).

„Aproximarea prin tangentă“

6.462 Pentru valori mari ale indicelui (argumentul este mai mic decît indicele).

Fie $x > 0$ și $p > 0$. Să punem $\frac{p}{x} = \operatorname{ch} \alpha$. Atunci, pentru valori mari ale lui p , sînt valabile dezvoltările

$$1. \quad J_p\left(\frac{p}{\operatorname{ch} \alpha}\right) \sim \frac{\exp(p \operatorname{th} \alpha - p \alpha)}{\sqrt{2 p \pi \operatorname{th} \alpha}} \left\{ 1 + \frac{1}{p} \left(\frac{1}{8} \operatorname{cth} \alpha - \frac{5}{24} \operatorname{cth}^3 \alpha \right) + \right. \\ \left. + \frac{1}{p^2} \left(\frac{9}{128} \operatorname{cth}^2 \alpha - \frac{231}{576} \operatorname{cth}^4 \alpha + \frac{1155}{3456} \operatorname{cth}^6 \alpha \right) + \dots \right\}. \quad W 269 (3)$$

$$2. N_p\left(\frac{p}{\operatorname{ch} \alpha}\right) \sim \frac{\exp(p\alpha - p \operatorname{th} \alpha)}{\sqrt{\frac{\pi}{2} p \operatorname{th} \alpha}} \left(1 - \frac{1}{p} \left(\frac{1}{8} \operatorname{cth} \alpha - \frac{5}{24} \operatorname{cth}^3 \alpha\right) + \right. \\ \left. + \frac{1}{p^2} \left(\frac{9}{128} \operatorname{cth}^3 \alpha - \frac{231}{576} \operatorname{cth}^4 \alpha + \frac{1155}{3456} \operatorname{cth}^6 \alpha\right) + \dots \right) \quad \text{W 270 (5)}$$

6.463 Pentru valori mari ale indicelui (argumentul este mai mare decât indicele).

Fie $x > 0$ și $p > 0$. Să punem $\frac{p}{x} = \cos \beta$. Atunci, pentru valori mari ale lui p , sint valabile dezvoltările.

$$1. J_p(p \sec \beta) \sim \sqrt{\frac{2}{p\pi \operatorname{tg} \beta}} \times \\ \times \left\{ \left[1 - \frac{1}{p^2} \left(\frac{9}{128} \operatorname{ctg}^3 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right] \cos \left(p \operatorname{tg} \beta - p\beta - \frac{\pi}{4} \right) + \right. \\ \left. + \left[\frac{1}{p} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \dots \right] \sin \left(p \operatorname{tg} \beta - p\beta - \frac{\pi}{4} \right) \right\} \quad \text{W 271 (4)}$$

$$2. N_p(p \sec \beta) \sim \sqrt{\frac{2}{p\pi \operatorname{tg} \beta}} \times \\ \times \left\{ \left[1 - \frac{1}{p^2} \left(\frac{9}{128} \operatorname{ctg}^3 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right] \sin \left(p \operatorname{tg} \beta - p\beta - \frac{\pi}{4} \right) - \right. \\ \left. - \left[\frac{1}{p} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \dots \right] \cos \left(p \operatorname{tg} \beta - p\beta - \frac{\pi}{4} \right) \right\} \quad \text{W 271 (5)}$$

$$3. H_p^{(1)}(p \sec \beta) \sim \frac{\exp \left[p i (\operatorname{tg} \beta - \beta) - \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2} p \operatorname{tg} \beta}} \left\{ 1 - \frac{i}{p} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \right. \\ \left. - \frac{1}{p^2} \left(\frac{9}{128} \operatorname{ctg}^3 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right\} \quad \text{W 271 (1)}$$

$$4. H_p^{(2)}(p \sec \beta) \sim \frac{\exp \left[-p i (\operatorname{tg} \beta - \beta) + \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2} p \operatorname{tg} \beta}} \left\{ 1 + \frac{i}{p} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \right. \\ \left. - \frac{1}{p^2} \left(\frac{9}{128} \operatorname{ctg}^3 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right\} \quad \text{W 271 (2)}$$

Formulele 6.463 nu sint aplicabile, cind $|x-p|$ este de ordinul de mărime al lui $x^{\frac{1}{3}}$. Pentru valori mici oarecare (de asemenea și pentru valori care nu sint neapărat mici) ale lui $|x-p|$, se pot folosi următoarele formule:

6.464 Fie $x > 0$ și $p > 0$. Punem

$$w = \sqrt{\frac{x}{p^3} - 1},$$

atunci

$$1. H_p^{(1)} = \\ = \frac{w}{\sqrt{3}} \exp \left\{ \left[\frac{\pi}{6} + p \left(w - \frac{w^3}{3} - \operatorname{arctg} w \right) \right] i \right\} H_{\frac{1}{3}}^{(1)} \left(\frac{p}{3} w^3 \right) + O \left(\frac{1}{|p|} \right)$$

$$2. H_p^{(2)} = \frac{w}{\sqrt{3}} \exp \left\{ \left[-\frac{\pi}{6} - p \left(w - \frac{w^3}{3} - \arctg w \right) \right] i \right\} H_{\frac{1}{3}}^{(2)} \left(\frac{p}{3} w^3 \right) + O \left(\frac{1}{|p|} \right) \\ \left[\arctg w = w - \frac{w^3}{3} + \frac{w^5}{5} - \dots \right]. \quad \text{MO 34}$$

Valoarea absolută a erorii $O\left(\frac{1}{|p|}\right)$ este aici mai mică decât $24 \sqrt{2} \left| \frac{1}{p} \right|$.

6.465 Pentru x real, p natural ($p=n$) și $n > 1$, au loc următoarele aproximări:

$$1. J_n(x) \approx \frac{1}{\pi} \sqrt{\frac{2(n-x)}{3x}} K_{\frac{1}{3}} \left\{ \frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\} \quad [n > x] \\ \text{(vezi și 6.449, 4.913); W 276 (1)}$$

$$\approx \frac{1}{2} e^{\frac{2}{3}\pi i} \sqrt{\frac{2(n-x)}{3x}} H_{\frac{1}{3}}^{(1)} \left\{ \frac{1}{3} \frac{[2(n-x)]^{\frac{3}{2}}}{\sqrt{x}} \right\} \quad [n > x]; \quad \text{MO 34}$$

$$\approx \frac{1}{\sqrt{3}} \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[\frac{[2(x-n)]^{\frac{3}{2}}}{3\sqrt{x}} \right] + J_{-\frac{1}{3}} \left[\frac{[2(x-n)]^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} \\ [x > n]. \\ \text{(vezi și 4.911, 4.912, 6.451 3., 6.451 4.). W 276 (2)}$$

$$2. N_n(x) \approx -\sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[\frac{[2(x-n)]^{\frac{3}{2}}}{3\sqrt{x}} \right] - \right. \\ \left. - J_{-\frac{1}{3}} \left[\frac{[2(x-n)]^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} \quad [x > n]. \quad \text{W 276 (3)}$$

Evaluarea erorii în formulele 6.465 nu s'a obținut până în prezent.

$$6.466 J_p^2(z) + N_p^2(z) \approx \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma\left(p+k+\frac{1}{2}\right)}{k! \Gamma\left(p-k+\frac{1}{2}\right)} \\ [|\arg z| < \pi] \quad \text{(vezi și 6.489 1.). W 250 (5)}$$

$$6.467 J_p^2(x) + J_{p+1}^2(x) \approx \frac{2}{\pi x} \quad [x \gg |p|]. \quad \text{W 223}$$

6.47 Funcții cilindrice al căror indice este egal cu un număr întreg plus $\frac{1}{2}$

Funcția $J_p(z)$.

6.471

$$1. J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{E\left(\frac{n}{2}\right)} \frac{(-1)^k (n+2k)!}{(2k)! (n-2k)! (2z)^{2k}} + \right. \\ \left. + \cos\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{E\left(\frac{n-1}{2}\right)} \frac{(-1)^k (n+2k+1)!}{(2k+1)! (n-2k-1)! (2z)^{2k+1}} \right\} \\ [n+1 \text{ natural}]. \quad \text{(cf. 6.461 1.). C 59 (6) W 66 (2)}$$

$$2. J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{E\left(\frac{n}{2}\right)} \frac{(-1)^k (n+2k)!}{(2k)! (n-2k)! (2z)^{2k}} - \right. \\ \left. - \sin\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{E\left(\frac{n-1}{2}\right)} \frac{(-1)^k (n+2k+1)!}{(2k+1)! (n-2k-1)! (2z)^{2k+1}} \right\}$$

[n+1 natural]

(cf. 6.461 1.).

C 59 (7) W 67 (5)

6.472

$$1. J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{n+k-1} (n+k)!}{k! (n-k)! (2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{n+k-1} (n+k)!}{k! (n-k)! (2z)^k} \right\}$$

[n+1 natural] C 59 (6) W 66 (1)

$$2. J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{n+k} (n+k)!}{k! (n-k)! (2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{n+k} (n+k)!}{k! (n-k)! (2z)^k} \right\}$$

[n+1 natural] C 59 (7) W 67 (4)

6.473

$$1. J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\sin z}{z} \right). \quad \text{C 58 (4)}$$

$$2. J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\cos z}{z} \right). \quad \text{C 58 (5)}$$

6.474 Cazuri particulare

$$1. J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \quad \text{D (809.01)}$$

$$2. J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z. \quad \text{D (809.21)}$$

$$3. J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right). \quad \text{D (809.03)}$$

$$4. J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\sin z - \frac{\cos z}{z} \right). \quad \text{D (809.23)}$$

$$5. J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left(\frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}. \quad \text{D (809.05)}$$

$$6. J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left(\frac{3}{z^2} - 1 \right) \cos z \right\}. \quad \text{D (809.25)}$$

Funcția $N_{n+\frac{1}{2}}(z)$

$$6.475 \quad N_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z); \quad N_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z). \quad \text{JE 227}$$

Funcțiile $H_{n+\frac{1}{2}}^{(1,2)}(z)$, $I_{n+\frac{1}{2}}(z)$, $K_{n+\frac{1}{2}}(z)$

6.476

$$1. H_{n-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k! (n-k-1)!} \frac{1}{(2iz)^k} \quad (\text{cf. 6.461 3.}).$$

$$2. H_{n-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} i^n e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k! (n-k-1)!} \frac{1}{(2iz)^k} \quad (\text{cf. 6.461 4.}).$$

6.477

$$I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[e^z \sum_{k=0}^n \frac{(-1)^k (n+k)!}{k! (n-k)! (2z)^k} + (-1)^{n+1} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k! (n-k)! (2z)^k} \right] \quad (\text{cf. 6.461 5.}). \quad C 60$$

$$6.478 \quad K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k! (n-k)! (2z)^k} \quad (\text{cf. 6.461 6.}). \quad C 60$$

În formulele 6.47, $n+1$ este un număr natural.

6.479 Cazuri particulare

$$1. N_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z. \quad 2. N_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$$

$$3. K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \quad W 95 (13)$$

$$4. H_{\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}. \quad 5. H_{\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{-iz}}{-i}. \quad MO 27$$

$$6. H_{-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}. \quad 7. H_{-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}. \quad MO 27$$

6.48-6.49 Relații funcționale

6.481 Formule de recurență

$$1. zZ_{p-1}(z) + zZ_{p+1}(z) = 2pZ_p(z). \quad C 56 (13) \quad W 56 (1) \quad W 79 (1) \quad W 88 (3)$$

$$2. Z_{p-1}(z) - Z_{p+1}(z) = 2 \frac{d}{dz} Z_p(z). \quad C 56 (12) \quad W 56 (2) \quad W 79 (2) \quad W 88 (4)$$

În construirea teoriei funcțiilor cilindrice, Sonin și Nielsen au definit funcțiile cilindrice ca funcții analitice de z care satisfac relațiile de recurență 6.481.

6.482 Consecințe din formulele de recurență

$$1. z \frac{d}{dz} Z_p(z) + pZ_p(z) = zZ_{p-1}(z). \quad C 56 (11) \quad W 56 (3) \quad W 79 (3) \quad W 88 (5)$$

$$2. z \frac{d}{dz} Z_p(z) - pZ_p(z) = -zZ_{p+1}(z). \quad C 56 (10) \quad W 56 (4) \quad W 79 (4) \quad W 88 (6)$$

$$3. \left(\frac{d}{dz} \right)^m (z^p Z_p(z)) = z^{p-m} Z_{p-m}(z). \quad C 56 (8) \quad W 57 (5) \quad W 89 (9)$$

$$4. \left(\frac{d}{dz}\right)^m (z^{-p} Z_p(z)) = (-1)^m z^{-p-m} Z_{p+m}(z).$$

W 89 (10) C 55 (5) W 57 (6)

$$5. Z_{-n}(z) = (-1)^n Z_n(z) \quad [n \text{ natural}].$$

(cf. 6.404).

6.483 Cazuri particulare

$$1. J_2(z) = \frac{2}{z} J_1(z) - J_0(z).$$

$$2. N_2(z) = \frac{2}{z} N_1(z) - N_0(z).$$

$$3. H_2^{(1,2)}(z) = \frac{2}{z} H_1^{(1,2)}(z) - H_0^{(1,2)}(z).$$

$$4. \frac{d}{dz} J_0(z) = -J_1(z).$$

$$5. \frac{d}{dz} N_0(z) = -N_1(z).$$

$$6. \frac{d}{dz} H_0^{(1,2)}(z) = -H_1^{(1,2)}(z).$$

6.484 Fiecare dintre perechile de funcții

$J_p(z)$ și $J_{-p}(z)$ ($p \neq 0, \pm 1, \pm 2, \dots$); $J_p(z)$ și $N_p(z)$; $H_p^{(1)}(z)$ și $H_p^{(2)}(z)$, care sînt soluții ale ecuației 6.401, precum și perechea de funcții

$$I_p(z) \text{ și } K_p(z)$$

sînt perechi de funcții liniar independente. Vronskianii acestor perechi sînt respectiv egali cu

$$\frac{2}{\pi z} \sin p\pi, \frac{2}{\pi z}, -\frac{4i}{\pi z}, -\frac{1}{z}.$$

C 52 (10) C 52 (11) C 52 (12) W 90 (1) W 90 (4)

6.485 Funcțiile $J_p(z)$, $N_p(z)$, $H_p^{(1,2)}(z)$, $I_p(z)$, $K_p(z)$, cu excepția lui $J_n(z)$ pentru n întreg, sînt neuniforme, $z=0$ fiind pentru ele un punct de ramificație. Ramurile acestor funcții, situate de ambele părți ale secțiunii $(-\infty, 0)$ sînt legate prin relațiile (de ocolire)

6.486

$$1. J_p(e^{m\pi i} z) = e^{mp\pi i} J_p(z). \quad \text{W 90 (1)}$$

$$2. N_p(e^{m\pi i} z) = e^{-mp\pi i} N_p(z) + 2i \sin m\pi \operatorname{ctg} p\pi J_p(z). \quad \text{W 90 (3)}$$

$$3. N_{-p}(e^{m\pi i} z) = e^{-mp\pi i} N_{-p}(z) + 2i \sin m\pi \operatorname{cosec} p\pi J_p(z). \quad \text{W 90 (4)}$$

$$4. I_p(e^{m\pi i} z) = e^{mp\pi i} I_p(z). \quad \text{W 95 (17)}$$

$$5. K_p(e^{m\pi i} z) = e^{-p\pi i} K_p(z) - i\pi \frac{\sin m\pi}{\sin p\pi} I_p(z) \quad [p \text{ neîntreg}]. \quad \text{W 95 (18)}$$

$$6. H_p^{(1)}(e^{m\pi i} z) = e^{-mp\pi i} H_p^{(1)}(z) - 2e^{-p\pi i} \frac{\sin m\pi}{\sin p\pi} J_p(z) = \frac{\sin(1-m)p\pi}{\sin p\pi} H_p^{(1)}(z) - e^{-p\pi i} \frac{\sin m\pi}{\sin p\pi} H_p^{(2)}(z). \quad \text{W 95 (5)}$$

$$7. H_p^{(2)}(e^{m\pi i} z) = e^{-mp\pi i} H_p^{(2)}(z) + 2e^{p\pi i} \frac{\sin m\pi}{\sin p\pi} J_p(z) = \frac{\sin(1+m)p\pi}{\sin p\pi} H_p^{(2)}(z) + e^{p\pi i} \frac{\sin m\pi}{\sin p\pi} H_p^{(1)}(z) \quad [m \text{ întreg}]. \quad \text{W 90 (6)}$$

$$8. H_p^{(1)}(e^{i\pi} z) = -H_p^{(2)}(z) = -e^{-i\pi p} H_p^{(2)}(z). \quad \text{MO 26}$$

$$9. H_p^{(2)}(e^{-i\pi} z) = -H_p^{(1)}(z) = -e^{i\pi p} H_p^{(1)}(z). \quad \text{MO 26}$$

$$10. \overline{H_p^{(2)}}(z) = H_p^{(1)}(\bar{z}). \quad \text{MO 26}$$

6.487

$$1. J_p(z) N_{p+1}(z) - J_{p+1}(z) N_p(z) = -\frac{2}{\pi z}. \quad \text{W 91 (12)}$$

$$2. I_p(z) K_{p+1}(z) + I_{p+1}(z) K_p(z) = \frac{1}{z}. \quad \text{W 95 (20)}$$

Vezi și 3.552.

Legătura cu funcțiile sferice, vezi 6.722.

Legătura cu polinoamele $C_n^p(t)$, vezi 6.944.

Legătura cu funcția hipergeometrică degenerată, vezi 7.335.

6.488

Pentru $p > 0$ și $x > 0$, produsul

$$x [J_p^2(x) + N_p^2(x)],$$

considerat ca funcție de x , este monoton descrescător, dacă $p > \frac{1}{2}$, și monoton crescător, dacă $0 < p < \frac{1}{2}$. MO 35

6.489

$$1. \frac{1}{\sqrt{x^2 - p^2}} > \frac{\pi}{2} [J_p^2(x) + N_p^2(x)] \geq \frac{1}{x} \quad \left[x \geq p \geq \frac{1}{2} \right]$$

(vezi și 4.406, 4.441, 6.466). MO 35

$$2. |J_n(nz)| \leq 1 \quad \left[\left| \frac{z \exp \sqrt{1-z^2}}{1 + \sqrt{1-z^2}} \right| < 1, n \text{ natural} \right]. \quad \text{MO 35}$$

Relații între funcțiile cilindrice de speța întâi,
a doua și a treia

$$6.491 \quad J_p(z) = \frac{N_{-p}(z) - N_p(z) \cos p\pi}{\sin p\pi} = H_p^{(1)}(z) - iN_p(z) =$$

$$= H_p^{(2)}(z) + iN_p(z) = \frac{1}{2} (H_p^{(1)}(z) + H_p^{(2)}(z))$$

(cf. 6.403 1., 6.405). W 89 (1) JE 228

$$6.492 \quad N_p(z) = \frac{J_{-p}(z) \cos p\pi - J_p(z)}{\sin p\pi} = iJ_p(z) - iH_p^{(1)}(z) =$$

$$= iH_p^{(2)}(z) - iJ_p(z) = \frac{i}{2} (H_p^{(2)}(z) - H_p^{(1)}(z))$$

(cf. 6.403 1., 6.405). W 89 (3) JE 228

6.493

$$1. H_p^{(1)}(z) = \frac{J_{-p}(z) - e^{-p\pi i} J_p(z)}{i \sin p\pi} = \frac{N_{-p}(z) - e^{-p\pi i} N_p(z)}{\sin p\pi} = J_p(z) + iN_p(z).$$

W 89 (5)

$$2. H_p^{(2)}(z) = \frac{e^{p\pi i} J_p(z) - J_{-p}(z)}{i \sin p\pi} = \frac{N_{-p}(z) - e^{p\pi i} N_p(z)}{\sin p\pi} = J_p(z) - iN_p(z)$$

(cf. 6.405). W 89 (6)

6.494

$$1. H_{-p}^{(1)}(z) = e^{p\pi i} H_p^{(1)}(z); \quad 2. H_{-p}^{(2)}(z) = e^{-p\pi i} H_p^{(2)}(z). \quad \text{W 89 (7)}$$

$$6.495 \quad K_p(z) = \frac{\pi}{2} \frac{I_{-p}(z) - I_p(z)}{\sin p\pi}$$

[p neîntreg], (vezi și 6.407). W 92 (6)

6.496 Formule de recurență și consecințele lor pentru funcțiile $I_p(z)$ și $K_p(z)$

1. $zI_{p+1}(z) - zI_{p-1}(z) = 2pI_p(z)$. W 93 (1)
2. $I_{p-1}(z) + I_{p+1}(z) = 2\frac{d}{dz}I_p(z)$. W 93 (2)
3. $z\frac{d}{dz}I_p(z) + pI_p(z) = zI_{p-1}(z)$. W 93 (3)
4. $z\frac{d}{dz}I_p(z) - pI_p(z) = zI_{p+1}(z)$. W 93 (4)
5. $\left(\frac{d}{zdz}\right)^m \{z^p I_p(z)\} = z^{p-m} I_{p-m}(z)$. W 93 (5)
6. $\left(\frac{d}{zdz}\right)^m \{z^{-p} I_p(z)\} = z^{-p-m} I_{p+m}(z)$. W 93 (6)
7. $I_{-n}(z) = I_n(z)$ [n — natural] W 93 (8)
8. $I_2(z) = -\frac{2}{z}I_1(z) + I_0(z)$.
9. $\frac{d}{dz}I_0(z) = I_1(z)$. W 93 (7)
10. $zK_{p-1}(z) - zK_{p+1}(z) = -2pK_p(z)$. W 93 (1)
11. $K_{p-1}(z) + K_{p+1}(z) = -2\frac{d}{dz}K_p(z)$. W 93 (2)
12. $z\frac{d}{dz}K_p(z) + pK_p(z) = -zK_{p-1}(z)$. W 93 (3)
13. $z\frac{d}{dz}K_p(z) - pK_p(z) = -zK_{p+1}(z)$. W 93 (4)
14. $\left(\frac{d}{zdz}\right)^m \{z^p K_p(z)\} = (-1)^m z^{p-m} K_{p-m}(z)$. W 93 (5)
15. $\left(\frac{d}{zdz}\right)^m \{z^{-p} K_p(z)\} = (-1)^m z^{-p-m} K_{p+m}(z)$. W 93 (6)
16. $K_{-p}(z) = K_p(z)$. W 93 (8)
17. $K_2(z) = \frac{2}{z}K_1(z) + K_0(z)$.
18. $\frac{d}{dz}K_0(z) = -K_1(z)$. W 93 (7)

6.497 Continuitate în raport cu indicele ¹

1. $\lim_{p \rightarrow n} N_p(z) = N_n(z)$ W 76
2. $\lim_{p \rightarrow n} H_p^{(1,2)}(z) = H_n^{(1,2)}(z)$ [n — întreg]. W 183
3. $\lim_{p \rightarrow n} K_p(z) = K_n(z)$ W 92

6.51 Ecuații diferențiale care duc la funcții cilindrice

Vezi și 6.401, 7.293, 7.295.

6.511

1. $\frac{1}{z}\frac{d}{dz}(zu') + \left(\beta^2 - \frac{p^2}{z^2}\right)u = 0$, $u = Z_p(\beta z)$. JE 237
2. $\frac{1}{z}\frac{d}{dz}(zu') + \left[(\beta\gamma z^{\gamma-1})^2 - \left(\frac{p\gamma}{z}\right)^2\right]u = 0$, $u = Z_p(\beta z^\gamma)$. JE 237
3. $u'' + \frac{1-2a}{z}u' + \left[(\beta\gamma z^{\gamma-1})^2 - \frac{a^2 - p^2\gamma^2}{z^2}\right]u = 0$, $u = z^a Z_p(\beta z^\gamma)$. JE 237
4. $u'' + \left[(\beta\gamma z^{\gamma-1})^2 - \frac{4p^2\gamma^2 - 1}{4z^2}\right]u = 0$, $u = \sqrt{z} Z_p(\beta z^\gamma)$. JE 237

¹ Continuitatea funcțiilor $J_p(z)$ și $I_p(z)$ în raport cu indicele rezultă direct din reprezentarea acestor funcții cu ajutorul seriilor.

$$5. \quad u'' + \left(\beta^2 - \frac{4p^2 - 1}{4z^2} \right) u = 0, \quad u = \sqrt{z} Z_p(\beta z). \quad \text{JE 237}$$

$$6. \quad u'' + \frac{1-2\alpha}{z} u' + \left(\beta^2 + \frac{\alpha^2 - p^2}{z^2} \right) u = 0, \quad u = z^\alpha Z_p(\beta z). \quad \text{JE 237}$$

$$7. \quad u'' + bz^m u = 0, \quad u = \sqrt{z} Z_{\frac{1}{m+2}} \left(\frac{2\sqrt{b}}{m+2} z^{\frac{m+2}{2}} \right). \quad \text{JE 238}$$

$$8. \quad u'' + \frac{1}{z} u' + 4 \left(z^2 - \frac{p^2}{z^2} \right) u = 0, \quad u = Z_p(z^2). \quad \text{W 111 (5)}$$

$$9. \quad u'' + \frac{1}{z} u' + \frac{1}{4z} \left(1 - \frac{p^2}{z} \right) u = 0; \quad u = Z_p(\sqrt{z}). \quad \text{W 111 (6)}$$

$$10. \quad u'' + \frac{1-p}{z} u' + \frac{1}{4} \frac{u}{z} = 0, \quad u = z^{\frac{p}{2}} Z_p(\sqrt{z}). \quad \text{W 111 (7)}$$

$$11. \quad u'' + \beta^2 \gamma^2 z^{2\beta-2} u = 0, \quad u = z^{\frac{1}{2}} Z_{\frac{1}{2\beta}}(\gamma z^\beta). \quad \text{W 111 (9)m}$$

$$12. \quad z^2 u'' + (2\alpha - 2\beta p + 1) z u' + [\beta^2 \gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta p)] u = 0, \\ u = z^{\beta p - \alpha} Z_p(\gamma z^\beta). \quad \text{W 110 (3)}$$

6.512

$$1. \quad u'' + (e^{2z} - p^2) u = 0, \quad u = Z_p(e^z). \quad \text{W 112 (21)}$$

$$2. \quad u'' + \frac{e^{\frac{2}{z}} - p^2}{z^4} u = 0, \quad u = z Z_p(e^{\frac{1}{z}}). \quad \text{W 112 (22)}$$

6.513

$$1. \quad u'' + \left(\frac{1}{z^2} - 2\operatorname{tg} z \right) u' - \left(\frac{p^2}{z^2} + \frac{\operatorname{tg} z}{z} \right) u = 0, \quad u = \sec z Z_p(z). \quad \text{JE 238}$$

$$2. \quad u'' + \left(\frac{1}{z^2} + 2\operatorname{ctg} z \right) u' - \left(\frac{p^2}{z^2} - \frac{\operatorname{ctg} z}{z} \right) u = 0, \quad u = \operatorname{cosec} z Z_p(z). \quad \text{JE 238}$$

6.514

$$1. \quad u'' + \frac{1}{z} u - \left(1 + \frac{p^2}{z^2} \right) u = 0, \quad u = Z_p(iz) = C_1 I_p(z) + C_2 K_p(z). \quad \text{JE 237}$$

$$2. \quad u'' + \frac{1}{z} u' - \left[\frac{1}{z} + \left(\frac{p}{2z} \right)^2 \right] u = 0, \quad u = Z_p(2i\sqrt{z}). \quad \text{JE 238}$$

$$3. \quad u'' + u' + \frac{1}{z^2} \left(\frac{1}{4} - p^2 \right) u = 0, \quad u = \sqrt{ze}^{-\frac{z}{2}} Z_p\left(\frac{iz}{2}\right). \quad \text{JE 238}$$

$$4. \quad u'' + \left(\frac{2p+1}{z} - k \right) u' - \frac{2p+1}{2z} k u = 0, \quad u = z^{-p} e^{\frac{1}{2} k z} Z_p\left(\frac{ikz}{2}\right). \quad \text{JE 238}$$

$$5. \quad u'' + \frac{1-p}{z} u' - \frac{1}{4} \frac{u}{z} = 0, \quad u = z^{\frac{p}{2}} Z\left(i\sqrt{z}\right). \quad \text{W 111 (8)}$$

$$6. \quad u'' \pm \frac{u}{\sqrt{z}} = 0, \quad u = \sqrt{z} Z_{\frac{2}{3}}\left(\frac{4}{3} z^{\frac{3}{4}}\right) \cdot \sqrt{z} Z_{\frac{2}{3}}\left(\frac{4}{3} iz^{\frac{3}{4}}\right) \quad \text{W 111 (10)}$$

$$7. \quad u'' \pm zu = 0; \quad u = \sqrt{z} Z_{\frac{1}{3}}\left(\frac{2}{3} z^{\frac{3}{2}}\right) \cdot \sqrt{z} Z_{\frac{1}{3}}\left(\frac{2}{3} iz^{\frac{3}{2}}\right) \quad \text{W 111 (10)}$$

$$8. \quad u'' - \left(c^2 + \frac{p(p+1)}{z^2} \right) u = 0, \quad u = \sqrt{z} Z_{p+\frac{1}{2}} \operatorname{ci}(z). \quad \text{W 108 (1)}$$

$$9. \quad u'' - \frac{2p}{z} u' - c^2 u = 0, \quad u = z^{p+\frac{1}{2}} Z_{p+\frac{1}{2}} \operatorname{ci}(z). \quad \text{W 109 (3) W 109 (4)}$$

$$10. \quad u'' - c^2 z^{2q-2} u = 0, \quad u = \sqrt{z} Z_{\frac{1}{2q}}\left(i \frac{c}{q} z^q\right). \quad \text{W 109 (5) W 109 (6)}$$

6.515

$$1. u'' + \frac{1}{z} u' + \left(i - \frac{p^2}{z^2}\right) u = 0, \quad u = Z_n(z \sqrt{i}). \quad \text{JE 238}$$

$$2. u'' + \left(\frac{1}{z} \mp 2i\right) u' - \left(\frac{p^2}{z^2} \pm \frac{i}{z}\right) u = 0, \quad u = e^{\pm iz} Z_p(z) \quad \text{JE 238}$$

$$3. u'' + \frac{1}{z} u' + se^{ia} u = 0, \quad u = Z_0(\sqrt{s} z e^{\frac{i}{2}a}). \quad \text{JE 238}$$

$$4. u'' + \left(se^{ia} + \frac{1}{4z^2}\right) u = 0, \quad u = \sqrt{z} Z_0(\sqrt{s} z e^{\frac{i}{2}a}). \quad \text{JE 238}$$

6.516

$$1. \frac{d^2}{dz^2} \left(z^4 \frac{d^2 u}{dz^2}\right) - z^2 u = 0, \quad u = \frac{1}{z} \{Z_2(2\sqrt{z}) + \bar{Z}_2(2i\sqrt{z})\}. \quad \text{W 122 (7)}$$

$$2. \frac{d^2}{dz^2} \left(z^{\frac{16}{5}} \frac{d^2 u}{dz^2}\right) - z^{\frac{8}{5}} u = 0, \quad u = z^{-\frac{7}{10}} \left\{Z_{\frac{5}{6}}\left(\frac{5}{3} z^{\frac{3}{5}}\right) + \bar{Z}_{\frac{5}{6}}\left(\frac{5}{3} iz^{\frac{3}{5}}\right)\right\}. \quad \text{W 122 (8)}$$

$$3. \frac{d^2}{dz^2} \left(z^{12} \frac{d^2 u}{dz^2}\right) - z^6 u = 0, \quad u = z^{-4} \{Z_{10}(2z^{-\frac{1}{2}}) + \bar{Z}_{10}(2iz^{-\frac{1}{2}})\}. \quad \text{W 122 (9)}$$

$$4. \frac{d^4 u}{dz^4} + \frac{2}{z} \frac{d^3 u}{dz^3} - \frac{2p^2 + 1}{z^2} \frac{d^2 u}{dz^2} + \frac{2p^2 + 1}{z^3} \frac{du}{dz} + \left(\frac{p^4 - 4p^2}{z^4} - 1\right) u = 0,$$

$$u = A_1 J_p(z) + A_2 N_p(z) + A_3 I_p(z) + A_4 K_p(z),$$

unde A_1, A_2, A_3, A_4 sînt constante

MO 29

6.52-6.53 Serii de funcții Bessel

6.521 Funcția generatoare a funcțiilor Bessel

$$1. \exp \frac{1}{2} \left(t - \frac{1}{t}\right) z = J_0(z) + \sum_{k=1}^{\infty} [t^k + (-t)^{-k}] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k$$

$|z| < |t|$. C 119 (12)

$$2. \exp \left(t - \frac{1}{t}\right) z = \left\{ \sum_{k=-\infty}^{\infty} t^k J_k(z) \right\} \left\{ \sum_{m=-\infty}^{\infty} t^m J_m(z) \right\}. \quad \text{W 40}$$

$$3. \exp(\pm iz \sin \varphi) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\varphi \pm$$

$$\pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\varphi. \quad \text{C 120 (13)}$$

$$4. \exp(iz \cos \varphi) = \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} (2k+1) i^k J_{k+\frac{1}{2}}(z) P_k(\cos \varphi); \quad \text{W 401 (1)}$$

$$= \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\varphi}; \quad \text{MO 27}$$

$$= 1 + 2 \sum_{k=1}^{\infty} i^k J_k(z) \cos k\varphi. \quad \text{MO 27}$$

$$5. \sqrt{\frac{i}{\pi}} e^{iz \cos 2\varphi} \int_{-\infty}^{\sqrt{2z} \cos \varphi} e^{-it^2} dt = \frac{1}{2} J_0(z) + \sum_{k=1}^{\infty} e^{\frac{i}{4} k\pi} J_{\frac{k}{2}}(z) \cos k\varphi. \quad \text{MO 28}$$

Seriele $\sum J_k(z)$

$$6.522 \quad J_0(z) + 2 \sum_{k=0}^{\infty} J_{2k}(z) = 1. \quad \text{W 44}$$

$$6.523^* \quad \sum_{k=0}^{\infty} \frac{(n+2k) \cdot (n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n. \quad \text{W 45}$$

$$1. \quad \sum_{k=0}^{\infty} \frac{(4k+1) \cdot (2k-1)!!}{2^k \cdot k!} J_{2k+\frac{1}{2}}(z) = \sqrt{2z}.$$

6.524

$$1. \quad \sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^p Q_{2k}^{(2p)} z^{2k} \quad [p=1, 2, 3, \dots]. \quad \text{W 46 (1)}$$

$$2. \quad \sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^p Q_{2k+1}^{(2p+1)} z^{2k+1} \quad [p=0, 1, 2, 3, \dots] \quad \text{W 46 (2)}$$

$$\left[\text{in formulele 6.524, } Q_k^{(p)} = \sum_{m=0}^{B\left(\frac{k-1}{2}\right)} \frac{(-1)^m \binom{m}{k} (k-2m)^p}{2^k \cdot k!} \right].$$

In particular,

$$3. \quad \sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2} (z + z^3). \quad \text{W 47 (4)}$$

$$4. \quad \sum_{k=1}^{\infty} (2k)^3 J_{2k}(z) = \frac{1}{2} z^2. \quad \text{W 47 (4)}$$

$$5. \quad \sum_{k=1}^{\infty} (2k)(2k+1)(2k+2) J_{2k+1}(z) = \frac{1}{2} z^2. \quad \text{W 47 (4)}$$

6.525

$$1. \quad \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) = \frac{\sin z}{2}. \quad \text{WW II 192}$$

$$2. \quad J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) = \cos z. \quad \text{WW II 192}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^{k+1} (2k)^2 J_{2k}(z) = \frac{z \sin z}{2}. \quad \text{W 32 (9)}$$

$$4. \quad \sum_{k=0}^{\infty} (-1)^k (2k+1)^2 J_{2k+1}(z) = \frac{z \cos z}{2}. \quad \text{W 32 (10)}$$

$$5. \quad J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\theta = \cos(z \sin \theta). \quad \text{C 120 (14) W 32}$$

$$6. \quad \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\theta = \frac{\sin(z \sin \theta)}{2}. \quad \text{C 120 (15) W 32}$$

$$7. \quad \sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2} \int_0^x J_0(t) dt \quad [x \text{ real}]. \quad \text{W 638}$$

6.526

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \left(\frac{2z+t}{2z} \right)^k J_{p+k}(z) = \left(\frac{z}{z+t} \right)^p J_p(z+t). \quad A(9140)$$

$$2. \sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}(x^2) = S(x). \quad MO 127m$$

$$3. \sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}(x^2) = C(x). \quad MO 127m$$

$$6.527 \quad \sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z \sin \theta) = (z \sin \theta)^{2n}. \quad W 47$$

Seriile $\sum a_k J_k(kx)$ și $\sum a_k J'_k(kx)$.

6.528

$$\left. \begin{aligned} 1. \sum_{k=1}^{\infty} J_k(kz) &= \frac{z}{2(1-z)} \\ 2. \sum_{k=1}^{\infty} (-1)^k J_k(kz) &= -\frac{z}{2(1+z)} \\ 3. \sum_{k=1}^{\infty} J_{2k}(2kz) &= \frac{z^2}{2(1-z^2)} \end{aligned} \right\} \left[\left| \frac{z \exp \sqrt{1-z^2}}{1 + \sqrt{1-z^2}} \right| < 1 \right]. \quad \begin{aligned} &W 615 (1) \\ &W 622 (1) \\ &MO 58 \end{aligned}$$

6.529

$$\begin{aligned} 1. \sum_{k=1}^{\infty} \frac{J'_k(kx)}{k} &= \frac{1}{2} + \frac{x}{4} & [0 \leq x < 1]. & MO 58 \\ 2. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{J'_k(kx)}{k} &= \frac{1}{2} - \frac{x}{4} & [0 \leq x < 1]. & MO 58 \\ 3. \sum_{k=1}^{\infty} k J'_k(kx) &= \frac{1}{2(1-x)^2} & [0 \leq x < 1]. & MO 58 \\ 4. \sum_{k=1}^{\infty} (-1)^{k-1} J'_k(kx) &= \frac{1}{2(1+x)^2} & [0 \leq x < 1]. & MO 58 \end{aligned}$$

Seriile $\sum a_k J_0(kx)$.

6.530. Dacă funcția $f(x)$ are, pe segmentul $[0 \leq x \leq \pi]$, derivate în raport cu x continuă și cu variație mărginită, atunci

$$1. f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \quad [0 < x < \pi],$$

unde

$$2. a_0 = 2f(0) + \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin \varphi) d\varphi.$$

$$3. \quad a_n = \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin \varphi) \cos nu \, d\varphi.$$

WW II 193

6.531 Exemple

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{m=1}^n \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \quad [2n\pi < x < 2(n+1)\pi].$$

MO 59

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} J_0(kx) = \frac{1}{2} \quad [0 < x < \pi].$$

C 124 (12)

$$3. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\{(2k-1)x\} = \frac{\pi^2}{8} - \frac{|x|}{2} \quad [-\pi < x < \pi];$$

C 124

$$= \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos \frac{\pi}{x} \quad [\pi < x < 2\pi]. \quad \text{MO 59}$$

$$4. \quad \sum_{k=1}^{\infty} e^{-kz} J_0(k \sqrt{x^2 + y^2}) =$$

$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{(2k\pi + z)^2 + x^2 + y^2}} + \frac{1}{\sqrt{(2k\pi - z)^2 + x^2 + y^2}} \right\}$$

$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1}\left(\frac{z}{r}\right) \quad [0 < r < 2\pi], \quad \text{MO 59}$$

unde $r = \sqrt{x^2 + y^2 + z^2}$, iar prin rădăcina pătrată se înțelege valoarea a cărei parte reală este pozitivă. În formula 6.531 4., prima egalitate are loc atunci când x și y sînt reali, iar $\operatorname{Re}(z) > 0$, ultima egalitate are loc pentru x, y și z reali.

Seriile $\sum a_k Z_0(kx) \sin kx$ și $\sum a_k Z_0(kx) \cos kx$.

6.532

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} +$$

$$+ \frac{1}{x \sqrt{1 - t^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}.$$

MO 59

$$2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} +$$

$$+ \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} -$$

$$- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}.$$

MO 59

$$3. \sum_{k=1}^{\infty} N_0(kx) \cos kxt = -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} -$$

$$\sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}. \quad \text{MO 60}$$

În formulele 6.532 $x > 0$, $0 \leq t < 1$, $2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi$, $m+1$ și $n+1$ sînt numere naturale.

6.533

$$1. \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} +$$

$$+ \sum_{l=1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}. \quad \text{MO 60}$$

$$2. \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} +$$

$$+ \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} -$$

$$- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}. \quad \text{MO 60}$$

$$3. \sum_{k=1}^{\infty} (-1)^k N_0(kx) \cos kxt = -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} -$$

$$- \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} -$$

$$- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}. \quad \text{MO 60}$$

În formulele 6.533 $x > 0$, $0 \leq t < 1$, $(2m-1)\pi < x(1-t) < (2m+1)\pi$, $(2n-1)\pi < x(1+t) < (2n+1)\pi$, m și n sînt numere naturale.

6.534

$$1. \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - (2l\pi - tx)^2}}. \quad \text{MO 60}$$

$$2. \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \sum_{l=0}^m \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} +$$

$$+ \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} -$$

$$- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l}. \quad \text{MO 60}$$

$$\begin{aligned}
 3. \quad \sum_{k=1}^{\infty} N_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) - \sum_{l=0}^m \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} + \\
 &+ \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\
 &- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\}. \quad \text{MO 61}
 \end{aligned}$$

În formulele 6.534, $x > 0$, $t > 1$, $2m\pi < x(t-1) < 2(m+1)\pi$, $2n\pi < x(t+1) < 2(n+1)\pi$, $m+1$ și $n+1$ sînt numere naturale.

6.535

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}. \quad \text{MO 61}$$

$$\begin{aligned}
 2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt &= \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} + \\
 &+ \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\
 &- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}. \quad \text{MO 61}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \sum_{k=1}^{\infty} (-1)^k N_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} - \\
 &- \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \\
 &- \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\
 &- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}.
 \end{aligned}$$

MO 61

În formulele 6.535 $x > 0$, $t > 1$, $(2m-1)\pi < x(t-1) < (2m+1)\pi$, $(2n-1)\pi < x(t+1) < (2n+1)\pi$, m și n sînt numere naturale.

6.536

$$\begin{aligned}
 1. \quad \sum_{k=1}^{\infty} K_0(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x\sqrt{1+t^2}} + \\
 &+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi} \right\} + \\
 &+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi} \right\}. \quad \text{MO 61}
 \end{aligned}$$

$$\begin{aligned}
2. \quad \sum_{k=1}^{\infty} (-1)^k K_0'(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \\
&+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi} \right\} + \\
&+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi} \right\} \\
&[x > 0, t \text{ real}]
\end{aligned}$$

MO 62

6.54 Descompunerea în produse de funcții cilindrice

„Teoreme de compunere“

6.540 Fie $r > 0$, $\rho > 0$, $\varphi > 0$, și $R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \varphi}$, adică fie r , ρ și R laturile unui triunghi avînd unghiul dintre laturile r și ρ egal cu φ . Fie, mai departe, $\rho < r$ și ψ unghiul opus laturii ρ , așa încît

$$1. \quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \rho e^{-i\varphi}}{r - \rho e^{i\varphi}}.$$

Dacă aceste condiții sînt îndeplinite, are loc „teorema de compunere“ a funcțiilor cilindrice:

$$2. \quad e^{i\psi} Z_p(mR) = \sum_{k=-\infty}^{\infty} J_k(m\rho) Z_{p+k}(mr) e^{ik\varphi}$$

[m număr complex oarecare].

W 394 (6)

Pentru $Z_p = J_p$ și pentru p întreg, condiția restrictivă $\rho < r$ devine superflua.

MO 31

6.541 Cazuri particulare

$$1. \quad J_0(mR) = J_0(m\rho) J_0(mr) + 2 \sum_{k=1}^{\infty} J_k(m\rho) J_k(mr) \cos k\varphi. \quad \text{W 391 (1)}$$

$$\begin{aligned}
2. \quad H_0^{(1,2)}(mR) &= J_0(m\rho) H_0^{(1,2)}(mr) + \\
&+ 2 \sum_{k=1}^{\infty} J_k(m\rho) H_k^{(1,2)}(mr) \cos k\varphi. \quad \text{MO 34}
\end{aligned}$$

$$\begin{aligned}
3. \quad J_0(z \sin \alpha) &= J_0^2\left(\frac{z}{2}\right) + 2 \sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right) \cos 2k\alpha; \\
&= \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k \cdot k!} J_{2k+\frac{1}{2}}(z) P_{2k}(\cos \alpha). \quad \text{MO 31}
\end{aligned}$$

6.542 Se mai numește „teoremă de compunere“ formula

$$1. \quad \frac{Z_p(mR)}{R^p} = 2^p m^{-p} \Gamma(p) \sum_{k=0}^{\infty} (p+k) \frac{J_{p+k}(m\rho)}{\rho^p} \frac{Z_{p+k}(mr)}{r^p} C_k^p(\cos \varphi).$$

[$p \neq -1, -2, -3, \dots$; condițiile pentru r, ρ, R, φ, m sînt aceleași ca și în formula 6.540; pentru $Z_p = J_p$ și pentru p întreg, formula 6.542 1. devine valabilă pentru r, ρ și φ oarecare].

W 398 (4)

6.543 Cazuri particulare

$$1. \frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\rho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(mp) H_{k+\frac{1}{2}}^{(1)}(mr) P_k(\cos \varphi). \quad \text{MO 31}$$

$$2. \frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\rho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(mp) H_{k+\frac{1}{2}}^{(2)}(mr) P_k(\cos \varphi). \quad \text{MO 31}$$

6.544 Teorema de compunere degenerată ($r \rightarrow \infty$)

$$e^{im\rho \cos \varphi} = \sqrt{\frac{\pi}{2mp}} \sum_{k=0}^{\infty} i^k (2k+1) J_{k+\frac{1}{2}}(mp) P_k(\cos \varphi); \quad \text{W 401 (1)}$$

$$= 2^p \Gamma(p) \sum_{k=0}^{\infty} (p+k) i^k (mp)^{-p} J_{p+k}(mp) C_p^p(\cos \varphi) \quad \text{W 401 (2)}$$

$[p \neq 0, -1, -2, \dots].$

6.545 Se numește teoremă de multiplicare formula

$$Z_p(\lambda z) = \lambda^p \sum_{k=0}^{\infty} \frac{1}{k!} Z_{p+k}(z) \left[\frac{1-\lambda^2}{2} z \right]^k \quad [|1-\lambda^2| < 1].$$

Dacă $Z_p \equiv J_p$, ea este valabilă pentru orice valori ale lui λ și z . MO 32

6.546

$$1. \sum_{k=0}^{\infty} \frac{(2n+2k) \cdot (2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \quad [n > 0]. \quad \text{W 47 (1)}$$

$$2. 2 \sum_{k=n}^{\infty} \frac{k \Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \quad [n > 0]. \quad \text{W 47 (2)}$$

$$3. J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z) = 1. \quad \text{W 41 (3)}$$

$$6.547 \quad \sum_{k=-\infty}^{\infty} Z_{p-k}(t) J_k(z) = Z_p(z+t) \quad [|z| < |t|]. \quad \text{W 158 (2)}$$

În particular,

$$\sum_{k=-\infty}^{\infty} J_k(z) J_{n-k}(z) = J_n(2z). \quad \text{W 41}$$

6.548

$$1. \sum_{k=-\infty}^{\infty} (-1)^k J_{-p+k}(t) J_k(z) = J_{-p}(z+t) \quad [|z| < |t|]. \quad \text{W 159}$$

$$2. \sum_{k=-\infty}^{\infty} Z_{p+k}(t) J_k(z) = Z_p(t-z) \quad [|z| < |t|]. \quad \text{W 159 (5)}$$

6.55 Rădăcini ale funcțiilor cilindrice

6.551 Pentru orice p real, funcția $J_p(z)$ are o infinitate de rădăcini reale; pentru $p > -1$, toate rădăcinile ei sînt reale. W 526 W 530

Funcția cilindrică $Z_p(z)$ nu are rădăcini multiple, cu excepția, poate, a originii coordonatelor. W 528

6.552 Toate rădăcinile funcției $N_0(z)$ a căror parte reală este pozitivă sînt reale. W 531

6.553 Dacă $-(2s+2) < p < -(2s+1)$, unde s este un număr natural sau zero, atunci $J_p(z)$ are exact $4s+2$ rădăcini complexe, dintre care două pur imaginare; dacă $-(2s+1) < p < -2s$, unde s este un număr natural, atunci funcția $J_p(z)$ are exact $4s$ rădăcini complexe, dintre care nici una nu este pur imaginară.

W 532

6.554 Dacă, pentru $p > 0$, x_p și x'_p sînt respectiv cele mai mici rădăcini pozitive ale funcțiilor $J_p(z)$ și $J'_p(z)$, atunci $x_p > p$, $x'_p > p$. Fie, în afară de aceasta, y_p cea mai mică rădăcină pozitivă a funcției $N_p(z)$; atunci $x_p' < y_p < x_p$.

W 534 W 536

Fie $z_{p,m}$ ($m=1, 2, 3, \dots$) rădăcinile funcției $z^{-p} J_p(z)$, ordonate în raport cu valorile absolute ale părților lor reale; aici se presupune că $p \neq -1, -2, -3, \dots$ în acest caz, pentru orice z ,

$$J_p(z) = \frac{\left(\frac{z}{2}\right)^p}{\Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_{p,m}^2}\right). \quad \text{W 550}$$

6.555 Numărul rădăcinilor funcției $z^{-p} J_p(z)$, cuprinse între axa imaginară și dreapta pentru care

$$\operatorname{Re}(z) = m\pi + \left[\frac{1}{2} \operatorname{Re}(p) + \frac{1}{4}\right] \pi, \quad \text{W 548}$$

6.556 Pentru $p \geq 0$, numărul rădăcinilor funcției $K_p(z)$, cuprinse în domeniul $\operatorname{Re}(z) < 0$, $|\arg z| < \pi$, este egal cu numărul par cel mai apropiat de $p - \frac{1}{2}$.

W 562

6.557 Rădăcinile mari ale funcțiilor $J_p(z) \cos \alpha - N_p(z) \sin \alpha$, unde p și α sînt numere reale, sînt date de dezvoltarea asimptotică

$$x_{p,m} \sim \left(m + \frac{1}{2}p - \frac{1}{4}\right) \pi - \alpha - \frac{4p^2 - 1}{8 \left[\left(m + \frac{1}{2}p - \frac{1}{4}\right) \pi - \alpha\right]} - \frac{(4p^2 - 1)(28p^2 - 31)}{384 \left[\left(m + \frac{1}{2}p - \frac{1}{4}\right) \pi - \alpha\right]^3} - \dots \quad \text{C 109 (24) W 558}$$

6.558 În particular, rădăcinile mari ale funcției $J_0(z)$ sînt, date de dezvoltarea

$$x_{0,m} \sim \frac{\pi}{4}(4m-1) + \frac{1}{2\pi(4m-1)} - \frac{31}{6\pi^3(4m-1)^3} + \frac{3779}{15\pi^5(4m-1)^5} - \dots$$

C 109 (25) W 556

Această serie poate fi folosită pentru calcularea tuturor rădăcinilor funcției $J_0(z)$ (cu excepția rădăcinii celei mai mici x_{01}), cu aproximație minimă de cinci zecimale.

6.559 Pentru calcularea rădăcinilor $x_{p,m}$, cele mai mici în valoare absolută, ale funcției $J_p(z)$, se poate folosi egalitatea

$$\sum_{m=1}^{\infty} \frac{1}{x_{p,m}^6} = \frac{429p^5 + 7640p^4 + 53752p^3 + 185430p^2 + 311387p + 202738}{2^{16}(p+1)^3(p+2)^4(p+3)^2(p+4)^2(p+5)(p+6)(p+7)(p+8)}.$$

C 112 (27)m W 554

6.56 Funcțiile Thompson și generalizările lor: $\operatorname{ber}_p(z)$, $\operatorname{bei}_p(z)$, $\operatorname{her}_p(z)$, $\operatorname{hei}_p(z)$, $\operatorname{ker}(z)$, $\operatorname{kei}(z)$

6.561 Definiție

$$\left. \begin{aligned} 1. \operatorname{ber}_p(z) + i \operatorname{bei}_p(z) &= J_p(z e^{\frac{3}{4}\pi i}). \\ 2. \operatorname{ber}_p(z) - i \operatorname{bei}_p(z) &= J_p(z e^{-\frac{3}{4}\pi i}). \end{aligned} \right\} \quad \text{W 96 (6)}$$

6.562

$$\left. \begin{aligned} 1. \operatorname{her}_p(z) + i \operatorname{hei}_p(z) &= H_p^{(1)}(ze^{\frac{3}{4}\pi i}) \\ 2. \operatorname{her}_p(z) - i \operatorname{hei}_p(z) &= H_p^{(1)}(ze^{-\frac{3}{4}\pi i}) \end{aligned} \right\} \quad (\text{vezi și 6.567}). \quad W 96 (7)$$

6.563

$$\left. \begin{aligned} 1. \operatorname{ber}_0(z) &\equiv \operatorname{ber}(z); \operatorname{bei}_0(z) \equiv \operatorname{bei}(z). \\ 2. \operatorname{ker}(z) &\equiv -\frac{\pi}{2} \operatorname{hei}_0(z); \operatorname{kei}(z) \equiv \frac{\pi}{2} \operatorname{her}_0(z). \end{aligned} \right\} \quad W. 96 (8)$$

Reprezentarea integrală, vezi 4.411, 4.471, 4.472.

Dezvoltarea în serie

6.564

$$1. \operatorname{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k} [(2k)!]^2}. \quad W 96 (3)$$

$$2. \operatorname{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2} [(2k+1)!]^2}. \quad W 96 (4)$$

$$3. \operatorname{ker}(z) = \left(\ln \frac{2}{z} - C \right) \operatorname{ber}(z) + \frac{\pi}{4} \operatorname{bei}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k} [(2k)!]^2} \sum_{m=1}^{2k} \frac{1}{m}. \quad W 96 (9)m D (824.3)$$

$$4. \operatorname{kei}(z) = \left(\ln \frac{2}{z} - C \right) \operatorname{bei}(z) - \frac{\pi}{4} \operatorname{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \sum_{m=1}^{2k+1} \frac{1}{m}. \quad W 96 (10)m D (824.4)$$

$$6.565 \operatorname{ber}_p^*(z) + \operatorname{bei}_p^*(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2p+4k}}{k! \Gamma(p+k+1) \Gamma(p+2k+1)}. \quad W 163 (6)$$

Dezvoltarea asimptotică

6.566

$$1. \operatorname{ber} z = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos \beta(z), \quad 2. \operatorname{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin \beta(z) \quad \left[|\arg z| < \frac{\pi}{4} \right] W 227 (1)$$

$$3. \operatorname{ker} z = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos \beta(-z), \quad 4. \operatorname{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin \beta(-z) \quad \left[|\arg z| < \frac{5}{4} \pi \right],$$

unde

$$\begin{aligned} \alpha(z) &\sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots \\ \beta(z) &\sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots \end{aligned} \quad W 227 (2)$$

6.567 Relații funcționale

1. $\ker(z) + i \operatorname{kei}(z) = K_0(z \sqrt{-i})$ (vezi și 6.562) W 96 (5)
2. $\ker(z) - i \operatorname{kei}(z) = K_0(z \sqrt{-i})$.

Integralele funcțiilor Thompson, vezi 5.255.

6.6 FUNCȚII MATHIEU**6.60 Definiție**

Se numesc *ecuație Mathieu*, atât ecuația

$$1. \frac{d^2 y}{dx^2} + (\lambda - 2h^2 \cos 2x)y = 0, \quad \text{St 28 (1) MO 63}$$

cât și ecuația

$$2. \frac{d^2 y}{dx^2} + (a^2 + b^2 \cos x)y = 0. \quad \text{MO 63}$$

6.61 Rezolvarea sub formă de serie trigonometrică

6.611 Există o soluție $y(x)$ a ecuației 6.60 2., care satisface relația

$$y(x + 2\pi) = \sigma y(x).$$

6.612 Dacă $y_1(x)$ și $y_2(x)$ sînt două soluții ale ecuației 6.60 2., definite prin condițiile inițiale

$$\begin{aligned} y_1(0) &= 1, & y_2(0) &= 0, \\ y_1'(0) &= 0, & y_2'(0) &= 1, \end{aligned}$$

atunci

$$\sigma^2 - \sigma [y_1(2\pi) + y_2'(2\pi)] + 1 = 0.$$

6.613 Punind în ecuația diferențială 6.60 2.

$$1. y = e^{\mu x} \sum_{k=-\infty}^{\infty} A_k e^{ikx} \quad [\sigma = e^{2\pi\mu}],$$

obținem, pentru infinitatea de coeficienți necunoscuți A_n , o infinitate de ecuații liniare omogene de forma

$$2. A_n [a^2 + (\mu + in)^2] + \frac{b^2}{2} (A_{n-1} + A_{n+1}) = 0$$

[n întreg].

Pentru ca acest sistem de ecuații să aibă o soluție diferită de zero, trebuie să fie satisfăcută condiția

$$3. \begin{vmatrix} \dots & M & 1 & M & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & M & 1 & M & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & N & 1 & N & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & M & 1 & M & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & M & 1 & M & \dots \end{vmatrix} = \Delta(\mu) = 0,$$

unde $M = \frac{b^2}{2[a^2 + (\mu - 2i)^2]}$, $N = \frac{b^2}{2(\mu^2 + a^2)}$. Din această ecuație se poate calcula, a și b fiind dați, indicele caracteristic μ al determinantului. Apoi, din formu-

lele de recurență 6.613 2. se pot determina (cu o precizie până la un factor constant) coeficienții A_n și, prin aceasta, se poate găsi soluția ecuației 6.60 2., sub forma seriei 6.613 1.

6.614 Indicele caracteristic μ poate fi determinat și din ecuația

$$\sin^2(i\pi\mu) = \Delta(0) \sin^2(\pi a),$$

unde $\Delta(0)$ este valoarea determinantului 6.613 3., pentru $\mu=0$.

6.615

1. Pentru μ pur imaginar, soluția lui 6.613 1. este stabilă; pentru μ real sau complex, această soluție este nestabilă.

2. Pentru $\mu = mi$ [$m+1$ natural], soluția lui 6.613 1. este periodică, având perioada 2π .

3. Pentru $\mu = (m + \frac{1}{2})i$ [$m+1$ natural], soluția lui 6.613 1. este periodică, având perioada 4π .

MO 63—64

6.62 Funcții periodice Mathieu

6.620 În general, ecuația lui Mathieu 6.60 1. nu are soluții periodice. Dacă h este real, atunci există o infinitate de valori proprii λ , cărora le corespund soluțiile periodice

$$y(x) = y(2\pi + x),$$

care nu sînt identic nule. Dacă h este diferit de zero, atunci nu există niciun fel de alte soluții periodice liniar independente. Soluțiile periodice ale ecuației Mathieu se numesc *funcții periodice Mathieu* sau *funcții Mathieu de speța întâi*, sau, pur și simplu, *funcții Mathieu*.

6.621 Ecuația Mathieu are patru tipuri de soluții periodice diferite:

$$\begin{aligned} 1. \quad ce_{2n}(x) &= \sum_{r=0}^{\infty} A_{2n,2r} \cos 2rx. & 2. \quad ce_{2n+1}(x) &= \sum_{r=0}^{\infty} A_{2n+1,2r+1} \cos(2r+1)x. \\ 3. \quad se_{2n}(x) &= \sum_{r=1}^{\infty} B_{2n,2r} \sin 2rx. & 4. \quad se_{2n+1}(x) &= \sum_{r=0}^{\infty} B_{2n+1,2n+1} \sin(2r+1)x \end{aligned}$$

[În 6.621 3., n este natural; în celelalte formule, $n+1$ este natural].

5. Coeficienții A și B depind de h^2 ; valorile proprii ale funcțiilor ce_{2n} , ce_{2n+1} , se_{2n} , se_{2n+1} se notează: a_{2n} , a_{2n+1} , b_{2n} , b_{2n+1} .

St 35 (1)–(4)

6.622 Soluțiile ecuației Mathieu se normalizează

$$\int_0^{2\pi} y^2 dx = \pi.$$

MO 65

6.623

$$\begin{aligned} 1. \quad \lim_{h \rightarrow 0} ce_0(x) &= \frac{1}{\sqrt{2}}. \\ 2. \quad \lim_{h \rightarrow 0} ce_n(x) &= \cos nx \quad [n \neq 0]. \\ 3. \quad \lim_{h \rightarrow 0} se_n(x) &= \sin nx. \end{aligned}$$

MO 65

6.624 Coeficienții $A_{2n,2r}$, $A_{2n+1,2r+1}$, $B_{2n,2r}$, $B_{2n+1,2r+1}$ (vezi 6.621) se obțin din formulele de recurență

1. $-\lambda A_{2n,0} + h^2 A_{2n,2} = 0$ [$r=0$].
2. $2h^2 A_{2n,0} + (4-\lambda) A_{2n,2} + h^2 A_{2n,4} = 0$ [$r=1$].
3. $h^2 A_{2n,2r-2} + (4r^2 - \lambda) A_{2n,2r} + h^2 A_{2n,2r+2} = 0$ [$r > 1$].
4. $h^2 A_{2n+1,1} + (1-\lambda) A_{2n+1,1} + h^2 A_{2n+1,3} = 0$ [$r=0$].
5. $h^2 A_{2n+1,2r-1} + [(2r+1)^2 - \lambda] A_{2n+1,2r+1} + h^2 A_{2n+1,2r+3} = 0$ [$r > 0$].
6. $(4-\lambda) B_{2n,2} + h^2 B_{2n,4} = 0$ St. 40 (1) [$r=1$].
7. $h^2 B_{2n,2r-2} + (4r^2 - \lambda) B_{2n,2r} + h^2 B_{2n,2r+2} = 0$ [$r > 1$].
8. $-h^2 B_{2n+1,1} + (1-\lambda) B_{2n+1,1} + h^2 B_{2n+1,3} = 0$ [$r=0$].
9. $h^2 B_{2n+1,2r-1} + [(2r+1)^2 - \lambda] B_{2n+1,2r+1} + h^2 B_{2n+1,2r+3} = 0$ [$r > 0$].

MO 65

6.625

Din egalitățile 6.622, 6.623 și 6.624, se pot determina coeficienții A și B , dacă se cunoaște λ . Să presupunem, de exemplu, că trebuie să determinăm coeficienții $A_{2n,2r}$ ai funcției $ce_{2n}(x)$. Din formulele de recurență, se obține ca o consecință relația

$$1. \begin{vmatrix} \lambda & -h^2 & 0 & 0 & 0 & \dots \\ -2h^2 & \lambda-4 & -h^2 & 0 & 0 & \dots \\ 0 & -h^2 & \lambda-16 & -h^2 & 0 & \dots \\ 0 & 0 & -h^2 & \lambda-36 & -h^2 & \dots \\ 0 & 0 & 0 & -h^2 & \lambda-64 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

St 40

Dacă h^2 este dat, atunci, din ecuația 6.625 1., se pot determina valorile proprii

$$2. \lambda = a_0, a_2, a_4, \dots \quad [|a_0| \leq |a_2| \leq |a_4| \leq \dots].$$

Punind, mai departe, $\lambda = a_{2n}$, se pot determina, din formulele de recurență 6.624 1.—6.624 3., coeficienții $A_{2n,2r}$, abstractie făcând de un factor de proporționalitate. Acesta din urmă se deduce din formula

$$3. 2A_{2n,0}^2 + \sum_{r=1}^{\infty} A_{2n,2r}^2 = \pi,$$

care rezultă din condițiile de normare (vezi 6.622) și din 6.623 1. și 6.623 2

MO 64—66

6.63 Funcții Mathieu de argument pur imaginar

6.630 Înlocuind în ecuația 6.60 1., pe x prin ix , ajungem la ecuația diferențială

$$1. \frac{d^2 y}{dx^2} + (-\lambda + 2h^2 \operatorname{ch} 2x) y = 0.$$

Soluțiile acestei ecuații se pot afla, substituind în funcțiile $ce_n(x)$ și $se_n(x)$ argumentul x prin ix . Funcțiile obținute în acest mod se numesc *funcții Mathieu adjuncate de speța întâi* și se notează

$$2. Ce_n(x) = ce_n(ix), \quad Se_n(x) = -ise_n(ix).$$

6.631 Dezvoltarea în serie a funcțiilor $Ce_n(x)$ și $Se_n(x)$

1. $Ce_{2n}^{(x)} = \sum_{r=0}^{\infty} A_{2n, 2r} \operatorname{ch} 2rx;$

$$= \frac{ce_{2n}^{(0)}}{A_{2n, 0}} \sum_{r=0}^{\infty} A_{2n, 2r} J_{2r}(2h \operatorname{sh} x);$$

$$= \frac{ce_{2n} \left(\frac{\pi}{2} \right)}{A_{2n, 0}} \sum_{r=0}^{\infty} A_{2n, 2r} (-1)^r J_{2r}(2h \operatorname{ch} x);$$

$$= \frac{ce_{2n}^{(0)} ce_{2n} \left(\frac{\pi}{2} \right)}{A_{2n, 0}^2} \sum_{r=0}^{\infty} (-1)^r A_{2n, 2r} \{J_r(he^{-x}) J_{r+2}(he^x) +$$

$$+ J_r(he^x) J_{r+2}(he^{-x})\}.$$
2. $Ce_{2n+1}(x) = \sum_{r=0}^{\infty} A_{2n+1, 2r+1} \operatorname{ch} (2r+1)x;$

$$= \frac{ce_{2n+1}^{(0)}}{hA_{2n+1, 1}} \operatorname{cth} x \sum_{r=0}^{\infty} (2r+1) A_{2n+1, 2r+1} J_{2r+1}(2h \operatorname{sh} x);$$

$$= -\frac{ce'_{2n+1} \left(\frac{\pi}{2} \right)}{hA_{2n+1, 1}} \sum_{r=0}^{\infty} (-1)^r A_{2n+1, 2r+1} J_{2r+1}(2h \operatorname{ch} x);$$

$$= -\frac{ce_{2n+1}^{(0)} ce'_{2n+1} \left(\frac{\pi}{2} \right)}{hA_{2n+1, 1}^2} \sum_{r=0}^{\infty} (-1)^r A_{2n+1, 2r+1} \{J_r(he^{-x}) J_{r+1}(he^x) +$$

$$+ J_r(he^x) J_{r+1}(he^{-x})\}.$$
3. $Se_{2n+2}(x) = \sum_{r=0}^{\infty} B_{2n+2, 2r+2} \operatorname{sh} (2r+2)x;$

$$= \frac{se'_{2n+2}^{(0)}}{h^2 B_{2n+2, 2}} \operatorname{cth} x \sum_{r=0}^{\infty} (2r+2) B_{2n+2, 2r+2} J_{2r+2}(2h \operatorname{sh} x);$$

$$= -\frac{se'_{2n+2} \left(\frac{\pi}{2} \right)}{h^2 B_{2n+2, 2}} \operatorname{th} x \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2n+2, 2r+2} J_{2r+2}(2h \operatorname{ch} x);$$

$$= -\frac{se'_{2n+2}^{(0)} se'_{2n+2} \left(\frac{\pi}{2} \right)}{h^2 B_{2n+2, 2}^2} \sum_{r=0}^{\infty} (-1)^r B_{2n+2, 2r+2} \{J_r(he^{-x}) J_{r+2}(he^x) -$$

$$- J_r(he^x) J_{r+2}(he^{-x})\}.$$

$$\begin{aligned}
4. \quad Se_{2n+1}(x) &= \sum_{r=0}^{\infty} B_{2n+1, 2r+1} \operatorname{sh}(2r+1)x; \\
&= \frac{se'_{2n+1}(0)}{hB_{2n+1, 1}} \sum_{r=0}^{\infty} B_{2n+1, 2r+1} J_{2r+1}(2h \operatorname{sh} x); \\
&= \frac{se_{2n+1}\left(\frac{\pi}{2}\right)}{hB_{2n+1, 1}} \operatorname{th} x \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2n+1, 2r+1} J_{2r+1}(2h \operatorname{ch} x); \\
&= \frac{se_{2n+1}(0) se_{2n+1}\left(\frac{\pi}{2}\right)}{hB_{2n+1}^2} \sum_{r=0}^{\infty} (-1)^r B_{2n+1, 2r+1} \times \\
&\quad \times \{J_r(he^{-x}) J_{r+1}(he^x) - J_r(he^x) J_{r+1}(he^{-x})\}
\end{aligned}$$

[$n+1$ este număr natural, accentul indică derivata],

MO 66 — 67

6.7-6.8 FUNCȚII SFERICE

6.70 Introducere

6.700 Funcțiile sferice sînt soluțiile ecuației diferențiale

$$1. \quad (1-z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[p(p+1) - \frac{q^2}{1-z^2} \right] u = 0 \quad (\text{cf. 6.701 1.})$$

în care p și q sînt constante complexe oarecare.

Această ecuație este un caz particular al ecuației hipergeometrice (a lui Riemann) (vezi 7.252). Punctele
 $+1, -1, \infty$

sînt, în general, puncte singulare, și anume puncte de ramificație ordinare.

Prezintă interes, pe de o parte, soluțiile ecuației care corespund valorilor reale ale variabilei independente z și care sînt situate pe segmentul $[-1 \leq z \leq +1]$, iar pe de altă parte, soluțiile corespunzătoare oricărei valori complexe a lui z , pentru care $\operatorname{Re}(z) > 1$. Acestea din urmă sînt funcții multiforme în planul z ; pentru stabilirea ramurilor uniforme ale acestor funcții, se face o secțiune de-a lungul axei reale de la $-\infty$ pînă la $+1$. Mai de parte, ne interesează soluțiile ecuației 6.700 1. pentru care p sau q , sau și p și q , sînt numere întregi. Cazul $q = 0$ are o deosebită importanță.

6.701 În conformitate cu aceasta, vom folosi următoarele notații.

Vom nota cu litera z orice mărime complexă; cu litera x , variabila reală, care variază pe segmentul $[-1 \leq x \leq +1]$; uneori vom nota $x = \cos \varphi$, unde φ este un număr real.

Vom nota cu simbolurile $P_p^q(z)$, $Q_p^q(z)$ soluțiile ecuației 6.700 1. care, pentru $|z| < 1$, sînt uniforme și regulate și, în particular univoc definite pentru $z = x$.

Vom nota cu simbolurile $\mathfrak{P}_p^q(z)$, $\mathfrak{Q}_p^q(z)$ soluțiile ecuației 6.700 1. care, pentru $\operatorname{Re}(z) > 1$, sînt uniforme și regulate; cînd aceste funcții nu pot fi

prelungite nemărginit fără să înceteze de a fi uniforme, se efectuează o secțiune de-a lungul axei reale, la stînga punctului $z = 1$. Valorile funcțiilor $\mathfrak{P}_p^q(z)$ și $\mathfrak{Q}_p^q(z)$ la limitele superioară și inferioară ale părții secționate, care este cuprinsă între punctele -1 și $+1$, se notează respectiv

$$\mathfrak{P}_p^q(x \pm i \cdot 0), \quad \mathfrak{Q}_p^q(x \pm i \cdot 0).$$

Literele n, m reprezintă numere naturale sau nule. Literele p, q reprezintă numere complexe oarecare, dacă nu se face nici o rezervă de altă natură.

Indicele superior, dacă este egal cu zero, se omite, adică se pune

$$\mathfrak{P}_p^0(z) = \mathfrak{P}_p(z), \quad \mathfrak{Q}_p^0(z) = \mathfrak{Q}_p(z), \quad P_p^0(z) = P_p(z), \quad Q_p^0(z) = Q_p(z).$$

Cele două funcții liniar independente

$$6.702 \quad \mathfrak{P}_p^q(z) = \frac{1}{\Gamma(1-q)} \left(\frac{z+1}{z-1} \right)^{\frac{q}{2}} F \left(-p, p+1; 1-q; \frac{1-z}{2} \right)$$

$$\left[\arg \frac{z+1}{z-1} = 0, \text{ dacă } z \text{ este real și mai mare decît } 1 \right]. \quad \text{MO 80 WW II 122.}$$

$$6.703 \quad \mathfrak{Q}_p^q(z) = \frac{e^{q\pi i} \Gamma(p+q+1) \Gamma\left(\frac{1}{2}\right)}{2^{p+1} \Gamma\left(p + \frac{3}{2}\right)} (z^2 - 1)^{\frac{q}{2}} z^{-p-q-1} \times$$

$$\times F \left(\frac{p+q+2}{2}, \frac{p+q+1}{2}; p + \frac{3}{2}; \frac{1}{z^2} \right)$$

[arg $(z^2 - 1) = 0$, atunci cînd z este real și mai mare decît 1; arg $z = 0$, atunci cînd z este real și pozitiv], H 109 (44) MO 80

care sînt soluțiile ecuației diferențiale 6.700 1., se numesc *funcții sferice* (sau *funcții Legendre adjuncțe*), respectiv de *speța întâi și de speța a doua*. Ele sînt definite, și anume univoc, în domeniile $|1-z| < 2$ și $|z| > 1$ respectiv, din care se exclude partea din axa reală situată între $-\infty$ și $+1$; cu ajutorul seriilor hipergeometrice, ele pot fi prelungite nemărginit în tot planul z , în care s-a făcut secțiunea menționată. Aceste expresii ale lui $\mathfrak{P}_p^q(z)$ și $\mathfrak{Q}_p^q(z)$

își pierd sensul, cînd $1-q$, respectiv $p + \frac{3}{4}$, sînt numere întregi și negative sau nule. MO 80

Cînd z este un număr real din segmentul $[-1, +1]$ ($z = x - \cos \varphi$), se iau ca soluții liniar independente ale ecuației funcțiile

$$6.704 \quad P_p^q(z) = e^{\frac{1}{2} q\pi i} \mathfrak{P}_p^q(\cos \varphi + i \cdot 0) = e^{-\frac{1}{2} q\pi i} \mathfrak{P}_p^q(\cos \varphi - i \cdot 0) = \\ = \frac{1}{\Gamma(1-q)} \left(\frac{1+x}{1-x} \right)^{q/2} F \left(-p, p+1; 1-q; \frac{1-x}{2} \right). \quad \text{H 90 (4)}$$

$$6.705 \quad Q_p^q(z) = \frac{1}{2} e^{-q\pi i} \left[e^{-\frac{1}{2} q\pi i} \mathfrak{Q}_p^q(x + i \cdot 0) + e^{\frac{1}{2} q\pi i} \mathfrak{Q}_p^q(x - i \cdot 0) \right] = \\ = \frac{\pi}{2 \sin q\pi} \left[P_p^q(x) \cos q\pi - \frac{\Gamma(p+q+1)}{\Gamma(p-q+1)} P_p^{-q}(x) \right]$$

(cf. 6.732 5.). H 110

Pentru $q = \pm m$ întreg, egalitatea din urmă își pierde sensul; pentru acest caz, cu ajutorul trecerii la limită, obținem

6.706

$$1. Q_p^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_p(x).$$

$$2. Q_p^{-m}(x) = (-1)^m \frac{\Gamma(p-m+1)}{\Gamma(p+m+1)} Q_p^m(x).$$

Pentru $p+q$ întreg și negativ, funcțiile $Q_p^q(z)$ sînt nedeterminate. De aceea, trebuie să excludem din ultimele formule cazurile cînd $p+q = -1, -2, -3, \dots$

Cînd $p+q \neq 0, \pm 1, \pm 2, \dots$ se consideră ca soluții liniar independente ale ecuației diferențiale funcțiile

$$\mathfrak{P}_p^{\pm q}(\pm z), \Omega_p^{\pm q}(\pm z), \mathfrak{P}_{p-1}^{\pm q}(\pm z), \Omega_{p-1}^{\pm q}(\pm z).$$

6.707 Totuși, definirea a două soluții liniar independente este posibilă în orice caz, și anume:

Dacă $p \pm q$ nu este număr întreg, ecuația diferențială **6.700** 1. are următoarele soluții:

$$1. \mathfrak{P}_p^{\pm q}(\pm z), \Omega_p^{\pm q}(\pm z), \mathfrak{P}_{p-1}^{\pm q}(\pm z), \Omega_{p-1}^{\pm q}(\pm z)$$

sau, respectiv, pentru $z = x = \cos \varphi$,

$$2. P_p^{\pm q}(\pm x), Q_p^{\pm q}(\pm x), P_{p-1}^{\pm q}(\pm x), Q_{p-1}^{\pm q}(\pm x).$$

Dacă $p \pm q$ nu este număr întreg, atunci soluțiile

3. $\mathfrak{P}_p^q(z), \Omega_p^q(z)$ și, respectiv, $P_p^q(x), Q_p^q(x)$ sînt liniar independente. Dacă $p \pm q$ este un număr întreg, dar q nu este un număr întreg, atunci soluțiile liniar independente ale ecuației **6.701** 1. sînt funcțiile

$$4. \mathfrak{P}_p^q(z), \mathfrak{P}_p^{-q}(z) \text{ și, respectiv, } P_p^q(x), P_p^{-q}(x).$$

Dacă $q = \pm m$, $p = n$ sau $p = -n-1$, atunci soluțiile liniar independente ale ecuației **6.700** 1., pentru $n \geq m$, sînt funcțiile

$$5. \mathfrak{P}_n^m(z), \Omega_n^m(z) \text{ și, respectiv, } P_n^m(x), Q_n^m(x).$$

Iar pentru $n < m$, funcțiile

$$6. \mathfrak{P}_n^{-m}(z), \Omega_n^m(z) \text{ și, respectiv, } P_n^{-m}(x), Q_n^m(x).$$

6.71 Reprezentări integrale

6.711

$$1. \mathfrak{P}_p^{-q}(z) = \frac{(z^2-1)^{\frac{q}{2}}}{2^q \sqrt{\pi} \Gamma\left(q+\frac{1}{2}\right)} \int_{-1}^1 \frac{(1-t^2)^{q-\frac{1}{2}}}{(z+t\sqrt{z^2-1})^{q-p}} dt$$

$$\left[\operatorname{Re}(q) > -\frac{1}{2}, |\arg(z \pm 1)| < \pi \right].$$

MO 88

$$2. \mathfrak{P}_p^m(z) = \frac{(p+1)(p+2)\dots(p+m)}{\pi} \int_0^\pi [z + \sqrt{z^2-1} \cos \varphi]^p \cos m\varphi d\varphi;$$

$$= (-1)^m \frac{p(p-1)\dots(p-m+1)}{\pi} \int_0^\pi \frac{\cos m\varphi d\varphi}{[z + \sqrt{z^2-1} \cos \varphi]^{p+1}}$$

$$\left[|\arg z| < \frac{\pi}{2}, \arg(z + \sqrt{z^2-1} \cos \varphi) = \arg z \text{ pentru } \varphi = \frac{\pi}{2} \right]$$

(cf. 6.842 1.) Sm III 483(15) WW II 123

$$3. \quad \mathfrak{D}_p^q(z) = \sqrt{\pi} \frac{e^{q\pi i} \Gamma(p+q+1)}{2^q \Gamma\left(q+\frac{1}{2}\right) \Gamma(p-q+1)} (z^2-1)^{\frac{q}{2}} \int_0^\infty \frac{\operatorname{sh} 2^q t \, dt}{(z + \sqrt{z^2-1} \operatorname{ch} t)^{p+q+1}}$$

$[\operatorname{Re}(p \pm q) > -1, |\arg(z \pm 1)| < \pi]$ (cf. 6.842 2.). MO 88

$$4. \quad \mathfrak{D}_p^q(z) = \frac{e^{q\pi i} \Gamma(p+1)}{\Gamma(p-q+1)} \int_0^\infty \frac{\operatorname{ch} q t \, dt}{(z + \sqrt{z^2-1} \operatorname{ch} t)^{p+1}}$$

$[\operatorname{Re}(p+q) > -1, p \neq -1, -2, -3, \dots; |\arg(z \pm 1)| < \pi]$. WWII134m MO 88

$$6.712 \quad \mathfrak{D}_p^q(z) = \frac{e^{q\pi i} \Gamma(p+q+1)}{2^{p+1} \Gamma(p+1)} (z^2-1)^{\frac{q}{2}} \int_{-1}^1 (1-t^2)^p (z-t)^{-p-q-1} dt$$

$[\operatorname{Re}(p+q) > -1, \operatorname{Re}(q) > -1, |\arg(z \pm 1)| < \pi]$ (cf. 6.841 2.) MO 88

6.713

$$1. \quad \mathfrak{D}_p^q(z) = \frac{e^{q\pi i} \Gamma\left(q+\frac{1}{2}\right)}{\sqrt{2\pi}} (z^2-1)^{\frac{q}{2}} \times$$

$$\times \left\{ \int_0^\pi \frac{\cos\left(p+\frac{1}{2}\right)t \, dt}{(z-\cos t)^{q+\frac{1}{2}}} - \cos p\pi \int_0^\infty \frac{e^{-\left(p+\frac{1}{2}\right)t} \, dt}{(z+\operatorname{ch} t)^{q+\frac{1}{2}}} \right\}$$

$\left[\operatorname{Re}(q) > -\frac{1}{2}, \operatorname{Re}(p+q) > -1, |\arg(z \pm 1)| < \pi \right]$. MO 89

$$2. \quad \mathfrak{P}_p^{-q}(z) = \frac{(z^2-1)^{\frac{q}{2}}}{2^p \Gamma(q-p) \Gamma(p+1)} \int_0^\infty \frac{\operatorname{sh} 2^{p+1} t}{(z + \operatorname{ch} t)^{p+q+1}} dt$$

$[\operatorname{Re}(z) > -1, |\arg(z \pm 1)| < \pi, \operatorname{Re}(p+1) > 0, \operatorname{Re}(q-p) > 0]$. MO 89

$$3. \quad \mathfrak{P}_p^{-q}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(p+q+1) \Gamma(q-p)} (z^2-1)^{\frac{q}{2}} \int_0^\infty \frac{\operatorname{ch}\left(p+\frac{1}{2}\right)t \, dt}{(z + \operatorname{ch} t)^{q+\frac{1}{2}}}$$

$[\operatorname{Re}(z) > -1, |\arg(z \pm 1)| < \pi, \operatorname{Re}(p+q) > -1, \operatorname{Re}(q-p) > 0]$. MO 89

6.714

$$1. \quad P_p^q(\cos \varphi) = \sqrt{\frac{2}{\pi}} \frac{\sin^q \varphi}{\Gamma\left(\frac{1}{2}-q\right)} \int_0^\pi \frac{\cos\left(p+\frac{1}{2}\right)t \, dt}{(\cos t - \cos \varphi)^{q+\frac{1}{2}}}$$

$\left[0 < \varphi < \pi, \operatorname{Re}(q) < \frac{1}{2} \right];$ (cf. 6.843).

MO 87

$$2. \quad P_p^{-q}(\cos \varphi) = \frac{\Gamma(2q+1) \sin^q \varphi}{2^q \Gamma(q+1) \Gamma(p+q+1) \Gamma(q-p)} \int_0^\infty \frac{t^{p+q} \, dt}{(1+2t \cos \varphi + t^2)^{q+\frac{1}{2}}}$$

$[\operatorname{Re}(p+q) > -1, \operatorname{Re}(q-p) > 0]$.

MO 89

$$3. \quad Q_p^q(\cos \varphi) = \frac{1}{2^{q+1}} \frac{\Gamma(p+q+1)}{\Gamma(p-q+1)} \frac{\sin^q \varphi}{\Gamma\left(q+\frac{1}{2}\right)} \int_0^\infty \left[\frac{\operatorname{sh}^{2q} t}{(\cos \varphi + i \sin \varphi \operatorname{ch} t)^{p+q+1}} + \frac{\operatorname{sh}^{2q} t}{(\cos \varphi - i \sin \varphi \operatorname{ch} t)^{p+q+1}} \right] dt$$

$$\left[\operatorname{Re}(p+q+1) > 0, \operatorname{Re}(p-q+1) > 0, \operatorname{Re}(q) > -\frac{1}{2} \right]. \quad \text{MO 89}$$

$$4. \quad P_p^q(\cos \varphi) = \frac{i}{2^q} \frac{\Gamma(p+q+1)}{\Gamma(p-q+1)} \frac{\sin^q \varphi}{\Gamma\left(q+\frac{1}{2}\right)} \int_0^\infty \left[\frac{\operatorname{sh}^{2q} t}{(\cos \varphi + i \sin \varphi \operatorname{ch} t)^{p+q+1}} - \frac{\operatorname{sh}^{2q} t}{(\cos \varphi - i \sin \varphi \operatorname{ch} t)^{p+q+1}} \right] dt$$

$$\left[\operatorname{Re}(p \pm q + 1) > 0, \operatorname{Re}(q) > -\frac{1}{2} \right]. \quad \text{MO 89}$$

6.715

$$1. \quad \mathfrak{P}_p^q(\operatorname{ch} \alpha) = \frac{\sqrt{2} \operatorname{sh}^q \alpha}{\sqrt{\pi} \Gamma\left(\frac{1}{2}-q\right)} \int_0^\alpha \frac{\operatorname{ch}\left(p+\frac{1}{2}\right) t \, dt}{(\operatorname{ch} \alpha - \operatorname{ch} t)^{q+\frac{1}{2}}} \quad \left[\alpha > 0, \operatorname{Re}(q) < \frac{1}{2} \right].$$

MO 87

$$2. \quad \Omega_p^q(\operatorname{ch} \alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{q\pi i} \operatorname{sh}^q \alpha}{\Gamma\left(\frac{1}{2}-q\right)} \int_\alpha^\infty \frac{e^{-(p+\frac{1}{2})t} \, dt}{(\operatorname{ch} t - \operatorname{ch} \alpha)^{q+\frac{1}{2}}}$$

$$\left[\alpha > 0, \operatorname{Re}(q) < \frac{1}{2}, \operatorname{Re}(p+q) > -1 \right]. \quad \text{MO 87}$$

Vezi și 4.431 3., 4.433, 4.466, precum și 6.742.

6.72 Serii asimptotice pentru valori mari ale lui $|p|$ 6.721 Pentru valori reale ale lui q , $|p| > 1$, $|p| \gg |q|$, $|\arg p| < \pi$, avem

$$1. \quad P_p^q(\cos \varphi) =$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(p+q+1) \sum_{k=0}^\infty \frac{\Gamma\left(q+k+\frac{1}{2}\right)}{\Gamma\left(q-k+\frac{1}{2}\right)} \frac{\cos\left[\left(p+k+\frac{1}{2}\right)\varphi - \frac{\pi}{4}(2k+1) + \frac{q\pi}{2}\right]}{k! \Gamma\left(p-k+\frac{3}{2}\right) (2 \sin \varphi)^{k+\frac{1}{2}}}$$

$[p+q \neq -1, -2, -3, \dots; p \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \text{pentru } \frac{\pi}{6} < \varphi < \frac{5\pi}{6}$
 această serie converge și pentru valorile complexe ale lui p și q ;
 în celelalte cazuri, ea reprezintă o dezvoltare asimptotică pentru $|p| \gg |q|$,
 $|p| \gg 1$, dacă $p > 0$, $q > 0$ și $0 < \varepsilon \leq \varphi \leq \pi - \varepsilon$].

MO 92

$$2. \quad Q_p^q(\cos \varphi) =$$

$$= \sqrt{\pi} \Gamma(p+q+1) \sum_{k=0}^\infty (-1)^k \frac{\Gamma\left(q+k+\frac{1}{2}\right)}{\Gamma\left(q-k+\frac{1}{2}\right)} \frac{\cos\left[\left(p+k+\frac{1}{2}\right)\varphi + \frac{\pi}{4}(2k+1) + \frac{q\pi}{4}\right]}{k! \Gamma\left(p-k+\frac{3}{2}\right) (2 \sin \varphi)^{k+\frac{1}{2}}}$$

$[p+q \neq -1, -2, -3, \dots; p \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \text{pentru } \frac{\pi}{6} < \varphi < \frac{5\pi}{6}$
 această serie converge, și pentru valori complexe ale lui p și q ; în
 celelalte cazuri, ea reprezintă o dezvoltare asimptotică pentru $|p| \gg |q|$, $|p| \gg 1$,
 dacă $p > 0$, $q > 0$, $0 < \varepsilon \leq \varphi \leq \pi - \varepsilon$].

MO 92

$$3. \quad P_q^p(\cos \varphi) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(p+q+1)}{\Gamma(p+\frac{3}{2})} \frac{\cos \left[\left(p + \frac{1}{2}\right) \varphi - \frac{\pi}{4} + \frac{q\pi}{2} \right]}{\sqrt{2 \sin \varphi}} \left[1 + O\left(\frac{1}{p}\right) \right] \\ \left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon, |p| \gg \frac{1}{\varepsilon} \right]. \quad \text{MO 92}$$

Pentru $p > 0, q > 0$ și $p > q$, rezultă, din formulele 6.721 1. și 6.721 2., că

$$4. \quad p^{-q} P_p^q(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \cos \left[\left(p + \frac{1}{2}\right) \varphi - \frac{\pi}{4} + \frac{q\pi}{2} \right] + O\left(\frac{1}{\sqrt{p^3}}\right).$$

$$5. \quad p^{-q} Q_p^q(\cos \varphi) = \sqrt{\frac{\pi}{2p \sin \varphi}} \cos \left[\left(p + \frac{1}{2}\right) \varphi + \frac{\pi}{4} + \frac{q\pi}{2} \right] + O\left(\frac{1}{\sqrt{p^3}}\right) \\ \left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon; p \gg \frac{1}{\varepsilon} \right]. \quad \text{MO 92}$$

6.722 Dacă φ este atât de apropiat de 0 sau de π , încît $p\varphi$ sau $p(\pi-\varphi)$ sînt mici în raport cu 1, atunci formulele asimptotice 6.721 devin inaplicabile; în acest caz, pentru $p > 0, q \geq 0, p \gg 1$ și pentru valori mici ale lui φ , se poate aplica următoarea dezvoltare asimptotică:

$$1. \quad \left[\left(p + \frac{1}{2}\right) \cos \frac{\varphi}{2} \right]^q P_p^{-q}(\cos \varphi) = \\ = J_q(\eta) + \sin^2 \frac{\varphi}{2} \left[\frac{J_{q+1}(\eta)}{2\eta} - J_{q+2}(\eta) + \frac{\eta}{6} J_{q+3}(\eta) \right] + O\left(\sin^4 \frac{\varphi}{2}\right),$$

unde $\eta = (2p+1) \sin \frac{\varphi}{2}$. În particular, rezultă de aici că

$$2. \quad \lim_{p \rightarrow \infty} p^q P_p^{-q} \left(\cos \frac{x}{p} \right) = J_q(x) \quad [x \geq 0, q \geq 0]. \quad \text{MO 93}$$

6.723 Comportarea funcțiilor $\mathfrak{P}_p^q(z)$ și $\Omega_p^q(z)$ pentru valori mari ale lui $|p|$ și pentru valori reale ale lui $z > \frac{3}{2\sqrt{2}}$ se vede din egalitățile

$$1. \quad \mathfrak{P}_p^q(\operatorname{ch} \alpha) = \frac{2^q}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(-p - \frac{1}{2}\right)}{\Gamma(-p-q)} \frac{e^{(q-p)\alpha} \operatorname{sh}^q \alpha}{(e^{2\alpha} - 1)^{q+\frac{1}{2}}} \times \right. \\ \times F\left(q + \frac{1}{2}, -q + \frac{1}{2}; p + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}}\right) + \\ \left. + \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p-q+1)} \frac{e^{(p+q+1)\alpha} \operatorname{sh}^q \alpha}{(e^{2\alpha} - 1)^{q+\frac{1}{2}}} F\left(q + \frac{1}{2}, -q + \frac{1}{2}; -p + \frac{1}{2}; \frac{1}{1 - e^{2\alpha}}\right) \right\} \\ \left[p \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots; \alpha > \frac{1}{2} \ln 2 \right]. \quad \text{MO 94}$$

$$2. \quad \Omega_p^q(\operatorname{ch} \alpha) = e^{\pi i} 2^q \sqrt{\pi} \frac{\Gamma(p+q+1)}{\Gamma\left(p + \frac{3}{2}\right)} \frac{e^{-(p+q+1)\alpha}}{(1 - e^{-2\alpha})^{q+\frac{1}{2}}} \operatorname{sh}^q \alpha \times \\ \times F\left(q + \frac{1}{2}, -q + \frac{1}{2}; p + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}}\right) \\ \left[p+q+1 \neq 0, -1, -2, \dots; \alpha > \frac{1}{2} \ln 2 \right]. \quad \text{MO 94}$$

Vezi și 6.776.

6.724 Inegalități.

$$\left. \begin{aligned}
 1. \quad |P_p^{\pm q}(\cos \varphi)| &< \sqrt{\frac{8}{p\pi}} \frac{\Gamma(p \pm q + 1)}{\Gamma(p + 1)} \frac{1}{\sin^{q + \frac{1}{2}} \varphi} \\
 2. \quad |Q_p^{\pm q}(\cos \varphi)| &< \sqrt{\frac{2\pi}{p}} \frac{\Gamma(p \pm q + 1)}{\Gamma(p + 1)} \frac{1}{\sin^{q + \frac{1}{2}} \varphi} \\
 3. \quad |P_q^{\pm m}(\cos \varphi)| &< \frac{2}{\sqrt{p\pi}} \frac{\Gamma(p \pm m + 1)}{\Gamma(p + 1)} \frac{1}{\sin^{m + \frac{1}{2}} \varphi} \\
 4. \quad |Q_q^{\pm m}(\cos \varphi)| &< \sqrt{\frac{\pi}{p}} \frac{\Gamma(p \pm m + 1)}{\Gamma(p + 1)} \frac{1}{\sin^{m + \frac{1}{2}} \varphi}
 \end{aligned} \right\} \begin{aligned}
 &[p \text{ și } q \text{ sînt numere} \\
 &\text{reale oarecare, satisfă-} \\
 &\text{cînd inegalitățile } p \geq 1, \\
 &p - q + 1 > 0, q \geq 0].
 \end{aligned}$$

MO 91-92

6.73-6.74 Relații funcționale

6.731

$$\begin{aligned}
 1. \quad (z^2 - 1) \frac{d\mathfrak{P}_p^q(z)}{dz} &= (p - q + 1) \mathfrak{P}_{p+1}^q(z) - (p + 1) z \mathfrak{P}_p^q(z) \\
 &\quad \text{(cf. 6.814 2., 6.852 1.).} \quad \text{MO 81} \\
 2. \quad (2p + 1) z \mathfrak{P}_p^q(z) &= (p - q + 1) \mathfrak{P}_{p+1}^q(z) + (p + q) \mathfrak{P}_{p-1}^q(z) \\
 &\quad \text{(cf. 6.814 1., 6.852 2.).} \quad \text{MO 81} \\
 3. \quad \mathfrak{P}_p^{q+2}(z) + 2(q + 1) \frac{z}{\sqrt{z^2 - 1}} \mathfrak{P}_p^{q+1}(z) &= (p - q)(p + q + 1) \mathfrak{P}_p^q(z). \quad \text{MO 82} \\
 4. \quad \mathfrak{P}_{p+1}^q(z) - \mathfrak{P}_{p-1}^q(z) &= (2p + 1) \sqrt{z^2 - 1} \mathfrak{P}_p^{q-1}(z). \quad \text{MO 82} \\
 5. \quad \mathfrak{P}_{-p-1}^q(z) = \mathfrak{P}_p^q(z) &\quad \text{(cf. 6.840 2., 6.852 4.).} \quad \text{MO 82}
 \end{aligned}$$

6.732

$$\begin{aligned}
 1. \quad (z^2 - 1) \frac{d\Omega_p^q(z)}{dz} &= (p - q + 1) \Omega_{p+1}^q(z) - (p + 1) z \Omega_p^q(z) \\
 &\quad \text{(cf. 6.852 3.).} \quad \text{MO 82} \\
 2. \quad (2p + 1) z \Omega_p^q(z) &= (p - q + 1) \Omega_{p+1}^q(z) + (p + q) z \Omega_{p-1}^q(z) \\
 &\quad \text{(cf. 6.852 4.).} \quad \text{MO 82} \\
 3. \quad \Omega_p^{q+2}(z) + 2(q + 1) \frac{z}{\sqrt{z^2 - 1}} \Omega_p^{q+1}(z) &= (p - q)(p + q + 1) \Omega_p^q(z). \quad \text{MO 82} \\
 4. \quad \Omega_{p-1}^q(z) - \Omega_{p+1}^q(z) &= -(2p + 1) \sqrt{z^2 - 1} \Omega_p^{q-1}(z). \quad \text{MO 82u} \\
 5. \quad e^{-q\pi i} \Omega_p^q(x \pm i0) &= e^{\pm \frac{1}{2} q\pi i} \left[Q_p^q(x) \mp i \frac{\pi}{2} P_p^q(x) \right]. \quad \text{MO 83}
 \end{aligned}$$

6.733

$$\begin{aligned}
 1. \quad (1 - x^2) \frac{dP_p^q(x)}{dx} &= (p + 1) x P_p^q(x) - (p - q + 1) P_{p+1}^q(x) \quad \text{(cf. 6.731 1.)} \\
 &= -px P_p^q(x) + (p + q) P_{p-1}^q(x); \\
 &= -\sqrt{1 - x^2} P_p^{q+1}(x) - qx P_p^q(x); \\
 &= (p - q + 1)(p + q) \sqrt{1 - x^2} P_p^{q-1}(x) + qx P_p^q(x). \quad \text{MO 82} \\
 2. \quad (2p + 1) x P_p^q(x) &= (p - q + 1) P_{p+1}^q(x) + (p + q) P_{p-1}^q(x) \\
 &\quad \text{(cf. 6.731 2.).} \quad \text{MO 82}
 \end{aligned}$$

$$3. P_p^{q+2}(x) + 2(q+1) \frac{x}{\sqrt{1-x^2}} P_p^{q+1}(x) + (p-q)(p+q+1) P_p^q(x) = 0$$

(cf. 6.731 3.). MO 82

$$4. P_{p-1}^q(z) - P_{p+1}^q(z) = (2p+1) \sqrt{1-x^2} P_p^{q-1}(z)$$

(cf. 6.731 4.). MO 82

$$5. P_{-p-1}^q(x) = P_p^q(x) \quad \text{cf. 6.731 5.).}$$

6.734

$$1. (p+q+1) z \Omega_p^q(z) + \sqrt{z^2-1} \Omega_p^{q+1}(z) = (p-q+1) \Omega_{p+1}^q(z). \quad \text{MO 82}$$

$$2. (p+q) \Omega_{p-1}^q(z) + \sqrt{z^2-1} \Omega_p^{q+1}(z) = (p-q) z \Omega_p^q(z). \quad \text{MO 82}$$

$$3. \Omega_{p-1}^q(z) - z \Omega_p^q(z) = -(p-q+1) \sqrt{z^2-1} \Omega_p^{q-1}(z). \quad \text{MO 82}$$

$$4. z \Omega_p^q(z) - \Omega_{p+1}^q(z) = -(p+q) \sqrt{z^2-1} \Omega_p^{q-1}(z). \quad \text{MO 82}$$

$$5. (p+q)(p+q+1) \Omega_{p-1}^q(z) + (2p+1) \sqrt{z^2-1} \Omega_p^{q+1}(z) =$$

$$= (p-q)(p-q+1) \Omega_{p+1}^q(z). \quad \text{MO 82}$$

6.735

$$1. (p+q+1) x P_p^q(x) + \sqrt{1-x^2} P_p^{q+1}(x) = (p-q+1) P_{p+1}^q(x). \quad \text{MO 83}$$

$$2. (p-q) x P_p^q(x) - (p+q) P_{p-1}^q(x) = \sqrt{1-x^2} P_p^{q+1}(x). \quad \text{MO 83}$$

$$3. P_{p-1}^q(x) - x P_p^q(x) = (p-q+1) \sqrt{1-x^2} P_p^{q-1}(x). \quad \text{MO 83}$$

$$4. x P_p^q(x) - P_{p+1}^q(x) = (p+q) \sqrt{1-x^2} P_p^{q-1}(x). \quad \text{MO 83}$$

$$5. (p-q)(p-q+1) P_{p+1}^q(x) = (p+q)(p+q+1) P_{p-1}^q(x) +$$

$$+ (2p+1) \sqrt{1-x^2} P_p^{q+1}(x). \quad \text{MO 83}$$

6.736

$$1. \mathfrak{P}_p^{-q}(z) = \frac{\Gamma(p-q+1)}{\Gamma(p+q+1)} \left[\mathfrak{P}_p^q(z) - \frac{2}{\pi} e^{-q\pi i} \sin q\pi \Omega_p^q(z) \right]. \quad \text{MO 83}$$

$$2. \mathfrak{P}_p^q(-z) = e^{p\pi i} \mathfrak{P}_p^q(z) - \frac{2}{\pi} \sin[(p+q)\pi] e^{-q\pi i} \Omega_p^q(z) \quad [\text{Im}(z) < 0]$$

(cf. 6.853 1.). MO 83

$$3. \mathfrak{P}_p^q(-z) = e^{-p\pi i} \mathfrak{P}_p^q(z) - \frac{2}{\pi} \sin[(p+q)\pi] e^{-q\pi i} \Omega_p^q(z) \quad [\text{Im}(z) > 0]$$

(cf. 6.853 2.). MO 83

$$4. \Omega_p^{-q}(z) = e^{-2q\pi i} \frac{\Gamma(p-q+1)}{\Gamma(p+q+1)} \Omega_p^q(z). \quad \text{MO 82}$$

$$5. \Omega_p^q(-z) = -e^{-p\pi i} \Omega_p^q(z) \quad [\text{Im}(z) < 0] \quad \text{(cf. 6.853 3.). MO 82}$$

$$6. \Omega_p^q(-z) = -e^{p\pi i} \Omega_p^q(z) \quad [\text{Im}(z) > 0] \quad \text{(cf. 6.853 4.). MO 82}$$

$$7. \Omega_p^q(z) \sin[(p+q)\pi] - \Omega_{p-1}^q(z) \sin[(p-q)\pi] = \pi e^{q\pi i} \cos q\pi \mathfrak{P}_p^q(z).$$

MO 83

6.737

$$1. P_p^{-q}(x) = \frac{\Gamma(p-q+1)}{\Gamma(p+q+1)} \left[\cos q\pi P_p^q(x) - \frac{2}{\pi} \sin q\pi Q_p^q(x) \right]. \quad \text{MO 84}$$

$$2. P_p^q(-x) = \cos(p+q)\pi P_p^q(x) - \frac{2}{\pi} \sin(p+q)\pi Q_p^q(x). \quad \text{MO 84}$$

$$3. Q_p^q(-x) = -\cos(p+q)\pi Q_p^q(x) + \frac{2}{\pi} \sin(p+q)\pi P_p^q(x). \quad \text{MO 83}$$

$$4. Q_{p-1}^q = \frac{\sin(p+q)\pi}{\sin(p-q)\pi} Q_p^q(x) - \frac{\pi \cos p\pi \cos q\pi}{\sin(p-q)\pi} P_p^q(x). \quad \text{MO 84}$$

6.738

$$1. \quad \mathfrak{D}_p^q(i \operatorname{ctg} \varphi) = \exp \left[i\pi \left(q - \frac{p+1}{2} \right) \right] \sqrt{\pi} \Gamma(p+q+1) \times \\ \times \sqrt{\frac{1}{2} \sin \varphi} P_{-q-\frac{1}{2}}^{-p-\frac{1}{2}}(\cos \varphi) \quad \left[0 < \varphi < \frac{\pi}{2} \right]. \quad \text{MO 83}$$

$$2. \quad \mathfrak{P}_q^p(i \operatorname{ctg} \varphi) = \sqrt{\frac{2}{\pi}} \exp \left[i\pi \left(p + \frac{1}{2} \right) \right] \frac{\sqrt{\sin \varphi}}{\Gamma(-p-q)} \mathfrak{D}_{-q-\frac{1}{2}}^{-p-\frac{1}{2}}(\cos \varphi - i \cdot 0) \\ \left[0 < \varphi < \frac{\pi}{2} \right]. \quad \text{MO 83}$$

$$6.739 \quad e^{-\alpha t} \mathfrak{D}_p^q(\operatorname{ch} \alpha) = \frac{\sqrt{\pi} \Gamma(p+q+1)}{\sqrt{2 \operatorname{sh} \alpha}} \mathfrak{P}_{-q-\frac{1}{2}}^{-p-\frac{1}{2}}(\operatorname{cth} \alpha) \\ [\operatorname{Re}(\operatorname{ch} \alpha) > 0]. \quad \text{MO 83}$$

6.741

$$1. \quad P_p^{-q}(x) \frac{dP_p^q(x)}{dx} - P_p^q(x) \frac{dP_p^{-q}(x)}{dx} = \frac{2 \sin q\pi}{\pi(1-x^2)}. \quad \text{MO 83}$$

$$2. \quad P_p^q(x) \frac{dQ_p^q(x)}{dx} - Q_p^q(x) \frac{dP_p^q(x)}{dx} = \frac{2^{2q}}{1-x^2} \frac{\Gamma\left(\frac{p+q+1}{2}\right) \Gamma\left(\frac{p+q}{2}+1\right)}{\Gamma\left(\frac{p-q+1}{2}\right) \Gamma\left(\frac{p-q}{2}+1\right)}. \\ \text{MO 83}$$

6.742

$$1. \quad \frac{\Gamma(p-q+1)}{\Gamma(p+q+1)} \left\{ \cos q\pi P_p^q(\cos \varphi) - \frac{2}{\pi} \sin q\pi Q_p^q(\cos \varphi) \right\} = \\ = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^q \varphi}{\Gamma\left(q+\frac{1}{2}\right)} \int_0^{\varphi} \frac{\cos\left(p+\frac{1}{2}\right)t dt}{(\cos t - \cos \varphi)^{\frac{1}{2}-q}} \quad \left[\operatorname{Re}(q) > -\frac{1}{2} \right]. \quad \text{MO 88}$$

$$2. \quad \frac{\Gamma(p-q+1)}{\Gamma(p+q+1)} \left\{ \cos p\pi P_p^q(\cos \varphi) - \frac{2}{\pi} \sin p\pi Q_p^q(\cos \varphi) \right\} = \\ = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^q \varphi}{\Gamma\left(q+\frac{1}{2}\right)} \int_{\varphi}^{\pi} \frac{\cos\left[\left(p+\frac{1}{2}\right)(t-\pi)\right] dt}{(\cos \varphi - \cos t)^{\frac{1}{2}-q}} \quad \left[\operatorname{Re}(q) > -\frac{1}{2} \right]. \quad \text{MO 88}$$

$$3. \quad P_p^q(\cos \varphi) \cos(p+q)\pi - \frac{2}{\pi} Q_p^q(\cos \varphi) \sin(p+q)\pi = \\ = \sqrt{\frac{2}{\pi}} \frac{\sin^q \varphi}{\Gamma\left(\frac{1}{2}-q\right)} \int_{\varphi}^{\pi} \frac{\cos\left[\left(p+\frac{1}{2}\right)(t-\pi)\right] dt}{(\cos \varphi - \cos t)^{q+\frac{1}{2}}} \quad \left[\operatorname{Re}(q) < \frac{1}{2} \right]. \quad \text{MO 88}$$

$$4. \quad \cos q\pi P_p^q(\cos \varphi) - \frac{2}{\pi} \sin q\pi Q_p^q(\cos \varphi) = \\ = \frac{1}{2^q \sqrt{\pi}} \frac{\Gamma(p+q+1)}{\Gamma(p-q+1)} \frac{\sin^q \varphi}{\Gamma\left(q+\frac{1}{2}\right)} \int_0^{\pi} \frac{\sin^{2q} t dt}{(\cos \varphi \pm \sin \varphi \cos t)^{q-p}} \\ \left[\operatorname{Re}(q) > -\frac{1}{2}, 0 < \varphi < \pi \right]. \quad \text{MO 88}$$

Integralele funcțiilor sferice, vezi 4.51.

6.75 Cazuri particulare și valori particulare

Cazuri particulare

6.751

$$1. P_p^m(x) = (-1)^m \frac{\Gamma(p+m+1) (1-x^2)^{\frac{m}{2}}}{2^m \Gamma(p-m+1) m!} \times \\ \times F\left(m-p, m+p+1; m+1; \frac{1-x}{2}\right). \quad \text{MO 84}$$

$$2. \mathfrak{P}_p^m(z) = \frac{\Gamma(p+m+1) (z^2-1)^{\frac{m}{2}}}{2^m m! \Gamma(p-m+1)} F\left(m-p, m+p+1; m+1; \frac{1-z}{2}\right). \\ \text{MO 84}$$

$$3. \mathfrak{Q}_{-n-\frac{3}{2}}^q(z) = \frac{e^{q\pi i} \Gamma\left(q+n+\frac{3}{2}\right)}{2^{n+\frac{3}{2}} (n+1)!} \times \\ \times (z^2-1)^{\frac{q}{2}} z^{2n-q+\frac{3}{2}} F\left(\frac{q+n+\frac{5}{2}}{2}, \frac{q+n+\frac{3}{2}}{2}; n+2; \frac{1}{z^2}\right). \quad \text{MO 84}$$

6.752

$$1. P_p^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_p(x). \quad \text{WW II 119m MO 84}$$

$$2. P_p^{-m}(x) = (-1)^m \frac{\Gamma(p-m+1)}{\Gamma(p+m+1)} P_p^m(x) = \\ = (1-x^2)^{-\frac{m}{2}} \int_x^1 \dots \int_x^1 P_p(x) (dx)^m. \quad \text{H 99m MO 85}$$

$$3. \mathfrak{P}_p^{-m}(z) = (z^2-1)^{-\frac{m}{2}} \int_1^z \dots \int_1^z P_p(z) (dz)^m. \quad \text{MO 85}$$

$$4. \mathfrak{Q}_p^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m}{dz^m} \mathfrak{Q}_p(z). \quad \text{WW II 119m MO 85}$$

$$5. \mathfrak{Q}_p^{-m}(z) = (-1)^m (z^2-1)^{-\frac{m}{2}} \int_z^\infty \dots \int_z^\infty \mathfrak{Q}_p(z) (dz)^m. \quad \text{MO 85}$$

Valori particulare (numerice) ale indicilor

6.753

$$1. P_0^q(\cos \varphi) = \frac{1}{\Gamma(1-q)} \operatorname{ctg}^q \frac{\varphi}{2}. \quad \text{MO 84}$$

$$2. P_p^{-1}(\cos \varphi) = -\frac{1}{p(p+1)} \frac{dP_p(\cos \varphi)}{d\varphi}. \quad \text{MO 84}$$

$$3. \mathfrak{P}_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0 \text{ pentru } m > n. \quad \text{MO 85}$$

6.754

$$1. \mathfrak{P}_{p-\frac{1}{2}}^{\frac{1}{2}}(\operatorname{ch} \alpha) = \sqrt{\frac{2}{\pi \operatorname{sh} \alpha}} \operatorname{ch} p\alpha. \quad \text{MO 85}$$

$$2. P_{p-\frac{1}{2}}^{\frac{1}{2}}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \cos p\varphi. \quad \text{MO 85}$$

$$3. P_{p-\frac{1}{2}}^{-\frac{1}{2}}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \frac{\sin p\varphi}{p}. \quad \text{MO 85}$$

$$4. Q_{p-\frac{1}{2}}^{\frac{1}{2}}(\operatorname{ch} \alpha) = i \sqrt{\frac{\pi}{2 \operatorname{sh} \alpha}} e^{-p\alpha}. \quad \text{MO 85}$$

6.755

$$1. P_p^{-p}(\cos \varphi) = \frac{1}{\Gamma(1+p)} \left(\frac{\sin \varphi}{2}\right)^p. \quad \text{MO 85}$$

$$2. \mathfrak{P}_p^{-p}(\operatorname{ch} \alpha) = \frac{1}{\Gamma(1+p)} \left(\frac{\operatorname{sh} \alpha}{2}\right)^p. \quad \text{MO 85}$$

Valori particulare ale funcțiilor sferice

6.756

$$1. P_p^q(0) = \frac{2^q \sqrt{\pi}}{\Gamma\left(\frac{p-q}{2}+1\right) \Gamma\left(\frac{-p-q+1}{2}\right)}. \quad \text{MO 84}$$

$$2. \frac{dP_p^q(0)}{dx} = \frac{2^{q+1} \sin \frac{1}{2}(p+q) \pi \Gamma\left(\frac{p+q}{2}+1\right)}{\sqrt{\pi} \Gamma\left(\frac{p-q+1}{2}\right)}. \quad \text{MO 84}$$

$$3. Q_p^q(0) = -2^{q-1} \sqrt{\pi} \sin \frac{1}{2}(p+q) \pi \frac{\Gamma\left(\frac{p+q+1}{2}\right)}{\Gamma\left(\frac{p-q}{2}+1\right)}. \quad \text{MO 84}$$

$$4. \frac{dQ_p^q(0)}{dx} = 2^q \sqrt{\pi} \cos \frac{1}{2}(p+q) \pi \frac{\Gamma\left(\frac{p+q}{2}+1\right)}{\Gamma\left(\frac{p-q+1}{2}\right)}. \quad \text{MO 84}$$

6.76 Derivate în raport cu indicele

$$\begin{aligned} 6.761 \quad \frac{\partial P_p^{-q}(x)}{\partial p} = & \frac{1}{\Gamma(q+1)} \left(\frac{1-x}{1+x}\right)^{\frac{q}{2}} \sum_{k=1}^{\infty} \frac{(-p)(1-p) \dots (n-1-p)(p+1)(p+2) \dots (p+n)}{(q+1)(q+2) \dots (q+n) 1 \cdot 2 \dots n} \times \\ & \times [\psi(p+n+1) - \psi(p-n+1)] \left(\frac{1-x}{2}\right)^n \\ & [p \neq 0, \pm 1, \pm 2, \dots; \operatorname{Re}(q) > -1]. \quad \text{MO 94} \end{aligned}$$

6.762

$$1. \left[\frac{\partial P_p(\cos \varphi)}{\partial p} \right]_{p=0} = 2 \ln \cos \frac{\varphi}{2}. \quad \text{MO 94}$$

$$2. \left[\frac{\partial P_p^{-1}(\cos \varphi)}{\partial p} \right]_{p=0} = -\operatorname{tg} \frac{\varphi}{2} - 2 \operatorname{ctg} \frac{\varphi}{2} \ln \cos \frac{\varphi}{2}. \quad \text{MO 94}$$

$$3. \left[\frac{\partial P_p^{-1}(\cos \varphi)}{\partial p} \right]_{p=1} = -\frac{1}{2} \operatorname{tg} \frac{\varphi}{2} \sin^2 \frac{\varphi}{2} + \sin \varphi \ln \cos \frac{\varphi}{2}. \quad \text{MO 94}$$

Legătura cu polinoamele $C_n^p(x)$, vezi 6.941—6.943.

Legătura cu funcția hipergeometrică, vezi 6.77.

6.77 Dezvoltarea în serie

Dezvoltarea în serie, vezi 6.721. Dezvoltarea funcțiilor sferice în serie se mai poate face, exprimându-le cu ajutorul funcției hipergeometrice.

6.771

$$1. \mathfrak{P}_p^q(z) = \left(\frac{z+1}{z-1}\right)^{\frac{q}{2}} \frac{1}{\Gamma(1-q)} F\left(-p, p+1; 1-q; \frac{1-z}{2}\right). \quad \text{MO 15}$$

$$2. \Omega_p^q(z) = \frac{e^{q\pi i}}{2^{p+1}} \frac{\Gamma(p+q+1)}{\Gamma\left(p+\frac{3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)(z^2-1)^{\frac{q}{2}}}{z^{p+q+1}} \times \\ \times F\left(\frac{p+q}{2}+1, \frac{p+q+1}{2}; p+\frac{3}{2}; \frac{1}{z^2}\right). \quad \text{MO 15}$$

Vezi și 6.702, 6.703, 6.704, 6.723, 6.751, 6.772.

Prelungirea analitică pentru $|z| \gg 1$.

Din teoremele asupra prelungerii analitice a seriilor hipergeometrice (vezi 7.254 și 7.255), rezultă următoarele formule:

6.772

$$1. \mathfrak{P}_p^q(z) = \frac{\sin(p+q)\pi}{2^{p+1}\sqrt{\pi}\cos p\pi} \frac{\Gamma(p+q+1)}{\Gamma\left(p+\frac{3}{2}\right)} \times \\ \times (z^2-1)^{\frac{q}{2}} z^{-p-q-1} F\left(\frac{p+q}{2}+1, \frac{p+q+1}{2}; p+\frac{3}{2}; \frac{1}{z^2}\right) + \\ + \frac{2^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(p-q+1)} (z^2-1)^{\frac{q}{2}} z^{p-q} F\left(\frac{q-p+1}{2}, \frac{q-p}{2}; \frac{1}{2}-p; \frac{1}{z^2}\right) \\ [2p \neq \pm 1, \pm 3, \pm 5, \dots; |z| > 1; |\arg(z \pm 1)| < \pi]. \quad \text{MO 85}$$

$$2. \mathfrak{P}_p^q(z) = \frac{\Gamma\left(-p-\frac{1}{2}\right)(z^2-1)^{-\frac{p+1}{2}}}{2^{p+1}\sqrt{\pi}\Gamma(-p-q)} \times \\ \times F\left(\frac{p-q+1}{2}, \frac{p+q+1}{2}; p+\frac{3}{2}; \frac{1}{1-z^2}\right) + \\ + \frac{2^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(p-q+1)} (z^2-1)^{\frac{p}{2}} F\left(\frac{q-p}{2}, -\frac{q+p}{2}; \frac{1}{2}-p; \frac{1}{1-z^2}\right) \\ [2p \neq \pm 1, \pm 3, \pm 5, \dots; |1-z^2| > 1; |\arg(z \pm 1)| < \pi]. \quad \text{MO 85}$$

$$3. \mathfrak{P}_p^q(z) = \frac{1}{\Gamma(1-q)} \left(\frac{z-1}{z+1}\right)^{-\frac{q}{2}} \left(\frac{z+1}{2}\right)^{-p} \times \\ \times F\left(-p, -p-q; 1-q; \frac{z-1}{z+1}\right) \quad \left[\left|\frac{z-1}{z+1}\right| < 1\right]. \quad \text{MO 86}$$

6.773

$$1. \Omega_p^q(z) = e^{q\pi i} \frac{\sqrt{\pi} \Gamma(p+q+1)}{2^{p+1} \Gamma\left(p+\frac{3}{2}\right)} (z^2-1)^{-\frac{p+1}{2}} \times \\ \times F\left(\frac{p+q+1}{2}, \frac{p-q+1}{2}; q+\frac{3}{2}; \frac{1}{1-z^2}\right) \\ [p+q \neq -1, -2, -3, \dots; |\arg(z \pm 1)| < \pi; |1-z^2| > 1]. \quad \text{MO 86}$$

$$2. \quad \mathfrak{D}_p^q(z) = \frac{1}{2} e^{q\pi i} \left\{ \Gamma(q) \left(\frac{z+1}{z-1} \right)^{\frac{q}{2}} F\left(-p, p+1; 1-q; \frac{1-z}{2}\right) + \right. \\ \left. + \frac{\Gamma(-q) \Gamma(p+q+1)}{\Gamma(p-q+1)} \left(\frac{z-1}{z+1} \right)^{\frac{q}{2}} F\left(-p, p+1; 1+q; \frac{1-z}{2}\right) \right\} \\ [|\arg(z \pm 1)| < \pi, |1-z| < 2]. \quad \text{MO 86}$$

$$6.774 \quad \mathfrak{P}_p^q(i \operatorname{ctg} \varphi) = \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(-p-\frac{1}{2})}{\Gamma(-p-q)} \times \\ \times e^{-i(p+1)\frac{\pi}{2}} \left(\operatorname{tg} \frac{\varphi}{2} \right)^{p+\frac{1}{2}} F\left(\frac{1}{2}+q, \frac{1}{2}-q; p+\frac{3}{2}; \sin^2 \frac{\varphi}{2}\right) + \\ + \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p-q+1)} e^{ip\frac{\pi}{2}} \left(\operatorname{ctg} \frac{\varphi}{2} \right)^{p+\frac{1}{2}} F\left(\frac{1}{2}+q, \frac{1}{2}-q; \frac{1}{2}-p; \sin^2 \frac{\varphi}{2}\right) \\ \left[2p \neq \pm 1, \pm 3, \pm 5, \dots; 0 < \varphi < \frac{\pi}{2} \right]. \quad \text{MO 86}$$

6.775

$$1. \quad P_p^q(x) = \frac{2^q \cos \frac{1}{2}(p+q)\pi \Gamma\left(\frac{p+q+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{p-q}{2}+1\right)} \times \\ \times (1-x^2)^{\frac{q}{2}} F\left(\frac{p+q+1}{2}, \frac{q-p}{2}; \frac{1}{2}; x^2\right) + \\ + \frac{2^{q+1} \sin \frac{1}{2}(p+q)\pi \Gamma\left(\frac{p+q+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{p-q+1}{2}\right)} x (1-x^2)^{\frac{q}{2}} F\left(\frac{p+q}{2}+1, \frac{p-q+1}{2}; \frac{3}{2}; x^2\right). \quad \text{MO 87}$$

$$2. \quad Q_p^q(x) = -\frac{\sqrt{\pi} \sin \frac{1}{2}(p+q)\pi \Gamma\left(\frac{p+q+1}{2}\right)}{2^{1-q} \Gamma\left(\frac{p-q}{2}+1\right)} \times \\ \times (1-x^2)^{\frac{q}{2}} F\left(\frac{p+q+1}{2}, \frac{q-p}{2}; \frac{1}{2}; x^2\right) + \\ + 2^q \sqrt{\pi} \frac{\cos \frac{1}{2}(p+q)\pi \Gamma\left(\frac{p+q+1}{2}\right)}{\Gamma\left(\frac{p-q+1}{2}\right)} x (1-x^2)^{\frac{q}{2}} \times \\ \times F\left(\frac{p+q}{2}+1, \frac{q-p+1}{2}; \frac{3}{2}; x^2\right). \quad \text{MO 87}$$

6.776 Pentru $|z| \gg 1$,

$$1. \quad \mathfrak{P}_p^q(z) = \left\{ \frac{2^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(p-q+1)} z^p + \frac{\Gamma\left(-p-\frac{1}{2}\right)}{2^{p+1} \sqrt{\pi} \Gamma(-p-q)} z^{-p-1} \right\} \left(1 + O\left(\frac{1}{z^2}\right) \right) \\ [2p \neq \pm 1, \pm 3, \pm 5, \dots; |\arg z| < \pi]. \quad \text{MO 87}$$

$$2. \quad \mathfrak{D}_p^q(z) = \sqrt{\pi} \frac{e^{q\pi i}}{2^{p+1}} \frac{\Gamma(p+q+1)}{\Gamma\left(p+\frac{3}{2}\right)} z^{-p-1} \left(1 + O\left(\frac{1}{z^2}\right) \right) \\ [2p \neq -3, -5, -7, \dots; |\arg z| < \pi]. \quad \text{MO 87}$$

6.777 Fie $\zeta = z + \sqrt{z^2 - 1}$. Prin această egalitate, variabila ζ este definită univoc în tot planul z , în care s-a făcut o secțiune de la $-\infty$ până la $+1$; aici se ia în considerare acea ramură a variabilei ζ pentru care, la valorile reale ale lui z mai mari decât 1, corespund valori ale lui ζ , deasemenea mai mari decât 1. În acest caz,

$$1. \quad \mathfrak{P}_p^q(z) = \frac{2^q \Gamma\left(-p - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(-p - q)} \frac{(z^2 - 1)^{\frac{q}{2}}}{\zeta^{p+q+1}} \times \\ \times F\left(\frac{1}{2} + q, p + q + 1; p + \frac{3}{2}; \frac{1}{\zeta^2}\right) + \\ + \frac{2^q}{\sqrt{\pi}} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p - q + 1)} \frac{(z^2 - 1)^{\frac{q}{2}}}{\zeta^{q-p}} F\left(\frac{1}{2} + q, q - p; \frac{1}{2} - p; \frac{1}{\zeta^2}\right) \\ [2p \neq \pm 1, \pm 3, \pm 5, \dots; |\arg(z - 1)| < \pi]. \quad \text{MO 86}$$

$$2. \quad \mathfrak{D}_p^q(z) = 2^q e^{q\pi i} \sqrt{\pi} \frac{\Gamma(p + q + 1)}{\Gamma\left(p + \frac{3}{2}\right)} \frac{(z^2 - 1)^{\frac{q}{2}}}{\zeta^{p+q+1}} \times \\ \times F\left(\frac{1}{2} + q, p + q + 1; p + \frac{3}{2}; \frac{1}{\zeta^2}\right) \quad [|\arg(z - 1)| < \pi]. \quad \text{MO 86}$$

6.78 Zerourile funcțiilor sferice

6.781 $P_p^{-q}(\cos \varphi)$, considerată ca funcție de p , are pentru $q \geq 0$ o infinitate de zerouri; ele sînt toate simple și reale. Dacă numărul p_0 este un zero al funcției $P_p^{-q}(\cos \varphi)$, atunci numărul $-p_0 - 1$ este, de asemenea, un zero al acestei funcții. MO 91

6.782 Dacă p și q sînt amîndoi reali și $q \leq 0$, sau dacă p și q sînt numere întregi, atunci funcția $\mathfrak{P}_p^q(t)$ nu are zerouri reale mai mari decât 1. Dacă p și q sînt amîndoi reali, dar $q < 0$, atunci funcția $\mathfrak{P}_p^q(t)$ nu are, pentru $q > p$, zerouri reale mai mari decât 1, în cazul cînd $\sin q \pi \sin(q - p) \pi > 0$, și are un singur zero de acest fel, în cazul cînd $\sin q \pi \sin(q - p) \pi < 0$; în sfîrșit, dacă $q \leq p$, atunci funcția $\mathfrak{P}_p^q(t)$ nu are zerouri reale mai mari decât 1 pentru $E(q)$ par, iar pentru $E(q)$ impar, ea are un singur zero.

6.783 Dacă $p > -\frac{3}{2}$ și $p + q + 1 > 0$, atunci funcția $\mathfrak{D}_p^q(t)$ nu are zerouri reale mai mari decât 1. MO 91

6.784 Funcția $\mathfrak{P}_{-\frac{1}{2} + i\lambda}$ are, pentru λ real, o infinitate de zerouri; toate zerourile ei sînt reale și mai mari decât unitatea.

6.785 Funcția $P_n(x)$ (n natural) are exact n zerouri reale, cuprinse în intervalul $-1, +1$.

6.786 Funcția $\mathfrak{D}_n(z)$, în care n este un număr natural, nu are zerouri pentru care $|\arg(z - 1)| < \pi$; funcția $Q_n(\cos \varphi)$ are exact $n + 1$ zerouri, cuprinse în intervalul $0 \leq \varphi \leq \pi$. MO 91

6.787 Pentru calcularea valorilor lui p , pentru care, la valori mici date ale lui φ , are loc egalitatea $P_p^{-q}(\cos \varphi) = 0$, servește următoarea formulă de aproximație:

$$p + \frac{1}{2} = \frac{j_q}{2 \sin \frac{\varphi}{2}} \left\{ 1 - \frac{\sin^2 \frac{\varphi}{2}}{6} \left(1 - \frac{4q-1}{j_q^2} \right) + O\left(\sin^4 \frac{\varphi}{2}\right) \right\}; \quad \text{MO 93}$$

aici j_q este orice rădăcină neidentică nulă a ecuației $J_q(z) = 0$ ($q \geq 0$). Dacă φ este apropiat de π , atunci, în locul formulei de mai sus, se pot folosi formulele

$$1. \quad p \approx q + k + \frac{\Gamma(2q+k+1)}{\Gamma(q)\Gamma(q+1)\Gamma(k+1)} \left(\frac{\pi-\varphi}{3} \right)^{2q} \quad [q > 0; k = 0, 1, 2, \dots]. \quad \text{MO 93}$$

$$2. \quad p \approx k + \frac{1}{2 \ln \left(\frac{2}{\pi-\varphi} \right)} \quad [q = 0, k = 0, 1, 2, \dots]. \quad \text{MO 93}$$

6.79 Serii de funcții sferice

6.791

$$1. \quad \frac{1}{z-t} = \sum_{k=0}^{\infty} (2k+1) \mathfrak{P}_k(t) \mathfrak{D}_k(t) \quad [|t + \sqrt{t^2-1}| < |z + \sqrt{z^2-1}|];$$

t trebuie să fie în interiorul elipsei care trece prin punctul z și are focarele în punctele ± 1 . MO 78

$$2. \quad \frac{1}{\sqrt{1-2tz+t^2}} \ln \frac{z-t+\sqrt{1-2tz+t^2}}{\sqrt{z^2-1}} = \sum_{k=0}^{\infty} t^k \mathfrak{D}_k(z) \quad [\operatorname{Re} z > 1, |t| < 1]. \quad \text{MO 78}$$

$$\begin{aligned} 6.792 \quad P_p^{-r}(\cos \varphi) P_p^{-s}(\cos \psi) &= \frac{\sin p\pi}{\pi} \sum_{k=1}^{\infty} (-1)^k \left[\frac{1}{p-k} - \frac{1}{p+k+1} \right] \times \\ &\times P_k^{-r}(\cos \varphi) P_k^{-s}(\cos \psi) \\ &[r \geq 0, s \geq 0, p \text{ real}, -\pi < \varphi \pm \psi < \pi]. \quad \text{MO 94} \end{aligned}$$

$$\begin{aligned} 6.793 \quad P_p^{-q}(\cos \varphi) &= \frac{\sin p\pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{p-k} - \frac{1}{p+k+1} \right) P_k^{-q}(\cos \varphi) \\ &[q \geq 0, 0 < \varphi < \pi]. \quad \text{MO 94} \end{aligned}$$

Teoreme de compunere

6.794

$$\begin{aligned} 1. \quad P_p(\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos \varphi) &= \\ &= P_p(\cos \phi_1) P_p(\cos \phi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_p^{-k}(\cos \phi_1) P_p^k(\cos \phi_2) \cos k\varphi; \\ &= P_p(\cos \phi_1) P_p(\cos \phi_2) + 2 \sum_{k=1}^{\infty} \frac{\Gamma(p-k+1)}{\Gamma(p+k+1)} P_p^k(\cos \phi_1) P_p^k(\cos \phi_2) \cos k\varphi \\ &[0 \leq \phi_1 < \pi, 0 \leq \phi_2 < \pi, \phi_1 + \phi_2 < \pi; \varphi \text{ real}] \\ &(\text{cf. 6.834, 6.864 1.}). \quad \text{MO 90} \end{aligned}$$

$$\begin{aligned}
2. \quad Q_p(\cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi) &= \\
&= P_p(\cos \psi_1) Q_p(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_p^{-k}(\cos \psi_1) Q_p^k(\cos \psi_2) \cos k\varphi \\
&\left[0 < \psi_1 < \frac{\pi}{2}, 0 < \psi_2 < \pi, 0 < \psi_1 + \psi_2 < \pi; \varphi \text{ real} \right] \\
&\text{(cf. 6.864 3.)}
\end{aligned}$$

MO 90

6.795

$$\begin{aligned}
1. \quad \mathfrak{P}_p(z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cos \varphi) &= \\
&= \mathfrak{P}_p(z_1) \mathfrak{P}_p(z_2) + 2 \sum_{k=1}^{\infty} (-1)^k \mathfrak{P}_p^k(z_1) \mathfrak{P}_p^{-k}(z_2) \cos k\varphi \\
&[\operatorname{Re}(z_1) > 0, \operatorname{Re}(z_2) > 0, |\arg(z_1 - 1)| < \pi, |\arg(z_2 - 1)| < \pi].
\end{aligned}$$

MO 91

$$\begin{aligned}
2. \quad \mathfrak{D}_p(x_1 x_2 - \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} \cos \varphi) &= \\
&= \mathfrak{P}_p(x_1) \mathfrak{D}_p(x_2) + 2 \sum_{k=1}^{\infty} (-1)^k \mathfrak{P}_p^{-k}(x_1) \mathfrak{D}_p^k(x_2) \cos k\varphi \\
&[1 < x_1 < x_2, p \neq -1, -2, -3, \dots; \varphi \text{ real}]
\end{aligned}$$

MO 91

$$\begin{aligned}
3. \quad \mathfrak{D}_n(x_1 x_2 + \sqrt{x_1^2 + 1} \sqrt{x_2^2 + 1} \operatorname{ch} \alpha) &= \\
&= \sum_{k=n+1}^{\infty} \frac{1}{(k-n-1)!(k+n)!} \mathfrak{D}_n^k(ix_1) \mathfrak{D}_n^k(ix_2) e^{i k \alpha} \\
&[x_1 > 0, x_2 > 0; \alpha > 0].
\end{aligned}$$

MO 91

$$\begin{aligned}
6.796 \quad P_p(-\cos \psi_1 \cos \psi_2 - \sin \psi_1 \sin \psi_2 \cos \varphi) &= \\
&= P_p(-\cos \psi_1) P_p(\cos \psi_2) + \\
&+ 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(p+k+1)}{\Gamma(p-k+1)} P_p^{-k}(-\cos \psi_1) P_p^k(\cos \psi_2) \cos k\varphi \\
&[0 < \psi_2 < \psi_1 < \pi; \varphi \text{ real}]. \quad \text{(cf. 6.864 2.)} \\
&\text{Vezi și 6.923 3.}
\end{aligned}$$

MO 91

6.81 Polinoame Legendre

6.810 Definiție. *Polinoamele Legendre* $P_n(z)$ sînt polinoame care satisfac ecuația 6.700 1., în care $q = 0$, adică ecuația

$$1. \quad (1-z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + n(n+1)u = 0.$$

Această ecuație are o soluție în formă de polinom, numai în cazul cînd n este întreg. Așadar, polinoamele Legendre sînt un tip particular de funcții sferice.

Polinoamele Legendre de gradul n au forma

$$2. \quad P_n(z) = \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n} (z^2 - 1)^n.$$

6.811 Forma dezvoltată a polinoamelor Legendre

$$\begin{aligned}
1. \quad P_n(z) &= \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k! (n-k)! (n-2k)!} z^{n-2k} = \\
&= \frac{(2n)!}{2^n (n!)^2} \left(z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} z^{n-4} - \dots \right); \\
&= \frac{(2n-1)!!}{n!} z^n P\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right). \quad \text{H 13 A (9001) MO 69}
\end{aligned}$$

2.
$$P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n \cdot n!} \left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots \right);$$

$$= (-1)^n \frac{(2n-1)!!}{2^n \cdot n!} F\left(-n, n + \frac{1}{2}; \frac{1}{2}; z^2\right). \quad A(9002) \text{ MO } 69$$
3.
$$P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n \cdot n!} \left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots \right);$$

$$= (-1)^n \frac{(2n+1)!!}{2^n \cdot n!} z F\left(-n, n + \frac{3}{2}; \frac{3}{2}; z^2\right). \quad A(9002) \text{ MO } 69$$
4.
$$P_n(\cos \varphi) = \frac{(2n-1)!!}{2^{n-1} \cdot n!} \left(\cos n\varphi + \frac{1}{1} \frac{n}{2n-1} \cos(n-2)\varphi + \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\varphi + \right.$$

$$\left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\varphi - \dots \right). \quad WW \text{ II } 92$$
5.
$$P_{2n}(\cos \varphi) = (-1)^n \frac{(2n-1)!!}{2^n \cdot n!} \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{2!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n \cdot n!}{(2n-1)!!} \cos^{2n} \varphi \right\}. \quad A(9011)$$
6.
$$P_{2n+1}(\cos \varphi) = (-1)^n \frac{(2n+1)!!}{2^n \cdot n!} \cos \varphi \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{3!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n \cdot n!}{(2n+1)!!} \cos^{2n} \varphi \right\}. \quad A(9012)$$
7.
$$P_n(z) = \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(n-k)! (k!)^2 2^{k+1}} [(1-z)^k + (-1)^n (1+z)^k]. \quad WW \text{ II } 128$$

6.812 Cazuri particulare

1. $P_0(x) = 1.$
2. $P_1(x) = x = \cos \varphi.$
3. $P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3 \cos 2\varphi + 1).$
4. $P_3(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{8}(5 \cos 3\varphi + 3 \cos \varphi).$
5. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = \frac{1}{64}(35 \cos 4\varphi + 20 \cos 2\varphi + 9).$
6. $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15) = \frac{1}{128}(63 \cos 5\varphi + 35 \cos 3\varphi + 30 \cos \varphi).$

6.813 Reprezentarea integrală

$$1. P_n(\cos \varphi) = \frac{2}{\pi} \int_{\varphi}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{\sqrt{2(\cos \varphi - \cos t)}} dt.$$

WW II 108

Relații funcționale

6.814 Formule de recurență

1. $(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0$
Sm 490 (37) WW II 98
2. $(z^2 - 1) \frac{dP_n}{dz} = n(zP_n(z) - P_{n-1}(z)) = \frac{n(n+1)}{2n+1} (P_{n+1}(z) - P_{n-1}(z)).$

WW II 99

6.815

$$1. \sum_{k=0}^n (2k+1) P_k(x) P_k(y) = (n+1) \frac{P_n(x) P_{n+1}(y) - P_n(y) P_{n+1}(x)}{y-x}. \quad \text{MO 70}$$

$$2. \sum_{k=0}^n (2n-4k-1) P_{n-2k-1}(z) = P'_n(z) \quad (\text{teorema de compunere})$$

[Insumarea se face pînă la primul termen de indice negativ].

$$3. \sum_{k=0}^n (2n-4k-3) P_{n-2k-2}(z) = z P'_n(z) - n P_n(z) \quad \text{MO 72}$$

[Insumarea se face pînă la primul termen de indice negativ].

Sm III 491 (42) WW II 28

$$E\left(\frac{n}{2}\right)$$

$$4. \sum_{k=1}^n (2n-4k+1) [k(2n-2k+1)-2] P_{n-2k}(z) = \\ = z^2 P''_n(z) - n(n-1) P_n(z). \quad \text{WW II 129}$$

$$5. \sum_{k=0}^m \frac{a_{m-k} a_k a_{n-k}}{a_{n+m-k}} \left(\frac{2n+2m-4k+1}{2n+2m-2k+1} \right) P_{n+m-2k}(z) = P_n(z) P_m(z) \\ \left[a_k = \frac{(2k-1)!!}{k!}, \quad m \leq n \right]. \quad \text{A (9036)}$$

6.816

$$1. P_n(\cos \varphi) = \frac{(2n-1)!!}{2^n n!} e^{\mp i n \varphi} F\left(\frac{1}{2}, -n; \frac{1}{2} - n; e^{\pm 2i\varphi}\right). \quad \text{MO 69}$$

$$2. P_n(\cos \varphi) = F\left(n+1, -n; 1; \sin^2 \frac{\varphi}{2}\right). \quad \text{MO 69}$$

$$3. P_n(\cos \varphi) = (-1)^n F\left(n+1, -n; 1; \cos^2 \frac{\varphi}{2}\right). \quad \text{WW II 103}$$

$$4. P_n(\cos \varphi) = \cos^n \varphi F\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1; -\tan^2 \varphi\right). \quad \text{H 23}$$

$$5. P_n(\cos \varphi) = \cos^{2n} \frac{\varphi}{2} F\left(-n, -n; 1; -\tan^2 \frac{\varphi}{2}\right) \quad \text{H 23 H 29 WW II 109m}$$

Vezi și 6.811 1., 6.811 2., 6.811 3.

Legătura cu celelalte funcții, vezi 6.943, 6.856, 7.131 2.

Integralele polinoamelor Legendre, vezi 4.52.

În legătură cu rădăcinile polinoamelor Legendre, vezi 6.785.

6.817 Inegalități

$$1. \text{Pentru } x > 1, \quad P_0(x) < P_1(x) < P_2(x) < \dots < P_n(x) < \dots \quad \text{MO 71}$$

$$2. \text{Pentru } x > -1, \quad P_0(x) + P_1(x) + \dots + P_n(x) > 0. \quad \text{MO 71}$$

$$3. [P_n(\cos \varphi)]^2 > \frac{\sin(2n+1)\varphi}{(2n+1)\sin \varphi} \quad [0 < \varphi < \pi]. \quad \text{MO 71}$$

$$4. \sqrt{n \sin \varphi} |P_n(\cos \varphi)| \leq 1. \quad \text{MO 71}$$

$$5. |P_n(\cos \varphi)| \leq 1. \quad \text{WW II 92}$$

6.82 Serii de polinoame Legendre

6.821 Funcția generatoare

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k P_k(z) \quad [|t| < \min |z \pm \sqrt{z^2-1}|]; \text{ Sm III 489 (31) WW II 91}$$

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) \quad [|t| > \max |z \pm \sqrt{z^2-1}|]. \quad \text{MO 70}$$

6.822

$$1. \quad z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z). \quad \text{MO 72}$$

$$2. \quad z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z). \quad \text{MO 72}$$

$$3. \quad \frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k \cdot k!} \right\}^2 P_{2k}(x) \quad [|x| < 1, (-1)!! \equiv 1]. \quad \text{MO 72 La 385 (15)}$$

$$4. \quad \frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!! (2k+1)!!}{2^{2k+1} \cdot k! (k+1)!} P_{2k+1}(x) \quad [|x| < 1, (-1)!! \equiv 1]. \quad \text{La 385 (17)}$$

$$5. \quad \sqrt{1-x^2} = \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!! (2k-1)!!}{2^{2k+1} \cdot k! (k+1)!} P_{2k}(x) \right\} \quad [|x| < 1, (-1)!! \equiv 1] \quad \text{La 385 (18)}$$

$$6.823 \quad \arcsin x = \frac{\pi}{2} \sum_{k=0}^{\infty} \left\{ \frac{(2k-1)!!}{2^k \cdot k!} \right\}^2 (P_{2k+1}(x) - P_{2k-1}(x)) \quad [|x| < 1, (-1)!! \equiv 1]. \quad \text{WW II 132}$$

6.824

$$1. \quad -\frac{1+\cos n\pi}{2(n^2-1)} P_0(\cos \theta) -$$

$$-\frac{1+\cos n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5) n^2 (n^2-2^2) \dots [n^2-(2k)^2]}{(n^2-1^2)(n^2-3^2) \dots [n^2-(2k+3)^2]} P_{2k+2}(\cos \theta) -$$

$$-\frac{3(1-\cos n\pi)}{2(n^2-2^2)} P_1(\cos \theta) -$$

$$-\frac{1-\cos n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3) (n^2-1^2) \dots [n^2-(2k-1)^2]}{(n^2-2^2)(n^2-4^2) \dots [n^2-(2k+2)^2]} P_{2k+1}(\cos \theta) = \cos n\theta.$$

$$\begin{aligned}
2. \quad & \frac{-\sin n\pi}{2(n^2-1)} P_0(\cos \theta) - \\
& - \frac{\sin n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5)n^2(n^2-2^2)\dots[n^2-(2k)^2]}{(n^2-1^2)(n^2-3^2)\dots[n^2-(2k+3)^2]} P_{2k+2}(\cos \theta) + \\
& + \frac{3\sin n\pi}{2(n^2-2^2)} P_1(\cos \theta) + \\
& + \frac{\sin n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3)(n^2-1^2)(n^2-3^2)\dots[n^2-(2k-1)^2]}{(n^2-2^2)(n^2-4^2)\dots[n^2-(2k+2)^2]} P_{2k+1}(\cos \theta) = \\
& = \sin n\theta. \quad A(9060.2)
\end{aligned}$$

$$\begin{aligned}
3. \quad & \frac{2^{n-1} \cdot n!}{(2n-1)!!} P_n(\cos \theta) + \\
& + n \sum_{k=1}^{\infty} (2n-4k+1) \frac{2^{n-2k-1} (n-k-1)! (2k-3)!!}{(2n-2k+1)!! k!} P_{n-2k}(\cos \theta) = \\
& = \cos n\theta. \quad A(9061.1)
\end{aligned}$$

$$\begin{aligned}
4. \quad & \frac{(2n-1)!! P_{n-1}(\cos \theta)}{2^{n-1} \cdot (n-1)!} - \\
& - \frac{n}{2^{n+2k+1}} \sum_{k=0}^{\infty} \frac{(2n+2k-1)!! (2k-1)!! (2n+4k+3)}{(n+k+1)! (k+1)!} P_{n+2k+1}(\cos \theta) = \\
& = \frac{4 \sin n\theta}{\pi}. \quad A(9061.2)
\end{aligned}$$

6.825

$$\begin{aligned}
1. \quad & \sum_{k=1}^{\infty} \frac{4k-1}{2^{2k} (2k-1)^2} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = 1 - \frac{2\theta}{\pi}. \\
2. \quad & \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k+1} (2k-1) (k+1)} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{1}{2} - \frac{2 \sin \theta}{\pi}. \quad A(9062.2) \\
3. \quad & \sum_{k=1}^{\infty} \frac{k(4k-1)}{2^{2k-1} (2k-1)} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = \frac{2 \operatorname{ctg} \theta}{\pi}. \quad A(9062.3) \\
4. \quad & \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{2}{\pi \sin \theta} - 1. \quad A(9062.4)
\end{aligned}$$

6.826

$$\begin{aligned}
1. \quad & \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) = \ln \frac{2 \operatorname{tg} \frac{\pi-\theta}{4}}{\sin \theta} = -\ln \sin \frac{\theta}{2} - \ln \left(1 + \sin \frac{\theta}{2} \right). \quad A(9063.2) \\
2. \quad & \sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos \theta) = \ln \frac{1 + \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} - 1. \quad A(9063.1)
\end{aligned}$$

$$\begin{aligned}
6.827 \quad & \sum_{k=0}^{\infty} \cos \left(k + \frac{1}{2} \right) \beta P_k(\cos \varphi) = \frac{1}{\sqrt{2} (\sec \beta - \cos \varphi)} \quad [0 \leq \beta < \varphi < \pi]; \\
& = 0 \quad [0 < \varphi < \beta < \pi]. \quad MO 72
\end{aligned}$$

6.828

$$1. \sum_{n=1}^{\infty} \frac{(-1)^n (4n+1) [(2n-1)!!]^2}{2^{3n} (n!)^3} P_{2n}(k') = \frac{4K}{\pi^2} - 1. \quad A(9064.1)$$

$$2. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1) [(2n+1)!!]^2}{(2n-1)(2n+2) 2^{3n} (n!)^3} P_{2n}(k') = \frac{4E}{\pi^2} - \frac{1}{2}. \quad A(9064.2)$$

$$3. \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{p-k} - \frac{1}{p+k+1} \right) P_k(\cos \varphi) = \frac{\pi}{\sin p\pi} P_p(\cos \varphi) \\ [p \text{ neîntreg} ; 0 \leq \varphi < \pi]. \quad MO 77$$

$$4. \sum_{k=0}^n (-1)^k \left(\frac{1}{p-k} - \frac{1}{p+k+1} \right) P_k(\cos \varphi) P_k(\cos \psi) = \\ = \frac{\pi}{\sin p\pi} P_p(\cos \varphi) P_p(\cos \psi) \\ [p \text{ neîntreg}, -\pi < \varphi + \psi < \pi, -\pi < \varphi - \psi < \pi]. \quad MO 77$$

Vezi și 6.531 4.

Seriile de produse dintre funcțiile Bessel și polinoamele Legendre, vezi 6.521 4., 6.541 4., 6.543 1., 6.543 2., 6.544.

6.83 Funcții sferice cu indici întregi (funcții Legendre adjuncte)

6.830 Pentru valori numerice întregi ale lui p și q , ecuația diferențială 6.700 1. (cu condiția $|p| > |q|$) are, în domeniul real, o soluție deosebit de simplă, și anume

$$u = P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

Funcțiile $P_n^m(x)$ se numesc *funcții Legendre adjuncte* sau *funcții sferice de speța întâi*. Numărul n se numește *gradul* funcției $P_n^m(x)$, iar numărul m este *ordinul* ei. Funcțiile

$$\cos^m \vartheta P_n^m(\cos \varphi), \sin^m \vartheta P_n^m(\cos \varphi),$$

care depind de unghiurile φ și ϑ , sînt, de asemenea, *funcții sferice de speța întâi*, și anume *tesseractale* (adică celulare) pentru $m < n$ și *spațiale* pentru $m = n$. Aceste funcții sînt periodice în raport cu unghiurile φ și ϑ , perioadele lor fiind respectiv egale cu π și 2π . Ele sînt uniforme și continue pe toată suprafața sferei de rază 1 $x_1^2 + x_2^2 + x_3^2 = 1$ ($x_1 = \sin \varphi \cos \vartheta$, $x_2 = \sin \varphi \sin \vartheta$, $x_3 = \cos \varphi$) și sînt soluția ecuației diferențiale

$$\frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial Y}{\partial \varphi} \right) + \frac{1}{\sin^2 \varphi} \frac{\partial^2 Y}{\partial \vartheta^2} + n(n+1)Y = 0.$$

6.831 Reprezentarea integrală

$$P_n^m(\cos \varphi) = \frac{(-1)^m (n+m)!}{\Gamma\left(m + \frac{1}{2}\right) \cdot (n-m)!} \sqrt{\frac{2}{\pi}} \sin^{-m} \varphi \times \\ \times \int_0^{\varphi} (\cos t - \cos \varphi)^{m-\frac{1}{2}} \cos\left(n + \frac{1}{2}\right) t dt. \quad MO 75$$

6.832 Dezvoltarea în serie

$$P_n^m(x) = \frac{(-1)^m (n+m)!}{2^m \cdot m! (n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ 1 - \frac{(n-m)(m+n+1)}{1! (m+1)} \frac{1-x}{2} + \right. \\ \left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2! (m+1)(m+2)} \left(\frac{1-x}{2}\right)^2 - \dots \right\}; \quad \text{MO 73}$$

$$= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} x^{n-m-2} + \right. \\ \left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-m-4} - \dots \right\}; \quad \text{MO 73}$$

$$= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} x^{n-m} F\left(\frac{m-n}{2}, \frac{n-m+1}{2}; \frac{1}{2} - n; \frac{1}{x^2}\right). \\ \text{MO 73}$$

6.833 Cazuri particulare

$$1. P_1^1(x) = -(1-x^2)^{\frac{1}{2}} = -\sin \varphi. \quad \text{MO 73}$$

$$2. P_2^1(x) = -3(1-x^2)^{\frac{1}{2}} x = -\frac{3}{2} \sin 2\varphi. \quad \text{MO 73}$$

$$3. P_2^2(x) = 3(1-x^2) = \frac{3}{2}(1 - \cos 2\varphi). \quad \text{MO 73}$$

$$4. P_3^1(x) = -\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1) = -\frac{3}{8}(\sin \varphi + 5 \sin 3\varphi). \quad \text{MO 73}$$

$$5. P_3^2(x) = 15(1-x^2)x = \frac{15}{4}(\cos \varphi - \cos 3\varphi). \quad \text{MO 73}$$

$$6. P_3^3(x) = -15(1-x^2)^{\frac{3}{2}} = -\frac{15}{4}(3 \sin \varphi - \sin 3\varphi). \quad \text{MO 73}$$

Relații funcționale

Formule de recurență, vezi 6.731.

$$6.834 P_n(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 \cos [\vartheta_1 - \vartheta_2]) =$$

$$= P_n \cos \varphi_1 P_n \cos \varphi_2 + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \varphi_1) P_n^m(\cos \varphi_2) \cos (\vartheta_1 - \vartheta_2)$$

(„teorema de compunere“). MO 74

6.835 Dacă

$$Y_{n_1}(\varphi, \vartheta) = a_0 P_{n_1}(\cos \varphi) + \sum_{m=1}^{n_1} (a_m \cos m\vartheta + b_m \sin m\vartheta) P_{n_1}^m(\cos \varphi),$$

$$Z_{n_2}(\varphi, \vartheta) = a_0 P_{n_2}(\cos \varphi) + \sum_{m=1}^{n_2} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) P_{n_2}^m(\cos \varphi),$$

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_{n_1}(\varphi, \vartheta) Z_{n_2}(\varphi, \vartheta) = 0,$$

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_n(\varphi, \vartheta) P_n[\cos \varphi \cos \psi + \sin \varphi \sin \psi \cos (\vartheta - \theta)] =$$

$$= \frac{4\pi}{2n+1} Y_n(\psi, \theta). \quad \text{MO 75}$$

$$6.836 \quad (\cos \varphi + i \sin \varphi \cos \vartheta)^n = P_n(\cos \varphi) + 2 \sum_{m=1}^n (-1)^m \frac{n!}{(n+m)!} \cos m\vartheta P_n^m(\cos \varphi).$$

MO 75

Integralele funcției $P_n^m(x)$, vezi 4.513, 4.514, 4.515, 4.517. 7.

6.84-6.85 Funcții Legendre

6.840 Ecuația diferențială

$$\frac{d}{dz} \left[(1-z^2) \frac{du}{dz} \right] + p(p+1)u = 0 \quad (\text{cf. 6.700 1.}),$$

în care parametrul p este un număr oarecare, are următoarele soluții liniar independente:

$$\begin{aligned} 1. \quad \mathfrak{P}_p(z) &= F\left(-p, p+1; 1; \frac{1-z}{2}\right). \\ 2. \quad \mathfrak{Q}_p(z) &= \frac{\Gamma(p+1) \Gamma\left(\frac{1}{2}\right)}{2^{p+1} \Gamma\left(p+\frac{3}{2}\right)} z^{-p-1} F\left(\frac{p+2}{2}, \frac{p+1}{2}; \frac{p+3}{2}; \frac{1}{z^2}\right). \end{aligned}$$

Sm III 518 (137)

Funcțiile $\mathfrak{P}_p(z)$ și $\mathfrak{Q}_p(z)$ se numesc *funcții Legendre*, respectiv de *speța întâi și a doua*. Dacă p nu este un număr întreg, atunci în punctele $z = -1$ și $z = \infty$, funcția $\mathfrak{P}_p(z)$ are singularități; dacă $p = n = 0, 1, 2, \dots$, atunci funcția $\mathfrak{P}_p(z)$ se transformă într-un polinom Legendre $P_n(z)$; pentru $p = -n = -1, -2, \dots$,

$$P_{-n-1}(z) = P_n(z).$$

3. Dacă $p \neq 0, 1, 2, \dots$, atunci funcția $\mathfrak{Q}_p(z)$ are singularități în punctele $z = \pm 1$ și $z = \infty$, aceste puncte joacă rolul de puncte de ramificație. Dacă $p = n = 0, 1, 2, \dots$, atunci funcția $\mathfrak{Q}_p(z)$ este uniformă pentru $|z| > 1$ și regulată pentru $z = \infty$.

4. În semiplanul drept,

$$\mathfrak{P}_p(z) = \left(\frac{1+z}{2}\right)^p F\left(-p, -p; 1; \frac{z-1}{z+1}\right) \quad [\operatorname{Re} z > 0].$$

5. Ecuațiile 6.840 1. și 6.840 4. definesc în mod univoc funcția $\mathfrak{P}_p(z)$, în interiorul unui cerc de rază 2, cu centrul în punctul $z = 1$ în semiplanul z drept

Pentru $z = x = \cos \varphi$, soluția ecuației 6.830 este funcția

$$6. \quad P_p(x) = \mathfrak{P}_p(x) = F\left(-p, p+1; 1; \sin^2 \frac{\varphi}{2}\right);$$

și, în general, au loc egalitățile

$$7. \quad \mathfrak{P}_p(z) = \mathfrak{P}_{-p-1}(z) = P_p(z) = P_{-p-1}(z).$$

8. Prin egalitatea 6.840 2., funcția $\mathfrak{Q}_p(z)$ este, pentru $|z| > 1$, definită univoc în planul z , în care s'a făcut o secțiune de la punctul $z = -\infty$ pînă la punctul $z = 1$. Cu ajutorul seriei hipergeometrice, funcția poate fi prelungită analitic în interiorul cercului de rază 1. Pe segmentul $[-1 \leq x \leq +1]$ al axei reale, funcția $Q_p(x)$ este dată de egalitatea

$$9. \quad Q_p(x) = \frac{1}{2} [\mathfrak{Q}_p(x+i \cdot 0) + \mathfrak{Q}_p(x-i \cdot 0)].$$

li 52 (53) WW II 113

Reprezentări integrale

6.841

$$1. \mathfrak{P}_p(z) = \frac{1}{2\pi i} \int_A^{(1+, z+)} \frac{(t^2-1)^p}{2^p (t-z)^{p+1}} dt. \quad \text{Sm III 514 (121) WW II 97}$$

[A este un punct pe axa reală în dreapta punctului $t = 1$ (și la dreapta lui z , dacă z este real): în punctul A, avem

$$\arg(t-1) = \arg(t+1) = 0 \text{ și } |\arg(t-z)| < \pi]. \quad \text{WW II 97}$$

$$2. \mathfrak{Q}_p(z) = \frac{1}{4i \sin p\pi} \int_A^{(1-, 1+)} \frac{(t^2-1)^p}{2^p (z-t)^{p+1}} dt$$

[p nu este un număr întreg, iar punctul A este extremitatea axei mari a elipsei în dreapta lui $t = 1$, construită în planul t și avînd focarele în punctele ± 1 , semi-axa a doua fiind atît de mică, încît punctul z este situat în afara ei. Conturul începe în punctul A, descrie drumul $(1-, -1+)$ și se întoarce în A; $|\arg z| \leq \pi$ și $|\arg(z-t)| \rightarrow \arg z$, atunci cînd $t \rightarrow 0$ pe contur; $\arg(t+1) = \arg(t-1) = 0$ în punctul A; z nu este situat pe axa reală între -1 și 1].

WW II 109

Pentru $p = n$ întreg,

$$3. \mathfrak{Q}_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1-t^2)^n (z-t)^{-n-1} dt. \quad \text{Sm III 517 (134) WW II 109}$$

6.842

$$1. \mathfrak{P}_p(z) = \frac{1}{\pi} \int_0^\pi \frac{d\varphi}{(z + \sqrt{z^2-1} \cos \varphi)^{p+1}} = \frac{1}{\pi} \int_0^\pi (z + \sqrt{z^2-1} \cos \varphi)^{p+1} d\varphi$$

$$\left[\operatorname{Re}(z) > 0 \text{ și } \arg\{z + \sqrt{z^2-1} \cos \varphi\} = \arg z \text{ pentru } \varphi = \frac{\pi}{2} \right].$$

WW II 105 WW II 106

$$2. \mathfrak{Q}_p(z) = \int_0^\infty \frac{d\varphi}{(z + \sqrt{z^2-1} \operatorname{ch} \varphi)^{p+1}}, \quad [\operatorname{Re}(p) > -1; \text{ dacă } p \text{ nu este}$$

un număr întreg, atunci $\arg\{(z + \sqrt{z^2-1}) \operatorname{ch} \varphi\}$, pentru $\varphi = 0$, are o valoare principală].

WW II 113

$$6.843 \quad P_p(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos\left(p + \frac{1}{2}\varphi\right)}{\sqrt{2(\cos \varphi - \cos \theta)}} d\varphi. \quad \text{WW II 108}$$

$$6.844 \quad \mathfrak{Q}_n(z) = 2_n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2-1)^{n+1}} = 2^n \int_z^\infty \frac{(t-z)^n}{(t^2-1)^{n+1}} dt;$$

$$= \frac{(-1)^n}{(2n-1)!} \frac{d^n}{dz^n} \left[(z^2-1)^n \int \frac{dt}{(t^2-1)^{n+1}} \right] \quad [\operatorname{Re}(z) > 1].$$

WW II 111-112 MO 78

$$6.845 \quad \mathfrak{Q}_n(z) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{z-t} dt \quad [|\arg(z-1)| < \pi]. \quad \text{WW II 114-115 MO 78}$$

Vezi și 4.431 3., 6.862

6.846 Serii trigonometrice

$$1. P_n(\cos \varphi) = \frac{2^{n+2}}{\pi} \frac{n!}{(2n+1)!!} \left[\sin(n+1)\varphi + \frac{1}{1} \cdot \frac{n+1}{2n+3} \sin(n+3)\varphi + \right. \\ \left. + \frac{1 \cdot 3}{1 \cdot 2} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \sin(n+5)\varphi + \dots \right] \quad [0 < \varphi < \pi]. \quad \text{MO 79}$$

$$2. Q_n(\cos \varphi) = 2^{n+1} \frac{n!}{(2n+1)!!} \left[\cos(n+1)\varphi + \frac{1}{1} \cdot \frac{n+1}{2n+3} \cos(n+3)\varphi + \right. \\ \left. + \frac{1 \cdot 3}{1 \cdot 2} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cos(n+5)\varphi + \dots \right] \quad [0 < \varphi < \pi]. \quad \text{MO 79}$$

Celelalte dezvoltări în serie ale funcțiilor Legendre ne sînt date de expresiile lor prin funcția hipergeometrică, vezi 6.840.

Cazuri particulare și valori particulare

6.847

$$1. Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = \text{arth } x. \quad \text{JE 207m}$$

$$2. Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1. \quad \text{JE 207m}$$

$$3. Q_2(x) = \frac{1}{4} (3x^2 - 1) \ln \frac{1+x}{1-x} - \frac{3}{2} x. \quad \text{JE 207m}$$

$$4. Q_3(x) = \frac{1}{4} (5x^3 - 3x) \ln \frac{1+x}{1-x} - \frac{5}{2} x^2 + \frac{2}{3}. \quad \text{JE 207m}$$

$$5. Q_4(x) = \frac{1}{16} (35x^4 - 30x^2 + 3) \ln \frac{1+x}{1-x} - \frac{35}{8} x^3 + \frac{55}{24} x. \quad \text{JE 207m}$$

$$6. Q_5(x) = \frac{1}{16} (63x^5 - 70x^3 + 15x) \ln \frac{1+x}{1-x} - \frac{63}{8} x^4 + \frac{49}{8} x^2 - \frac{8}{15}. \quad \text{JE 207m}$$

6.848

$$1. P_p(1) = 1. \quad \text{MO 79}$$

$$2. P_p(0) = -\frac{\sin p\pi}{\sqrt{\pi^2}} \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(-\frac{p}{2}\right). \quad \text{MO 79}$$

$$6.849 \quad Q_p(0) = \frac{1}{4\sqrt{\pi}} (1 - \cos p\pi) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(-\frac{p}{2}\right). \quad \text{MO 79}$$

Relații funcționale

6.851

$$1. Q_p(x) = \frac{\pi}{2 \sin p\pi} [\cos p\pi P_p(x) - P_p(-x)] \quad [p \neq 0, \pm 1, \pm 2, \dots]$$

MO 76

$$2. Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{p-1}(x) \quad [p=n=0, 1, 2, \dots]$$

unde

$$3. W_{n-1}(x) = \sum_{k=0}^{2E\left(\frac{n-1}{2}\right)} \frac{2(n-2k)-1}{(2k+1)(n-k)} P_{n-2k-1}(x) = \sum_{k=1}^n \frac{1}{k} P_{k-1}(x) P_{n-k}(x)$$

și

$$W_{-1}(x) \equiv 0$$

(vezi și 6.859).

Sm III 516 (131) MO 76

6.852

1. $(z^2 - 1) \frac{d}{dz} \mathfrak{P}_p(z) = (p+1) [\mathfrak{P}_{p+1}(z) - z \mathfrak{P}_p(z)]$. WW II 99 m
2. $(2p+1) z \mathfrak{P}_p(z) = (p+1) \mathfrak{P}_{p+1}(z) + p \mathfrak{P}_{p-1}(z)$. WW II 98
3. $(z^2 - 1) \frac{d}{dz} \mathfrak{Q}_p(z) = (p+1) [\mathfrak{Q}_{p+1}(z) - z \mathfrak{Q}_p(z)]$. WW II 112 m
4. $(2p+1) z \mathfrak{Q}_p(z) = (p+1) \mathfrak{Q}_{p+1}(z) + p \mathfrak{Q}_{p-1}(z)$. WW II 112
5. $P_{-p-1}(z) = P_p(z)$. MO 75

6.853

1. $\mathfrak{P}_p(-z) = e^{p\pi i} \mathfrak{P}_p(z) - \frac{2}{\pi} \sin p\pi \mathfrak{Q}_p(z) \quad [\operatorname{Im}(z) < 0]$. MO 77
2. $\mathfrak{P}_p(-z) = e^{-p\pi i} \mathfrak{P}_p(z) - \frac{2}{\pi} \sin p\pi \mathfrak{Q}_p(z) \quad [\operatorname{Im}(z) > 0]$. MO 77
3. $\mathfrak{Q}_p(-z) = -e^{-p\pi i} \mathfrak{Q}_p(z) \quad [\operatorname{Im}(z) < 0]$. MO 77
4. $\mathfrak{Q}_p(-z) = -e^{p\pi i} \mathfrak{Q}_p(z) \quad [\operatorname{Im}(z) > 0]$. MO 77

6.854

1. $\mathfrak{Q}_p(x \pm i \cdot 0) = Q_p(x) \mp \frac{\pi i}{2} P_p(x)$. MO 77
2. $\mathfrak{Q}_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z)$. MO 77

6.855

1. $\mathfrak{Q}_p(z) - \mathfrak{Q}_{-p-1}(z) = \pi \operatorname{ctg} p\pi P_p(z) \quad [\sin p\pi \neq 0]$. MO 77
2. $Q_{-p-1}(\cos \varphi) = Q_p(\cos \varphi) - \pi \operatorname{ctg} p\pi P_p(\cos \varphi) \quad [\sin p\pi \neq 0]$. MO 77
3. $Q_p(-\cos \varphi) = -\cos p\pi Q_p(\cos \varphi) + \frac{\pi}{2} \sin p\pi P_p(\cos \varphi)$. MO 77

6.856

1. $\mathfrak{Q}_n(z) = \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \ln \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1}$. MO 79
2. $Q_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \ln \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x}$. MO 79

6.857.

1. $P_p(x) = P_p(\cos \varphi) = F\left(-p, p+1; 1; \sin^2 \frac{\varphi}{2}\right) \quad (\text{cf. 6.840 6.}).$
MO 78
2. $\mathfrak{P}_p(z) = \frac{\operatorname{tg} p\pi}{2^{p+1} \sqrt{\pi}} \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)} z^{-p-1} F\left(\frac{p}{2}+1, \frac{p+1}{2}; p+\frac{3}{2}; \frac{1}{z^2}\right) +$
 $+ \frac{2^p}{\sqrt{\pi}} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} z^p F\left(\frac{1-p}{2}, -\frac{p}{2}; \frac{1}{2}-p; \frac{1}{z^2}\right).$ MO 78

Vezi și 6.840

Integralele funcțiilor Legendre, vezi 4.52.

6.858 Inegalități

1. $|P_p(\cos \varphi) - P_{p+2}(\cos \varphi)| \leq 2C_0 \sqrt{\frac{1}{p\pi}}$. MO 78
2. $|Q_p(\cos \varphi) - Q_{p+2}(\cos \varphi)| < C_0 \sqrt{\frac{\pi}{p}}$. MO 78

[$0 \leq \varphi \leq \pi$, $p > 1$, C_0 este un număr care nu depinde de valorile lui p și φ].Zerourile funcțiilor Legendre de speța a doua, vezi 6.784, 6.785, 6.786.
Dezvoltarea funcțiilor Legendre, cu ajutorul funcțiilor sferice, vezi 6.794, 6.795, 6.796.

6.859 Ecuația diferențială care duce la funcția $W_{n-1}(x)$ (vezi 6.852 3.) (x)

$$(1-x^2) \frac{d^2 W_{n-1}}{dx^2} - 2x \frac{dW_{n-1}}{dx} + (n+1)nW_{n-1} = 2 \frac{dP_n(x)}{dx}. \quad \text{MO 76}$$

6.86 Funcții conice

6.860

Dacă, în ecuația diferențială 6700 1., care definește funcțiile sferice, vom pune

$$\rho = -\frac{1}{2} + i\lambda,$$

unde λ este un parametru real, vom obține ecuația diferențială a așa-numitelor *funcții conice*. Funcțiile conice sînt un caz particular al funcțiilor sferice. Dar funcțiile sferice

$$P_{-\frac{1}{2}+i\lambda}(x), Q_{-\frac{1}{2}+i\lambda}(x)$$

au anumite particularități care ne obligă să le trecem într-o clasă specială, aceea a funcțiilor conice. Cea mai importantă dintre aceste particularități este următoarea:

6.861 Funcțiile

$$P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\varphi}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 \cdot 4^2} \sin^4 \frac{\varphi}{2} + \dots$$

sînt reale, pentru φ real, iar

$$P_{-\frac{1}{2}+i\lambda}(x) \equiv P_{-\frac{1}{2}-i\lambda}(x). \quad \text{MO 95}$$

6.862 Reprezentări integrale

$$1. P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = \frac{2}{\pi} \int_0^\varphi \frac{\operatorname{ch} \lambda u \, du}{\sqrt{2(\cos u - \cos \varphi)}} = \frac{2}{\pi} \operatorname{ch} \lambda \pi \int_0^\infty \frac{\cos \lambda u \, du}{\sqrt{2(\cos \varphi + \operatorname{ch} u)}}. \quad \text{MO 95}$$

$$2. Q_{-\frac{1}{2}+i\lambda}(\cos \varphi) = \pm i \operatorname{sh} \lambda \pi \int_0^\infty \frac{\cos \lambda u \, du}{\sqrt{2(\operatorname{ch} u + \cos \varphi)}} + \int_0^\infty \frac{\operatorname{ch} \lambda u \, du}{\sqrt{2(\operatorname{ch} u - \cos \varphi)}}. \quad \text{MO 95}$$

Relații funcționale (vezi și 6.73).

$$6.863 P_{-\frac{1}{2}+i\lambda}(-\cos \varphi) = \frac{\operatorname{ch} \lambda \pi}{\pi} [Q_{-\frac{1}{2}+i\lambda}(\cos \varphi) + Q_{-\frac{1}{2}-i\lambda}(\cos \varphi)]. \quad \text{MO 95}$$

6.864

$$1. P_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) = \\ = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + \\ + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_k^{-\frac{1}{2}+i\lambda}(\cos \psi) P_k^{-\frac{1}{2}+i\lambda}(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2) \dots [4\lambda^2 + (2k-1)^2]}$$

$$\left[0 < \vartheta < \frac{\pi}{2}, 0 < \psi < \pi, 0 < \psi + \vartheta < \pi \right] \quad (\text{cf. 6.794 1.}). \quad \text{MO 95}$$

$$\begin{aligned}
 2. \quad P_{-\frac{1}{2}+i\lambda}(-\cos\psi\cos\vartheta-\sin\psi\sin\vartheta\cos\varphi) = \\
 = P_{-\frac{1}{2}+i\lambda}(\cos\psi)P_{-\frac{1}{2}+i\lambda}(-\cos\vartheta) + \\
 + 2\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos\psi) P_{-\frac{1}{2}+i\lambda}^k(-\cos\vartheta) \cos m\varphi}{(4\lambda^2+1)(4\lambda^2+3^2)\dots[4\lambda^2+(2k-1)^2]} \\
 \left[0 < \psi < \frac{\pi}{2} < \vartheta, \psi + \vartheta < \pi\right] \quad (\text{cf. } 6.796). \quad \text{MO } 95
 \end{aligned}$$

$$\begin{aligned}
 3. \quad Q_{-\frac{1}{2}+i\lambda}(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi) = P_{-\frac{1}{2}+i\lambda}(\cos\psi)Q_{-\frac{1}{2}+i\lambda}(\cos\vartheta) + \\
 + 2\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos\psi) Q_{-\frac{1}{2}+i\lambda}^k(\cos\vartheta) \cos m\varphi}{(4\lambda^2+1)(4\lambda^2+3^2)\dots[4\lambda^2+(2k-1)^2]} \\
 \left[0 < \psi < \frac{\pi}{2} < \vartheta, \psi + \vartheta < \pi\right] \quad (\text{cf. } 6.794 \text{ 2.}). \quad \text{MO } 96
 \end{aligned}$$

Zerourile funcțiilor conice, vezi 6.784.

6.87 Funcțiile torului (sau inelului)

6.870

Se numesc *funcții ale torului* soluțiile ecuației diferențiale

$$1. \quad \frac{d^2 u}{d\eta^2} + \frac{\text{ch } \eta}{\text{sh } \eta} \frac{du}{d\eta} - \left(n^2 - \frac{1}{4} + \frac{m^2}{\text{sh}^2 \eta}\right) u = 0,$$

care este una dintre formele funcțiilor sferice. În particular, servesc ca soluții ale ecuației 6.870 1. funcțiile

$$\mathfrak{P}_{n-\frac{1}{2}}^m(\text{ch } \eta), \quad \Omega_{n-\frac{1}{2}}^m(\text{sh } \eta). \quad \text{MO } 96$$

Pentru funcțiile torului, sînt deosebit de importante următoarele formule care se obțin drept consecințe din formulele pentru funcțiile sferice citate mai sus:

6.871 Reprezentări integrale

$$\begin{aligned}
 1. \quad \mathfrak{P}_{n-\frac{1}{2}}^m(\text{ch } \eta) = \\
 = \frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)} \frac{(\text{sh } \eta)^m}{2^m \sqrt{\pi} \Gamma\left(m+\frac{1}{2}\right)} \int_0^\pi \frac{\sin^{2m} \varphi d\varphi}{(\text{ch } \eta + \text{sh } \eta \cos \varphi)^{n+m+\frac{1}{2}}} = \\
 = \frac{(-1)^m}{2\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)} \int_0^{2\pi} \frac{\cos m\varphi d\varphi}{\text{ch } \eta + \text{sh } \eta \cos \varphi}^{n+\frac{1}{2}}. \quad \text{MO } 96
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \Omega_{n-\frac{1}{2}}^m(\text{ch } \eta) = (-1)^m \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)} \int_0^\infty \frac{\text{ch } mt dt}{(\text{ch } \eta + \text{sh } \eta \text{ch } t)^{n+\frac{1}{2}}}; \quad [n \geq m] \\
 = (-1)^m \frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{\text{in cth } \frac{\eta}{2}} (\text{ch } \eta - \text{sh } \eta \text{ch } t)^{n-\frac{1}{2}} \text{ch } mt dt. \quad \text{MO } 96
 \end{aligned}$$

6.872 Relații funcționale

$$1. \quad \mathfrak{D}_{n-\frac{1}{2}}^m(\operatorname{ch} \eta) = (-1)^m \frac{2^m \Gamma\left(n+m+\frac{1}{2}\right) \sqrt{\pi}}{\Gamma(n+1)} \operatorname{sh}^m \eta e^{-(n+m+\frac{1}{2})\eta} \times \\ \times F\left(m+\frac{1}{2}, n+m+\frac{1}{2}; n+1; e^{-2\eta}\right). \quad \text{MO 96}$$

$$2. \quad \mathfrak{P}_{n-\frac{1}{2}}^{-m}(\operatorname{ch} \eta) = \frac{2^{-m}}{\Gamma(m+1)} (1-e^{-2\eta})^m e^{-(n+\frac{1}{2})\eta} \times \\ \times F\left(m+\frac{1}{2}, n+m+\frac{1}{2}; 2m+1; 1-e^{-2\eta}\right). \quad \text{MO 96}$$

6.873 Dezvoltarea asimptotică a funcției $\mathfrak{P}_{n-\frac{1}{2}}(\operatorname{ch} \eta)$ pentru valori mari ale lui n

$$\mathfrak{P}_{n-\frac{1}{2}}(\operatorname{ch} \eta) = \frac{\Gamma(n) e^{(n-\frac{1}{2})\eta}}{\sqrt{\pi} \Gamma\left(n+\frac{1}{2}\right)} \times \\ \times \left[\frac{2 \Gamma^2\left(n+\frac{1}{2}\right)}{\pi n! \Gamma(n)} \ln(4e^\eta) \cdot e^{-2n\eta} F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; e^{-2\eta}\right) + A + B \right],$$

unde

$$A = 1 + \frac{1}{2^2} \frac{1 \cdot (2n-1)}{1 \cdot (n-1)} e^{-2\eta} + \frac{1}{2^4} \frac{1 \cdot 3 \cdot (2n-1)(2n-3)}{1 \cdot 2 \cdot (n-1)(n-2)} e^{-4\eta} + \dots + \\ + \frac{1}{2^{2n-2}} \cdot \left(\frac{(2n-1)!!}{(n-1)!} \right)^2 e^{-2(n-1)\eta}.$$

$$B = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi^3} \Gamma(n)} \sum_{k=1}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right) \Gamma\left(n+k+\frac{1}{2}\right)}{\Gamma(n+k+1) \Gamma(k+1)} \times \\ \times (u_{n+k} + u_k - v_{n+k-\frac{1}{2}} - v_{k-\frac{1}{2}}) e^{-2(n+k)\eta};$$

aici

$$u_r = \sum_{s=1}^r \frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^r \frac{2}{2s-1} \quad [r \text{ natural}]. \quad \text{MO 97}$$

6.9 POLINOAME $C_n^p(t)$

6.90 Definiție

Polinoamele $C_n^p(t)$ de gradul n sint coeficienții lui α^n în dezvoltarea în serie de puteri a funcției

$$(1-2t\alpha+\alpha^2)^{-p} = \sum_{n=0}^{\infty} C_n^p(t) \alpha^n. \quad \text{WW II 127}$$

Așadar, polinoamele $C_n^p(t)$ servesc ca o generalizare a polinoamelor Legendre.

6.91 Reprezentarea integrală

$$C_n^p(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2p+n)}{n! \Gamma(2p)} \frac{\Gamma\left(\frac{2p+1}{2}\right)}{\Gamma(p)} \int_0^\pi (t + \sqrt{t^2-1} \cos \varphi)^n \sin^{2p-1} \varphi d\varphi. \quad \text{MO 99}$$

6.92 6.92 Relații funcționale

6.921 Exprimarea prin funcția hipergeometrică

$$1. \quad C_n^p(t) = \frac{\Gamma(2p+n)}{\Gamma(n+1) \Gamma(2p)} F\left(2p+n, -n; p+\frac{1}{2}; \frac{1-t}{2}\right); \quad \text{MO 97}$$

$$= \frac{2^n \Gamma(p+n)}{n! \Gamma(p)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-p-n; \frac{1}{t^2}\right). \quad \text{MO 99}$$

$$2. \quad C_{2n}^p(t) = \frac{(-1)^n}{(p+n) B(p, n+1)} F\left(-n, n+p; \frac{1}{2}; t^2\right). \quad \text{MO 99}$$

$$3. \quad C_{2n+1}^p(t) = \frac{(-1)^n 2t}{B(p, n+1)} F\left(-n, n+p+1; \frac{3}{2}; t^2\right). \quad \text{MO 99}$$

6.922 Formule de recurență

$$1. \quad (n+2) C_{n+2}^p(t) = 2(p+n+1) t C_{n+1}^p(t) - (2p+n) C_n(t). \quad \text{MO 98}$$

$$2. \quad n C_n^p(t) = 2p [t C_{n-1}^{p+1}(t) - C_{n-2}^{p+1}(t)]. \quad \text{WW II 128}$$

$$3. \quad (2p+n) C_n^p(t) = 2p [C_{n-1}^{p+1}(t) - t C_{n-2}^{p+1}(t)]. \quad \text{WW II 128}$$

$$4. \quad n C_n^p(t) = (2p+n-1) t C_{n-1}^p(t) - 2p(1-t^2) C_{n-2}^p(t). \quad \text{WW II 128}$$

6.923

$$1. \quad C_n^p(t) = \frac{(-1)^n}{2^n} \frac{\Gamma(2p+n) \Gamma\left(\frac{2p+1}{2}\right)}{\Gamma(2p) \Gamma\left(\frac{2p+1}{2}+n\right)} \frac{(1-t^2)^{\frac{1}{2}-p}}{n!} \frac{d^n}{dt^n} [(1-t^2)^{p+n-\frac{1}{2}}].$$

WW II 127

$$2. \quad C_n^p(\cos \varphi) = \sum_{\substack{k, l=0 \\ k+l=n}}^n \frac{\Gamma(p+k) \Gamma(p+l)}{k! l! [\Gamma(p)]^2} \cos(k-l) \varphi. \quad \text{MO 99}$$

$$3. \quad C_n^p(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) =$$

$$= \frac{\Gamma(2p-1)}{[\Gamma(p)]^2} \sum_{k=0}^n \frac{2^{2k} (n-k)! [\Gamma(p+k)]^2}{\Gamma(2p+n+k)} (2p+2k-1) \sin^k \psi \sin^k \vartheta \times$$

$$\times C_{n-k}^{p+\frac{k}{2}}(\cos \psi) C_{n-k}^{p+\frac{k}{2}}(\cos \vartheta) C_k^{p-\frac{1}{2}}(\cos \varphi)$$

$$[\psi, \vartheta, \varphi \text{ reali}; p \neq \frac{1}{2}] \text{ „teorema de compunere”}$$

(vezi și 6.794–6.796).

WW II 136m

$$4. \quad \lim_{p \rightarrow 0} \Gamma(p) C_n^p(\cos \varphi) = \frac{2 \cos n\varphi}{n}. \quad \text{MO 98}$$

Ortogonalitate, vezi 7.104, 4.60.

6.93 Derivate

$$6.931 \quad \frac{d^k}{dt^k} C_n^p(t) = 2^k \frac{\Gamma(p+k)}{\Gamma(p)} C_{n-k}^{p+k}(t). \quad \text{MO 99}$$

¹ Această egalitate servește la definirea funcțiilor generalizate $C_n^p(t)$, al căror indice n poate fi oarecare,

In particular,

$$6.932 \quad \frac{dC_n^p(t)}{dt} = 2pC_{n-1}^{p+1}(t).$$

WW II 128

Integralele polinoamelor $C_n^p(x)$, vezi 4.6

6.94 Legătura cu alte funcții

$$6.941 \quad C_n^p(t) = \frac{\Gamma(2p+n) \Gamma\left(p + \frac{1}{2}\right)}{\Gamma(2p) \Gamma(n+1)} \left\{ \frac{1}{4} (t^2 - 1) \right\}^{\frac{1}{4} - \frac{p}{2}} P_{p+n-\frac{1}{2}}^{\frac{1}{2}-p}(t). \quad \text{MO 98}$$

$$6.942 \quad C_{n-m}^{m+\frac{1}{2}}(t) = \frac{1}{(2m-1)!} \frac{d^m P_n(t)}{dt^m} = (-1)^m \frac{(1-t^2)^{\frac{m}{2}} m! 2^m}{(2m)!} P_n^m(t) \quad \text{MO 98 WW II 127}$$

[m + 1 natural]

$$6.943 \quad C_n^{\frac{1}{2}}(t) = P_n(t).$$

$$6.944 \quad J_{p-\frac{1}{2}}(r \sin \vartheta \sin \alpha) (r \sin \vartheta \sin \alpha)^{-p+\frac{1}{2}} e^{-ir \cos \vartheta \cos \alpha} =$$

$$= \sqrt{2} \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)} \sum_{k=0}^{\infty} (p+k) i^{-k} \frac{J_{p+k}(r)}{r^{p+k}} \frac{C_k^p(\cos \vartheta) C_k^p(\cos \alpha)}{r^p C_k^p(1)}. \quad \text{MO 99}$$

$$6.945 \quad \lim_{p \rightarrow \infty} p^{-\frac{n}{2}} C_n^{\frac{p}{2}}\left(t \sqrt{\frac{2}{p}}\right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(t). \quad \text{MO 99 m}$$

Vezi și 6.921.

6.95 Cazuri particulare și valori particulare

1. $C_n^1(\cos \varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi}. \quad \text{MO 99}$
2. $C_0^0(\cos \varphi) = 1. \quad \text{MO 98}$
3. $C_0^p(t) \equiv 1. \quad \text{MO 98}$
4. $C_n^p(1) = \binom{2p+n-1}{n}. \quad \text{MO 98}$

6.96 Ecuația diferențială care duce la polinoamele $C_n^p(t)$

$$y'' + \frac{(2p+1)t}{t^2-1} y' - \frac{n(2p+n)}{t^2-1} y = 0 \quad (\text{cf. 7.274}). \quad \text{WW II 127}$$

Seriile de produse ale funcțiilor Bessel cu polinoamele $C_n^p(x)$, vezi 6.542, 6544.

7.1 POLINOAME ORTOGONALE

7.10 Introducere

7.101 Fie $w(x)$ o funcție reală nenegativă de variabilă reală x și fie (a, b) un interval fix pe axa X . Să presupunem, mai departe, că, pentru $n = 0, 1, 2, \dots$, integrala

$$\int_a^b x^n w(x) dx$$

există și că, în afară de aceasta integrala

$$\int_a^b w(x) dx$$

este pozitivă. În acest caz, există un șir de polinoame $p_0(x), p_1(x), \dots, p_n(x), \dots$ definite în mod univoc prin următoarele condiții:

1. $p_n(x)$ este un polinom de gradul n , coeficientul lui x^n în acest polinom fiind pozitiv.
2. Polinoamele $p_0(x), p_1(x), \dots$ sînt ortogonale și normate, adică

$$\int_a^b p_n(x) p_m(x) w(x) dx = \begin{cases} 0 & \text{pentru } n \neq m \\ 1 & \text{pentru } n = m \end{cases}$$

În acest caz, se spune că polinoamele $p_n(x)$ formează un sistem de polinoame ortogonale în intervalul (a, b) cu ponderea $w(x)$.

7.102 Dacă q_n este coeficientul lui x^n în polinomul $p_n(x)$, atunci

$$\sum_{k=0}^n p_k(x) p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y}.$$

7.103 Între trei polinoame ortogonale succesive oarecare, există relația

$$p_n(x) = (A_n x + B_n) p_{n-1}(x) - C_n p_{n-2}(x) \quad [n = 2, 3, 4, \dots].$$

În această formulă, A_n, B_n, C_n sînt constante și avem

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2} \quad \text{MO 102}$$

7.104 Exemple de sisteme normate de polinoame ortogonale

Notăția și denumirea	Intervalul	Ponderea
$\left(n + \frac{1}{2}\right)^{\frac{1}{2}} P_n(x)$ (Legendre)	$(-1, +1)$	1
$2^p \Gamma(p) \left[\frac{(n+p) \cdot n!}{2n! \Gamma(2p+n)} \right]^{\frac{1}{2}} C_n^{(p)}(x)$ (Gegenbauer)	$(-1, +1)$	$(1-x^2)^{p-\frac{1}{2}}$
$\sqrt{\frac{\pi}{n}} T_n(x)$ (Cebîșev)	$(-1, +1)$	$(1-x^2)^{-\frac{1}{2}}$
$2^{-\frac{n}{2}} \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} H_n(x)$ (Hermite)	$(-\infty, \infty)$	e^{-x^2}
$\left[\frac{q(q+1) \dots (q+n-1) (p+2n) \Gamma(p+n)}{n! \Gamma(q) \Gamma(p-q+n+1)} \right]^{\frac{1}{2}} \times$ $\times G(p, q, x)$ (Jacobi)	$(0, 1)$	$x^{q-1} (1-x)^{p-q}$
$[n! \Gamma(p+n+1)]^{-\frac{1}{2}} L_n^p(x)$ (Laguerre)	$(0, \infty)$	$x^p e^{-x}$

(cf. 4.520 1., 4.60. 4.71 — 4.741).

MO 103m

1. $\varepsilon_n = 2$ pentru $n = 1, 2, 3, \dots$; $\varepsilon_0 = 1$.

7.11 Polinoamele $T_n(x)$ și $U_n(x)$ ale lui Cebîșev

7.110 Definiție

1. Polinoamele Cebîșev de speța întâi

$$T_n(x) = \cos(n \arccos x) = \frac{1}{2} [(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n] = \\ = x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (1-x^2)^2 - \binom{n}{6} x^{n-6} (1-x^2)^3 + \dots$$

Na 66 Na 71

2. Polinoamele Cebîșev de speța a doua

$$U_n(x) = \sin(n \arccos x) = \frac{1}{2i} [(x + i\sqrt{1-x^2})^n - (x - i\sqrt{1-x^2})^n] = \\ = \sqrt{1-x^2} \left[\binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} (1-x^2) + \binom{n}{5} x^{n-5} (1-x^2)^2 - \dots \right].$$

MO 103

Relații funcționale

7.111 Formule de recurență

$$1. T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

Na 358

$$2. U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0.$$

Ortogonalitate, vezi 4.71, 7.104.

7.112 Legătura cu celelalte funcții

$$1. T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right).$$

MO 104

$$2. T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}.$$

MO 104

$$3. U_{n+1}(x) = (-1)^n \frac{n+1}{(2n+1)!!} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}.$$

MO 104

Vezi și 7.131 3.

7.113 Cazuri particulare

$$1. T_0(x) = 1.$$

$$7. U_0(x) = 0.$$

$$2. T_1(x) = x.$$

$$8. U_1(x) = \sqrt{1-x^2}.$$

$$3. T_2(x) = 2x^2 - 1.$$

$$9. U_2(x) = \sqrt{1-x^2} \cdot 2x.$$

$$4. T_3(x) = 4x^3 - 3x.$$

$$10. U_3(x) = \sqrt{1-x^2} (4x^2 - 1).$$

$$5. T_4(x) = 8x^4 - 8x^2 + 1.$$

$$11. U_4(x) = \sqrt{1-x^2} (8x^3 - 4x).$$

$$6. T_5(x) = 16x^5 - 20x^3 + 5x.$$

$$12. U_5(x) = \sqrt{1-x^2} (16x^4 - 12x^2 + 1).$$

7.114 Valori particulare

$$1. T_n(1) = 1.$$

$$5. U_n(1) = 0.$$

$$2. T_n(-1) = (-1)^n.$$

$$6. U_n(-1) = 0.$$

$$3. T_{2n}(0) = (-1)^n.$$

$$7. U_{2n}(0) = 0.$$

$$4. T_{2n+1}(0) = 0.$$

$$8. U_{2n+1}(0) = (-1)^n.$$

7.115 Funcția generatoare

$$1. \frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2 \sum_{k=1}^{\infty} T_k(x) t^k.$$

MO 104

$$2. \frac{1}{1-2tx+t^2} = \frac{1}{\sqrt{1-x^2}} \sum_{k=0}^{\infty} U_{k+1}(x) t^k.$$

MO 104

7.116 Zerouri. Polinoamele $T_n(x)$ și $U_n(x)$ au numai zerouri simple reale; toate aceste zerouri sînt situate în intervalul $(-1, +1)$. Extremitățile acestui interval — punctele $x = \pm 1$ — sînt zerouri ale funcției $U_n(x)$.

7.117 Funcțiile $T_n(x)$ și $U_n(x)$ sînt două soluții liniar independente ale ecuației diferențiale

$$(1-x^2) \frac{d^2 u}{dx^2} - x \frac{dy}{dx} + n^2 y = 0. \quad \text{Na 69 (58)}$$

7.118 Dintre toate polinoamele de gradul n avînd primul coeficient egal cu 1, deviază cel mai puțin de la valoarea zero pe segmentul $[-1, +1]$ polinomul

$$\frac{1}{2^{n-1}} T_n(x). \quad \text{Na 63}$$

7.12 Polinoamele $H_n(x)$ ale lui Hermite

7.120 Definiție

$$1. H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \quad \text{Sm III 567 (14)}$$

sau

$$2. H_n(x) = 2^n x^n - 2^{n-1} \binom{n}{2} x^{n-2} + 2^{n-2} \cdot 1 \cdot 3 \cdot \binom{n}{4} x^{n-4} - \\ - 2^{n-3} \cdot 1 \cdot 3 \cdot 5 \cdot \binom{n}{6} x^{n-6} + \dots \quad \text{MO 105m}$$

7.121 Reprezentare integrală

$$1. H_n(x) = \frac{\sqrt{2^n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2} dt. \quad \text{MO 106m}$$

Relații funcționale

7.122 Formule de recurență

$$1. \frac{\partial H_n(x)}{\partial x} = 2n H_{n-1}(x). \quad \text{Sm III 569 (22)}$$

$$2. H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x). \quad \text{Sm III 570 (23)}$$

Ortogonalitate, vezi 4.72, 7.104.

7.123 Legătura cu alte funcții

$$1. H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1\left(-n; \frac{1}{2}; x^2\right). \quad \text{MO 106m}$$

$$2. H_{2n+1}^{(x)} = (-1)^n 2 \frac{(2n+1)!}{n!} x \cdot {}_1F_1\left(-n; \frac{3}{2}; x^2\right). \quad \text{MO 106m}$$

Legătura cu polinoamele $C_n^p(x)$, vezi 6.945

Legătura cu polinoamele Laguerre, vezi 7.142

Legătura cu funcțiile cilindrului parabolic, vezi 7.353

7.124 Inegalități

$$|H_n(x)| \leq 2^{\frac{n}{2} - E\left(\frac{n}{2}\right)} \frac{n!}{\left[E\left(\frac{n}{2}\right)\right]!} e^{2x} \sqrt{E\left(\frac{n}{2}\right)} \quad [x > 0]. \quad \text{MO 106m}$$

7.125 Dezvoltare asimptotică

$$1. H_{2n}(x) = (-1)^n 2^{2n} \cdot (2n-1)!! e^{\frac{x^2}{2}} \left[\cos \sqrt{4n+1} x + O\left(\frac{1}{\sqrt{n}}\right) \right].$$

Sm III 579

$$2. H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} \cdot (2n-1)!! \sqrt{2n+1} e^{\frac{x^2}{2}} \left[\sin \sqrt{4n+3} x + O\left(\frac{1}{\sqrt{n}}\right) \right].$$

Sm III 579

7.126 Cazuri particulare și valori particulare

$$1. H_0(x) = 1.$$

$$2. H_1(x) = 2x.$$

$$3. H_2(x) = 4x^2 - 2.$$

$$4. H_3(x) = 8x^3 - 12x.$$

$$5. H_4(x) = 16x^4 - 48x^2 + 12.$$

$$6. H_{2n}(0) = (-1)^n 2^n \cdot (2n-1)!!$$

Sm III 570 (24)

$$7. H_{2n+1}(0) = 0.$$

Serii de polinoame Hermite

7.127 Funcția generatoare

$$1. \exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x).$$

Sm III 569 (21)

$$2. \frac{1}{e} \operatorname{sh} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x).$$

MO 106m

$$3. \frac{1}{e} \operatorname{ch} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x).$$

MO 106m

$$4. e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x).$$

MO 106m

$$5. e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x).$$

MO 106m

7.128 „Teorema de compunere“

$$1. \frac{\left(\sum_{k=1}^n a_k^2\right)^{\frac{n}{2}}}{n!} H_n \left(\frac{\sum_{k=1}^n a_k x_k}{\sqrt{\sum_{k=1}^n a_k^2}} \right) = \sum_{m_1+m_2+\dots+m_n=n} \prod_{k=1}^n \left\{ \frac{a_{k m_k}}{m_k!} H_{m_k}(x_k) \right\}.$$

MO 106m

Caz particular

$$2. 2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}).$$

MO 107m

7.129 Polinoamele Hermite sînt soluții ale ecuației diferențiale

$$1. \frac{d^2 u_n}{dx^2} - 2x \frac{du_n}{dx} + 2n u_n = 0;$$

Sm III 566 (9)

alte soluții ale acestei ecuații diferențiale sînt funcțiile

$$2. \quad u_{2n} = (-1)^n A x {}_1F_1\left(\frac{1}{2} - n; \frac{3}{2}; x^2\right),$$

$$3. \quad u_{2n+1} = (-1)^n B {}_1F_1\left(-\frac{1}{2} - n; \frac{1}{2}; x^2\right)$$

[A și B sînt constante arbitrare].

MO 107

7.13 Polinoamele $G_n(p, q; x)$ ale lui Jacobi

7.130 Definiție

$$G_n(p, q; x) = \frac{x^{1-q} (1-x)^{q-p}}{q(q+1) \dots (q+n-1)} \frac{d^n}{dx^n} [x^{q+n-1} (1-x)^{p+n-q}]. \quad \text{CH 83}$$

7.131 Legătura cu celelalte funcții

$$1. \quad G_n(p, q; x) = F(p+n, -n; q; x) =$$

$$= 1 + \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{(p+n)(p+n+1) \dots (p+n+k-1)}{q(q+1) \dots (q+k-1)} x^k. \quad \text{CH 83 MO 107}$$

$$2. \quad P_n(x) = G_n\left(1, 1; \frac{1-x}{2}\right) = F\left(-n, n+1; 1; \frac{1-x}{2}\right). \quad \text{CH 83 MO 108}$$

$$3. \quad T_n(x) = G_n\left(0, \frac{1}{2}; \frac{1-x}{2}\right) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right). \quad \text{CH 83 MO 108}$$

$$4. \quad C_n^p(x) = (-1)^n \binom{2p+n-1}{n} G_n\left(2p, p+\frac{1}{2}; \frac{1+x}{2}\right) =$$

$$= (-1)^n \binom{2p+n-1}{n} F\left(-n, 2p+n; p+\frac{1}{2}; \frac{1+x}{2}\right). \quad \text{MO 108}$$

Ortogonalitate, vezi 4.73, 7.104.

7.132 Cazuri particulare

$$1. \quad G_0(p, q; x) = 1.$$

$$2. \quad G_1(p, q; x) = 1 - \frac{p+1}{q} x.$$

$$3. \quad G_2(p, q; x) = 1 - 2 \frac{p+2}{q} x + \frac{(p+2)(p+3)}{q(q+1)} x^2.$$

$$4. \quad G_3(p, q; x) = 1 - 3 \frac{p+3}{q} x + 3 \frac{(p+3)(p+4)}{q(q+1)} x^2 -$$

$$- \frac{(p+3)(p+4)(p+5)}{q(q+1)(q+2)} x^3.$$

CH 83

7.133 Funcția generatoare

$$\frac{(1-x)^{1-q} (1+x)^{q-p} (t-1 + \sqrt{1-2tx+t^2})^{q-1} (t+1 - \sqrt{1-2tx+t^2})^{p-q}}{t^{p-1} \sqrt{1-2tx+t^2}} =$$

$$= \sum_{k=0}^{\infty} \binom{q+k-1}{k} G_k\left(p, q; \frac{1-x}{2}\right) t^k. \quad \text{CH 84}$$

7.134 Zerouri. Toate cele n zerouri ale polinoamelor Jacobi sînt simple, reale și situate în intervalul $0 \leq x \leq 1$, dacă p și q sînt reali și, în același timp, $q > 0$, iar $p > q-1$. MO 108

7.135 Polinoamele Jacobi sînt unica soluție rațională întreaga a ecuației diferențiale (hipergeometrice)

$$x(1-x) \frac{d^2 u}{dx^2} + [q - (p+1)x] \frac{du}{dx} + n(p+n)u = 0. \quad \text{MO 108}$$

7.14 Polinoame Laguerre

7.140 Definiție

$$1. L_n^{(p)}(x) = x^{-p} e^x \frac{d^n}{dx^n} (e^{-x} x^{n+p}) =$$

$$= \sum_{k=0}^n (-1)^k \binom{n+p}{n-k} n(n-1) \dots (k+1) x^k$$

[$n+1$ — natural].

Sm III 573 (33) MO 109m

$$2. L_n^{(0)}(x) = L_n(x)$$

Relații funcționale

7.141 Formule de recurență

$$1. \frac{d}{dx} L_n^{(p)}(x) = -n L_{n-1}^{(p+1)}(x).$$

Sm III 575 (42)

$$2. x L_n^{(p)}(x) = (n+p) L_n^{(p-1)}(x) - L_{n+1}^{(p-1)}(x).$$

Sm III 575 (43)

$$3. L_n^{(p-1)}(x) = L_n^{(p)}(x) - n L_{n-1}^{(p)}(x).$$

Sm III 575 (44)

$$4. n^2(n-1) L_n^{(p)}(x) = (n-1)(-x+2n+p-1) L_{n-1}^{(p)}(x) - (n+p-1) L_{n-2}^{(p)}(x) \quad (n=2, 3, 4, \dots).$$

MO 109m

7.142 Legătura cu alte funcții

$$1. L_n^{(p)}(x) = (p+1)(p+2) \dots (p+n) {}_1F_1(-n; p+1; x).$$

MO 109m

$$2. H_{2n}(x) = (-1)^n 2^{2n} L_n^{(-\frac{1}{2})}(x^2).$$

Sm III 576 (47)

$$3. H_{2n+1}(x) = (-1)^n 2^{2n+1} L_n^{(\frac{1}{2})}(x^2).$$

Sm III 577 (48)

Vezi și 7.339.

Ortogonalitate, vezi 4.741, 7.104.

Integralele polinoamelor Laguerre, vezi 4.742..

7.143 Cazuri particulare

$$1. L_0^{(n)}(x) = 1.$$

$$2. L_1(x) = 1 - x.$$

$$3. L_2(x) = 2 - 4x + x^2.$$

$$4. L_3(x) = 6 - 18x + 9x^2 - x^3.$$

$$5. L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4.$$

$$6. L_5(x) = 120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5.$$

$$7. L_n^{-n}(x) = (-1)^n x^n.$$

MO 109m

MO 109m

Serii de polinoame Laguerre

7.144

$$1. \frac{\exp\left(\frac{xt}{1-t}\right)}{(1-t)^{p+1}} = \sum_{k=0}^{\infty} L_k^{(p)}(x) \frac{t^k}{k!} \quad [|t| < 1].$$

MO 109m

$$2. (1+t)^p e^{-xt} = \sum_{k=0}^{\infty} L_k^{(p-k)}(x) \frac{t^k}{k!} \quad [|t| < 1].$$

MO 110m

$$7.145 \quad \sum_{k=0}^{\infty} \frac{L_k^{(p)}(x)}{\Gamma(k+p+1)} \frac{t^k}{k!} = e^t (xt)^{-\frac{p}{2}} J_p(2\sqrt{xt}) \quad [p > -1]. \quad \text{MO 109m}$$

În particular, pentru $p=0$,

$$1. \quad \sum_{k=0}^{\infty} \frac{L_k(x)}{(k!)^2} t^k = e^t J_0(2\sqrt{xt}). \quad \text{MO 109m}$$

$$7.146 \quad 1 + \sum_{k=1}^{\infty} \left(\frac{L_k(x)}{k!} - \frac{L_{k-1}(x)}{(k-1)!} \right)^2 = e^x. \quad \text{MO 110m}$$

7.147

$$1. \quad \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(p-q+k)}{\Gamma(p-q)} L_{n-k}^{(q)}(x) = L_n^{(p)}(x). \quad \text{MO 110m}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{(2n-2k)!}{(n-k)!} \frac{\Gamma(1+p+n)}{\Gamma(1+p-k)} L_{2k}^{(2p)}(2x) = \frac{1}{n!} [L_n^{(p)}(x)]^2. \quad \text{MO 110m}$$

$$3. \quad \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(1+p+n)}{\Gamma(1+p+k)} (xy)^k L_{n-k}^{(p+2k)}(x+y) = L_n^{(p)}(x) L_n^{(p)}(y). \quad \text{MO 110m}$$

7.148 „Teoreme de compunere“

$$1. \quad L_n^{(p_1+p_2+\dots+p_k+k-1)}(x_1+x_2+\dots+x_k) = \sum_{(m_1+m_2+\dots+m_k=n)} \frac{n}{m_1!m_2!\dots m_k!} L_{m_1}^{(p_1)}(x_1) L_{m_2}^{(p_2)}(x_2) \dots L_{m_k}^{(p_k)}(x_k). \quad \text{MO 110m}$$

$$2. \quad L_n^{(p)}(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L_n^{(p+k)}(x). \quad \text{MO 110m}$$

7.149 Polinoamele Laguerre satisfac ecuația diferențială

$$x \frac{d^2 u}{dx^2} + (p-x+1) \frac{du}{dx} + nu = 0. \quad \text{Sm III 574 (34)}$$

7.2 FUNCȚII HIPERGEOMETRICE

7.20 Definiție

7.200 Se numește *serie hipergeometrică* seria

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1) \beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha+1)(\alpha+2) \beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots$$

7.201 Seria hipergeometrică se întrerupe, dacă α sau β devin egali cu un număr întreg negativ sau cu zero. Dacă $\gamma = -n$ (n este un număr negativ sau zero), atunci seria hipergeometrică este nedeterminată, dacă nici α , nici β nu sînt egali cu $-m$ ($m < n$, m natural). Totuși

$$1. \quad \lim_{\gamma \rightarrow -n} \frac{F(\alpha, \beta; \gamma; z)}{\Gamma(c)} = \frac{\alpha(\alpha+1) \dots (\alpha+n) \beta(\beta+1) \dots (\beta+n)}{(n+1)!} {}_{n+1}F(\alpha+n+1, \beta+n+1; n+2; z).$$

7.202 Excluzînd valorile menționate ale parametrilor α, β, γ , seria hipergeometrică este convergentă în cercul $|z|=1$. Aici au loc următoarele condiții de convergență:

1. $1 > \operatorname{Re}(\alpha + \beta - \gamma) \geq 0$. Seria este convergentă în tot cercul de rază unu, cu excepția punctului $z=1$.
2. $\operatorname{Re}(\alpha + \beta - \gamma) < 0$. Seria este (absolut) convergentă în tot cercul de rază unu, inclusiv punctul $z=1$.
3. $\operatorname{Re}(\alpha + \beta - \gamma) \geq 1$. Seria este divergentă în tot cercul de rază unu.

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7.21 Reprezentări integrale

$$7.211 \quad F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt$$

$$[\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0].$$

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$$7.212 \quad F(p, n+p; n+1; z^2) = \frac{z^{-n}}{2\pi} n B(p, n) \int_0^{2\pi} \frac{\cos nt \, dt}{(1-2z \cos t + z^2)^p}$$

$$[n=0, 1, 2, \dots; p \neq 0, -1, -2, \dots].$$

MO 16

$$7.213 \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t dt,$$

$|\arg(-z)| < \pi$, drumul de integrare alegîndu-se în așa fel, încît polurile funcțiilor $\Gamma(\alpha+t)$, $\Gamma(\beta+t)$ să fie situate în stînga drumului, iar polurile funcției $\Gamma(-t)$, în dreapta drumului.

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$$7.214 \quad F\left(-m, \frac{p+m}{2}; 1 - \frac{p+m}{2}; -1\right) = \frac{(-2)^m (p+m)}{\sin p\pi} \int_0^\pi \cos^m \varphi \cos p\varphi d\varphi$$

$$[m+1 \text{ natural}; 0 < \operatorname{Re}(p) < 1]$$

MO 16

7.22 Exprimarea funcțiilor elementare cu ajutorul funcției hipergeometrice

7.221

$$1. \quad F(-n, \beta; \beta; -z) = (1+z)^n \quad [\beta \text{ arbitrar}] \quad \text{Ga 127 I m}$$

$$2. \quad F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n + (t-z)^n}{2t^n}. \quad \text{Ga 127 II}$$

$$3. \quad \lim_{\omega \rightarrow \infty} F\left(-n, \omega; 2\omega; -\frac{z}{t}\right) = \frac{(t+z)^n + t^n}{2t^n}. \quad \text{Ga 127 III}$$

$$4. \quad F\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}. \quad \text{Ga 127 IV}$$

$$5. \quad F\left(1-n, 1; 2; -\frac{z}{t}\right) = \frac{(t+z)^n - t^n}{nzt^{n-1}}. \quad \text{Ga 127 V}$$

$$6. \quad F(1, 1; 2; -z) = \frac{\ln(1+z)}{z}. \quad \text{Ga 127 VI}$$

$$7. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = -\frac{\ln \frac{1+z}{1-z}}{2z}. \quad \text{Ga 127 VII}$$

8. $\lim_{k \rightarrow \infty} F\left(1, k; 1; \frac{z}{k}\right) = 1 + z \lim_{k \rightarrow \infty} F\left(1, k; 2; \frac{z}{k}\right) =$
 $= 1 + z + \frac{z^2}{2} \lim_{k \rightarrow \infty} F\left(1, k; 3; \frac{z}{k}\right) = \dots = e^z.$ Ga 127 VIII
9. $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z + e^{-z}}{2} = \operatorname{ch} z.$ Ga 127 IX
10. $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z - e^{-z}}{2z} = \frac{\operatorname{sh} z}{z}.$ Ga 127 X
11. $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; -\frac{z^2}{4kk'}\right) = \frac{\sin z}{z}.$ Ga 127 XI
12. $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; -\frac{z^2}{4kk'}\right) = \cos z.$ Ga 127 XII
13. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z}.$ Ga 127 XIII
14. $F\left(1, 1; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z \cos z}.$ Ga 127 XIV
15. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{z}{\operatorname{tg} z}.$ Ga 127 XV
16. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z}.$ Ga 127 XVI
17. $F\left(\frac{n+2}{2}, -\frac{n-2}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z \cos z}.$ Ga 127 XVII
18. $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}.$ Ga 127 XVIII
19. $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}.$ Ga 127 XIX
20. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz.$ Ga 127 XX
21. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}.$ Ga 127 XXI
22. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \frac{\cos nz}{\cos^n z}.$ Ga 127 XXII
23. $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \cos nz \cos^n z.$ Ga 127 XXIII
24. $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z} \quad \left[|z| \leq \frac{1}{2}, |z(1-z)| \leq \frac{1}{4}\right].$
25. $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z.$
26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z} \quad (\text{cf. 7.221 13.}).$
27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\operatorname{Arctg} z}{z} \quad (\text{cf. 7.221 15., precum și 1.644 1.}).$
28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{Arctg} z}{z} \quad (\text{cf. 7.221 26.}).$
29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz} \quad (\text{cf. 7.221 16.}).$

$$30. \quad F\left(1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz \sqrt{1-z^2}} \quad (\text{cf. 7.221 17.}).$$

$$31. \quad F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z) \quad (\text{cf. 7.221 20.}).$$

$$32. \quad F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}} \quad (\text{cf. 7.221 21.}).$$

Reprezentarea funcțiilor speciale prin funcția hipergeometrică, vezi:

pentru integralele eliptice complete, 6.113 1., 6.114 1.;
 pentru integralele funcțiilor cilindrice, 4.415, 4.432, 4.433 1.;
 pentru polinoamele Legendre 6.811, 6.816;
 (toate aceste serii hipergeometrice sînt cu un număr finit de termeni, adică
 aceste serii se transformă în sume finite);
 pentru funcțiile Legendre, 6.840, 6.857;
 pentru funcțiile sferice, 6.702, 6.703, 6.751, 6.77, 6.872, 6.873, 6.921;
 pentru polinoamele Cebîșev, 7.112 1.;
 pentru polinoamele Jacobi, 7.131 1.;
 pentru polinoamele Gegenbauer; $C_n^p(x)$, 7.131 4.;
 pentru integralele funcțiilor cilindrului parabolic, 4.824.

7.222 Valori particulare

$$1. \quad F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad \text{Ga 147 (48) F II 793}$$

$$2. \quad F(\alpha, \beta; \gamma; 1) = F(-\alpha, -\beta; \gamma - \alpha - \beta; 1) \quad [R(\gamma) > 0]; \quad \text{Ga 148 (49)}$$

$$= \frac{1}{F(-\alpha, \beta; \gamma - \alpha; 1)} \quad [R(\gamma - \beta) > 0]; \quad \text{Ga 148 (50)}$$

$$= \frac{1}{F(\alpha, -\beta; \gamma - \beta; 1)} \quad [R(\gamma - \alpha) > 0]; \quad \text{Ga 148 (51)}$$

$$3. \quad F\left(1, 1; \frac{3}{2}; \frac{1}{2}\right) = \frac{\pi}{2}.$$

7.23 Formule de transformare și prelungirea analitică a funcțiilor definite prin serii hipergeometrice

7.230 Seria $F(\alpha, \beta, \gamma; z)$ definește o funcție analitică, avînd, în general, singularități (în cazul general, puncte de ramificație) în punctele $z=0, 1, \infty$. Secționăm planul z de-a lungul axei reale, de la punctul $z=1$ pînă la punctul $z=\infty$, adică vom cere ca, pentru $|z| \geq 1$, să aibă loc inegalitatea $|\arg(-z)| < \pi$. Atunci, seria $F(\alpha, \beta, \gamma; z)$ va admite, în planul secționat, o prelungire analitică univocă, care (dacă $\gamma+1$ nu este un număr natural, iar $\alpha-\beta$ și $\gamma-\alpha-\beta$ nu sînt numere întregi) se efectuează cu ajutorul formulelor de mai jos. Aceste formule ne permit să calculăm valorile lui F în domeniul dat, și în cazul cînd $|z| > 1$. La acestea se mai adaugă alte cîteva formule de transformare, care, dacă sînt date relații corespunzătoare între α, β, γ , pot servi, de asemenea, ca prelungire analitică.

Formule de transformare

7.231

$$1. F(a, \beta; \gamma; z) = (1-z)^{-a} F\left(a, \gamma-\beta; \gamma; \frac{z}{z-1}\right); \quad \text{Ga 218 (91)}$$

$$= (1-z)^{-\beta} F\left(\beta, \gamma-a; \gamma; \frac{z}{z-1}\right); \quad \text{Ga 218 (92)}$$

$$= (1-z)^{\gamma-a-\beta} F(\gamma-a, \gamma-\beta; \gamma; z).$$

$$2. F(a, \beta; \gamma; z) = \frac{\Gamma(\gamma) \Gamma(\gamma-a-\beta)}{\Gamma(\gamma-a) \Gamma(\gamma-\beta)} F(a, \beta; a+\beta-\gamma+1; 1-z) +$$

$$+ z^{1-\gamma} (1-z)^{\gamma-a-\beta} \frac{\Gamma(\gamma) \Gamma(a+\beta-\gamma)}{\Gamma(a) \Gamma(\beta)} F(1-a, 1-\beta; \gamma-a-\beta+1; 1-z). \quad \text{Ga 213 (87)}$$

7.232

$$1. F(a, \beta; \gamma; z) = (1-z)^{-a} \frac{\Gamma(\gamma) \Gamma(\beta-a)}{\Gamma(\beta) \Gamma(\gamma-a)} F\left(a, \gamma-\beta; a-\beta+1; \frac{z}{1-z}\right) +$$

$$+ (1-z)^{-\beta} \frac{\Gamma(\gamma) \Gamma(a-\beta)}{\Gamma(a) \Gamma(\gamma-\beta)} F\left(\beta, \gamma-a; \beta-a+1; \frac{1}{1-z}\right). \quad \text{MO 13}$$

$$2. F(a, \beta; \gamma; z) =$$

$$= \frac{\Gamma(\gamma) \Gamma(\beta-a)}{\Gamma(\beta) \Gamma(\gamma-a)} (-1)^a z^{-a} F\left(a, a+1-\gamma; a+1-\beta; \frac{1}{z}\right) +$$

$$+ \frac{\Gamma(\gamma) \Gamma(a-\beta)}{\Gamma(a) \Gamma(\gamma-\beta)} (-1)^\beta z^{-\beta} F\left(\beta, \beta+1-\gamma; \beta+1-a; \frac{1}{z}\right). \quad \text{Ga 220 (93)}$$

$$7.233 \quad F\left(2a, 2\beta; a+\beta+\frac{1}{2}; z\right) = F\left[a, \beta; a+\beta+\frac{1}{2}; 4z(1-z)\right]$$

$$\left[|z| \leq \frac{1}{2}, |z(1-z)| \leq \frac{1}{4}\right]. \quad \text{WW II 85}$$

7.234

$$1. F(a, \beta; 2\beta; z) = \left(1 - \frac{z}{2}\right)^{-a} F\left[\frac{a}{2}, \frac{a+1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{2-z}\right)\right]. \quad \text{MO 13}$$

$$2. F(2a, 2a+1-\gamma; \gamma; z) = (1+z)^{-2a} F\left(a, a+\frac{1}{2}; \gamma; \frac{4z}{(1+z)^2}\right) \quad \text{Ga 225 (100)}$$

$$3. F\left(a, a+\frac{1}{2}-\beta; \beta+\frac{1}{2}; z^2\right) = (1+z)^{-2a} F\left(a, \beta; 2\beta; \frac{4z}{(1+z)^2}\right). \quad \text{Ga 225 (101)}$$

$$7.235. \quad F\left(a, \beta; a+\beta+\frac{1}{2}; \sin^2 \varphi\right) = F\left(2a, 2\beta; a+\beta+\frac{1}{2}; \sin^2 \frac{\varphi}{2}\right)$$

$$\left[x = \sin^2 \frac{\varphi}{2} \text{ real}, \frac{1-\sqrt{2}}{2} < x < \frac{1}{2}\right]. \quad \text{MO 13}$$

7.236

Notăm

$$A = \frac{\Gamma\left(a+\beta+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right) \Gamma\left(\beta+\frac{1}{2}\right)}, \quad B = \frac{\Gamma\left(a+\beta+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}\right)}{\Gamma(a) \Gamma(\beta)};$$

atunci

$$1. F\left(2a, 2\beta; a+\beta+\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) =$$

$$= AF\left(a, \beta; \frac{1}{2}; z\right) + B\sqrt{z} F\left(a+\frac{1}{2}, \beta+\frac{1}{2}; \frac{3}{2}; z\right). \quad \text{Ga 227 (106)}$$

$$2. F\left(2a, 2\beta; a+\beta+\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) =$$

$$= AF\left(a, \beta; \frac{1}{2}; z\right) - B\sqrt{z} F\left(a+\frac{1}{2}, \beta+\frac{1}{2}; \frac{3}{2}; z\right). \quad \text{Ga 228 (107)}$$

$$\begin{aligned}
 3. \quad & \frac{\left(\alpha - \frac{1}{2}\right) \left(\beta - \frac{1}{2}\right)}{\alpha + \beta - \frac{1}{2}} A \sqrt{z} F\left(\alpha, \beta; \frac{3}{2}; z\right) = \\
 & = F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1 + \sqrt{z}}{2}\right) - \\
 & - F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1 - \sqrt{z}}{2}\right). \quad \text{Ga 229 (110)}
 \end{aligned}$$

7.237 Formulele de recurență ale lui Gauss

1. $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z] F(\alpha, \beta; \gamma; z) +$
 $+ (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) + \gamma(\gamma - 1)(z - 1) F(\alpha, \beta; \gamma - 1; z) = 0.$
2. $(2\alpha - \gamma - \alpha z + \beta z) F(\alpha, \beta; \gamma; z) + (\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) +$
 $+ \alpha(z - 1) F(\alpha + 1, \beta; \gamma; z) = 0.$
3. $(2\beta - \gamma - \beta z + \alpha z) F(\alpha, \beta; \gamma; z) + (\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) +$
 $+ \beta(z - 1) F(\alpha, \beta + 1; \gamma; z) = 0.$
4. $\gamma F(\alpha, \beta - 1; \gamma; z) - \gamma F(\alpha - 1, \beta; \gamma; z) + (\alpha - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0.$
5. $\gamma(\alpha - \beta) F(\alpha, \beta; \gamma; z) - \alpha(\gamma - \beta) F(\alpha + 1, \beta; \gamma + 1; z) +$
 $+ \beta(\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) = 0.$
6. $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta; \gamma + 1; z) -$
 $- \alpha\beta z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
7. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) -$
 $- \alpha(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
8. $\gamma F(\alpha, \beta; \gamma; z) + (\beta - \gamma) F(\alpha + 1, \beta; \gamma + 1; z) -$
 $- \beta(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
9. $\gamma(\gamma - \beta z - \alpha) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) +$
 $+ \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
10. $\gamma(\gamma - \alpha z - \beta) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) +$
 $+ \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
11. $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha, \beta + 1; \gamma; z) + \alpha z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
12. $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha + 1, \beta; \gamma; z) + \beta z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
13. $\gamma[\alpha - (\gamma - \beta)z] F(\alpha, \beta; \gamma; z) - \alpha\gamma(1 - z) F(\alpha + 1, \beta; \gamma; z) +$
 $+ (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0.$
14. $\gamma[\beta - (\gamma - \alpha)z] F(\alpha, \beta; \gamma; z) - \beta\gamma(1 - z) F(\alpha, \beta + 1; \gamma; z) +$
 $+ (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0.$
15. $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta + 1; \gamma + 1; z) +$
 $+ \alpha(\gamma - \beta)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
16. $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha + 1, \beta; \gamma + 1; z) +$
 $+ \beta(\gamma - \alpha)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
17. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \beta) F(\alpha, \beta; \gamma + 1; z) - \beta F(\alpha, \beta + 1; \gamma + 1; z) = 0.$
18. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta; \gamma + 1; z) - \alpha F(\alpha + 1, \beta; \gamma + 1; z) = 0.$

7.24 Seria hipergeometrică generalizată

Seria

$$1. {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{\alpha_1^{[k]} \alpha_2^{[k]} \dots \alpha_p^{[k]}}{\beta_1^{[k]} \beta_2^{[k]} \dots \beta_q^{[k]}} \cdot \frac{z^k}{k!},$$

unde

$$2. \alpha^{[k]} = \alpha(\alpha+1) \dots (\alpha+k-1); \alpha^{[0]} = 1,$$

$$3. \beta^{[k]} = \beta(\beta+1) \dots (\beta+k-1); \beta^{[0]} = 1$$

se numește *seria hipergeometrică generalizată* (vezi și 7.31).

$$4. {}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z).$$

MO 14

MO 15

7.25 Ecuația diferențială hipergeometrică

7.251 Seria hipergeometrică este una dintre soluțiile ecuației diferențiale

$$z(1-z) \frac{d^2u}{dz^2} + [\gamma - (\alpha + \beta + 1)z] \frac{du}{dz} - \alpha\beta u = 0,$$

WW II 67

numită *hipergeometrică*.Rezolvarea ecuației diferențiale
hipergeometrice

7.252 Ecuația diferențială hipergeometrică 7.251 are două soluții liniar independente. Aceste soluții pot fi nemărginit prelungite analitic în tot planul z , cu excepția, poate, a trei puncte: $z=0, 1$ și ∞ . În general, punctele $z=0, 1, \infty$ sînt punctele de ramificație, cel puțin, pentru una din ramurile fiecăreia dintre soluțiile ecuației diferențiale hipergeometrice. Raportul $w(z)$ a două soluții liniar independente oarecare satisface ecuația diferențială

$$2 \frac{w''}{w'} - 3 \left(\frac{w''}{w'} \right)^2 = \frac{1-\alpha_1^2}{z^2} + \frac{1-\alpha_2^2}{(z-1)^2} + \frac{\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - 1}{z(z-1)},$$

unde

$$\alpha_1^2 = (1-\gamma)^2, \quad \alpha_2^2 = (\gamma - \alpha - \beta)^2, \quad \alpha_3^2 = (\alpha - \beta)^2.$$

Dacă α, β, γ sînt reali, atunci funcția $w(z)$ reprezintă semiplanul superior ($\text{Im}(z) > 0$) sau inferior ($\text{Im}(z) < 0$) pe triunghiul curbiliniu ale cărui unghiuri sînt $\pi\alpha_1, \pi\alpha_2, \pi\alpha_3$. Virfurile acestui triunghi sînt imaginile punctelor $z=0, z=1, z=\infty$.

7.253 În interiorul cercului de rază unu $|z| < 1$, soluțiile liniar independente $u_1(z)$ și $u_2(z)$ ale ecuației diferențiale hipergeometrice sînt date de următoarele formule:

1. Dacă γ nu este un număr întreg, atunci

$$u_1 = F(\alpha, \beta; \gamma; z),$$

$$u_2 = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z).$$

2. Dacă $\gamma = 1$, atunci

$$u_1 = F(\alpha, \beta; 1; z),$$

$$u_2 = F(\alpha, \beta; 1; z) \ln z +$$

$$+ \sum_{k=1}^{\infty} z^k \frac{\alpha^{[k]} \beta^{[k]}}{(k!)^2} \{ \psi(\alpha+k) - \psi(\alpha) + \psi(\beta+k) - \psi(\beta) + 2\psi(k+1) - 2\psi(1) \}.$$

(vezi 7.24 2.).

3. Dacă $\gamma = m+1$ (m natural) și, în același timp, atât α cât și β nu sînt numere pozitive mai mici sau egale cu m , atunci

$$u_1 = F(\alpha, \beta; m+1; z),$$

$$u_2 = F(\alpha, \beta; m+1; z) \ln z +$$

$$+ \sum_{k=1}^{\infty} z^k \frac{\alpha^{[k]} \beta^{[k]}}{(1+m)^{[k]} k!} \{h(k) - h(0)\} - \sum_{k=1}^m \frac{(k-1)! (-m)^{[k]}}{(1-\alpha)^{[k]} (1-\beta)^{[k]}} z^{-k} \quad (\text{v. 7.242.}),$$

unde

$$h(n) = \psi(\alpha+n) + \psi(\beta+n) - \psi(m+1+n) - \psi(n+1) \\ [n+1 \text{ natural}].$$

4. Fie $\gamma = m+1$ (m natural) și, în același timp, α sau β egal $m'+1$, unde $0 \leq m' < m$. Atunci, de exemplu, pentru $\alpha = m'+1$, obținem

$$u_1 = F(1+m', \beta; 1+m; z),$$

$$u_2 = z^{-m} F(1+m'-m, \beta-m; 1-m; z).$$

În acest caz, u_2 este un polinom în z^{-1} .

5. Dacă $\gamma = 1-m$ (m natural) și, în același timp, atât α cât și β sînt diferiți de numerele $0, -1, -2, \dots, 1-m$, atunci

$$u_1 = z^m F(\alpha+m, \beta+m; 1+m; z),$$

$$u_2 = z^m F(\alpha+m, \beta+m; 1+m; z) \ln z +$$

$$+ z^m \sum_{k=1}^{\infty} z^k \frac{(\alpha+m)^{[k]} (\beta+m)^{[k]}}{(1+m)^{[k]} k!} \{h^*(k) - h^*(0)\} - \\ - \sum_{k=1}^{\infty} \frac{(k-1)! (-m)^{[k]}}{(1-\alpha-m)^{[k]} (1-\beta-m)^{[k]}} z^{m-k} \quad (\text{vezi 7.242.}),$$

unde

$$h^*(n) = \psi(\alpha+m+n) + \psi(\beta+m+n) - \psi(1+m+n) - \psi(1+n).$$

Remarcăm că

$$\psi(\alpha+n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha+1} + \dots + \frac{1}{\alpha+n-1} \quad (\text{cf. 6.3583.})$$

și că, pentru $\alpha = -\lambda$, unde λ este un număr natural, și $n = \lambda+1, \lambda+2, \dots$, expresia

$$\alpha^{[k]} [\psi(\alpha+n) - \psi(\alpha)]$$

din formulele 7.253 2. — 7.253 5. trebuie înlocuită cu expresia

$$(-1)^{\lambda} \lambda! (n-\lambda-1)!.$$

6. Fie $\gamma = 1-m$ (m natural) și, în același timp, α sau β egal cu un număr întreg $-m'$, unde m' este unul dintre numerele $0, 1, \dots, m-1$. Fie, de exemplu, $\alpha = -m'$. Atunci

$$u_1 = F(-m', \beta; 1-m; z),$$

$$u_2 = F(-m', m; \beta+m; 1+m; z).$$

7. Pentru $\gamma = \frac{1}{2}(\alpha+\beta+1)$,

$$u_1 = F\left(\alpha, \beta; \frac{1}{2}(\alpha+\beta+1); z\right),$$

$$u_2 = F\left(\alpha, \beta; \frac{1}{2}(\alpha+\beta+1); 1-z\right)$$

sînt două soluții liniar independente ale ecuației diferențiale hipergeometrice, cu condiția ca α , β și γ să fie diferiți atît de zero, cît și de numerele negative întregi.

MO 17-19

Prelungirea analitică a soluției regulate în punctul $z=0$

7.254 Formulele 7.253 fac posibilă prelungirea analitică în domeniul $|z|>1$, $|\arg(-z)|<\pi$ a funcției $F(\alpha, \beta; \gamma; z)$, definită prin seria hipergeometrică în interiorul cercului $|z|<1$. Aici se presupune că $\alpha-\beta$ nu este un număr întreg. Dacă $\alpha-\beta$ este un număr întreg, de exemplu, dacă

$$\beta = \alpha + m \quad [m \text{ natural}],$$

atunci, pentru $|z|>1$, $|\arg(-z)|<\pi$, avem

$$\begin{aligned} 1. \quad & \frac{\Gamma(\alpha) \Gamma(\alpha+m)}{\Gamma(\gamma)} F(\alpha, \alpha+m; \gamma; z) = \\ & = \frac{\sin \pi(\gamma-\alpha)}{\pi} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha+k) \Gamma(1-\gamma+\alpha+k) \Gamma(m-k)}{k!} (-z)^{-\alpha-k} + \right. \\ & \left. + (-z)^{-\alpha-m} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+m+k) \Gamma(1-\gamma+\alpha+m+k)}{k! (k+m)!} g(k) z^{-k} \right\}, \end{aligned}$$

unde

$$\begin{aligned} 2. \quad & g(n) = \ln(-z) + \pi \operatorname{ctg} \pi(\gamma-\alpha) + \psi(n+1) + \psi(n+m+1) - \\ & - \psi(\alpha+m+n) - \psi(1-\gamma+\alpha+m+n). \end{aligned}$$

Pentru $m=0$, trebuie să punem $\sum_{k=0}^{m-1} = 0$.

7.255 Această formulă își pierde sensul, atunci cînd α , γ sau $\alpha-\gamma+1$ este egal cu unul dintre numerele $0, -1, -2, \dots$. În acest din urmă caz, avem:

1. Dacă α este un număr întreg și negativ sau zero, iar γ nu este un număr întreg, atunci $F(\alpha, \alpha+m; \gamma; z)$ este un polinom în z .

2. Fie γ un număr întreg și negativ, sau zero, iar α diferit de un număr întreg. Să luăm atunci $\gamma = -\lambda$, unde $\lambda = 0, 1, 2, \dots$. Atunci

$$\frac{\Gamma(\alpha+\lambda+1) \Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)} z^{\lambda+1} F(\alpha+\lambda+1, \alpha+\lambda+m+1; \lambda+2; z)$$

este o soluție a ecuației hipergeometrice, regulată în punctul $z=0$. Această soluție este egală cu membrul al doilea al formulei 7.254 1., dacă, în aceasta și în formula 7.254 2., se înlocuiește γ prin λ .

3. Dacă $\alpha-\gamma+1$ este un număr întreg și negativ sau zero, iar α și γ sînt diferiți de numere întregi, atunci putem folosi formula

$$F(\alpha, \alpha+m; \gamma; z) = (1-z)^{\gamma-2\alpha-m} F(\gamma-\alpha-m, \gamma-\alpha; \gamma; z)$$

și să ținem seama în ea de membrul al doilea al formulei 7.254 1., dacă $\gamma-\alpha-m>0$; dacă $\gamma-\alpha-m \leq 0$, atunci membrul al doilea al acestei expresii este un polinom înmulțit cu o putere al lui $1-z$.

4. Dacă α , β și γ sînt numere întregi, atunci ecuația diferențială hipergeometrică are întotdeauna o soluție, regulată pentru $z=0$ și avînd forma

$$R_1(z) + \ln(1-z) R_2(z),$$

unde $R_1(z)$ și $R_2(z)$ sînt funcții raționale de z . Pentru a putea scrie soluția sub această formă, trebuie să exprimăm funcția $F(\alpha, \beta; \gamma; z)$ cu ajutorul formulelor 7.237 1. — 7.237 3. Dacă $\gamma = -\lambda$, unde $\lambda+1$ este un număr natural, atunci formulele 7.237 1. și 7.237 2. nu se scriu pentru funcția $F(\alpha, \beta; \gamma; z)$, ci pentru funcția $z^{\lambda+1} \times F(\alpha+\lambda+1, \beta+\lambda+1, \lambda+2, z)$.

Prin aplicarea succesivă a formulelor menționate, valorile pozitive ale parametrilor pot fi reduse la 2, 1 și 0. Mai departe, din formulele

$$F(1, 1; 2; z) = -z^{-1} \ln(1-z),$$

$$F(0, \beta; \gamma; z) = F(\alpha, 0; \gamma; z) = 1$$

se obține forma menționată a soluției.

MO 19-20

7.26 Ecuația diferențială a lui Riemann

7.260 Ecuația diferențială hipergeometrică este un caz particular al ecuației diferențiale a lui Riemann.

$$1. \frac{d^2u}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right] \frac{du}{dz} + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right] \frac{u}{(z-a)(z-b)(z-c)} = 0.$$

WW I 284

Coefficienții acestei ecuații au poliuri în punctele a, b, c , iar numerele $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ se numesc *exponenții* care corespund acestor poliuri. Exponenții $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ sînt legați prin următoarea relație:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 1 = 0.$$

WW I 283

2. Ecuațiile diferențiale 7.260 1. se scriu schematic

$$3. \quad u = P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' \end{Bmatrix}.$$

Punctele singulare ale ecuației sînt scrise în schemă în prima linie; exponenții care le corespund sînt scriși în linia a doua și a treia, iar variabila independentă este scrisă în coloana a patra.

WW I 284

7.261 Formule de transformare pentru ecuația P a lui Riemann.

Sînt valabile următoarele două formule de transformare:

$$1. \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' \end{Bmatrix} = P \begin{Bmatrix} a & b & c \\ \alpha+k & \beta-k-l & \gamma+l & z \\ \alpha'+k & \beta'-k-l & \gamma'+l \end{Bmatrix}. \quad \text{WW I 284}$$

$$2. \quad P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' \end{Bmatrix} = P \begin{Bmatrix} a_1 & b_1 & c_1 \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' \end{Bmatrix}.$$

WW I 284

Prima dintre aceste formule arată: dacă

$$u = P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\},$$

atunci funcția

$$u_1 = \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l u$$

satisface ecuația diferențială de ordinul doi care are aceleași puncte singulare ca și ecuația 7.261 2. și exponenții egali cu $\alpha+k$, $\alpha'+k$; $\beta-k-l$, $\beta'-k-l$; $\gamma+l$, $\gamma'+l$. A doua formulă de transformare transformă ecuația diferențială cu singularități în punctele a , b , c , cu exponenții α , α' ; β , β' ; γ , γ' și cu variabila independentă z într-o ecuație diferențială cu aceiași exponenți, cu punctele singulare a_1 , b_1 , c_1 și cu variabila independentă z_1 . Variabila z_1 este legată de variabila z prin transformarea rațională liniară

$$z = \frac{Az_1 + B}{Cz_1 + D} \quad [AD - BC \neq 0].$$

Aceeași transformare rațională liniară leagă și punctele a_1 , b_1 , c_1 , de punctele a , b , c . WW I 285 MO 20

Rezolvarea ecuației diferențiale a lui Riemann

7.262 Cu ajutorul aplicării succesive a celor două formule de transformare 7.261 1. și 7.261 2., ecuația diferențială a lui Riemann se transformă în ecuația diferențială hipergeometrică; așadar, soluția ecuației diferențiale a lui Riemann se poate exprima prin funcția hipergeometrică.

Pentru $k = -\alpha$, $l = -\gamma$ și $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$ avem:

$$\begin{aligned} 1. \quad u &= P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\} = \\ &= \left(\frac{z-a}{z-b} \right)^{\alpha} \left(\frac{z-c}{z-b} \right)^{\gamma} P \left\{ \begin{matrix} a & b & c \\ 0 & \beta + \alpha + \gamma & 0 \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma \end{matrix} \quad z \right\} = \\ &= \left(\frac{z-a}{z-b} \right)^{\alpha} \left(\frac{z-c}{z-b} \right)^{\gamma} P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \beta + \alpha + \gamma & 0 \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma \end{matrix} \quad \frac{(z-a)(c-b)}{(z-b)(c-a)} \right\}. \quad \text{MO 23} \end{aligned}$$

Așadar, această soluție se exprimă în modul următor, cu ajutorul seriei hipergeometrice:

$$2. \quad u = \left(\frac{z-a}{z-b} \right)^{\alpha} \left(\frac{z-c}{z-b} \right)^{\gamma} F \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)} \right).$$

Dacă constantele a, b, c ; α, α' ; β, β' ; γ, γ' sint permutate în mod corespunzător, ecuația lui Riemann nu se schimbă. În felul acesta, se obține un ansamblu de 24 soluții ale ecuațiilor diferențiale care (cu condiția ca niciuna dintre diferențele $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ să nu fie un număr întreg) au următoarea formă:

7.263

1. $u_1 = \left(\frac{z-a}{z-b}\right)^a \left(\frac{z-c}{z-b}\right)^{\gamma} F \left\{ a+\beta+\gamma, a+\beta'+\gamma; 1+a-a'; \frac{(a-b)(z-a)}{(a-a)(z-b)} \right\}$
2. $u_2 = \left(\frac{z-a}{z-b}\right)^{a'} \left(\frac{z-c}{z-b}\right)^{\gamma} F \left\{ a'+\beta+\gamma, a'+\beta'+\gamma; 1+a'-a; \frac{(a-b)(z-a)}{(a-a)(z-b)} \right\}$
3. $u_3 = \left(\frac{z-a}{z-b}\right)^a \left(\frac{z-c}{z-b}\right)^{\gamma'} F \left\{ a+\beta+\gamma', a+\beta'+\gamma'; 1+a-a'; \frac{(a-b)(z-a)}{(a-a)(z-b)} \right\}$
4. $u_4 = \left(\frac{z-a}{z-b}\right)^{a'} \left(\frac{z-c}{z-b}\right)^{\gamma'} F \left\{ a'+\beta+\gamma', a'+\beta'+\gamma'; 1+a'-a; \frac{(a-b)(z-a)}{(a-a)(z-b)} \right\}$

7.264

1. $u_5 = \left(\frac{z-b}{z-c}\right)^{\beta} \left(\frac{z-a}{z-c}\right)^{\alpha} F \left\{ \beta+\gamma+\alpha, \beta+\gamma'+\alpha; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$
2. $u_6 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha} F \left\{ \beta'+\gamma+\alpha, \beta'+\gamma'+\alpha; 1+\beta'-\beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$
3. $u_7 = \left(\frac{z-b}{z-c}\right)^{\beta} \left(\frac{z-a}{z-c}\right)^{\alpha'} F \left\{ \beta+\gamma+\alpha', \beta+\gamma'+\alpha'; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$
4. $u_8 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha'} F \left\{ \beta'+\gamma+\alpha', \beta'+\gamma'+\alpha'; 1+\beta'-\beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$

7.265

1. $u_9 = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta} F \left\{ \gamma+\alpha+\beta, \gamma+\alpha'+\beta; 1+\gamma-\gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$
2. $u_{10} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta} F \left\{ \gamma'+\alpha+\beta, \gamma'+\alpha'+\beta; 1+\gamma'-\gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$
3. $u_{11} = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta'} F \left\{ \gamma+\alpha+\beta', \gamma+\alpha'+\beta'; 1+\gamma-\gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$
4. $u_{12} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta'} F \left\{ \gamma'+\alpha+\beta', \gamma'+\alpha'+\beta'; 1+\gamma'-\gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$

7.266

1. $u_{13} = \left(\frac{z-a}{z-c}\right)^a \left(\frac{z-b}{z-c}\right)^{\beta} F \left\{ a+\gamma+\beta, a+\gamma'+\beta; 1+a-a'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$
2. $u_{14} = \left(\frac{z-a}{z-c}\right)^{a'} \left(\frac{z-b}{z-c}\right)^{\beta} F \left\{ a'+\gamma+\beta, a'+\gamma'+\beta; 1+a'-a; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$
3. $u_{15} = \left(\frac{z-a}{z-c}\right)^a \left(\frac{z-b}{z-c}\right)^{\beta'} F \left\{ a+\gamma+\beta', a+\gamma'+\beta'; 1+a-a'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$
4. $u_{16} = \left(\frac{z-a}{z-c}\right)^{a'} \left(\frac{z-b}{z-c}\right)^{\beta'} F \left\{ a'+\gamma+\beta', a'+\gamma'+\beta'; 1+a'-a; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$

7.267

1. $u_{17} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^a = F \left\{ \gamma+\beta+\alpha, \gamma+\beta'+\alpha; 1+\gamma-\gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$
2. $u_{18} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^a = F \left\{ \gamma'+\beta+\alpha, \gamma'+\beta'+\alpha; 1+\gamma'-\gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$
3. $u_{19} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{a'} = F \left\{ \gamma+\beta+\alpha', \gamma+\beta'+\alpha'; 1+\gamma-\gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$
4. $u_{20} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{a'} = F \left\{ \gamma'+\beta+\alpha', \gamma'+\beta'+\alpha'; 1+\gamma'-\gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$

7.268

1. $u_{21} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma} = F\left\{\beta + \alpha + \gamma, \beta + \alpha' + \gamma; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$
2. $u_{22} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma} = F\left\{\beta' + \alpha + \gamma, \beta' + \alpha' + \gamma; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$
3. $u_{23} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma'} = F\left\{\beta + \alpha + \gamma', \beta + \alpha' + \gamma'; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$
4. $u_{24} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma'} = F\left\{\beta' + \alpha + \gamma', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$

WW II 68-69

7.27 Scrierea unor ecuații diferențiale de ordinul doi cu ajutorul schemei lui Riemann

7.271 Ecuația hipergeometrică (vezi 7.251)

$$u = P \begin{Bmatrix} 0 & \infty & 1 \\ 0 & \alpha & 0 \\ 1-\gamma & \beta & \gamma-\alpha-\beta \end{Bmatrix} z. \quad \text{WW II 78}$$

7.272 Ecuația lui Legendre, care definește funcțiile $P_n^m(z)$ [n și m întregi] (vezi 6.700 1.)

$$1. \quad u = P \begin{Bmatrix} 0 & \infty & 1 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{Bmatrix} \frac{1-z}{2}. \quad \text{WW II 120}$$

$$2. \quad u = P \begin{Bmatrix} 0 & \infty & 1 \\ -\frac{1}{2}n & \frac{1}{2}m & 0 \\ \frac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} \end{Bmatrix} \frac{1}{1-z^2}. \quad \text{WW II 134}$$

7.273 Funcția $P_n^m\left(1 - \frac{z^2}{2n^2}\right)$ satisface ecuația

$$u = P \begin{Bmatrix} 4n^2 & \infty & 0 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{Bmatrix} z^3. \quad \text{WW II 168}$$

Funcția $J_m(z)$ satisface forma limită a acestei ecuații, care se obține pentru $n \rightarrow \infty$.7.274 Ecuația care definește polinoamele $C_n^p(z)$ (vezi 6.96):

$$u = P \begin{Bmatrix} -1 & \infty & 1 \\ \frac{1}{2}-p & n+2p & \frac{1}{2}-p \\ 0 & -n & 0 \end{Bmatrix} z. \quad \text{WW II 135}$$

7.275 Ecuația lui Bessel (vezi 6.401 1.) este forma limită a ecuațiilor

$$1. \quad u = P \begin{Bmatrix} 0 & \infty & c \\ n & ic & \frac{1}{2} + ic \\ -n & -ic & \frac{1}{2} - ic \end{Bmatrix} z, \quad \text{WW II 181}$$

$$2. \quad u = e^{iz} P \begin{pmatrix} 0 & \infty & c \\ n & \frac{1}{2} & 0 \\ -n & \frac{3}{2} - 2ic & 2ic - 1 \end{pmatrix} z, \quad \text{WW II 181m}$$

$$3. \quad u = P \begin{pmatrix} 0 & \infty & c^2 \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 \end{pmatrix} z^2, \quad \text{WW II 181}$$

care se obțin pentru $c \rightarrow \infty$.

7.3 FUNCȚIA HIPERGEOMETRICĂ DEGENERATĂ

7.30 Introducere

7.301 Funcția hipergeometrică degenerată se obține în urma trecerii la limită pentru $c \rightarrow +\infty$ în soluția ecuației diferențiale a lui Riemann.

$$P \begin{pmatrix} 0 & \infty & c \\ \frac{1}{2} + \mu & -c & c - \lambda \\ \frac{1}{2} - \mu & 0 & \lambda \end{pmatrix} z. \quad \text{WW II 139}$$

7.302 Ecuația care se obține în urma acestei treceri la limită are forma

$$1. \quad \frac{d^2 u}{dz^2} + \frac{du}{dz} + \left(\frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0. \quad \text{WW II 139}$$

Ecuația 7.302 1. are următoarele două soluții liniar independente:

$$2. \quad z^{\frac{1}{2} + \mu} e^{-z} {}_1F_1 \left(\frac{1}{2} + \mu - \lambda; 2\mu + 1; z \right),$$

$$3. \quad z^{\frac{1}{2} - \mu} e^{-z} {}_1F_1 \left(\frac{1}{2} - \mu - \lambda; -2\mu + 1; z \right),$$

care sînt definite pentru toate valorile lui $\mu \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$ MO 111

7.31 Funcția ${}_1F_1(\alpha; \gamma; z)$

7.310 Seria

$${}_1F_1(\alpha; \gamma; z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} + \dots$$

se numește, de asemenea, funcție hipergeometrică degenerată. MO 111

7.311 Reprezentare integrală

$$1. \quad {}_1F_1(\alpha; \gamma; z) = \frac{2^{1-\gamma} e^{\frac{1}{2}z}}{\Gamma(\alpha, \gamma-\alpha)} \int_{-1}^1 (1-t)^{\gamma-\alpha-1} (1+t)^{\alpha-1} e^{\frac{1}{2}zt} dt$$

$$[0 < \operatorname{Re}(\alpha) < \operatorname{Re}(\gamma)].$$

MO 114

$$2. {}_1F_1(\alpha; \gamma; z) = \frac{1}{\Gamma(\alpha, \gamma - \alpha)} z^{1-\gamma} \int_0^z e^t t^{\alpha-1} (z-t)^{\gamma-\alpha-1} dt$$

$$[0 < \operatorname{Re}(\alpha) < \operatorname{Re}(\gamma)]. \quad \text{MO 114}$$

$$3. {}_1F_1(-\nu; \alpha + 1; z) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \nu + 1)} e^z z^{-\frac{\alpha}{2}} \int_0^\infty e^{-t} t^{\nu + \frac{\alpha}{2}} J_\alpha(2\sqrt{zt}) dt$$

$$[\operatorname{Re}(\alpha + \nu + 1) > 0, |\arg z| < \frac{\pi}{2}] \quad (\text{cf. 5.257 4.}) \quad \text{MO 115}$$

Relații funcționale

7.312

$$1. {}_1F_1(\alpha; \gamma; z) = e^z {}_1F_1(\gamma - \alpha; \gamma; -z). \quad \text{MO 112}$$

$$2. \frac{z}{\gamma} {}_1F_1(\alpha + 1; \gamma + 1; z) = {}_1F_1(\alpha + 1; \gamma; z) - {}_1F_1(\alpha; \gamma; z).$$

$$3. \alpha {}_1F_1(\alpha + 1; \gamma + 1; z) = (\alpha - \gamma) {}_1F_1(\alpha; \gamma + 1; z) + \gamma {}_1F_1(\alpha; \gamma; z). \quad \text{MO 112}$$

$$4. \alpha {}_1F_1(\alpha + 1; \gamma; z) =$$

$$= (z + 2\alpha - \gamma) {}_1F_1(\alpha; \gamma; z) + (\gamma - \alpha) {}_1F_1(\alpha - 1; \gamma; z). \quad \text{MO 112}$$

$$7.313 \quad \frac{d {}_1F_1(\alpha; \gamma; z)}{dz} = \frac{\alpha}{\gamma} {}_1F_1(\alpha + 1; \gamma + 1; z). \quad \text{MO 112}$$

$$7.314 \quad \lim_{\gamma \rightarrow -n} \frac{1}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; z) = z^{n+1} \binom{\alpha+n}{n+1} {}_1F_1(\alpha + n + 1; n + 2; z)$$

$$[n = 0, 1, 2, \dots]. \quad \text{MO 112}$$

7.315

$$1. {}_1F_1(\alpha; \alpha; z) = e^z. \quad \text{MO 15}$$

$$2. {}_1F_1(\alpha; 2\alpha; 2z) = 2^{\alpha - \frac{1}{2}} \exp\left[\frac{1}{4}(1 - 2\alpha)\pi i\right] \times$$

$$\times \Gamma\left(\alpha + \frac{1}{2}\right) e^z z^{\frac{1}{2} - \alpha} J_{\alpha - \frac{1}{2}}\left(ze^{\frac{\pi}{2}i}\right). \quad \text{MO 112}$$

$$3. {}_1F_1\left(p + \frac{1}{2}; 2p + 1; 2iz\right) = \Gamma(p + 1) \left(\frac{z}{2}\right)^{-p} e^{iz} J_p(z). \quad \text{MO 15}$$

Reprezentarea funcțiilor speciale prin funcția hipergeometrică degenerată

${}_1F_1(\alpha; \gamma; z)$, vezi

pentru integrala de probabilitate, 7.336;

pentru integralele funcțiilor cilindrice, 4.434;

pentru polinoamele Hermite, 7.123, 7.129;

pentru polinoamele Laguerre, 7.142 1;

pentru funcțiile cilindrului parabolic, 7.340;

pentru funcția $M_{\lambda, \mu}(z)$, 7.320 2., 7.320 3.;

pentru funcția $W_{p, q}(z)$ 7.339.

7.316 Funcția ${}_1F_1(\alpha; \gamma; z)$ este soluția ecuației diferențiale

$$1. z \frac{d^2 F}{dz^2} + (\gamma - z) \frac{dF}{dz} - \alpha F = 0. \quad \text{MO 111}$$

Această ecuație are două soluții liniar independente:

$$2. {}_1F_1(\alpha; \gamma; z) \text{ și}$$

$$3. z^{1-\gamma} {}_1F_1(\alpha - \gamma + 1; 2 - \gamma; z). \quad \text{MO 112}$$

7.32-7.33 Funcțiile $M_{\lambda, \mu}(z)$ și $W_{\lambda, \mu}(z)$ ale lui Whittaker

7.320 Efectuind în ecuația 7.302 1. schimbarea de variabilă $u = e^{-\frac{z}{2}}W$, ajungem la ecuația

$$1. \frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) W = 0. \quad \text{MO 115}$$

Ecuația 7.320 1. are următoarele două soluții liniar independente:

$$2. M_{\lambda, \mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z\right),$$

$$3. M_{\lambda, -\mu}(z) = z^{-\mu + \frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(-\mu - \lambda + \frac{1}{2}; -2\mu + 1; z\right). \quad \text{MO 115}$$

Pentru obținerea soluțiilor valabile și pentru $2\mu = \pm 1, \pm 2, \pm 3, \dots$, se introduce funcția Whittaker

$$4. W_{\lambda, \mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda, -\mu}(z), \quad \text{WW II 152}$$

care, pentru 2μ tinzând către un număr întreg, servește și ea ca soluție a ecuației 7.320 1.

Pentru funcțiile $M_{\lambda, \mu}(z)$ și $W_{\lambda, \mu}(z)$, $z = 0$ este un punct de ramificație, iar $z = \infty$ este un punct esențial singular. De aceea, vom considera aceste funcții numai pentru $|\arg z| < \pi$.

Funcțiile $W_{\lambda, \mu}(z)$ și $W_{-\lambda, \mu}(-z)$ sînt soluții liniar independente ale ecuației 7.320 1.

Reprezentări integrale

7.321

$$M_{\lambda, \mu}(z) = \frac{z^{\mu + \frac{1}{2}}}{2^{2\mu} B\left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}\right)} \int_{-1}^1 (1+t)^{\mu - \lambda - \frac{1}{2}} (1-t)^{\mu + \lambda - \frac{1}{2}} e^{\frac{1}{2}zt} dt, \quad \text{WW II 159}$$

dacă integrala este convergentă. Vezi și 4.433 3., 4.81 1., 5.257 2.

7.322

$$1. W_{\lambda, \mu}(z) = \frac{z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}}}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} \int_0^\infty e^{-zt} t^{\mu - \lambda - \frac{1}{2}} (1+t)^{\mu + \lambda - \frac{1}{2}} dt. \quad \text{MO 118}$$

$$2. W_{\lambda, \mu}(z) = \frac{z^{\frac{1}{2}} e^{-\frac{z}{2}}}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} \int_0^\infty t^{\mu - \lambda - \frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu + \lambda - \frac{1}{2}} dt \quad \text{WW II 143}$$

$$\left[\operatorname{Re}(\mu - \lambda) > -\frac{1}{2}, |\arg z| < \pi \right].$$

$$7.323 \quad W_{\lambda, \mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(u-\lambda) \Gamma\left(-u-\mu+\frac{1}{2}\right) \Gamma\left(-u+\mu+\frac{1}{2}\right)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right) \Gamma\left(-\lambda-\mu+\frac{1}{2}\right)} z^u du$$

[drumul de integrare se alege în așa fel, încît polurile funcției $\Gamma(u-\lambda)$ să nu coincidă cu polurile funcțiilor $\Gamma\left(-u-\mu+\frac{1}{2}\right)$ și $\Gamma\left(-u+\mu+\frac{1}{2}\right)$].

MO 118

Vezi și 4.515.2.

$$7.324 \quad W_{\mu, \frac{1}{2}+\mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^{\infty} (1+t)^{2\mu} e^{-zt} dt = \\ = z^{-\mu} e^{-\frac{1}{2}z} \int_z^{\infty} t^{2\mu} e^{-t} dt \quad [\operatorname{Re}(z) > 0].$$

WW II 160

7.325

$$1. \quad W_{\lambda, \mu}(x) W_{-\lambda, \mu}(x) = -x \int_0^{\infty} \operatorname{th} 2t \frac{t}{2} \{J_{2\mu}(x \operatorname{sh} t) \sin(\mu-\lambda)\pi + \\ + N_{2\mu}(x \operatorname{sh} t) \cos(\mu-\lambda)\pi\} dt \quad \left[|\operatorname{Re}(\mu)| - \operatorname{Re}(\lambda) < \frac{1}{2}; x > 0\right].$$

MO 119

$$2. \quad W_{x, \mu}(z_1) W_{\lambda, \mu}(z_2) = \frac{(z_1 z_2)^{\mu+\frac{1}{2}} \exp\left[-\frac{1}{2}(z_1+z_2)\right]}{\Gamma(1-x-\lambda)} \times \\ \times \int_0^{\infty} e^{-t} t^{-x-\lambda} (z_1+t)^{-\frac{1}{2}+x-\mu} (z_2+t)^{-\frac{1}{2}+\lambda-\mu} \\ \times F\left(\frac{1}{2}-x+\mu, \frac{1}{2}-\lambda+\mu; 1-x-\lambda; \theta\right) dt,$$

unde

$$\theta = \frac{t(z_1+z_2+t)}{(z_1+t)(z_2+t)}$$

$$[z_1 \neq 0, z_2 \neq 0, |\arg z_1| < \pi, |\arg z_2| < \pi, \operatorname{Re}(x+\lambda) < 1].$$

MO 119

Vezi și 4.441, 5.215, 5.216.2.

7.326 Dezvoltări în serie

$$M_{0, \mu}(z) = z^{\frac{1}{2}+\mu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{4k} k! (\mu+1)(\mu+2) \dots (\mu+k)} \right\}. \quad \text{WW II 141}$$

Dezvoltări asimptotice

7.327 Pentru valori mari ale lui $|z|$,

$$W_{\lambda, \mu}(z) \sim e^{-\frac{z}{2}} z^{\lambda} \left(1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] \left[\mu^2 - \left(\lambda - \frac{3}{2}\right)^2\right] \dots \left[\mu^2 - \left(\lambda - k + \frac{1}{2}\right)^2\right]}{k! z^k} \right)$$

$$[|\arg z| \leq \pi - \alpha < \pi].$$

WW II 147

7.328 Pentru valori mari ale indicelui $|\lambda|$,

$$M_{\lambda, \mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu+1) \lambda^{-\mu-\frac{1}{4}} z^{\frac{1}{4}} \cos\left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi\right). \quad \text{MO 118}$$

7.329

$$1. W_{\lambda, \mu} \sim - \left(\frac{4z}{\lambda} \right)^{\frac{1}{4}} e^{-\lambda + 1 \ln \lambda} \sin \left(2 \sqrt{\lambda z} - \lambda \pi - \frac{\pi}{4} \right). \quad \text{MO 118}$$

$$2. W_{-\lambda, \mu} \sim \left(\frac{z}{4\lambda} \right)^{\frac{1}{4}} e^{\lambda - 1 \ln \lambda - 2 \sqrt{\lambda z}}. \quad \text{MO 118}$$

[Formulele 7.328 și 7.329 sînt aplicabile pentru

$$|\lambda| \gg 1, \quad |\lambda| \gg |z|, \quad |\lambda| \gg |\mu|, \quad z \neq 0, \quad |\arg \sqrt{z}| < \frac{3\pi}{4} \text{ și } |\arg \lambda| < \frac{\pi}{2}]. \quad \text{MO 118}$$

Relații funcționale

7.331

$$1. M_{n+\mu+\frac{1}{2}, \mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{1}{2}z}}{(2\mu+1)(2\mu+2)\dots(2\mu+n+1)} \frac{d^n}{dz^n} (z^{n+2\mu} e^{-z})$$

$$[n=0, 1, 2, \dots; 2\mu \neq -1, -2, -3, \dots]. \quad \text{MO 117}$$

$$2. z^{-\frac{1}{2}-\mu} M_{\lambda, \mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda, \mu}(-z) [2\mu \neq -1, -2, -3, \dots]. \quad \text{WW II 140}$$

7.332

$$1. W_{\lambda, \mu}(z) = W_{\lambda, -\mu}(z). \quad \text{MO 116}$$

$$2. W_{-\lambda, \mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu+\lambda\right)} M_{-\lambda, \mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu+\lambda\right)} M_{-\lambda, -\mu}(-z).$$

$$|\arg(-z)| < \frac{3}{2}\pi. \quad \text{WW II 152}$$

7.333

$$1. M_{\lambda, \mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{i\pi\lambda} W_{-\lambda, \mu}(e^{i\pi}z) +$$

$$+ \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda, \mu}(z)$$

$$\left[-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}; 2\mu \neq -1, -2, \dots\right]. \quad \text{MO 117}$$

$$2. M_{\lambda, \mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{-i\pi\lambda} W_{-\lambda, \mu}(e^{-i\pi}z) +$$

$$+ \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[-i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda, \mu}(z)$$

$$\left[-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi; 2\mu \neq -1, -2, \dots\right]. \quad \text{MO 117}$$

7.334 Formule de recurență

$$1. W_{\mu, \lambda}(z) = \sqrt{z} W_{\mu-\frac{1}{2}, \lambda-\frac{1}{2}}(z) + \left(\frac{1}{2} + \lambda - \mu\right) W_{\mu-1, \lambda}(z); \quad \text{WW II 159}$$

$$2. \quad W_{\mu, \lambda}(z) = \sqrt{z} W_{\mu - \frac{1}{2}, \lambda + \frac{1}{2}}(z) + \left(\frac{1}{2} - \lambda - \mu\right) W_{\mu - 1, \lambda}(z). \quad \text{WW II 159}$$

$$3. \quad z \frac{d}{dz} W_{\lambda, \mu}(z) = \left(\lambda - \frac{1}{2} z\right) W_{\lambda, \mu}(z) - \left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] W_{\lambda - 1, \mu}(z). \\ \text{WW II 159}$$

$$4. \quad \left[\left(\mu + \frac{1-z}{2}\right) W_{\lambda, \mu}(z) - z \frac{d}{dz} W_{\lambda, \mu}(z)\right] \left(\mu + \frac{1}{2} + \lambda\right) = \\ = \left[\left(\mu + \frac{1+z}{2}\right) W_{\lambda, \mu+1}(z) + z \frac{d}{dz} W_{\lambda, \mu+1}(z)\right] \left(\mu + \frac{1}{2} - \lambda\right). \quad \text{MO 117}$$

$$5. \quad \left(\frac{3}{2} + \lambda + \mu\right) \left(\frac{1}{2} + \lambda + \mu\right) z W_{\lambda, \mu}(z) = z(z + 2\mu + 1) \frac{d}{dz} W_{\lambda+1, \mu+1}(z) + \\ + \left[\frac{1}{2} z^2 + \left(\mu - \lambda - \frac{1}{2}\right) z + 2\mu^2 + 2\mu + \frac{1}{2}\right] W_{\lambda+1, \mu+1}(z). \quad \text{MO 117}$$

Legătura cu alte funcții

7.335

$$1. \quad M_{0, \mu}(z) = 2^{2\mu} \Gamma(\mu + 1) \sqrt{z} I_{\mu}\left(\frac{z}{2}\right). \quad \text{MO 125}$$

$$2. \quad W_{0, \mu}(z) = \sqrt{\frac{z}{\pi}} K_{\mu}\left(\frac{z}{2}\right). \quad \text{MO 125}$$

$$7.336 \quad \Phi(x) = 1 - \frac{e^{-\frac{x^2}{2}}}{\sqrt{\pi x}} W_{-\frac{1}{4}, \frac{1}{4}}(x^2) = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right).$$

WW II 144 MO 126

$$7.337 \quad \text{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln \frac{1}{z}}} W_{-\frac{1}{2}, 0}(-\ln z). \quad \text{WW II 145}$$

$$7.338 \quad W_{\lambda, \mu}(z) = \frac{(-1)^{2\mu} z^{\mu + \frac{1}{2}} e^{-\frac{1}{2} z}}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right) \Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \times \\ \times \left\{ \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - \lambda + \frac{1}{2}\right)}{k! (2\mu + k)!} z^k \left[\psi(k+1) + \psi(2\mu + k+1) - \psi\left(\mu + k - \lambda + \frac{1}{2}\right) - \ln z \right] + \right. \\ \left. + (-z)^{2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu - k) \Gamma\left(k - \mu - \lambda + \frac{1}{2}\right)}{k!} (-z)^k \right\} \\ \left[|\arg z| < \frac{3\pi}{2}; \quad 2\mu + 1 \text{ natural} \right]. \quad \text{MO 116}$$

7.339 Fie $\lambda - \mu - \frac{1}{2} = l$, unde $l + 1$ este un număr natural. Atunci

$$W_{l+\mu+\frac{1}{2}, \mu}(z) = (-1)^l z^{\mu + \frac{1}{2}} e^{-\frac{1}{2} z} (2\mu + 1)(2\mu + 2) \dots (2\mu + l) {}_1F_1(-l; 2\mu + 1; z) = \\ = (-1)^l z^{\mu + \frac{1}{2}} e^{-\frac{1}{2} z} L_l^{(2\mu)}(z). \quad \text{MO 116}$$

7.34-7.35 Funcțiile cilindrului parabolic, $D_p(z)$

7.340 Funcțiile

$$D_p(z) = 2^{\frac{1}{4} + \frac{p}{2}} z^{-\frac{1}{2}} W_{\frac{1}{4} + \frac{p}{2}, -\frac{1}{4}}\left(\frac{z^2}{2}\right) =$$

$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} {}_1F_1\left(-\frac{p}{2}; \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{p}{2}\right)} {}_1F_1\left(\frac{1-p}{2}; \frac{3}{2}; \frac{z^2}{2}\right) \right\}$$

MO 120

se numesc funcțiile cilindrului parabolic.

Reprezentări integrale

7.341

$$1. D_p(z) = \frac{1}{\sqrt{\pi}} 2^{\frac{p+1}{2}} e^{-\frac{\pi}{2} p i} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2 + 2ixz} dx$$

[Re $p > -1$; pentru $x < 0$; $\arg t^p = p\pi i$]. MO 122.

$$2. D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^{\infty} e^{-xz - \frac{x^2}{2}} x^{-p-1} dx \quad [\operatorname{Re}(p) < 0] \quad (\text{cf. 5.221 3}).$$

MO 122

7.342

$$1. D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4} z^2 (0+)} \int_{\infty}^{-\infty} e^{-zt - \frac{1}{2} t^2} (-t)^{-p-1} dt$$

[$|\arg(-t)| \leq \pi$]. WW II 157

$$2. D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2} + 1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4} z^2 t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)} dt$$

[$|\arg z| < \frac{\pi}{4}$; $|\arg(1+t)| \leq \pi$]. WW II 161

$$3. D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4} z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2} t - \frac{1}{2} p\right) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

[$|\arg z| < \frac{3}{4} \pi$; p nu este un număr întreg și pozitiv]. WW II 161

$$4. D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4} z^2} \int_{\infty}^{(0-)} \frac{\Gamma\left(\frac{1}{2} t - \frac{1}{2} p\right) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

[pentru toate valorile lui $\arg z$; drumul de integrare înconjoară polurile funcției $\Gamma(-t)$, dar nu înconjoară polurile funcției $\Gamma\left(\frac{1}{2} t - \frac{1}{2} p\right)$]. WW II 161

7.343

$$1. D_n(z) = (-1)^n \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} (\sqrt{n})^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt + \int_0^{\infty} \left[e^{\frac{1}{2}n(1-t^2)} t^n - e^{-n(t-1)^2} \right] \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt - \int_{-\infty}^0 e^{-n(t-1)^2} \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt \right\}$$

[n natural].

WW II 162

$$2. D_n(z) = (-1)^n 2^{n+2} (2\pi)^{-\frac{1}{2}} e^{\frac{1}{4}z^2} \int_0^{\infty} t^n e^{-2t^2} \frac{\cos(2zt)}{\sin(2zt)} dt$$

[n natural, $\mu = E\left(\frac{n}{2}\right)$; se consideră cosinusul sau sinusul, după cum n este un număr par sau impar].

WW II 162

7.344

$$1. D_{-p-1}[(1+i)z] = \frac{e^{\frac{iz^2}{2}}}{2^{\frac{p-1}{2}} \Gamma\left(\frac{p+1}{2}\right)} \int_0^{\infty} \frac{e^{-iz^2x^2} x^p}{(1+x^2)^{1+\frac{p}{2}}} dx$$

[$\operatorname{Re}(p) > -1$, $\operatorname{Re}(iz^2) \geq 0$].

MO 122

$$2. D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma\left(-\frac{p}{2}\right)} \int_1^{\infty} e^{-\frac{i}{2}z^2x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}} dx$$

[$\operatorname{Re}(p) < 0$; $\operatorname{Re}(iz^2) \geq 0$].

Vezi și 5.216 1.

MO 122

7.345

$$1. D_p(x) D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^{\infty} \operatorname{cth}^{\frac{p+1}{2}} \frac{t}{2} \frac{1}{\sqrt{\operatorname{sh} t}} \sin \frac{x^2 \operatorname{sh} t + p\pi}{2} dt$$

[x real, $\operatorname{Re}(p) < 0$].

MO 122

$$2. D_p\left(ze^{\frac{\pi}{4}i}\right) D_p\left(ze^{-\frac{\pi}{4}i}\right) = \frac{1}{\Gamma(-p)} \int_0^{\infty} \operatorname{cth}^p t \exp\left(-\frac{z^2 \operatorname{sh} 2t}{2}\right) \frac{dt}{\operatorname{sh} t}$$

[$|\arg z| < \frac{\pi}{4}$; $\operatorname{Re}(p) < 0$].

MO 122

Vezi și 4.423.

7.346 Dezvoltări asimptotice. Dacă $|z| \gg 1$, $|z| \gg p$, atunci

$$1. D_p(z) \sim e^{\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$

[$|\arg z| < \frac{3}{4}\pi$].

MO 121

$$2. D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{\frac{z^2}{4}} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right) \left[\frac{\pi}{4} < \arg z < \frac{5}{4}\pi \right]. \quad \text{MO 121}$$

$$3. D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{\frac{z^2}{4}} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right) \left[-\frac{\pi}{4} > \arg z > -\frac{5}{4}\pi \right]. \quad \text{MO 121}$$

Relații funcționale

7.347 Formule de recurență

$$1. D_{p+1}(z) - zD_p(z) + pD_{p-1}(z) = 0. \quad \text{WW II 157}$$

$$2. \frac{d}{dz} D_p(z) + \frac{1}{2} zD_p(z) - pD_{p-1}(z) = 0. \quad \text{WW II 157}$$

$$3. \frac{d}{dz} D_p(z) - \frac{1}{2} zD_p(z) + D_{p+1}(z) = 0. \quad \text{MO 121}$$

7.348 Relații liniare

$$1. D_p(z) = \frac{\Gamma(p+1)}{\sqrt{2\pi}} [e^{\frac{\pi}{2}pi} D_{-p-1}(iz) + e^{-\frac{\pi}{2}pi} D_{-p-1}(-iz)];$$

$$2. = e^{-p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-\frac{\pi}{2}(p+1)i} D_{-p-1}(iz);$$

$$3. = e^{p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{\frac{\pi}{2}(p+1)i} D_{-p-1}(-iz). \quad \text{MO 121}$$

$$7.349 D_p[(1+i)x] + D_p[-(1+i)x] =$$

$$= \frac{2^{1+\frac{p}{2}}}{\Gamma(-p)} \exp \left[-\frac{i}{2} \left(x^2 + \frac{\pi}{2} \right) \right] \int_0^\infty \frac{\cos xt}{i^{p+1}} e^{-\frac{i}{4}t^2} dt$$

[x real; $-1 < \operatorname{Re}(p) < 0$]. MO 122

$$7.351 D_n(z) = (-1)^n e^{\frac{z^2}{4}} \frac{d^n}{dz^n} \left(e^{-\frac{z^2}{2}} \right) \quad [n=0, 1, 2, \dots]. \quad \text{WW II 157}$$

$$7.352 D_p(ax+by) = \exp \frac{(bx-ay)^2}{4} \left(\frac{a}{\sqrt{a^2+b^2}} \right)^p \times$$

$$\times \sum_{k=0}^{\infty} \binom{p}{k} D_{p-k}(\sqrt{a^2+b^2}x) D_k(\sqrt{a^2+b^2}y) \left(\frac{b}{a} \right)^k$$

[$a > b > 0, x > 0, y > 0, \operatorname{Re}(p) \geq 0$]
[„teorema de compunere“]. MO 124

Legătura cu alte funcții

$$7.353 D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n \left(\frac{z}{\sqrt{2}} \right). \quad \text{MO 123m}$$

7.354

$$1. D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right) \right]. \quad \text{MO 123}$$

$$2. D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{ z \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right) \right] - \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} \right\}. \quad \text{MO 123}$$

7.355. Ecuațiile diferențiale care duc la funcțiile cilindrului parabolic

$$1. \frac{d^2 u}{dz^2} + \left(p + \frac{1}{2} - \frac{1}{4} z^2 \right) u = 0,$$

$$u = D_p(z), D_p(-z), D_{-p-1}(iz), D_{-p-1}(-iz)$$

(aceste patru soluții nu sînt liniar independente, vezi 7.348). WW II 153 MO 120

$$2. \frac{d^2 u}{dz^2} + (x^2 + \lambda) u = 0, \quad u = D_{-\frac{1+\lambda}{2}} [\pm (1+i)x]. \quad \text{MO 123}$$

$$3. \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (p+1) u = 0, \quad u = e^{-\frac{z^2}{4}} D_p(z). \quad \text{MO 123}$$

7.4 FUNCȚIA ζ A LUI RIEMANN: $\zeta(z, q)$ ȘI $\zeta(z)$

7.41 Definiție și reprezentări integrale

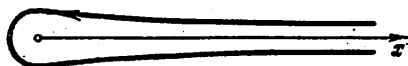
$$7.411 \quad \zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1-e^{-t}} dt; \quad \text{WW II 45}$$

$$= \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2 \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \left[\sin\left(z \arctg \frac{t}{q}\right) \right] \frac{dt}{e^{2\pi t} - 1}$$

$$[0 < q < 1, \operatorname{Re}(z) > 1].$$

WW II 50

$$7.412 \quad \zeta(z, q) = -\frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0+)} \frac{(-\theta)^{z-1} e^{-q\theta}}{1-e^{-\theta}} d\theta.$$

Această relație este valabilă pentru toate valorile lui z , cu excepția lui

$z = 1, 2, 3, \dots$ Se presupune că drumul de integrare (vezi figura) nu trece prin punctele $2n\pi i$ [n natural].

WW II 46

Vezi și 3.626 3., 3.653 4., 3.654 1., 3.655 4., 3.665 6.

7.413

$$1. \zeta(z) = \frac{1}{(1-2^{1-z}) \Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 46}$$

$$2. \zeta(z) = \frac{2^z}{(2^z - 1) \Gamma(z)} \int_0^\infty \frac{t^{z-1} e^t}{e^{2t} - 1} dt \quad [\operatorname{Re}(z) > 1]. \quad \text{WW II 46}$$

$$3. \zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left[\frac{1}{z(z-1)} + \int_1^{\infty} \left(t^{\frac{1-z}{2}} + t^{\frac{z}{2}}\right)t^{-1} \sum_{k=1}^{\infty} e^{-k^2 \pi t} dt \right] \quad \text{WW II 54}$$

$$4. \zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^{\infty} (1+t^2)^{-\frac{z}{2}} \sin(z \operatorname{arctg} t) \frac{dt}{e^{\pi t} + 1}. \quad \text{WW II 62}$$

$$5. \zeta(z) = \frac{2^{z-1}}{2^z-1} \frac{z}{z-1} + \frac{2}{2^z-1} \int_0^{\infty} \left(\frac{1}{4} + t^2\right)^{-\frac{z}{2}} \sin(z \operatorname{arctg} 2t) \frac{dt}{e^{2\pi t} - 1}. \quad \text{WW II 62}$$

Vezi și 3.233 2., 3.324 1., 3.327 1., 3.327 2., 3.651 6.

7.42 Dezvoltarea în serie sau sub formă de produs infinit

7.421

$$1. \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z} \quad [\operatorname{Re}(z) > 1]. \quad \text{WW II 44}$$

$$2. \zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin \frac{\pi z}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi q n}{n^{1-z}} + \cos \frac{\pi z}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi q n}{n^{1-z}} \right] \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 49}$$

$$3. \zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$$

$$\text{unde } F_n(z) = \frac{1}{1-z} \left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}} \right) - \frac{1}{(n+1+q)^z} = z \int_n^{n+1} \frac{(t-n) dt}{(t+q)^{z+1}}. \quad [\operatorname{Re}(z) > 1, N \text{ natural}]. \quad \text{WW II 55}$$

7.422

$$1. \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad [\operatorname{Re}(z) > 1]. \quad \text{WW II 44}$$

$$2. \zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \quad [\operatorname{Re}(z) > 0]. \quad \text{WW II 46}$$

7.423

$$1. \zeta(m) = \prod \frac{1}{1-p^{-m}}, \quad \left. \begin{array}{l} 1. \\ 2. \end{array} \right\} \begin{array}{l} \text{Înmulțirea și sumarea se efectuează} \\ \text{pentru toate numerele } p \text{ prime.} \end{array} \quad \begin{array}{l} \text{WW II 53} \\ \text{WW II 63} \end{array}$$

$$2. \ln \zeta(z) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k p^{kz}}.$$

$$7.424 \quad \frac{\zeta'(z)}{\zeta(z)} = - \sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z},$$

unde $\Lambda(k) = 0$, atunci cînd k nu este puterea unui număr prim, și $\Lambda(k) = \ln p$ atunci cînd k este o putere a unui număr prim $[\operatorname{Re}(z) > 1]$. WW II 63

7.43 Relații funcționale

$$7.431 \quad \zeta(-z, q) = -\frac{B_{z+2}^{(q)}}{(z+1)(z+2)} \quad [z \text{ natural sau zero}]. \quad \text{WW II 47}$$

$$7.432 \quad \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k, q) = \ln \frac{e^{-qz} \Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{q^k}{k(q+k)} \quad [|z| < q]. \quad \text{WW II 59}$$

7.433

$$1. \quad \lim_{z \rightarrow 1} \frac{\zeta(z, q)}{\Gamma(1-z)} = -1. \quad \text{WW II 46}$$

$$2. \quad \lim_{z \rightarrow 1} \left\{ \zeta(z, q) - \frac{1}{z-1} \right\} = -\phi(q). \quad \text{WW II 51}$$

$$3. \quad \left\{ \frac{d}{dz} \zeta(z, q) \right\}_{z=0} = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi. \quad \text{WW II 52}$$

7.434

$$\zeta(z, 1) = \zeta(z).$$

7.435

$$1. \quad \zeta(z) = \frac{1}{2^z-1} \zeta\left(z, \frac{1}{2}\right) \quad [\operatorname{Re}(z) > 1]. \quad \text{WW II 46}$$

$$2. \quad 2^z \Gamma(1-z) \zeta(1-z) \sin \frac{\pi z}{2} = \pi^{1-z} \zeta(z). \quad \text{WW II 57}$$

$$3. \quad 2^{1-z} \Gamma(z) \zeta(z) \cos \frac{\pi z}{2} = \pi^z \zeta(1-z). \quad \text{WW II 49}$$

$$4. \quad \Gamma\left(\frac{z}{2}\right) \pi^{-\frac{z}{2}} \zeta(z) = \Gamma\left(\frac{1-z}{2}\right) \pi^{\frac{z-1}{2}} \zeta(1-z). \quad \text{WW II 49}$$

$$7.436 \quad \lim_{z \rightarrow 1} \left\{ \zeta(z) - \frac{1}{z-1} \right\} = C. \quad \text{WW II 52}$$

$$7.437 \quad \text{Fie } z = \frac{1}{2} + it; \text{ atunci } \Xi(t) = \frac{(z-1) \Gamma\left(\frac{z}{2} + 1\right)}{\sqrt{\pi^z}} \zeta(z) = \Xi(-t)$$

este o funcție pară în raport cu t , avind coeficienți reali în dezvoltarea după puterile lui t^2 . JE 368

7.44 Puncte singulare și zerouri

7.441

1. $z=1$ este unicul punct singular al funcției $\zeta(z, q)$. WW II 46

2. Funcția $\zeta(z)$ are zerouri simple în punctele $-2n$ (n natural). Toate celelalte zerouri ale funcției $\zeta(z)$ sînt situate în banda $0 \leq \operatorname{Re}(z) \leq 1$.

3. Ipoteza lui Riemann: toate zerourile funcției $\zeta(z)$, din banda $0 \leq \operatorname{Re}(z) \leq 1$, sînt situate pe dreapta $\operatorname{Re}(z) = 1/2$. S-a demonstrat că, pe această dreaptă, se află o infinitate de zerouri ale funcției $\zeta(z)$.

7.442 Valori particulare

WW II 53

$$1. \quad \zeta(2m) = \frac{2^{2m-1} \pi^{2m} |B_{2m}|}{(2m)!}, \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} [m \text{ natural}] \quad \text{WW II 49}$$

$$2. \quad \zeta(1-2m) = -\frac{B_{2m}}{2m}, \quad \text{WW II 47}$$

$$3. \quad \zeta(-2m) = 0 \quad \text{WW II 47}$$

$$4. \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi. \quad \text{WW II 52}$$

Tabela valorilor funcției ζ , vezi 8.3.

7.5 NUMERE ȘI POLINOAME BERNOULLI

7.51 Numere Bernoulli

7.510 Numerele B_n , care apar drept coeficienți ai lui $\frac{t^n}{n!}$ în dezvoltarea funcției

$$1. \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

se numesc *numerele Bernoulli*. Așadar, funcția $\frac{t}{e^t - 1}$ este funcția generatoare a numerelor Bernoulli. Gh 56 (79) F II 721m

7.511 Reprezentări integrale

$$1. \quad B_{2n} = (-1)^{n-1} 4n \int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1} dx \quad (\text{cf. 3.233 3.}) \quad \text{F II 721m}$$

$$2. \quad B_{2n} = (-1)^{n-1} \pi^{2n} \int_0^{\infty} \frac{x^{2n}}{\text{sh}^2 x} dx.$$

$$3. \quad B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^{\infty} x^{2n-2} \ln(1 - e^{-2\pi x}) dx.$$

Vezi și 3.324 2., 3.651.

Proprietăți și relații funcționale

7.512 Formula de recurență (notație simbolică)

$$B^n = (B + 1)^n; \quad B^0 = B_0 = 1.$$

Pentru calcul, trebuie, după dezvoltarea binomului din membrul al doilea, să transformăm toate puterile în indici, adică

$$B_n = \sum_{k=0}^n \binom{n}{k} B_k, \quad B_0 = 1. \quad \text{Gh 51 (62)}$$

7.513 Toate numerele Bernoulli sînt numere raționale.

7.514 Orice număr B_n poate fi scris sub forma

$$B_n = C_n - \sum \frac{1}{k+1},$$

unde C_n este un număr întreg oarecare, iar suma se extinde la toți $k > 0$, care satisfac condiția $k+1$ prim, k fiind un divizor al lui n . Gh 64

7.515 Toate numerele Bernoulli cu indice impar sînt nule, în afară de $B_1 = -\frac{1}{2}$, adică $B_{2n+1} = 0$ [n natural] Gh 52 F II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=2}^{2n-2} \frac{2n \cdot (2n-1) \dots (2n-2k+2)}{k!} B_k.$$

$$7.516 \quad B_{2n} = \frac{(-1)^{n-1} (2n)!}{2^{2n-1} \pi^{2n}} \zeta(2n).$$

Gh 56 (79) F II 521 m

$$7.517 \quad B_{2n} = (-1)^{n-1} \frac{2 \cdot (2n)!}{(2\pi)^{2n}} \frac{1}{\prod_{p=1}^{\infty} \left(1 - \frac{1}{p^{2n}}\right)} \quad (\text{cf. } 7.423)$$

(produsul se extinde la toate numerele prime p)

Legătura cu funcția ζ a lui Riemann, vezi 7.442.

Legătura cu numerele Euler, vezi 7.535.

Tabela valorilor numerelor Bernoulli, vezi 8.21.

7.518 Inegalitate (notație simbolică)

$$|(B - \theta)^n| \leq |B^n| \quad [0 < \theta < 1].$$

Ce 337

7.52 Polinoame Bernoulli

7.520 Se numesc *polinoame Bernoulli* $B_n(x)$ polinoamele de forma

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$$

sau, simbolic,

$$B_n(x) = (B + x)^n.$$

7.521 Funcția generatoare

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (\text{cf. } 1.213). \quad \text{Gh 51 (61)m}$$

7.522 Dezvoltare în serie

$$B_{2n}(x) = \frac{(-1)^{n-1} 2 \cdot (2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}} \quad [0 < x < 1]. \quad \text{Gh 71}$$

7.523 Relații funcționale și proprietăți

$$1. B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m \quad [n \text{ și } m \text{ naturali}]$$

(vezi și 0.121).

Gh 51 (65)

$$2. \Delta B_n(x) = B_n(x+1) - B_n(x) = nx^{n-1}.$$

Gh 65 (90)

$$3. B'_n(x) = n B_{n-1}(x).$$

Gh 66

$$4. B_n(1-x) = (-1)^n B_n(x).$$

Gh 66

$$7.524 \quad B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \quad [\text{„teorema de multiplicare“}]. \quad \text{Gh 67}$$

7.525 Diferențele

$$B_n(x) - B_n$$

pentru n impar, sînt egale cu zero pe segmentul $[0, 1]$, numai în punctele $0, \frac{1}{2}, 1$, schimbîndu-și semnul în punctul $x = \frac{1}{2}$. Pentru n par, aceste diferențe se anulează la extremitățile segmentului $[0, 1]$, iar în interiorul acestui segment, își păstrează semnul, modulul fiind maxim în punctul $x = \frac{1}{2}$,

7.526 În intervalul $(0, 1)$, polinoamele

$$B_{2n}(x) - B_{2n}, \quad B_{2n+2}(x) - B_{2n+2}$$

sînt de semn contrar.

Gh 87

7.527 Cazuri particulare

$$\left. \begin{aligned} 1. \quad B_1(x) &= x - \frac{1}{2}. \\ 2. \quad B_2(x) &= x^2 - x + \frac{1}{6}. \\ 3. \quad B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \\ 4. \quad B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}. \\ 5. \quad B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{8}x^3 - \frac{1}{6}x. \end{aligned} \right\}$$

Gh 70

7.528 Valori particulare

$$\left. \begin{aligned} 1. \quad B_n(0) &= B_n. \\ 2. \quad B_n(1) &= (-1)^n B_n. \end{aligned} \right\}$$

Gh 76

7.53 Numere Euler

7.530 Numerele E_n , care apar drept coeficienții lui $\frac{t^n}{\pi t}$ în dezvoltarea funcției

$$\frac{1}{\operatorname{ch} t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

se numesc *numerele Euler*. Așadar, funcția $\frac{1}{\operatorname{ch} t}$ este funcția generatoare a numerelor Euler

Ce 330

7.531 Formula de recurență (notație simbolică)

$$(E+1)^n + (E-1)^n = 0, \quad E_0 = 1.$$

Ce 329

Proprietățile numerelor lui Euler

7.532 Numerele lui Euler sînt numere întregi.

7.533 Numerele lui Euler cu indice impar sînt nule, iar cele două numere cu indici pari vecine sînt de semn contrar, adică

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0.$$

Ce 329

7.534 Dacă $\alpha, \beta, \gamma, \dots$ sînt divizori ai numărului $n-m$, atunci diferența $E_{2n} - E_{2m}$ se împarte cu acele dintre numerele $2\alpha + 1, 2\beta + 1, 2\gamma + 1, \dots$ care sînt prime.

Ce 330

7.535 Legătura cu numerele Bernoulli (notație simbolică)

$$1. \quad E_{n-1} = \frac{(4B-1)^n - (4B-3)^n}{2n}.$$

Ce 330

$$2. \quad B_n = \frac{n(E+1)^{n-1}}{2^n(2^n-1)}.$$

Ce 330

$$3. \quad \left(B - \frac{1}{4}\right)^{2n+1} = \frac{2n+1}{4^{2n+1}} E_{2n}.$$

Ce 341

Tabela valorilor numerelor Euler, vezi 8.22

8. TABELE NUMERICE

8.1 FUNCȚIA $L(x)$ A LUI LOBACEVSKI *

Argu- mentul x în grade	Argu- mentul x în radiani	Valoarea lui $L(x)$	Argu- mentul x în grade	Argu- mentul x în radiani	Valoarea lui $L(x)$	Argu- mentul x în grade	Argu- mentul x în radiani	Valoarea lui $L(x)$
0	0,0000	0,0000000	31	0,5411	0,02722	61	1,0647	0,23076
1	0,0175	0,0000009	32	0,5585	0,03000	62	1,0821	0,24367
2	0,0349	0,0000071	33	0,5760	0,03297	63	1,0996	0,25716
3	0,0524	0,0000239	34	0,5934	0,03614	64	1,1170	0,27124
4	0,0698	0,0000567	35	0,6109	0,03952	65	1,1345	0,28596
5	0,0873	0,0001108	36	0,6283	0,04311	66	1,1519	0,30132
6	0,1047	0,0001916	37	0,6458	0,04692	67	1,1694	0,31737
7	0,1222	0,0003044	38	0,6632	0,05096	68	1,1868	0,33413
8	0,1396	0,0004546	39	0,6807	0,05524	69	1,2043	0,35165
9	0,1571	0,0006476	40	0,6981	0,05976	70	1,2217	0,36997
10	0,1745	0,0008888	41	0,7156	0,06455	71	1,2392	0,38912
11	0,1920	0,001184	42	0,7330	0,06959	72	1,2566	0,40916
12	0,2094	0,001538	43	0,7505	0,07491	73	1,2741	0,43013
13	0,2269	0,001957	44	0,7679	0,08052	74	1,2915	0,45210
14	0,2443	0,002446	45	0,7854	0,08641	75	1,3090	0,47514
15	0,2618	0,003011	46	0,8029	0,09262	76	1,3265	0,49931
16	0,2793	0,003658	47	0,8203	0,09914	77	1,3439	0,52471
17	0,2967	0,004392	48	0,8378	0,10598	78	1,3614	0,55142
18	0,3142	0,005220	49	0,8552	0,11316	79	1,3788	0,57957
19	0,3316	0,006146	50	0,8727	0,12070	80	1,3963	0,60929
20	0,3491	0,007177	51	0,8901	0,12859	81	1,4137	0,64074
21	0,3665	0,008319	52	0,9076	0,13687	82	1,4312	0,67412
22	0,3840	0,009578	53	0,9250	0,14553	83	1,4486	0,70967
23	0,4014	0,010960	54	0,9425	0,15460	84	1,4661	0,74771
24	0,4189	0,012471	55	0,9599	0,16409	85	1,4835	0,78866
25	0,4363	0,014119	56	0,9774	0,17401	86	1,5010	0,83312
26	0,4538	0,015908	57	0,9948	0,18438	87	1,5184	0,88198
27	0,4712	0,017846	58	1,0123	0,19522	88	1,5359	0,93677
28	0,4887	0,019938	59	1,0297	0,20655	89	1,5533	1,00068
29	0,5061	0,022193	60	1,0472	0,21839	90	1,5708	1,08879
30	0,5236	0,024617						

*) Tabela a fost calculată de N. V. Tomantova, sub îndrumarea lui B. L. Laptev.

8.2 NUMERE BERNOULLI ȘI NUMERE EULER

8.21 Numere Bernoulli

$$\begin{array}{ll}
 B_0 = 1, & B_{18} = \frac{43\,867}{798}, \\
 B_1 = -\frac{1}{2}, & B_{20} = -\frac{174\,611}{330}, \\
 B_2 = \frac{1}{6}, & B_{22} = \frac{854\,513}{138}, \\
 B_4 = -\frac{1}{30}, & B_{24} = -\frac{236\,364\,091}{2730}, \\
 B_6 = \frac{1}{42}, & B_{26} = \frac{8\,553\,103}{6}, \\
 B_8 = -\frac{1}{30}, & B_{28} = -\frac{23\,749\,461\,029}{870}, \\
 B_{10} = \frac{5}{66}, & B_{30} = \frac{8\,615\,841\,276\,005}{14\,322}, \\
 B_{12} = -\frac{691}{2730}, & B_{32} = -\frac{7\,709\,321\,041\,217}{510}, \\
 B_{14} = \frac{7}{6}, & B_{34} = \frac{2\,577\,867\,858\,367}{6}, \\
 B_{16} = -\frac{3617}{510}, &
 \end{array}$$

8.22 Numere Euler

$$\begin{array}{ll}
 E_0 = 1, & E_{12} = 2\,702\,765, \\
 E_2 = -1, & E_{14} = -199\,360\,981, \\
 E_4 = 5, & E_{16} = 19\,391\,512\,145, \\
 E_6 = -61, & E_{18} = -2\,404\,879\,675\,441, \\
 E_8 = 1385, & E_{20} = 370\,371\,188\,237\,525, \\
 E_{10} = -50\,521, &
 \end{array}$$

Numerele lui Bernoulli și ale lui Euler cu indici impari (înafară de B_1) sînt nule.

8.3 FUNCȚIA ζ A LUI RIEMANN

$$\begin{array}{ll}
 \zeta(0) = -\frac{1}{2}, & \zeta(6) = \frac{\pi^6}{925} = 1,017\,343\,0620\dots, \\
 \zeta(1) = \infty, & \zeta(7) = \frac{\pi^7}{2\,995,286\dots} = 1,008\,349\,277\,4\dots, \\
 \zeta(2) = \frac{\pi^2}{6} = 1,644\,934\,066\,8\dots, & \zeta(8) = \frac{\pi^8}{9\,450} = 1,004\,077\,356\,2\dots, \\
 \zeta(3) = \frac{\pi^3}{25,794\,86\dots} = 1,202\,056\,903\,2\dots, & \zeta(9) = \frac{\pi^9}{29\,749,35\dots} = 1,002\,008\,392\,8\dots, \\
 \zeta(4) = \frac{\pi^4}{90} = 1,082\,323\,233\,7\dots, & \zeta(10) = 1,000\,994\,575\,1\dots, \\
 \zeta(5) = \frac{\pi^5}{295,121\,5\dots} = 1,036\,927\,755\,1\dots, & \zeta(11) = 1,000\,494\,183\,6\dots
 \end{array}$$

8.4 COEFICIENTI NUMERICI UZUALI

$$8.41 \quad \frac{(2n-1)!!}{2^n \cdot n!}$$

$$\frac{1}{2} = 0,500\,000\,000\,0$$

$$\frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8} = 0,375\,000\,000\,0$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} = \frac{5}{16} = 0,312\,500\,000\,0$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} = \frac{35}{128} = 0,273\,437\,500\,0$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} = \frac{63}{256} = 0,246\,093\,750\,0$$

$$\frac{1 \cdot 3 \dots 9 \cdot 11}{2 \cdot 4 \dots 10 \cdot 12} = \frac{231}{1024} = 0,225\,585\,937\,5$$

$$\frac{1 \cdot 3 \dots 11 \cdot 13}{2 \cdot 4 \dots 12 \cdot 14} = \frac{429}{2048} = 0,209\,472\,656\,2$$

$$\frac{1 \cdot 3 \dots 13 \cdot 15}{2 \cdot 4 \dots 14 \cdot 16} = \frac{6\,435}{32\,768} = 0,196\,380\,615\,2$$

$$\frac{1 \cdot 3 \dots 15 \cdot 17}{2 \cdot 4 \dots 16 \cdot 18} = \frac{12\,155}{65\,536} = 0,185\,470\,581\,1$$

$$\frac{1 \cdot 3 \dots 17 \cdot 19}{2 \cdot 4 \dots 18 \cdot 20} = \frac{46\,189}{262\,144} = 0,176\,197\,052\,0$$

$$\frac{1 \cdot 3 \dots 19 \cdot 21}{2 \cdot 4 \dots 20 \cdot 22} = \frac{88\,179}{524\,288} = 0,168\,188\,095\,1$$

$$\frac{1 \cdot 3 \dots 21 \cdot 23}{2 \cdot 4 \dots 22 \cdot 24} = \frac{676\,039}{4\,194\,304} = 0,161\,180\,257\,8$$

$$\frac{1 \cdot 3 \dots 23 \cdot 25}{2 \cdot 4 \dots 24 \cdot 26} = \frac{1\,300\,075}{8\,388\,608} = 0,154\,398\,101\,8$$

$$\frac{1 \cdot 3 \dots 25 \cdot 27}{2 \cdot 4 \dots 26 \cdot 28} = \frac{5\,014\,575}{33\,554\,432} = 0,149\,446\,010\,5$$

$$\frac{1 \cdot 3 \dots 27 \cdot 29}{2 \cdot 4 \dots 28 \cdot 30} = \frac{9\,694\,845}{67\,108\,684} = 0,144\,479\,346\,3$$

$$8.42 \quad \frac{(2n-1)!!}{2^n \cdot n! (2n+1)}$$

$$\frac{1}{2 \cdot 3} = \frac{1}{6} = 0,166\,666\,666\,7$$

$$\frac{1 \cdot 3}{2 \cdot 4 \cdot 5} = \frac{3}{40} = 0,075\,000\,000\,0$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} = \frac{5}{112} = 0,044\,642\,857\,1$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} = \frac{35}{1\,152} = 0,030\,381\,944\,4$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} = \frac{63}{2\,816} = 0,022\,372\,159\,1$$

$$\frac{1 \cdot 3 \dots 9 \cdot 11}{2 \cdot 4 \dots 10 \cdot 12 \cdot 13} = \frac{231}{13\,312} = 0,017\,352\,764\,4$$

$$\frac{1 \cdot 3 \dots 11 \cdot 13}{2 \cdot 4 \dots 12 \cdot 14 \cdot 15} = \frac{429}{30\,270} = 0,013\,964\,843\,7$$

$$\frac{1 \cdot 3 \dots 13 \cdot 15}{2 \cdot 3 \dots 14 \cdot 16 \cdot 17} = \frac{6\,435}{557\,056} = 0,011\,551\,800\,9$$

$$\frac{1 \cdot 3 \dots 15 \cdot 17}{2 \cdot 4 \dots 16 \cdot 18 \cdot 19} = \frac{12\,155}{1\,245\,184} = 0,009\,761\,609\,5$$

$$\frac{1 \cdot 3 \dots 17 \cdot 19}{2 \cdot 4 \dots 18 \cdot 20 \cdot 21} = \frac{46\,189}{5\,505\,024} = 0,008\,390\,335\,8$$

$$\frac{1 \cdot 3 \dots 19 \cdot 21}{2 \cdot 4 \dots 20 \cdot 22 \cdot 23} = \frac{88\,179}{12\,058\,624} = 0,007\,312\,527\,5$$

$$\frac{1 \cdot 3 \dots 21 \cdot 23}{2 \cdot 4 \dots 22 \cdot 24 \cdot 25} = \frac{676\,039}{104\,857\,600} = 0,006\,447\,210\,5$$

$$\frac{1 \cdot 3 \dots 23 \cdot 25}{2 \cdot 4 \dots 24 \cdot 26 \cdot 27} = \frac{1\,300\,075}{226\,492\,416} = 0,005\,740\,026\,5$$

$$\frac{1 \cdot 3 \dots 25 \cdot 27}{2 \cdot 4 \dots 26 \cdot 28 \cdot 29} = \frac{5\,014\,575}{973\,078\,528} = 0,005\,153\,309\,7$$

$$\frac{1 \cdot 3 \dots 27 \cdot 29}{2 \cdot 4 \dots 28 \cdot 30 \cdot 31} = \frac{9\,694\,845}{2\,080\,374\,784} = 0,004\,660\,148\,3$$

$$8.43 \quad \frac{(2n-1)!!}{2^{n+1} \cdot (n+1)!}$$

$$\frac{1}{2} = 0,500\,000\,000\,0$$

$$\frac{1}{2 \cdot 4} = \frac{1}{8} = 0,125\,000\,000\,0$$

$$\frac{1 \cdot 3}{2 \cdot 4 \cdot 6} = \frac{1}{16} = 0,062\,500\,000\,0$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} = \frac{5}{128} = 0,039\,062\,500\,0$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} = \frac{7}{256} = 0,027\,343\,750\,0$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} = \frac{21}{1024} = 0,020\,507\,812\,5$$

$$\frac{1 \cdot 3 \dots 9 \cdot 11}{2 \cdot 4 \dots 12 \cdot 14} = \frac{33}{2048} = 0,016\,113\,281\,2$$

$$\frac{1 \cdot 3 \dots 11 \cdot 13}{2 \cdot 4 \dots 14 \cdot 16} = \frac{429}{32\,768} = 0,013\,092\,041\,0$$

$$\frac{1 \cdot 3 \dots 13 \cdot 15}{2 \cdot 4 \dots 16 \cdot 18} = \frac{715}{65\,536} = 0,010\,910\,034\,2$$

$$\frac{1 \cdot 3 \dots 15 \cdot 17}{2 \cdot 4 \dots 18 \cdot 20} = \frac{2\,431}{262\,144} = 0,009\,273\,529\,1$$

$$\frac{1 \cdot 3 \dots 17 \cdot 19}{2 \cdot 4 \dots 20 \cdot 22} = \frac{4\,199}{524\,288} = 0,008\,008\,956\,9$$

$$\frac{1 \cdot 3 \dots 19 \cdot 21}{2 \cdot 4 \dots 22 \cdot 24} = \frac{29\,393}{4\,194\,304} = 0,007\,007\,837\,3$$

$$\frac{1 \cdot 3 \dots 21 \cdot 23}{2 \cdot 4 \dots 24 \cdot 26} = \frac{52\,003}{8\,388\,608} = 0,006\,199\,240\,7$$

$$\frac{1 \cdot 3 \dots 23 \cdot 25}{2 \cdot 4 \dots 26 \cdot 28} = \frac{185\,725}{33\,554\,432} = 0,005\,535\,036\,3$$

$$\frac{1 \cdot 3 \dots 25 \cdot 27}{2 \cdot 4 \dots 28 \cdot 30} = \frac{334\,305}{67\,108\,864} = 0,004\,981\,532\,7$$

$$8.44 \quad \frac{(2n-1)!!}{2^{n+1} \cdot (n+1)! (2n+3)}$$

$$\frac{1}{2 \cdot 3} = \frac{1}{6} = 0,166\,666\,666\,7$$

$$\frac{1}{2 \cdot 4 \cdot 5} = \frac{1}{40} = 0,025\,000\,000\,0$$

$$\frac{1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 7} = \frac{1}{112} = 0,008\,928\,571\,4$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} = \frac{5}{1152} = 0,004\,340\,277\,8$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} = \frac{7}{2\,816} = 0,002\,485\,795\,4$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 13} = \frac{21}{13\,812} = 0,001\,577\,524\,0$$

$$\frac{1 \cdot 3 \dots 9 \cdot 11}{2 \cdot 4 \dots 12 \cdot 14 \cdot 15} = \frac{33}{30\,270} = 0,001\,074\,218\,7$$

$$\frac{1 \cdot 3 \dots 11 \cdot 13}{2 \cdot 4 \dots 14 \cdot 16 \cdot 17} = \frac{429}{557\,056} = 0,000\,770\,120\,1$$

$$\frac{1 \cdot 3 \dots 13 \cdot 15}{2 \cdot 4 \dots 16 \cdot 18 \cdot 19} = \frac{715}{1\,245\,184} = 0,000\,574\,212\,3$$

$$\frac{1 \cdot 3 \dots 15 \cdot 17}{2 \cdot 4 \dots 18 \cdot 20 \cdot 21} = \frac{2\,431}{5\,505\,024} = 0,000\,441\,596\,6$$

$$\frac{1 \cdot 3 \dots 17 \cdot 19}{2 \cdot 4 \dots 20 \cdot 22 \cdot 23} = \frac{4\,199}{12\,058\,624} = 0,000\,348\,215\,5$$

$$\frac{1 \cdot 3 \dots 19 \cdot 21}{2 \cdot 4 \dots 22 \cdot 24 \cdot 25} = \frac{29\,393}{104\,857\,600} = 0,000\,280\,313\,5$$

$$\frac{1 \cdot 3 \dots 21 \cdot 23}{2 \cdot 4 \dots 24 \cdot 26 \cdot 27} = \frac{52\,003}{226\,492\,416} = 0,000\,229\,601\,1$$

$$\frac{1 \cdot 3 \dots 23 \cdot 25}{2 \cdot 4 \dots 26 \cdot 28 \cdot 29} = \frac{185\,725}{978\,078\,528} = 0,000\,190\,863\,3$$

$$\frac{1 \cdot 3 \dots 25 \cdot 27}{2 \cdot 4 \dots 28 \cdot 30 \cdot 31} = \frac{334\,305}{2\,080\,374\,784} = 0,000\,160\,694\,8$$

8.5 CONSTANTA EULER ȘI CONSTANTA CATALAN

Constanta Euler

$$C = 0,577\,215\,664\,901\,532\,5 \dots$$

Constanta Catalan

$$G = 0,915\,965\,594 \dots$$

Alte cîteva tabele numerice sînt cuprinse în text, și anume

$$S(n) = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^n} \quad \text{v. } 0,235.$$

$$\sigma(n) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^n} \quad \text{v. } 0,234.$$

$$s(n) = \sum_{k=1}^{\infty} \frac{k^n}{k!} \quad \text{v. } 0,248.$$

INDEXUL FUNCȚIILOR SPECIALE ȘI AL NOTAȚIILOR LOR*

Notația	Denumirea funcției și formulele în care apare această funcție
$\text{am}(u, k)$	Amplitudinea eliptică 6.141; 2.814, 6.145, 6.146
B_n	Numere Bernoulli 7.510—7.518; 0.121, 0.122, 0.132, 0.233, 1.213, 1.411, 1.518, 3.233, 3.324, 3.536, 3.651, 3.666, 3.763, 3.767, 6.344, 6.362, 6.531, 7.442, 7.535, 8.21.
$B_n(x)$	Polinoame Bernoulli 7.520—7.527; 1.443, 7.431.
$B(x, y)$	Funcția B 6.37—6.38; 3.159, 3.171, 3.172, 3.177, 3.178, 3.181, 3.182, 3.184, 3.185, 3.191, 3.192, 3.194, 3.195, 3.212, 3.245, 3.313, 3.421, 3.427, 3.428, 3.429, 3.454, 3.455, 3.518, 3.519, 3.537, 3.628, 3.651, 3.775, 4.433, 4.81, 5.218, 6.323, 6.326, 6.390—6.395, 6.921, 7.211, 7.212, 7.311, 7.321.
$\beta(x)$	6.390—6.395; 3.151, 3.195, 3.211, 3.322, 3.334, 3.421—3.423, 3.644, 3.757, 3.832, 5.218, 5.222, 6.358, 6.396, 6.397.
$\text{ber}_p(z)$, $\text{bei}_p(z)$	Funcția Thomson; partea reală și partea imaginară a funcțiilor Bessel de $ze^{\frac{3}{4}\pi i}$ (pentru z real). 6.561; 5.255, 6.565.
$\text{ber}(z)$, $\text{bei}(z)$	6.566; 5.255, 6.563, 6.564.
C	Constanta lui Euler 6.360—6.362; 0.131, 0.132, 0.264, 3.161, 3.162, 3.196, 3.197, 3.225, 3.228, 3.262, 3.263, 3.266, 3.278, 3.286, 3.289—3.292, 3.553, 3.572, 3.619, 3.625, 3.629, 3.643, 3.689, 3.711, 3.713, 3.714, 3.717, 3.72, 3.741, 3.753, 3.758, 3.762, 3.781, 3.783, 3.785, 3.791, 3.842, 3.862, 4.225, 4.235, 4.321, 4.322, 4.332, 4.471, 4.48, 5.239, 6.246, 6.212, 6.213, 6.214, 6.220, 6.222, 6.321, 6.322, 6.325, 6.326, 6.342, 6.343, 6.351—6.354, 6.355, 6.358, 6.359, 6.403, 6.454, 6.456, 6.532—6.536, 6.564, 7.432, 7.436, 8.5.

* Acest index cuprinde numai numerele tipărite în text cu litere grase.

Notafia	Denumirea funcției și formulele în care apare această funcție
$C(x)$	Cosinusul integralei Fresnel 6.23; 3.563, 4.401, 4.25, 5.129, 5.134, 5.226, 5.246, 5.247, 5.249, 6.528.
$C_n^p(t)$	Polinoame Gegenbauer 6.90—6.96; 4.6, 6.542, 6.544, 7.104, 7.274, 7.131.
$ce_{2n}(x)$ $ce_{2n+1}(x)$	Funcții periodice Mathieu 6.62; 6.631.
$C_{2n}(x)$, $C_{2n+1}(x)$	Funcții Mathieu de argument pur imaginar 6.63.
$ci(x)$	Cosinus integral 6.22; 2.522, 2.523, 2.525, 2.526, 2.84, 3.217, 3.323, 3.522, 3.523, 3.526, 3.527, 3.568, 3.724, 4.23, 4.48, 5.141, 5.211, 5.235, 5.238, 5.244.
cu	Cosinus eliptic 6.143; 2.81, 6.14—6.15, 6.169, 6.191, 6.194.
$D(\varphi, k)$	6.111; 6.112.
$D(k) \equiv D$	6.112; 2.82, 6.115.
$D_p(z)$	Funcția cilindrului parabolic 7.340—7.355; 3.574, 3.575, 4.423, 4.82, 5.216, 5.221, 5.253, 5.266, 5.317.
du	Delta amplitudinii 6.143; 2.81, 6.14—6.15, 6.168, 6.191, 6.194.
e_1, e_2, e_3	6.162, 6.163, 6.164, 6.169, 6.193.
E_n	Numere Euler 7.53; 0,233, 1.411, 8.22.
$E(\varphi, k)$	Integrala eliptică de speța a doua 6.111—6.112; 2.471, 2.481, 2.483, 2.485, 2.487, 2.488, 2.814, 3.134, 3.138, 3.145, 3.146, 3.149, 3.416, 3.837, 3.866, 4.12, 4.121—4.123, 6.117, 6.118, 6.121, 6.123, 6.125, 6.127, 6.194.
$E(k) \equiv E$ } $E(k') \equiv E'$ }	Integrala eliptică completă de speța a doua 6.112; 2.82, 3.413, 3.414, 3.416, 4.516, 3.517, 3.521, 3.636, 3.766, 3.822, 4.121, 4.122, 4.13, 4.435, 6.114, 6.117, 6.118, 6.121—6.124, 6.126, 6.198, 6.828.
$Ei(x)$	Funcția exponențială integrală 6.210; 2.324, 2.325, 3.215, 3.221—3.223, 3.225, 3.226, 3.511, 3.518, 3.523, 3.527, 3.533, 3.542, 3.554, 3.563, 3.568, 3.612, 3.642, 3.657, 3.691, 3.692, 3.711, 3.721, 3.782, 3.786, 3.863, 3.871, 4.22, 4.232, 4.48, 5.111, 5.112, 5.211, 5.242, 5.112, 6.117, 6.118, 6.212, 6.214—6.218, 6.223

Notăția	Denumirea funcției și formulele în care apare această funcție
$F(\varphi, k)$	Integrala eliptică de speța întâi 6.111; 2.471, 2.472—2.474, 2.476, 2.481—2.489, 3.131, 3.133—3.137, 3.141, 3.144, 3.147, 3.148, 3.416, 3.679, 3.831, 3.837, 3.866, 4.11, 4.111, 4.112, 6.112, 6.117, 6.118, 6.121, 6.123, 6.125, 6.127.
$F(\alpha, \beta; \gamma; z)$	Funcția hipergeometrică 7.200—7.255; 1.641, 1.644, 1.645, 3.181, 4.403, 4.415, 4.432—4.434, 4.824, 4.900, 5.256, 5.263—5.265, 6.113, 6.114, 6.452, 6.702—6.704, 6.723, 6.751, 6.752, 6.771—6.775, 6.777, 6.811, 6.816, 6.832, 6.840, 6.857, 6.872, 6.873, 6.921, 7.112, 7.131, 7.262—7.268, 7.291, 7.325.
${}_1F_1(\alpha; \gamma; z)$	Funcția hipergeometrică degenerată 7.310—7.320; 4.434, 5.257, 5.316, 7.123, 7.129, 7.142, 7.336, 7.339, 7.340.
g_2, g_3	Invariantii funcției $\wp(u)$ 6.161; 2.831, 6.162, 6.164, 6.165, 6.167, 6.169, 6.173.
$gd x$	Gudermanian 1.49; 1.482 2.331.
$G_n(p, q; x)$	Polinoame Jacobi 7.130—7.135; 4.73, 7.104, 7.131.
G	Constanta lui Catalan 0.234; 3.322, 3.513, 3.514, 3.515, 3.615—3.617, 3.622, 3.624, 3.661, 3.663, 3.667, 3.669, 3.821, 3.831, 3.833, 3.834 4.13, 8.5.
$\Gamma(z)$	Funcția Γ 6.310—6.339; 3.132, 3.142, 3.164, 3.175, 3.183, 3.319, 3.195, 3.214, 3.215, 3.225, 3.227, 3.228, 3.233, 3.237, 3.247, 3.265, 3.266, 3.267, 3.287, 3.324—3.329, 3.343, 3.411, 3.427, 3.434, 3.442, 3.443, 3.473, 3.483, 3.473, 3.484, 3.491, 3.518, 3.519, 3.525, 3.529, 3.533, 3.562, 3.565, 3.573, 3.575, 3.579, 3.584, 3.613, 3.619, 3.629, 3.652—3.656, 3.672, 13, 3.675, 3.714, 3.723, 3.759, 3.832, 3.85, 3.932—3.944, 3.946, 4.213, 4.222, 4.232, 4.242, 4.252, 4.311, 4.403, 4.412, 4.415, 4.416, 4.417, 4.419, 4.423, 4.432—4.435, 4.44, 4.453, 4.454, 4.466, 4.511, 4.516, 4.60, 4.61, 4.73, 4.823, 4.824, 5.116, 5.207, 5.213, 5.214, 5.216, 5.221, 5.229, 5.237, 5.253, 5.256, 5.257, 5.262, 5.264—5.266, 6.234, 6.357, 6.381, 6.402, 6.412, 6.413, 6.417, 6.419, 6.422, 6.425, 6.426, 6.432—6.435, 6.437, 6.441, 6.442, 6.444, 6.445, 6.446, 6.448, 6.450—6.453, 6.455, 6.461, 6.466, 6.542, 6.544, 6.546, 6.565, 6.702 6.706, 6.711—6.715, 6.721, 6.723, 6.724, 6.736—6.753, 6.755, 6.756, 6.772—6.777, 6.787, 6.796, 6.831, 6.648, 6.857, 6.871—6.873, 6.91, 6.921, 6.923, 6.931, 6.941—6.944, 7.104, 7.145, 7.147, 7.213,

Notația	Denumirea funcției și formulele în care apare această funcție
$\ln \Gamma(z)$	7.222, 7.231, 7.232, 7.236, 7.254, 7.255, 7.311, 7.314, 7.315, 7.320, 7.322, 7.323, 7.325, 7.328, 7.332, 7.333, 7.335, 7.338, 7.340—7.342, 7.344—7.346, 7.348, 7.349, 7.411, 7.421, 7.432—7.435, 7.437. 6.341—6.344; 3.239, 3.241, 3.244, 3.245, 3.332, 3.635, 3.636, 3.638, 3.639, 3.675, 3.686—3.688, 3.712, 3.716, 3.731, 3.732, 3.841, 4.321, 4.322, 4.331, 4.332, 5.219, 6.351.
$H_n(x)$	Polinoame Hermite 7.120—7.129; 4.72, 5.148, 5.149, 5.207, 5.261, 6.945, 7.104, 7.142, 7.353.
$H(u), H_1(u)$	6.191—6.193.
$H_p^{(1)}(z), H_p^{(2)}(z)$	Funcții Hankel de speța 1 și 2. 6.401; 6.405; 4.419, 5.146, 5.214, 5.254, 6.405, 6.407, 6.431—6.439, 6.461, 6.463, 6.464, 6.465, 6.476, 6.479, 6.483—6.486, 6.491, 6.492, 6.494, 6.497, 6.541, 6.543, 6.562.
$H_0^{(1)}(z), H_0^{(2)}(z)$	4.423, 4.431, 5.122, 5.254, 6.483, 6.541.
$H_1^{(1)}(z), H_1^{(2)}(z)$	6.483; 6.436.
$hei_p(z), hei_p(z)$	Funcții Thompson 6.562, 6.563.
$I_p(z)$	Funcții Bessel de argument imaginar 6.406; 0.246, 3.227, 3.473, 3.941, 3.947, 4.422, 4.431, 4.433, 4.436, 4.454, 5.214, 5.233, 5.253, 5.254, 5.257—5.259, 5.314, 6.439, 6.441—6.443, 6.455, 6.456, 6.461, 6.477, 6.484—6.487, 6.495, 6.496, 6.514, 6.516, 7.335.
$I_0(z)$	0.246, 3.217, 3.321, 4.414, 4.435, 4.463, 6.457, 6.496,
$I_1(z)$	0.246, 3.321, 4.435, 6.457, 6.496.
$J_p(z)$	Funcții Bessel 6.401—6.419; 0.247, 2.851, 3.481, 3.482, 3.483, 3.485, 3.565, 3.584, 3.585, 4.4, 4.61, 4.62, 4.911, 4.912, 5.142—5.144, 5.147, 5.206, 5.252, 5.253, 5.256—5.258, 5.3, 6.4—6.5, 6.631, 6.722, 6.787, 6.944, 7.145, 7.273, 7.315, 7.325.
$J_0(z)$	0.246, 3.471, 3.524, 3.528, 3.585, 3.59, 4.403—4.405, 4.411, 4.414, 4.431, 4.434, 4.435, 4.451, 4.452, 4.453, 4.461—4.464, 4.467, 4.471, 4.48, 5.113, 5.127, 5.135—5.137, 5.206, 5.252, 5.253, 5.258, 6.419, 6.451, 6.454, 6.483, 6.515, 6.521, 6.522, 6.525, 6.530, 6.531—6.535, 6.541, 6.553—6.555, 6.558, 7.145.

Notăția	Denumirea funcției și formulele în care apare această funcție
$J_1(z)$	0,246, 3.524, 3.528, 4.435, 4.451, 4.471, 4.472, 4.481, 5.206, 5.258, 6.451, 6.454, 6.483.
$J_{\frac{1}{3}}(z), J_{-\frac{1}{3}}(z)$	5.121, 6.451, 6.465.
$J_{n+\frac{1}{2}}(z)$	6.471—6.475, 4.401, 5.252, 6.256, 6.523, 6.526, 6.543, 6.544.
$K(k) \equiv K$ $K(k') \equiv K'$	Integrala eliptică completă de speța întâi 6.112; 2.82, 3.149, 3.412—3.416, 3.441, 3.517, 3.521, 3.665, 3.674, 3.678, 3.752, 3.762, 3.766, 3.822, 3.921, 3.922, 4.11—4.13, 4.435, 5.252, 6.113, 6.117—6.119, 6.121—6.124, 6.126, 6.128, 6.129, 6.142, 6.146, 6.147, 6.151, 6.169, 6.191, 6.192, 6.195—6.197, 6.828.
$K_p(z)$	Funcții Hankel modernizate 6.606; 0.126, 3.216, 3.227, 3.282, 3.336, 3.472, 3.529, 3.585, 4.406, 4.411, 4.412, 4.419, 4.421, 4.425, 4.433, 4.435, 4.437, 4.44, 4.466, 4.512, 4.515, 4.913, 5.116, 5.145, 5.146, 5.214, 5.217, 5.221, 5.253, 5.254, 5.258, 5.259, 5.314, 6.444—6.448, 6.456, 6.461, 6.478, 6.479, 6.484—6.487, 6.495—6.497, 6.514, 6.516, 6.556, 7.335.
$K_0(z)$	3.528, 3.552, 3.717, 4.411, 4.414, 4.421, 4.425, 4.441, 4.451, 4.463, 5.01, 5.113, 5.114, 5.122, 5.135, 5.138, 5.141, 5.236, 5.254, 5.259, 6.457, 6.496, 6.536, 6.567.
$K_1(z)$	5.227, 5.254, 6.496.
$K_{\frac{1}{3}}(z)$	5.121, 6.449, 6.465.
$\ker(z), \ker_i(z)$	Funcții Thompson 6.563; 4.411, 4.471, 4.472, 5.255, 6.564, 6.566, 6.567.
$L(x)$	Funcția Lobacevski 6.24; 3.222, 3.325, 3.326, 3.615—3.618, 3.762, 3.764, 3.769, 8.1.
$L_n^{(p)}(z)$	Polinoame Laguerre generalizate 7.140—7.149; 4.741, 4.742, 5.149, 5.257, 5.262, 5.315, 7.104, 7.339.
$li(x)$	Logaritm integral 6.210; 2.624, 3.221, 3.611, 3.641, 4.21, 6.211, 6.213, 6.218, 7.337.
$M_{\lambda, \mu}(z)$	Funcții Whittaker 7.320; 4.433, 4.81, 5.257, 5.264, 7.321, 7.326, 7.328, 7.331, 7.332, 7.333, 7.335.
$N_p(z)$	Funcția Neumann 6.401, 6.403; 3.585, 4.402, 4.406, 4.419, 4.422, 4.425, 4.433, 4.441, 5.254, 6.404,

Notația	Denumirea funcției și formulele în care apare această funcție
$N_0(z)$	6.421—6.426, 6.453, 6.461—6.463, 6.465, 6.466, 6.479, 6.484—6.493, 6.497, 6.516, 6.557, 7.325. 3.471, 3.528, 3.552, 4.411, 4.421, 4.452, 4.464, 4.48, 5.113, 5.135, 5.137, 5.139, 5.141, 5.254, 6.405, 6.421, 6.454, 6.483, 6.532—6.535, 6.552.
$N_1(z)$	6.454, 6.483.
$N_{n+\frac{1}{2}}(z)$	6.475.
$\wp(u)$	Funcția eliptică Weierstrass 6.160—6.169; 2.83, 2.831, 6.177, 6.193.
$P_p^q(z), \wp_p^q(z)$	Funcții sferice 6.7, 6.830—6.836; 3.456, 4.433, 4.466, 4.51. 6.94, 6.941—6.943, 7.272, 7.273.
$P_p(z), \wp_p(z)$	Polinoame și funcții Legendre 6.70, 6.810—6.828; 3.426, 3.452, 4.52, 5.147, 5.262, 6.521, 6.531, 6.541, 6.543. 6.544, 6.840, 6.859, 6.943, 7.104, 7.131.
$\wp_{n+\frac{1}{2}}(z)$	Funcțiile -torului 6.870—6.873; 4.512.
$P_{-\frac{1}{2}+u}(z)$	Funcții conice 6.860—6.864.
$\Pi(x)$	„Unghiul de paralelism“ Lobacevski 1.48; 3.851.
$\Pi(\varphi, n, k)$	Integrala eliptică de speța a treia 6.111; 2.481, 2.483, 2.485, 2.487—2.489, 6.116, 6.194.
$P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma; z \\ a' & \beta' & \gamma' \end{Bmatrix}$	Ecuția diferențială a lui Riemann (schema) 7.260—7.275, 7.301.
$\Phi(x)$	Integrala de probabilitate 6.23; 3.213, 3.224, 3.273—3.276, 3.564, 4.24, 5.212, 5.219, 5.221, 5.233, 5.245, 5.251, 7.336, 7.354.
$\psi(z)$	Funcția ψ a lui Euler 6.351—6.359; 3.161, 3.162, 3.167, 3.179, 3.196, 3.197, 3.211, 3.228, 3.243, 3.262, 3.263, 3.266, 3.281, 3.341, 3.423, 3.432, 3.433, 3.438, 3.533, 3.619, 3.625, 3.628, 3.629, 3.634, 3.645, 3.658, 3.665, 3.723, 3.731, 3.753, 3.758, 3.762, 4.33, 5.219, 5.237, 6.360, 6.390, 6.397, 6.403, 6.456, 6.457, 6.761, 7.253, 7.254, 7.338, 7.433.
$Q_p^q(z), \Omega_p^q(z)$	Funcții sferice de speța a doua 6.7; 3.329, 4.433, 4.511, 6.524.
$Q_p(z), \Omega_p(z)$	Funcții Legendre adjuncate de speța a doua 6.70, 6.840—6.859; 3.191, 3.329, 4.431 4.511.

Notația	Denumirea funcției și formulele în care apare această funcție
$S(x)$	Sinus integral Fresnel 6.23; 3.563, 4.401, 4.25, 5.129, 5.134, 5.246, 5.247, 5.248, 6.526.
$se_{2n}(x), se_{2n+1}(x)$	Funcții periodice Mathieu 6.62; 6.631.
$Se_{2n}(x), Se_{2n+1}(x)$	Funcții Mathieu de argument pur imaginar 6.63.
$si(x)$	Sinus integral 6.22; 2.522, 2.523, 2.525, 2.526, 2.84, 3.217, 3.223, 3.522, 3.523, 3.526, 3.527, 3.568, 3.724, 3.751, 3.792, 4.23, 4.245, 4.254, 4.48, 4.141, 5.211, 5.226, 5.235, 5.238, 5.243.
snz	Sinus eliptic 6.143; 2.81, 6.14—6.15. 6.169, 6.191, 6.194.
$\sigma(u)$	Funcția σ a lui Weierstrass 6.17; 2.83, 6.193.
$T_n(x)$	Polinoame Cebîșev de speța întâi 7.110—7.118; 4.71, 7.104, 7.131.
$\vartheta_1(u), \vartheta_2(u), \vartheta_3(u), \vartheta_4(u)$	Funcții ϑ eliptice 6.180—6.189; 5.241, 6.191, 6.192.
$\Theta(u), \Theta_1(u)$	Funcții Θ Jacobi 6.191—6.196.
$U_n(x)$	Polinoame Cebîșev de speța doua 7.110—7.117; 4.71.
$W_{\lambda, \mu}(z)$	Funcția Whittaker 7.322—7.325; 3.225, 3.227, 4.24, 4.441, 4.515, 5.215, 5.216, 5.265, 7.320, 7.327, 7.328, 7.332—7.340.
$Z_p^1(z), \mathfrak{B}_p(z)$	Funcții cilindrice 6.401; 2.85, 6.481, 6.482, 6.511—6.516, 6.540, 6.542, 6.545, 6.547, 6.548, 6.551.
$\zeta(u)$	Funcția ζ a lui Weierstrass 6.17; 2.83, 6.193.
$\zeta(z, p)$	Funcția ζ a lui Riemann 7.411; 3.237, 3.579, 3.626, 3.653—3.655, 3.665, 3.772, 7.412, 7.421, 7.431—7.434, 7.441.
$\zeta(z) \equiv \zeta(z, 1)$	7.413; 0.233, 3.233, 3.324, 3.327, 3.579, 3.651, 3.767, 3.884, 6.321, 6.342, 6.353, 6.360, 6.378, 6.393, 7.422—7.424, 7.434—7.442, 7.516, 8.3.

LISTA NOTAȚIILOR FOLOSITE

În secțiunile 3, 4 și 5, numerele naturale se notează cu primele litere ale alfabetului $a, b, c, \dots, h$. Litera k (atunci când nu servește ca indice de sumare) indică un număr situat pe segmentul $[0, 1]$. Aceste notații sînt folosite în integralele care se reduc la cele eliptice. Aici numărul $\sqrt{1-k^2}$ se notează cu k' .

Funcția exponențială; alegerea diferitelor notații este dictată exclusiv de considerente grafice.

Funcția rațională,
Partea reală a numărului complex $z = x + iy$.

Partea imaginară a numărului complex $z = x + iy$.
Conjugatul numărului complex $z = x + iy$.

Argumentul numărului complex $z = x + iy$;

$$\sin(\arg z) = \frac{y}{\sqrt{x^2 + y^2}}; \quad \cos(\arg z) = \frac{x}{\sqrt{x^2 + y^2}}$$

Semnul numărului real x ; $\operatorname{sgn} x = +1$ pentru $x > 0$; $\operatorname{sgn} x = -1$ pentru $x < 0$.

Partea întreagă a numărului real x .

Limita funcției $f(x)$, în vecinătatea lui x_0 , pentru valorile lui $x > x_0$.

Integrală pe un contur; drumul de integrare: pornind din punctul a , se apropie de punctul b (în linie dreaptă, dacă nu există alte indicații), ocolește printr-un mic cerc în sensul pozitiv punctul b și revine în punctul a , parcurgînd drumul inițial în sens contrar.

Integrală pe un contur, definită la fel ca cea precedentă, cu singura deosebire că punctul b este ocolit în sens negativ.

Integrală curbilinie, luată de-a lungul curbei C .

$$= 1 \cdot 2 \dots n.$$

$$= 1.$$

$$= 1 \cdot 3 \dots (2n+1).$$

$$= \frac{p(p-1) \dots (p-n+1)}{1 \cdot 2 \dots n} \quad (n \text{ natural}).$$

$$= 1.$$

$$e^x \equiv \exp(x)$$

$$\begin{aligned} R(x) \\ \operatorname{Re}(z) \equiv x \end{aligned}$$

$$\begin{aligned} \operatorname{Im}(z) \equiv y \\ \bar{z} \equiv x - iy \end{aligned}$$

$$\arg z$$

$$\operatorname{sgn} x$$

$$E(x)$$

$$f(x_0+0) = \lim_{s \rightarrow 0} f(x_0+s)$$

$$f(x_0-0) = \lim_{s \rightarrow 0} f(x_0-s), \quad s > 0.$$

$$f(x_0 \pm 0) = \lim_{s \rightarrow 0} f(x_0 \pm is), \quad s > 0.$$

$$\int_a^{(b+)} z$$

$$\int_a^{(b-)} z$$

$$\begin{aligned} \int_C \\ n! \\ 0! \\ (2n+1)!! \\ \binom{p}{n} \\ \binom{p}{0} \end{aligned}$$

BIBLIOGRAFIE*

Cifrul

- A Adams E., *Smithsonian Mathematical Formulae*, Washington, 1922.
- Be Bertrand J., *Traité de calcul différentiel et de calcul intégral, 2-e partie. Calcul intégral. Intégrales définies et indéfinies*, Paris, Gauthier-Villars, 1870.
- BH Bierens de Haan David, *Nouvelles tables d'intégrales définies*, Amsterdam, 1867.
- Br₀₈ Bromwich T. J., *T'a. An Introduction to the Theory of Infinite Series*, London, MacMillan & Co, 1908.
- Br₂₈ Idem, ediția a II-a, 1926.
- Ce Чезаро Э., *Элементарный учебник алгебраического анализа ичисления бесконечно малых (Cesaro Ernesto, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung mit zahlreichen Übungsbeispielen)*, перев. с нем., ОНТИ, Л.-М., ч. 1, 1936.
- CH Курант Р. и Гильберт Д., *Методы математической физики (Courant R. u. Hilbert D., Methoden der mathematischen Physik)*, перев. с нем., т. 1, ГТТИ, М.-Л., 1951.
- C Кузьмин Р. О., *Бесселевы функции*, ГТТИ, М.-Л., 1935.
- D Двайт Г. Б., *Таблицы интегралов и другие математические формулы*, ИЛ, М., 1948.
- ED Эфрос А. М. и Данилевский А. М., *Операционное исчисление и контурные интегралы*. ОНТИ-ГНТИУ, Харьков, 1937.
- E Эйлер Леонард, *Введение в анализ бесконечно малых (Euler Leonhard, Introductio in analysin infinitorum)*, перев. с латинского, ОНТИ-ГТТИ, М.-Л., 1936.
- F II Фихтенгольц Г. М., *Курс дифференциального и интегрального исчисления*, т. II, Гостехиздат, М.-Л., 1948.
- F III Idem, т. III, 1949.
- Ga Gauss Carl Friedrich, *Werke, Dritter Band*. Göttingen, 1876.
- Gh Гелфонд А. О., *Исчисление конечных разностей*, ч. I. ОНТИ-ГТТИ, М.-Л., 1936¹.
- Go I, Гурса Э., *Курс математического анализа (Goursat H., Cours d'analyse mathématique)*, перев. с франц., т. I, ч. I, ГТТИ, М., 1933.
- Go II, Idem, т. II, ч. I.
- GC Сборник задач по высшей математике под ред. Н. М. Гюнтера и Р. О. Кузьмина, т. III, ГТТИ, М.-Л., 1947².
- H Hobson E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, 1931.
- JE Янке Е. и Эмде Ф., *Таблицы функций с формулами и кривыми. (Janke E. u. Emde F., Funktionentafeln mit Formeln und Kurven)*, перев. с нем., Гостехиздат, М.-Л., 1948.
- Jl Jolley L., *Summation of Series*, London, Chapman and Hall Ltd, 1925.
- J Журавский А. М., *Справочник по эллиптическим функциям*, изд. АН СССР, М.-Л., 1941.

* După cifrul care desemnează cartea, indicațiile bibliografice conțin numere. Numerele necuprinse în paranteze indică pagina; numerele în paranteze rotunde, numărul formulei; cifrele în paranteze drepte, numărul tabelului.

¹ Traducerea în limba română a acestui lucrări (ediția 1932) apare la Editura Tehnică, 1956.

² Traducerea în limba română (în ediția a II-a), Editura Tehnică, 1952.

- Kr Кречмар В. А., Задачник по алгебре, изд. 2-е, Гостехиздат, М.-Л., 1930
- La Laskà W., Sammlung von Formeln der reinen und angewandten Mathematik, Braunschweig, Friedrich Vieweg und Sohn, 1888.
- Li Lindman C. E., Examen des Nouvelles tables d'intégrales définies de M. Bierens de Haan, Amsterdam, 1867. Kongl. Svenska Vetenskaps-Akademiens Handlingar, Bandet 24, Nr. 5, Stockholm, 1891.
- Lol Лобачевский Н. И., Полное собрание сочинений, Гостехиздат, М.-Л., т. I, 1946.
- Lolli Idem, т. III, 1951.
- LoV Idem, т. V, sub tipar.
- MO Magnus W. und Oberhettinger F., Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, Spengler-Verlag, Berlin-Göttingen-Heidelberg, 1948.
- Na Натансон С. Н., Конструктивная теория функций, Гостехиздат, М.-Л., 1949.
- NH Nielsen Niels, Handbuch der Theorie der Gammafunktion, Leipzig, Teubner, 1906.
- NI Nielsen Niels, Theorie des Integrallogarithmus und verwandter Transzendenten, Leipzig, Teubner, 1906.
- No Новоселов С. И., Обратные тригонометрические функций. Пособие для учителей. изд. 3-е. Учпедгиз, М.-Л., 1950.
- P Peirce B. O., A Short Table of Integrals, Third edition, Boston, Ginn and Co., 1929.
- SI Сикорский Ю. С., Элементы теории эллиптических функций с приложениями к механике, ГТТИ, 1936.
- Smli Смирнов В. И., Курс высшей математики, т. III, ч. 2, изд. 4, Гостехиздат, М.-Л., 1949.¹
- St Стретт М. Д. О., Функций Ляме, Матье и родственные им в физике и технике, перев. с нем., ГНТУ, Харьков-Киев, 1935.
- T Тимофеев А. Ф., Интегрирование функций, Гостехиздат, М.-Л., 1947.
- W Ватсон Г. Н., Теория бесселевых функций (Watson G. N., A Treatise of the Theory of Bessel Functions), перев. с англ., ч. I, ИЛ, М., 1949.
- WWI Уиттекер Е. Т. и Ватсон Г. Н., Курс современного анализа (Whittaker E. T. and Watson G. N., A Course of Modern Analysis), перев. с англ., ГТТИ, М.-Л., ч. I, 1933.
- WWII Idem, ч. II, 1934.

¹ Traducerea în limba română (ediția 1954), Editura Tehnică, 1955.

DIVERSE NOTAȚII ALE NUMERELOR SPECIALE ȘI ALE FUNCȚIILOR SPECIALE

Integrale și funcții eliptice

Uneori, argumentele în notarea integralelor eliptice se scriu în alt mod decât s-a arătat în text, și anume se scriu $F(k, \varphi)$ și $E(k, \varphi)$.

Se întâlnesc următoarele notații prescurtate:

$$\begin{array}{lll} \operatorname{ns} u = \frac{1}{\operatorname{sn} u}, & \operatorname{nc} u = \frac{1}{\operatorname{cn} u}, & \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}, \\ \operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u}, & \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u}, & \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}, \\ \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u}, & \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u}, & \operatorname{nd} u = \frac{1}{\operatorname{dn} u}. \end{array}$$

Funcția ζ a lui Jacobi

$$\operatorname{zn}(u, k) = \int_0^u \operatorname{dn}^2(t, k) dt - \frac{E(k)}{K(k)} u = E(k, \operatorname{am}(u, k)) - \frac{E(k)}{K(k)} u.$$

$$\operatorname{zn}(u, k) = -\frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin \frac{n\pi u}{K} = \frac{\pi}{K} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi u}{K}}{\operatorname{sh} \frac{n\pi K'}{K}}$$

$$\left(\left| \operatorname{Im} \left(\frac{\pi u}{2K} \right) \right| < \frac{\pi}{2} \operatorname{Im} \tau \right).$$

$$\operatorname{zn}(u + 2K) = \operatorname{zn} u; \quad \operatorname{zn}(u + 2iK') = \operatorname{zn} u - \frac{\pi i}{K}.$$

$$\operatorname{zn}(u + v) = \operatorname{zn} u + \operatorname{zn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v). \quad \text{WW II 368}$$

$$z(u, k) = i \left\{ \frac{\operatorname{sn}(u, k') \operatorname{dn}(u, k')}{\operatorname{cn}(u, k')} - \frac{\pi u}{2KK'} - \operatorname{zn}(u, k') \right\}. \quad \text{WW II 369}$$

Zerourile funcției $\operatorname{zn}(u, k)$ sînt numerele nK , unde $n = 0, 1, 2, \dots$. Funcția $\operatorname{zn} u$ se notează uneori $Z(u)$.

Funcțiile eliptice Weierstrass

$\eta_1 = \zeta(\omega_1) = -\frac{1}{12\omega_1\pi^2\theta_1''}$, unde θ'' și θ' sînt valorile derivatelor lui $\theta(u, \tau)$ în raport cu u , pentru $u = 0$. (vezi și 6.193 3). MO 154

$$\eta_2 = \zeta(\omega_2). \quad \text{MO 154}$$

$$\eta_3 = \zeta(\omega_1 + \omega_2) = \eta_1 + \eta_2. \quad \text{MO 154}$$

$$\sigma_r(u) = \frac{\exp(-\eta_r u) \sigma(u + \omega_r)}{\sigma(\omega_r)},$$

unde $r = 1, 2, 3$ și $\omega_3 = \omega_1 + \omega_2$.

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Funcția θ

Diversi autori folosesc diferite sisteme de notații pentru funcțiile θ . Schema de mai jos dă principalele sisteme de notații. Simbolurile din aceeași coloană desemnează aceeași funcție.

$\theta_1(u)$	$\theta_2(u)$	$\theta_3(u)$	$\theta(u) = \theta_4(u)$	Jacobi și manualul de față
$\theta_1\left(\frac{u}{\pi}\right)$	$\theta_2\left(\frac{u}{\pi}\right)$	$\theta_3\left(\frac{u}{\pi}\right)$	$\theta_1\left(\frac{u}{\omega}\right)$	Tennery și Molk
$\theta_1\left(\frac{\omega}{\pi} u\right)$	$\theta_2\left(\frac{\omega}{\pi} u\right)$	$\theta_3\left(\frac{\omega}{\pi} u\right)$	$\theta\left(\frac{\omega}{\pi} u\right)$	Briot și Bouquet
$\theta_1\left(\frac{u}{\pi}\right)$	$\theta_2\left(\frac{u}{\pi}\right)$	$\theta_3\left(\frac{u}{\pi}\right)$	$\theta_0\left(\frac{u}{\pi}\right)$	Weierstrass
$\theta\left(\frac{u}{\pi}\right)$	$\theta_1\left(\frac{u}{\pi}\right)$	$\theta_3\left(\frac{u}{\pi}\right)$	$\theta_2\left(\frac{u}{\pi}\right)$	Jordan

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Funcția exponențială integrală
și funcțiile înrudite

În textul principal, funcțiile $Ei(x)$ și $li(x)$ se definesc numai pentru valori reale ale argumentului, cu ajutorul integralelor. Dar se vede ușor că aceste definiții se pot extinde la valorile complexe ale argumentului, dacă aceste integrale se iau pe un drum curbiliniu, ocolind punctul 0, sau dacă se pornește de la seriile 6.213 și 6.214. Aici funcția $Ei(x)$, pentru $x > 0$, este egală cu $\frac{1}{2}[Ei(x+i \cdot 0) + Ei(x-i \cdot 0)]$. Pentru această funcție se mai întâlnește notația $\bar{E}i(x)$.

$$Si(x) = si(x) + \frac{\pi}{2} = \int_0^x \frac{\sin t}{t} dt. \quad Ci(x) = ci(x).$$

$$Shi(x) = \int_0^x \frac{\sinh t}{t} dt; \quad Chi(x) = C + \ln x + \int_0^x \frac{\cosh t - 1}{t} dt.$$

$$\operatorname{erf}(x) = \Phi(x). \quad \operatorname{erfc}(x) = 1 - \Phi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

$$L(x) = \int_x^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \operatorname{erfc}(x).$$

După S. N. Bernștein, integrala de probabilitate Φ_B se definește în felul următor:

$$\Phi_B(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t} dt = \frac{1}{2} \Phi\left(\frac{x}{\sqrt{2}}\right).$$

În afară de definiția menționată în text a integralelor lui Fresnel, mai există și următoarele:

$$S^*(x) = \int_0^x \sin \frac{\pi}{2} t^2 dt = S\left(\sqrt{\frac{\pi}{2}} x\right); \quad C^*(x) = \int_0^x \cos \frac{\pi}{2} t^2 dt = C\left(\sqrt{\frac{\pi}{2}} x\right).$$

Integralele euleriene și funcțiile înrudite

$\Pi(x) = x! = \Gamma(x+1)$; cînd folosim această notație, punctul de plecare pentru calcularea variabilei independente x în funcția $\psi(z)$ se deplasează pe axa reală spre stînga cu 1 și se notează cu $\Psi(x-1) = \psi(x)$.

Constanta lui Euler se notează uneori cu litera γ ; uneori însă, cu litera γ se notează expresia e^C .

Generalizarea ulterioară a factorialelor sînt așa numitele „facultăți”, sau funcții factoriale $u^{y/x} \equiv (u, x)^y$, definite prin egalitățile

$$u^{y/x} \equiv (u, +x)^y = x^y \frac{u}{u+x} \prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^y \frac{u+n}{u+(y+n)x},$$

$$u^{y/-x} \equiv (u, -x)^y = x^y \frac{u-(y-1)x}{u+x} \prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^y \frac{u-(y-n-1)x}{u+(n+1)x},$$

unde u , x și y pot lua orice valori, reale sau complexe. Literele u , x , y se numesc respectiv baza, pasul și exponentul funcției factoriale.

Funcțiile factoriale se exprimă în modul următor prin funcția Γ :

$$u^{y/x} \equiv (u, +x)^y = x^y \frac{\Gamma\left(\frac{u}{x} + y\right)}{\Gamma\left(\frac{u}{x}\right)},$$

$$u^{y/-x} \equiv (u, -x)^y = x^y \frac{\Gamma\left(\frac{u}{x} + 1\right)}{\Gamma\left(\frac{u}{x} + 1 - y\right)}.$$

Subliniem următoarea convenție în notarea funcțiilor factoriale:

$$u^{y+z/x} \equiv u^{(y+z)/x}.$$

Pentru y natural,

$$u^{y/x} = u(u+x)(u+2x) \dots [u+(y-1)x].$$

Această expresie se mai notează

$$[x(y-1)+u] \underbrace{11\dots 1}_{\text{de } x \text{ ori}}.$$

În particular,

$$1^{n/1} = n! = 1 \cdot 2 \cdot 3 \dots n,$$

$$1^{n/2} = (2n-1)!! = 1 \cdot 3 \cdot 5 \dots (2n-1),$$

$$2^{n/2} = (2n)!! = 2 \cdot 4 \dots 2n = 2^n n!$$

Notarea valorilor inverse ale integralelor lui Euler:

$$E_c(z) = \frac{1}{\Gamma(z)}.$$

Funcția lui I. M. Rijk:

$$s(x+1, y+1) = \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} = \frac{1}{(x+y+1)B(x+1, y+1)}.$$

Pentru $x \equiv m$, $y \equiv n$ numere naturale,

$$s(m+1, n+1) = \binom{m+n}{m} = \binom{m+n}{n}$$

și, deci, funcția lui Rijk este o generalizare a noțiunii de „coeficient binomial”. Amănunte despre proprietățile funcției lui Rijk, vezi în edițiile I și II ale acestei cărți, precum și în cartea lui I. M. Rijk „Funcții speciale”.

Funcțiile cilindrice

Funcția $N_p(z)$ se notează adesea $Y_p(z)$. În afară de aceasta, se mai întâlnesc alte definiții ale funcțiilor cilindrice de speța a doua, legate de funcțiile $N_p(z)$ prin următoarele relații:

$$Y_p(z) = \frac{\pi e^{p\pi i}}{\cos p\pi} N_p(z) \quad \text{W 78 (1)}$$

$$Y^{(n)}(z) = \frac{1}{2} \pi N_n(z) + (\ln 2 - C) J_n(z) \quad [n+1 \text{ natural}], \quad \text{W 85 (8)}$$

$$\zeta_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z); \quad \zeta_n^{(2)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(2)}(z). \quad \text{MO 32}$$

Ca funcție de argument pur imaginar se ia uneori funcția

$$K_p^*(z) = \frac{1}{2} \pi \operatorname{ctg} p\pi \{I_{-p}(z) - I_p(z)\} = K_p(z) \cos p\pi. \quad \text{WW}$$

Simbolurile $Ai(x)$ și $W(x)$ indică, adeseori, așa-numitele integrale ale lui Airy

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt,$$

$$W(x) = \int_0^\infty \cos \frac{1}{2} \pi (t^3 - xt) dt \quad (\text{vezi și 4,9})$$

De funcțiile Bessel se leagă următoarele polinoame:

Polinoamele $O_k(t)$ în raport cu t (polinoamele Neumann) se definesc ca coeficienți în dezvoltarea funcției

$$\frac{1}{t-z} = O_0(t) + 2 \sum_{k=1}^{\infty} O_k(t) J_k(z).$$

Polinoamele $Q_k(t)$ în raport cu $\frac{1}{t}$ (tot polinoame Neumann) se definesc ca coeficienți în dezvoltarea funcției

$$\frac{1}{t^2 - z^2} = Q_0(t) J_0^2(z) + 2 \sum_{k=1}^{\infty} Q_k(t) J_k^2(z).$$

Polinoamele Lommel $R_{m,p}(z)$ [$m=0, 1, 2, \dots, p$, un număr oarecare], definite prin egalitatea

$$R_{m,p}(z) = p(p+1) \dots (p+m-1) \left(\frac{z}{2}\right)^{-m} {}_2F_3\left(\frac{1-m}{2}, -\frac{m}{2}; p, -m, 1-p-m; -z^2\right).$$

Polinoamele Lommel se exprimă în felul următor prin funcțiile Bessel:

$$\frac{2 \sin p\pi}{\pi z} R_{m,p}(z) = J_{p+m}(z) J_{-p+1}(z) + (-1)^m J_{-p-m}(z) J_{p-1}(z).$$

Funcțiile sferice

Polinoamele Legendre se notează adeseori.

$$X_n(x) = P_n(x).$$

Se întâlnesc următoarele notații ale funcțiilor sferice (de speța întâi):

$$P^{pq}, P_{pq}, R_p^q, R_{pq}.$$

Unii autori notează prin simbolul $P_n^m(x)$ funcțiile care, în notațiile manualului nostru, se scriu

$$(-1)^m P_n^m(x).$$

În acest caz, formulele 6.830 și 6.752 1. devin

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

În schimb, funcțiile notate în manualul nostru cu $P_n^m(x)$ sînt notate de acești autori cu $T_n^m(x)$

Polinoamele ortogonale

Polinoamele $H_n(x)$ ale lui Hermite sînt definite de unii autori prin egalitatea

$$H_n^*(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

Funcțiile $H_n^*(x)$ se leagă de polinoamele $H_n(x)$, introduse în manualul de față, prin relația

$$H_n^*(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right).$$

Unii autori numesc polinoame Laguerre polinoamele care, în notațiile adoptate în manualul nostru, se scriu $\frac{1}{n!} L_n^{(p)}(x)$.

N. I. Sonin a folosit, în locul polinoamelor Laguerre, polinoame care se deosebesc de polinoamele Laguerre printr-un factor constant. El le-a notat $T_p^{(n)}(x)$:

$$T_p^{(n)}(x) = \frac{(-1)^n}{n! \Gamma(p+n+1)} L_n^{(p)}(x).$$

N. I. Sonin a arătat că

$$|T_p^{(n)}(x)| < n! p! e^{\frac{x}{2}} \quad [p \text{ și } n+1 \text{ numere naturale, } x > 0].$$

Numerele și polinoamele Bernoulli

Numerele lui Bernoulli se notează întotdeauna cu simbolul B_n , dar diferiți autori dau acestui simbol un sens diferit, cum ar fi:

1. Toate numerele lui Bernoulli coincid cu cele menționate în text, dar $B_1 = \frac{1}{2}$ (nu $-\frac{1}{2}$, ca în textul nostru).

$$2. |B_n| = B_n^*.$$

$$3. B_{2n} = B_n^{**}.$$

$$4. |B_{2n}| = B_n^{***}.$$

$$5. |B_{2n}| = B_{2n-1}^{**}.$$

Steluțele marchează numerele lui Bernoulli, ale căror valori sînt diferite de valorile adoptate în ediția de față. În primele două ediții, s-a adoptat valoarea B_n^* .

Polinoamele Bernoulli se notează, uneori, prin simbolul $\varphi_n(x)$. Unii autori numesc polinoame Bernoulli polinoamele

$$\frac{B_n(x) - B_n}{n+1}.$$

Numerele și polinoamele Euler

La fel cu numerele lui Bernoulli, numerele lui Euler au valori diferite la diverși autori; în particular, în primele două ediții, se înțelegeau prin E_n numere egale cu $|E_{2n}|$.

Unii autori numesc polinoamele

$$\frac{2}{n+1} \left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right]$$

polinoame de ordinul n ale lui Euler.

ANEXĂ*

La 2.821

Formula de recurență

$$(2n+1)^2 \int K k^{2n+1} dk = k^{2n} [E - (2n+1)k'^2 K] + 4n \int K k^{2n-1} dk.$$

$$(2n+1)(2n+3) \int E k^{2n+1} dk = -k^{2n} \left\{ [(2n+1)k'^2 - 2] E + k'^2 K \right\} + 4n^3 \int E k^{2n-1} dk.$$

Cazuri particulare

$$\int K k dk = E - k'^2 K \quad \text{cf. 2.821 3.}$$

$$3 \int E k dk = (1+k^2) E - k'^2 K \quad \text{cf. 2.821 4.}$$

$$9 \int K k^3 dk = (4+k^2) E - k'^2 (4+3k^2) K.$$

$$45 \int E k^3 dk = (4+k^2+9k^4) E - k'^2 (3+4k^2) K.$$

La 2.856

Formula de recurență

$$\int z^{m+1} Z_p(z) dz = z^{m+1} Z_{p+1}(z) + (m-p) z^m Z_p(z) - (m^2-p^2) \int z^{m-1} Z_p(z) dz. \quad \text{W 147 (4)}$$

* Această Anexă a fost adăugată de autor la ediția în limba română. — (Nota Editurii Tehnice).

Pentru $p=0$,

$$\int Z_0(z) dz = Z_{0i}(z),$$

unde Z_{0i} este o funcție nouă, ale cărei valori sînt tabelate în traducerea în limba rusă a lucrării lui Watson¹.

$$\int z Z_0(z) dz = z Z_1(z) \quad \text{cf. 2.856 2.}$$

$$\int z^2 Z_0(z) dz = z^2 Z_1(z) + z Z_0(z) - Z_{10}(z).$$

$$\int z^3 Z_0(z) dz = z^3 Z_1(z) + 2 z^2 Z_0(z) - 4 z Z_1(z).$$

Pentru $p=1$,

$$\int Z_1(z) dz = -Z_0(z) \quad \text{cf. 2.856 1.}$$

$$\int z Z_1(z) dz = -z Z_0(z) + Z_{10}(z).$$

$$\int z^2 Z_1(z) dz = -z^2 Z_0(z) + 2 z Z_1(z).$$

$$\int z^3 Z_1(z) dz = -z^3 Z_0(z) + 3 z^2 Z_1(z) + 3 z Z_0(z) - 3 Z_{10}(z).$$

¹ In volumul II. Valorile funcțiilor $\frac{1}{2} \int_0^x J_0(x) dx$ și $\frac{1}{2} \int_0^x Y_0(x) dx = \frac{1}{2} \int_0^x N_0(x) dx$

sînt date în tabela VIII. p. 104, iar valorile funcțiilor $\int_x^\infty K_0(x) dx$ și $\int_0^x I_0(x) dx$, în tabela X, pp. 215-219.