N6 • Developing proportional reasoning

Mathematical goals

To help learners to:

- reflect on the reasoning they currently use when solving proportion problems;
- examine proportion problems and appreciate their multiplicative structure;
- create their own variants of proportion problems.

Starting points

Proportional reasoning is notoriously difficult for many learners. Many have difficulty in recognising the multiplicative structures that underlie proportion problems. Instead, they use addition methods, or informal methods using doubling, halving and adding. This session aims to expose and build on this prior learning.

Learners are given four direct proportion problems to solve, taken from different areas of the mathematics curriculum. They then compare their methods for solving these with methods produced by other learners. This leads to a discussion that compares the use of more primitive informal methods that use adding, doubling and halving with the use of more sophisticated methods that use multiplication.

Materials required

- OHT 1 – *Solving proportion problems in one step*;
- OHT 2 – *Solving proportion problems in two steps*;
- an OHT of 'Paint prices' from Sheet 1 – *Problems* (page 1).

For each learner you will need:

- Sheet 1 – *Problems* (pages 1 and 2);
- Sheet 2 – *Sample work*;
- Sheet 3 – *Making up your own questions*;
- a calculator.

Time needed

At least 1 hour. If time is short, fewer problems may be tackled.

Suggested approach

Beginning the session

Give out copies of Sheet 1 – *Problems* (pages 1 and 2). Allow learners time to tackle the problems individually or in pairs. They may use calculators if they wish. Do not comment on learners' solution strategies at this stage as the purpose of the session is to compare different strategies.

Working in groups

Hand out copies of Sheet 2 – *Sample work*. Invite learners to assess these pieces of work.

They should attempt to:

- correct the work;
- write advice to the learner, identifying and explaining the errors that have been made and how the solution strategies can be improved.

The pieces of work illustrate the following issues:

Recipe

The learner has answered part 1 correctly, but part 2 incorrectly. For both parts, the learner has adopted an addition strategy. In part 1, she has reasoned that, because $10 = 4 + 4 +$ half of 4, then the quantities in the recipe may be deduced by doubling, then adding one half as much again. This is a correct strategy and is one that is helpful when working out calculations mentally. When a calculator is available, however, it is not the most efficient method.

In the second problem, the learner has followed the same strategy but has then added 'tops and bottoms'.

Paint prices

In part 1 the learner has chosen, wrongly, to divide rather than to multiply. This may be because the learner believes that 'division will make numbers smaller' and so $15 \div 0.6$ will be less than 15. (Multiplication has been rejected on the grounds that it will result in a number greater than 15.)

In parts 2 and 3, a 'halving and adding' strategy has been correctly used.

Part 4 is correct; a multiplying strategy has been correctly used.

Enlarging a photograph to make a poster

In these two commonly-found answers, an additive strategy has been incorrectly used.

Advertising

The percentages have been correctly calculated using an addition strategy. With a calculator, a more efficient strategy would have been to simply calculate the missing entries by multiplying or dividing by 3.6

Whole group discussion

Place an OHT of the 'Paint prices' problem from Sheet 1 – *Problems* onto the overhead projector and ask learners to share their comments about the sample work. Invite learners to suggest better ways of doing the problems, assuming that a calculator is available. In particular, challenge them to find a simple and efficient way of obtaining each answer using a calculator. For example, the missing paint prices can be found quite easily by multiplying every capacity by 15.

Ask learners if they noticed that the problems all have something in common.

Explain the following ideas, asking learners to contribute to each step. Write their responses on the board.

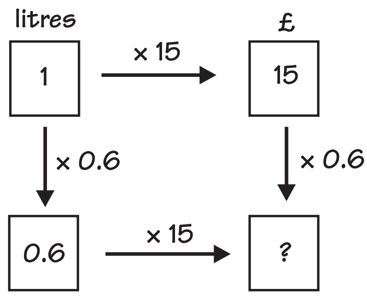
> All the problems we have looked at in this session have involved two quantities. Can you help me list them?
>
> | | |
> |---|---|
> | Number of pancakes | Amount of ingredients |
> | Quantity of paint | Cost of paint |
> | Height of poster | Width of poster |
> | Percentage of money spent | Angle in the pie chart |
>
> These are proportional situations. If we double the first quantity, we double the second. If we plotted a graph of the two quantities, we would get a straight line through the origin.
>
> Can you suggest some more pairs of quantities that are proportional?
>
> Can you suggest some that are not?

Learners may suggest other proportional pairs such as distance travelled and time taken (assuming constant speed), and others that are not, such as speed of travel and time taken (this is inversely proportional).

Show OHT 1 – *Solving proportion problems in one step* (or draw the diagram on the board). Explain that you will show a method for working out proportional problems using this diagram.

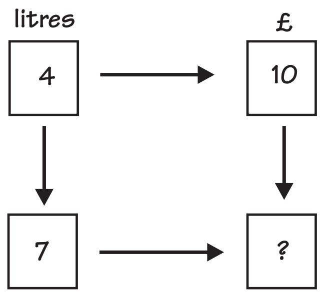
Look at the paint problem. The two quantities are amount bought (litres) and cost (£).

1 litre costs £15. (Fill in the top two boxes in the diagram).
We need to know how much 0.6 litres costs. (Fill in the remaining quantities in the diagram, explaining the reasoning.)



Explain how proportion problems are really just multiplication problems and show how the multipliers may be found and used. The vertical multipliers are dimensionless. The horizontal ones are rates (here £ per litre).

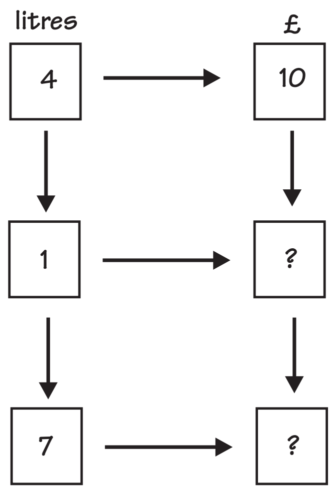
Create a new problem by changing the numbers, for example as shown here. Ask learners to state the new problem in their own words and to suggest the multipliers which, this time, are not so obvious. Some may see that the rows involve multiplying by $\frac{10}{4}$ or 2.5, and the columns by $\frac{7}{4}$ or 1.75.



In the same way, create additional problems using different sets of numbers, some of which use multipliers that are less than 1.

Learners who struggle may like to use two steps (using OHT 2 – *Solving proportion problems in two steps*), first dividing by 4 to find the price of 1 litre, then multiplying by 7 to find the price of 7 litres, as shown here.

Ask learners to return to the original problems and, using this diagram method, to solve any they could not do earlier.



Reviewing and extending learning

Ask learners to tell you how they can recognise when a situation is a direct proportion and when it is not.

> When one value is zero, so is the other.
>
> When one variable doubles, so does the other.
>
> The graph of one variable against the other is a straight line through the origin.
>
> The formula is $y = ax$.

What learners might do next

Issue learners with Sheet 3 – *Making up your own questions*. This sheet contains a selection of problems with numbers missing. Learners need to:

- decide which situations are direct proportions and which are not;
- write their own numbers in the spaces;
- solve the problems that are created, using both their own informal methods and the diagram method.

Encourage learners to use at least two sets of numbers for each problem. One set should make the problem quite easy (but not trivial), and one set should make it quite difficult.

If some learners are more able, ask them to write variable names for each space (e.g. x, y and z) and to write solutions for each situation using these letters.

Further ideas

This session involves analysing and comparing different methods for solving problems. This is a powerful idea that may be used in any topic. Why not give your learners a completed examination paper to mark and comment on? This is very useful if the answers that you provide reveal common misconceptions.

BLANK PAGE FOR NOTES

N6 Sheet 1 – *Problems* (page 1)

RECIPE

Pancake mix

For 4 pancakes you will need:

6 dessertspoons of flour

$\frac{1}{4}$ litre milk

1 pinch salt

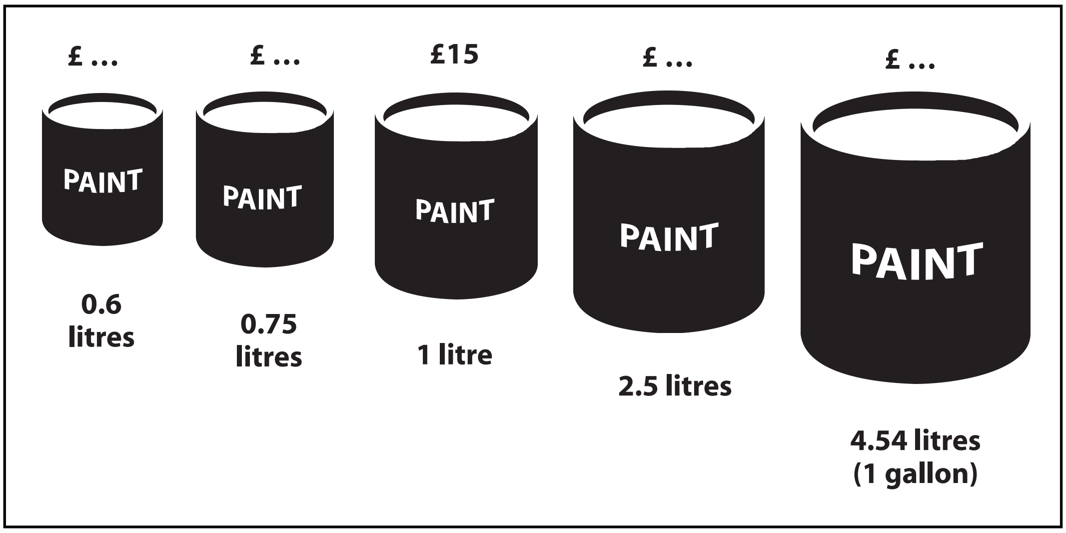
1 egg

You want to make 10 pancakes.

1. How much flour do you need?
2. How much milk do you need?

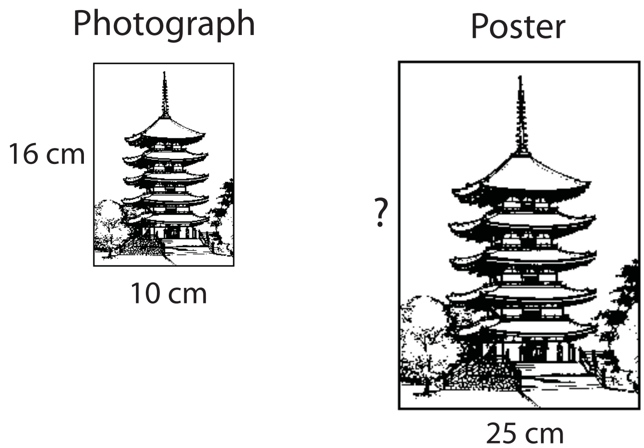
PAINT PRICES

Calculate the missing prices of the paint cans below.
The prices are proportional to the amount of paint in the can.



N6 Sheet 1 – *Problems* (page 2)

ENLARGING A PHOTOGRAPH TO MAKE A POSTER

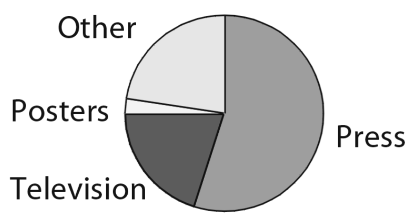


1. The photograph is enlarged to make a poster.
   The photograph is 10 cm wide and 16 cm high.
   The poster is 25 cm wide.
   How high is the poster?

2. In the poster, the building is 30 cm tall.
   How tall is it in the photograph?

ADVERTISING

	% spent	Angle in pie chart
Press	55	
Television	20	72°
Posters		9°
Other		



The pie chart shows the proportion spent on advertising in different media in one year.

Calculate the missing entries in the table.

N6 Sheet 2 – *Sample work*

(i) Mark the answers right or wrong.

(ii) Find the causes of the mistakes.

(iii) Write down your advice to the learner, explaining how the work should be improved, even when the answer is right.

Recipe

① 10 pancakes $= 6 + 6 + 3 = 15$ spoons of flour

② $= \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{3}{16}$ pints of milk.

Paint prices

1 0·6 litres $15 \div 0{\cdot}6 = 25$p

2. 0·75 litres $= \frac{1}{2} + \frac{1}{4}$

$= £7{\cdot}50 + £3{\cdot}75 = £11{\cdot}25$

3. 2·5 litres $= 2 + \frac{1}{2}$

$= £30 + £7{\cdot}50 = £37{\cdot}50$

4. 4·54 litres $= 15 \times 4{\cdot}54$

$= 68{\cdot}1$

Enlarging a photograph to make a poster

The poster is 15cm bigger

(1) $16 + 15 = 31$ cm high.

(2) $30 - 15 = 15$ cm high.

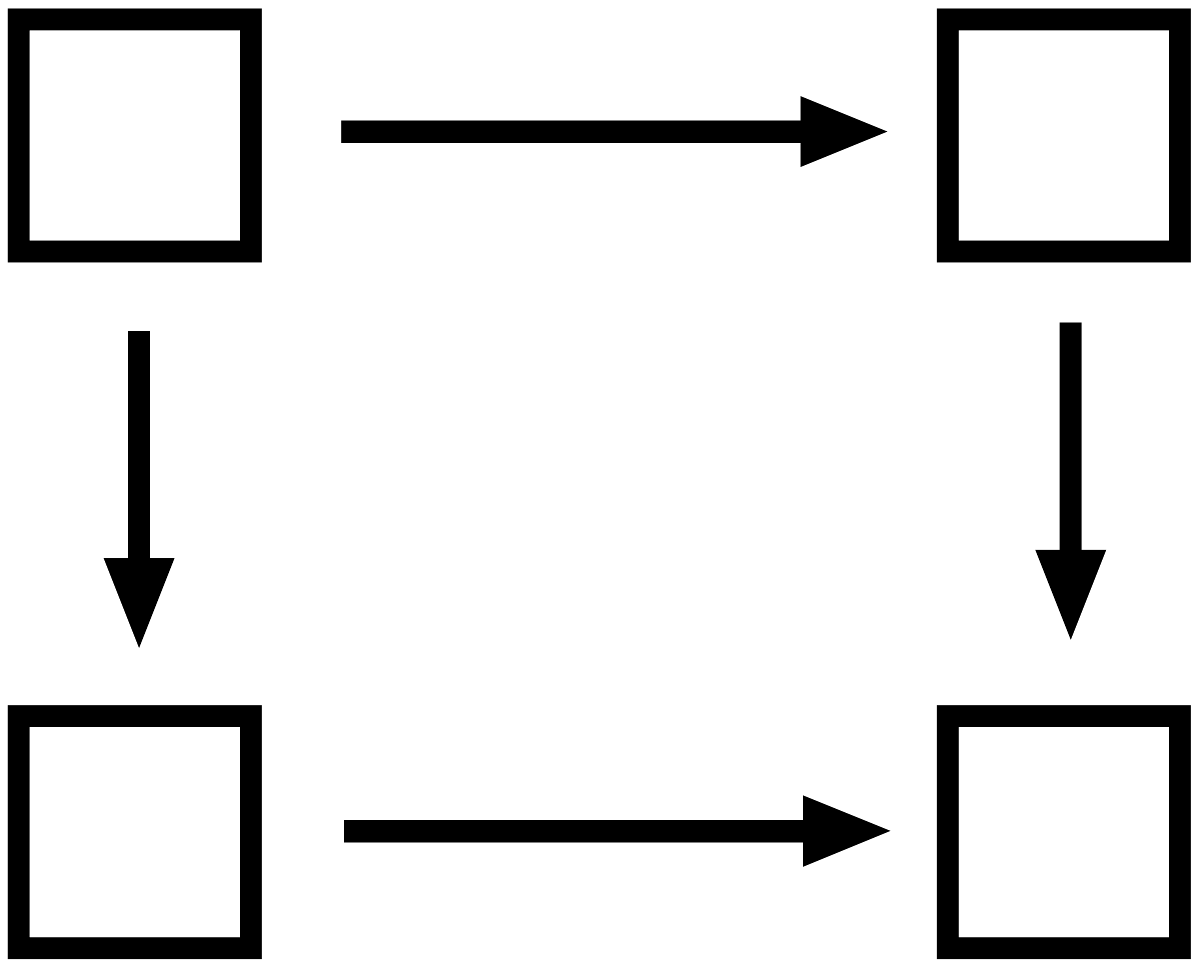
Advertising

20% ⇒ 72°
10% ⇒ 36°
5% ⇒ 18°

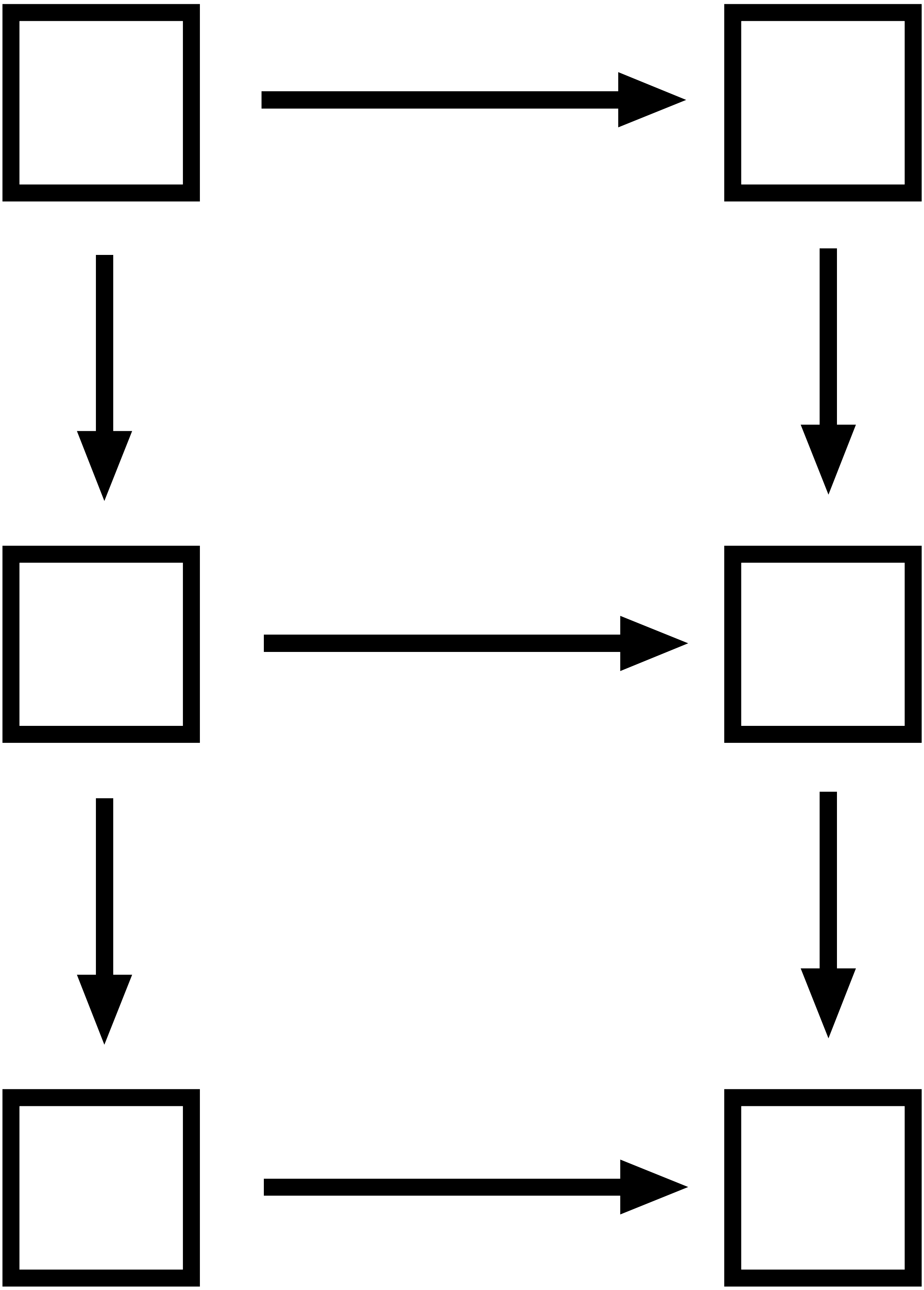
Press $= 72° + 72° + 36° + 18° \Rightarrow 198°$
Posters $= 9° \Rightarrow 2{\cdot}5\%$
Other $= 100 - 77{\cdot}5 = 22{\cdot}5\%$
$= 20\% + 2{\cdot}5\%$
$\Rightarrow 72° + 9° = 81°$

$$\begin{array}{r} 55 \; + \\ 20 \\ 2{\cdot}5 \\ \hline 77{\cdot}5 \end{array}$$

N6 OHT 1 – *Solving proportion problems in one step*



N6 OHT 2 – *Solving proportion problems in two steps*



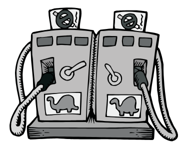
N6 Sheet 3 – *Making up your own questions*

CYCLE

It takes .... minutes
to cycle ...... miles.

At the same speed, how long
does it take to cycle ...... miles?

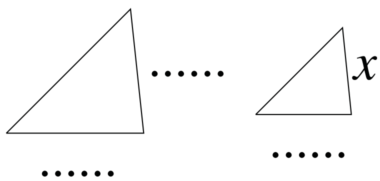
PETROL



..... litres cost £ .....

How much will ...... litres cost?

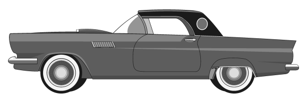
TRIANGLES



These two triangles are similar.

Calculate the length marked x.

DRIVING



If I drive at ...... miles per hour, the
journey will take .... hours.

How long will it take if I drive at
.... miles per hour?

LINE

A straight line passes through the
points (0,0) and (...... , ...).

It also passes through the point
(...... , y).

Calculate the value of y.

FIRE

It would take ..... minutes to vacate a
building if we put in ..... fire escapes.

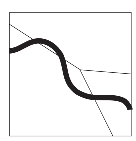
How long would it take with ..... fire
escapes?

MONEY

£ ..... is worth the same as ..... dollars.

If I change £ ....., how many dollars
will I get?

MAP



A road ..... cm long on a map is
..... km long in real life.

A river is ..... cm long on the map.
How long is the real river?