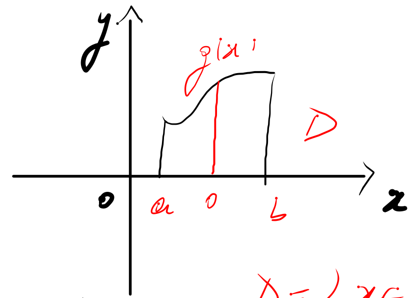
INTEGRALI

In 2 dimensioni (in R^2)

$$\int_D f(x,y)\,dx\,dy$$



$D = \{ x \in R^2 : \; 0 \le y \le g(x), \; a \le x \le b \}$

- Per fili

$$\int_D f(x,y)\,\underbrace{dx\,dy}_{dA} = \int_a^b dx \int_0^{g(x)} f(x,y)\,dy$$

- D simmetrie di tipo polare

CORRETTO (*)

$\varphi^{-1} : \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$

trasformazione

$$\int_D f(x,y)\,dx\,dy =$$

$$= \int_D f(u,v)\,|\det(J\varphi)|\,du\,dv$$

Coord. polari

$$dx\,dy = \rho\,d\rho\,d\theta$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

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ω5

$$I = \int_{\Omega} xy \, dx \, dy$$

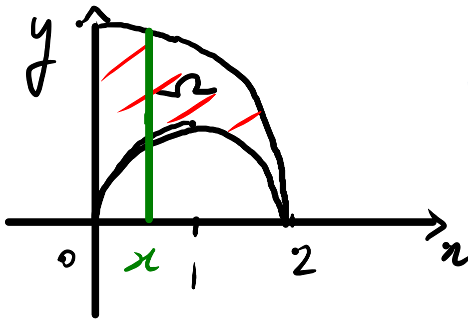
$$\Omega = \{ \underline{x} \in \mathbb{R}^2 : \ |x| \le 2, \quad x^2+y^2-2x \ge 0, \quad x \ge 0, y \ge 0 \}$$

$x^2+y^2 \le 4$

$y = \sqrt{4-x^2}$

$(x-1)^2+y^2 = 1$

$y = \sqrt{2x-x^2}$



$$\int_{\Omega} xy \, dx \, dy = \int_0^2 dx \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} xy \, dy =$$

$$= \int_0^2 x \, dx \left[\frac{y^2}{2} \right]_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} = \int_0^2 x \left[\frac{4-\cancel{x^2}}{2} - \frac{2x-\cancel{x^2}}{2} \right] dx$$

$$= \int_0^2 x \left[2-x \right] dx = \int_0^2 (2x-x^2) \, dx = \left[x^2 - \frac{1}{3} x^3 \right]_0^2 =$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

(d)

In $\mathbb{R}^3$ (in 3 dimensioni)

$$\int_D f(x,y,z) \underbrace{dx\,dy\,dz}_{dV}$$

- Per fili
- Coordinate cilindriche

$$\varphi : \begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = z \end{cases}$$

$$J\varphi = \begin{bmatrix} \cos\vartheta & -\rho\sin\vartheta & 0 \\ \sin\vartheta & \rho\cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det J\varphi = \rho\, d\rho\, d\vartheta$$

$$dx\,dy\,dz = dV = \rho\, d\rho\, d\vartheta\, dz$$

BARICENTRO $\quad G(x_G, y_G, z_G)$

$$G \begin{cases} x_G = \dfrac{\int_D x \, dm}{m} \\ y_G = \dfrac{\int_D y \, dm}{m} \\ z_G = \dfrac{\int_D z \, dm}{m} \end{cases} \qquad dm = \delta \, dV \qquad m = \int_D \delta \, dV$$

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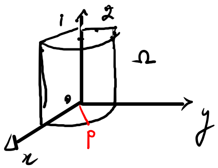
$$\Omega = \{ \underline{x} \in \mathbb{R}^3 : \; x^2+y^2 \leq 1 , \; 0 \leq z \leq 1 , \; x \geq 0, y \geq 0 \}$$



$\delta(x,y) = x^2+y^2$ densità

$$m = \int_\Omega \delta(x,y) \, dx dy dz \underset{\substack{\text{coordinate} \\ \text{cilindriche}}}{=} \int_\Omega \rho^2 \, \rho \, d\rho d\vartheta \, dz = \int_\Omega \rho^3 d\rho \, d\vartheta \, dz =$$

$$= \int_0^1 dz \int_0^{\pi/2} d\vartheta \int_0^1 \rho^3 d\rho = \frac{\pi}{2} \cdot \left[\frac{\rho^4}{4} \right]_0^1 = \frac{\pi}{8}$$

$$x_G = \frac{\int_\Omega x \, \delta(x,y) \, dV}{m} \underset{\substack{\text{coord.} \\ \text{cilindriche}}}{=} \frac{8}{\pi} \int_\Omega \rho \cos\vartheta \cdot \rho^2 \cdot \rho \, d\rho d\vartheta dz =$$

$$= \frac{8}{\pi} \int_0^1 dz \int_0^{\pi/2} \cos\vartheta \, d\vartheta \int_0^1 \rho^4 \, d\rho =$$

$$= \frac{8}{\pi} \left[\sin\vartheta \right]_0^{\pi/2} \left[\frac{1}{5} \rho^5 \right]_0^1 = \frac{8}{\pi} \cdot \frac{1}{5} = \frac{8}{5\pi}$$

Analogamente $\quad y_G = \dfrac{8}{5\pi} \quad$ (per simmetria)

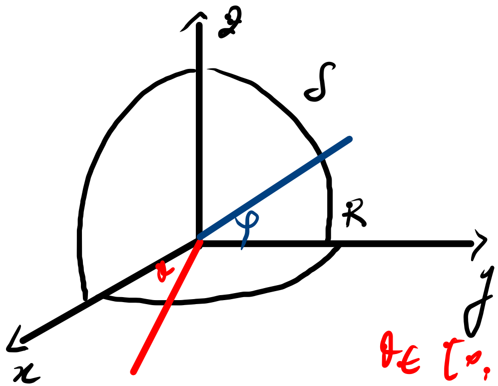
$$z_G = \frac{\int_\Omega z \, \delta(x,y) \, dV}{m} = \frac{8}{\pi} \int_\Omega z \cdot \rho^2 \cdot \rho \, d\rho d\vartheta \, dz =$$

$$= \frac{8}{\pi} \int_0^1 z \, dz \int_0^{\pi/2} d\vartheta \int_0^1 \rho^3 d\rho =$$

$$= \frac{8}{\pi} \left[\frac{z^2}{2} \right]_0^1 \frac{\pi}{2} \left[\frac{1}{4} \rho^4 \right]_0^1 = \cancel{\frac{8}{\pi}} \cdot \frac{1}{2} \cdot \cancel{\frac{\pi}{2}} \cdot \cancel{\frac{1}{4}} = \frac{1}{2}$$

$G\left(\dfrac{8}{5\pi} , \dfrac{8}{5\pi} , \dfrac{1}{2} \right) \qquad$ (c)

Coordinate sferiche



$\vartheta \in [0, 2\pi]$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$S: \quad x^2+y^2+z^2 = R^2$$

$$S: \begin{cases} x = R\cos\varphi\cos\vartheta \\ y = R\cos\varphi\sin\vartheta \\ z = R\sin\varphi \end{cases}$$

$$\varphi: \begin{cases} x = \rho\cos\varphi\cos\vartheta \\ y = \rho\cos\varphi\sin\vartheta \\ z = \rho\sin\varphi \end{cases}$$

coordinate sferiche

$$J_\varphi = \begin{bmatrix} \cos\varphi\cos\vartheta & -\rho\sin\varphi\cos\vartheta & -\rho\cos\varphi\sin\vartheta \\ \cos\varphi\sin\vartheta & -\rho\sin\varphi\sin\vartheta & \rho\cos\varphi\cos\vartheta \\ \sin\varphi & \rho\cos\varphi & 0 \end{bmatrix}$$

$$\det J_\varphi = \sin\varphi\left(-\rho^2\sin\varphi\cos\varphi\cos^2\vartheta - \rho^2\cos\varphi\sin\varphi\sin^2\vartheta\right) +$$

$$-\rho\cos\varphi\left(\rho\cos^2\varphi\cos^2\vartheta + \rho\cos^2\varphi\sin^2\vartheta\right)$$

$$= -\rho^2\sin^2\varphi\cos\varphi - \rho^2\cos^3\varphi = -\rho^2\cos\varphi$$

CORRETTO

$$|\det J_\varphi| = \rho^2\cos\varphi$$

$$\boxed{dV = dx\,dy\,dz = \rho^2\cos\varphi\,d\rho\,d\vartheta\,d\varphi}$$

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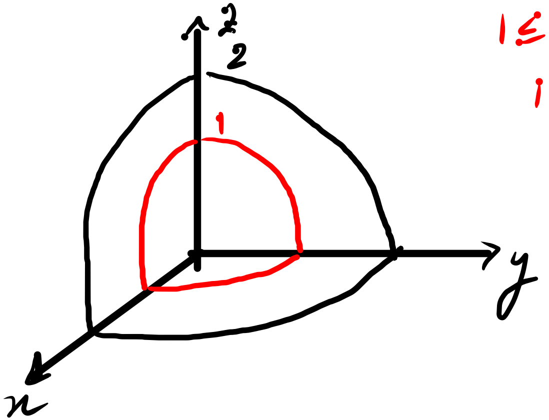
$$I = \int_{\Omega} (x^2+y^2+z^2)\sqrt{x^2+y^2}\,dx\,dy\,dz$$

$$\Omega = \{ \underline{x} \in R^3 : 1 \le |\underline{x}|^2 \le 4 \;, \quad z \ge 0 \}$$



$1 \le |\underline{x}| \le 2$

$1 \le \rho \le 2$

$0 \le \varphi \le \pi/2$

coord. sferiche

$$I = \int_{\Omega'} \rho^2 \sqrt{\rho^2 - \rho^2 \sin^2\varphi}\;\; \rho^2 \cos\varphi \; d\rho\, d\theta\, d\varphi =$$

CORRETTO

$\cos\varphi$

$$= \int_0^{2\pi} d\theta \int_{\Omega'} \rho^5 \sqrt{1-\sin^2\varphi}\; d\rho\, d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_1^2 \rho^5\, d\rho \int_0^{\pi/2} \cos^2\varphi =$$

$$= 2\pi \left[\frac{1}{6}\rho^6\right]_1^2 \left[\frac{\varphi + \sin\varphi\cos\varphi}{2}\right]_0^{\pi/2} = 2\pi \cdot \frac{1}{6}(64-1)\cdot\frac{\pi}{4} = \frac{21}{4}\pi^2$$

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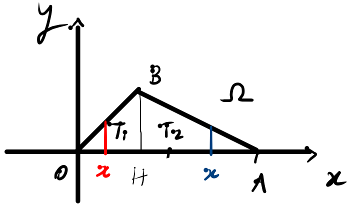


$O(0,0)$

$A(3a, 0)$

$B(a, a)$

$H(a, 0)$

$$\int_{\Omega} (x^2+y^2)\,dx\,dy = \int_{T_1} (x^2+y^2)\,dx\,dy + \int_{T_2} (x^2+y^2)\,dx\,dy$$

r_{OB} : $y = x$

r_{AB} : $y = \frac{a}{-2a}(x-3a) = -\frac{1}{2}x + \frac{3}{2}a$

$$\int_{T_1} (x^2+y^2)\,dx = \int_0^x \left(\int_0^a (x^2+y^2)\,dx \right) dy = \int_0^a x^2\,dx \int_0^x dy + \int_0^a dx \int_0^x y^2\,dy =$$

$$= \int_0^a x^3\,dx + \int_0^a \frac{x^3}{3}\,dx = \frac{1}{4}a^4 + \frac{1}{12}a^4 = \frac{a^4}{3}$$

$$\int_{T_2} (x^2+y^2)\,dx = \int_0^{-\frac{x}{2}+\frac{3}{2}a} \left(\int_a^{3a} x^2+y^2\,dx \right) dy =$$

$$= \int_a^{3a} x^2\,dx \int_0^{-\frac{x}{2}+\frac{3}{2}a} dy + \int_a^{3a} dx \int_0^{-\frac{x}{2}+\frac{3}{2}a} y^2\,dy =$$

$$= \int_a^{3a} x^2\left(-\frac{x}{2}+\frac{3}{2}a\right) dx + \int_a^{3a} \left[\frac{1}{3}y^3\right]_0^{-\frac{x}{2}+\frac{3}{2}a} dx =$$

$$= \left[-\frac{x^4}{8} + \frac{1}{2}a x^3\right]_a^{3a} + \frac{1}{24}\int_a^{3a} \left(-x^3 + 27a^3 + 9ax^2 - 27a^2x\right) dx =$$

$$= -\frac{81a^4}{8} + \frac{27a^4}{2} - \left(-\frac{a^4}{8} + \frac{1}{2}a^4\right) + \frac{1}{24}\left[-\frac{1}{4}x^4 + 27a^3x + 3ax^3 - \frac{27}{2}a^2x^2\right]_a^{3a}$$

e si conclude ...

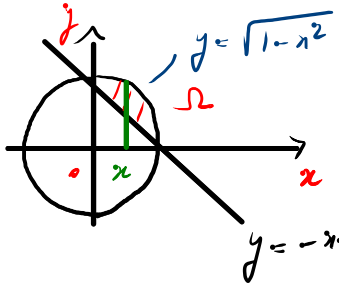
02/02/22

es 4.

$$I = \int_{\Omega} xy \, dx dy$$

$$\Omega = \{ \underline{x} \in \mathbb{R}^2 : \ 0 \le |x| \le 1 , \quad x + y \ge 1 \}$$

$$y \ge -x+1$$



$$I = \int_{\Omega} xy \, dx dy = \int_0^1 x \left(\int_{-x+1}^{\sqrt{1-x^2}} y \, dy \right) dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{-x+1}^{\sqrt{1-x^2}} dx =$$

$$= \int_0^1 x \left(\frac{\cancel{1} - x^2}{2} - \frac{x^2 - 2x + \cancel{1}}{2} \right) dx =$$

$$= \int_0^1 x \left(-x^2 + x \right) dx =$$

$$= \int_0^1 \left(-x^3 + x^2 \right) dx = -\frac{1}{4} + \frac{1}{3} = \frac{1}{12} \quad (f)$$

Oss $\quad xy \underset{x,y \ge 0}{\le} \dfrac{x^2+y^2}{2}$

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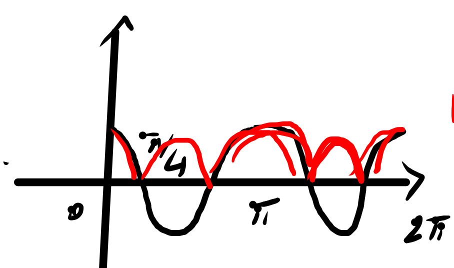
$$\Omega = \left\{ \underline{x} \in \mathbb{R}^3 : \ x^2+y^2 \le R^2, \ 0 \le z \le \frac{|x^2-y^2|}{R} \right\}$$

$$\mathrm{vol}(\Omega) = \int_\Omega dx\,dy\,dz \underset{\substack{\text{coord.}\\ \text{cilindriche}}}{=} \int_0^1 \rho \int_0^{2\pi} d\theta \left(\int_0^{\frac{\rho^2}{R}|\cos\theta|} dz \right) d\rho$$

$$\frac{|x^2-y^2|}{R} = \frac{\rho^2\,|\cos^2\theta - \sin^2\theta|}{R} = \frac{\rho^2}{R}\,|\cos(2\theta)|$$

$$\mathrm{vol}(\Omega) = \left(\int_0^1 \frac{\rho^3}{R}\, d\rho \right) \left(\int_0^{2\pi} |\cos(2\theta)|\, d\theta \right) =$$

$$= \frac{1}{4R} \cdot 8\int_0^{\pi/4} \cos(2\theta)\, d\theta = \frac{2}{R} \cdot \frac{1}{2} \Big[\sin(2\theta)\Big]_0^{\pi/4} =$$



$|\cos(2\theta)|$

$$= \frac{1}{R}$$