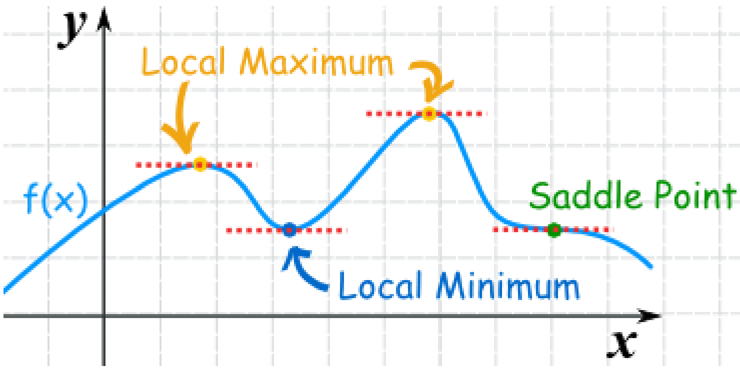
Finding Maxima and Minima using 2nd Derivatives Test

Link:

http://www.mathsisfun.com/calculus/maxima-minima.html

Where is a function at a high or low point? Calculus can help!

In a smoothly changing function a low point (a *minimum*) or high point (a *maximum*) are where the function **flattens out** :



(But not all flat parts are maxima or minima, we can also have a **saddle point**)

Where does it flatten out? Where the **slope is zero**.

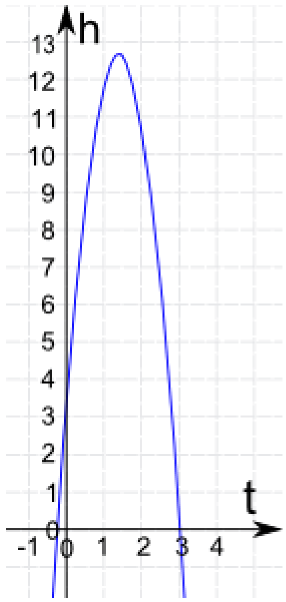
Where is the slope zero? The **Derivative** tells us!

(You might like to read about derivatives first.)

Example: A ball is thrown in the air. Its height at any time t is given by:

$h = 3 + 14t - 5t^2$

What is its maximum height?



Using derivatives we can find the slope of that function:

$$\frac{d}{dt} h = 0 + 14 - 5(2t)$$
$$= 14 - 10t$$

(See below this example for how we found that derivative.)

Now find when the **slope is zero**:

$14 - 10t = 0$
$10t = 14$
$t = 14 / 10 =$ **1.4**

The slope is zero at **t = 1.4 seconds**

And the height at that time is:

$h = 3 + 14 \times 1.4 - 5 \times 1.4^2$
$h = 3 + 19.6 - 9.8 =$ **12.8**

And so:

The maximum height is **12.8 m** (at t = 1.4 s)

A Quick Refresher on Derivatives

A derivative basically finds the slope of a function.

In the previous example we took this:

$h = 3 + 14t - 5t^2$

and came up with this derivative:

$\frac{d}{dt} h = 0 + 14 - 5(2t)$
$= 14 - 10t$

Which tells us the **slope** of the function at any time **t**

There are **rules** we can follow to find derivatives. We used these rules:

- The slope of a **constant** value (like 3) is 0
- The slope of the **line** 14t is 14
- A **square** function like t^2 has a slope of 2t, so $5t^2$ has a slope of 5(2t) = 10t
- And then we added them up

Learn more at Derivative Rules

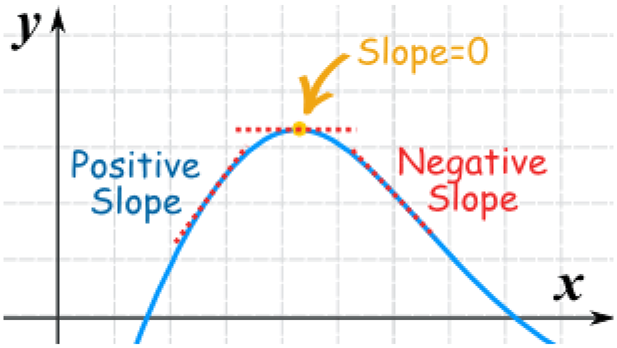
How Do We Know it is a Maximum (or Minimum)?

We saw it on the graph! But otherwise ... derivatives come to the rescue again.

Take the **derivative of the slope** (the second derivative of the original function):

The Derivative of 14 − 10t is **−10**

This means the slope is continually getting smaller (−10): travelling from left to right the slope starts out positive (the function rises), goes through zero (the flat point), and then the slope becomes negative (the function falls):



A slope that gets smaller (and goes though 0) means a maximum.

This is called the **Second Derivative Test**

On the graph above I showed the slope before and after, but in practice we do the test **at the point where the slope is zero**:

Second Derivative Test

When a function's **slope is zero at x**, and the **second derivative at x** is:

- less than 0, it is a local maximum
- greater than 0, it is a local minimum
- equal to 0, then the test fails (there may be other ways of finding out though)

"Second Derivative: less than 0 is a maximum, greater than 0 is a minimum"

Example: Find the maxima and minima for:

$y = 5x^3 + 2x^2 - 3x$

The derivative (slope) is:

$\frac{d}{dx} y = 15x^2 + 4x - 3$

Which is quadratic with zeros at:

- $x = -3/5$
- $x = +1/3$

Could they be maxima or minima? (Don't look at the graph yet!)

The second derivative is **y'' = 30x + 4**

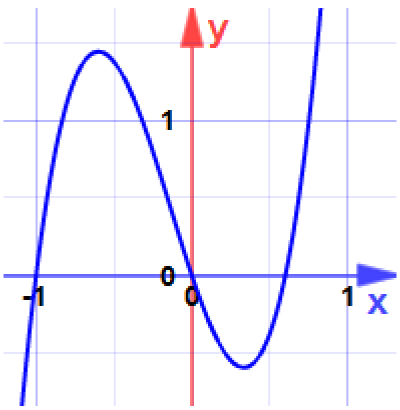
At x = −3/5:

y'' = 30(−3/5) + 4 = −14
it is less than 0, so −3/5 is a local maximum

At x = +1/3:

y'' = 30(+1/3) + 4 = +14
it is greater than 0, so +1/3 is a local minimum

(Now you can look at the graph.)



Words

A high point is called a **maximum** (plural ***maxima***).

A low point is called a **minimum** (plural ***minima***).

The general word for maximum or minimum is **extremum** (plural **extrema**).

We say **local** maximum (or minimum) when there may be higher (or lower) points elsewhere but not nearby.

One More Example

Example: Find the maxima and minima for:

$y = x^3 - 6x^2 + 12x - 5$

The derivative is:

$$\frac{d}{dx} y = 3x^2 - 12x + 12$$

Which is quadratic with only one zero at **x = 2**

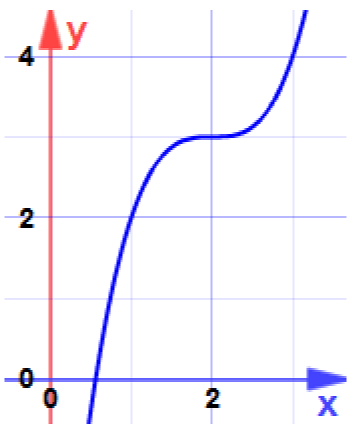
Is it a maximum or minimum?

The second derivative is **y'' = 6x − 12**

At x = 2:

y'' = 6(2) − 12 = 0
it is 0, so the test fails

And here is why:



It is a **saddle point** ... the slope does become zero, but it is neither a maximum or minimum.

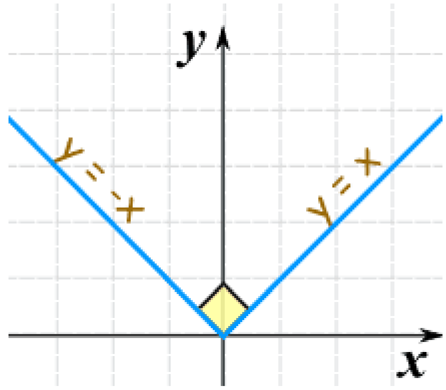
Must Be Differentiable.

And there is an important technical point:

The function must be **differentiable** (the derivative must exist at each point in its domain).

Example: How about the function f(x) = |x| (absolute value) ?

|x| looks like this:



At x=0 it has a very pointy change!

In fact it is not differentiable there (as shown on the differentiable page).

So we can't use this method for the absolute value function.

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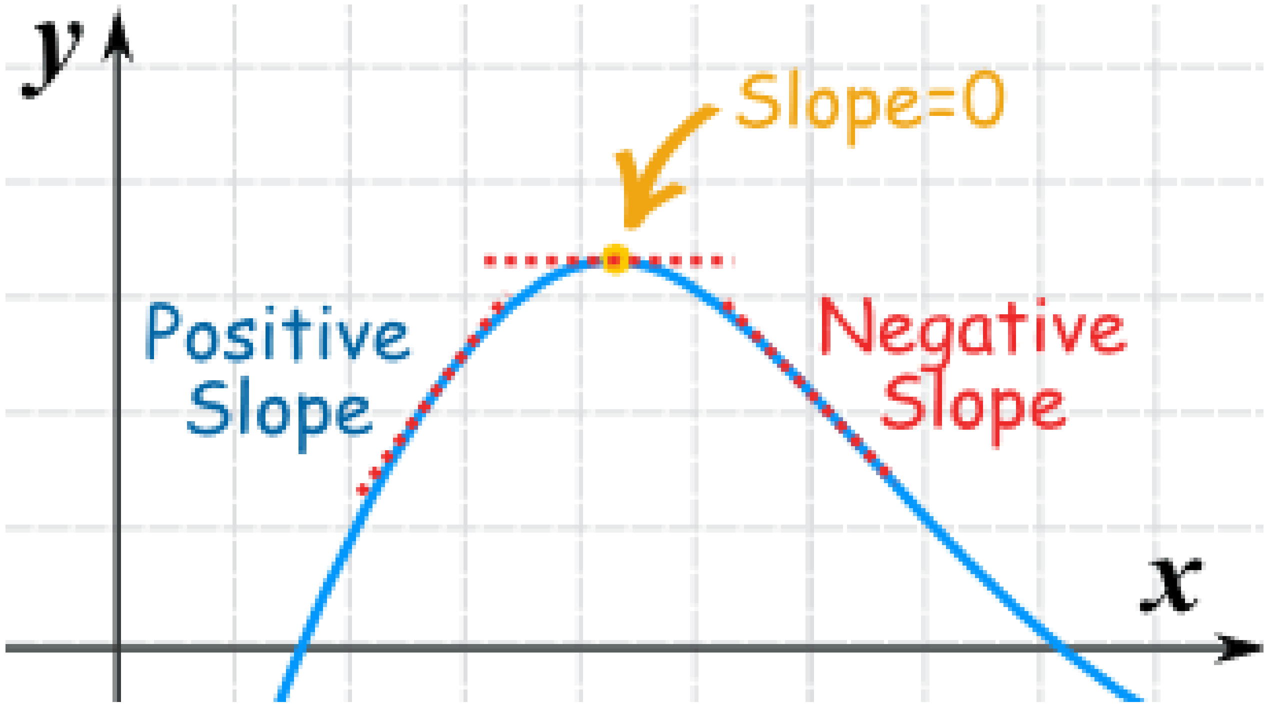

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