CAMPI VETTORIALI

Def $\quad F: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ campo vettoriale

F è irrotazionale se $\underline{\nabla} \wedge \underline{F} = \underline{0}$

Prop $\quad D$ semplicemente connesso (= senza buchi)

F irrotazionale se F conservativo

Def $\quad F$ è conservativo se $\exists\, U: D \to \mathbb{R}$

tale che $\quad \underline{F} = \nabla \underline{U}$

Alternativamente

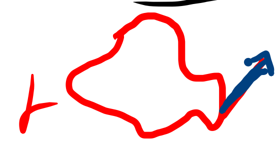
$$\underline{F} \cdot d\underline{s} = d\underline{U}$$

Def **Integrale di linea** (**lavoro**)

γ curva differenziabile $\qquad \gamma : [0,1] \to \mathbb{R}$

$$\int_\gamma \underline{F} \cdot d\underline{s} = \int_0^1 \underline{F}(x(t), y(t), z(t)) \cdot \gamma'(t)\, dt$$

lavoro

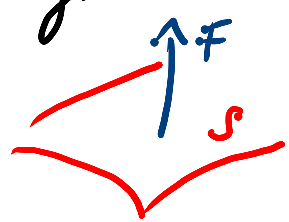


Oss F conservativo $\qquad \int_\gamma \underline{F} \cdot d\underline{s} = U(\underline{x}(1)) - U(\underline{x}(0))$

γ chiusa: $\oint_\gamma \underline{F} \cdot d\underline{s} = 0 \qquad (\underline{x}(1) = \underline{x}(0))$

Def S superficie differenziabile $\qquad S : D \to \mathbb{R}^3$

$$\Phi_S(F) = \int_S \underline{F} \cdot \underline{N}\, ds =$$



flusso

$$= \int_S F(u,v) \cdot \underbrace{\sqrt{\det G}\, du\, dv}_{dS}$$

Def

$$\underline{\nabla} \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \mathrm{div}(F) \in \mathbb{R}$$

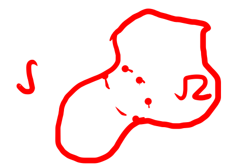
<u>divergenza di F</u>

$$\underline{\nabla} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \mathrm{rot}(F) \in \mathbb{R}^3$$

<u>rotore di F</u>

<u>Teor</u> (Gauss-Green)

$S = \partial \Omega$ S chiusa



Allora: $\displaystyle\int_S \underline{F} \cdot \underline{N} \, dS = \int_\Omega \underline{\nabla} \cdot \underline{F} \, dV$

<u>Teor</u> (Stokes)

$\gamma = \partial S$ γ chiusa

Allora: $\displaystyle\int_\gamma \underline{F} \cdot d\underline{l} = \int_S \underline{\nabla} \wedge \underline{F} \cdot \underline{N} \, dS$

13/01/22

n. 3

$$\underline{F} = \begin{pmatrix} 4x + 3xe^y z \\ 2x - 3y - 3e^y z \\ 5\sin y + 2z \end{pmatrix}$$

$$\Sigma : x^2 + y^2 + z^2 + 6x - 4y - 2z + 13 = 0$$

$$\phi = \int_\Sigma \underline{F} \cdot \underline{N}\, dS \underset{\text{Gauss}}{=} \int_{\overline{\Sigma}} \underline{\nabla} \cdot \underline{F}\, dV$$

$$\underline{\nabla} \cdot \underline{F} = 4 + \cancel{3e^y z} - 3 - \cancel{3e^y z} + 2 =$$

$$= 3.$$

$$\Sigma : (x-3)^2 + (y+2)^2 + (z-1)^2 = 1 \qquad \Sigma = \partial\overline{\Sigma}$$

$$R = \frac{1}{2}\sqrt{36 + 16 + 4 - 52} = \frac{2}{2} = 1$$

$$\overline{\Sigma} : (x-3)^2 + (y+2)^2 + (z-1)^2 \leq 1$$

$$\left(\varphi : \begin{cases} x - 3 = \rho\cos\varphi\cos\vartheta \\ y + 2 = \rho\cos\varphi\sin\vartheta \\ z - 1 = \rho\sin\varphi \end{cases} \qquad dV = \rho^2\cos\varphi\, d\rho\, d\varphi\, d\vartheta \right)$$

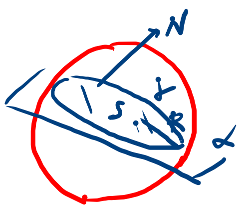
$$\phi = \int_{\overline{\Sigma}} 3\, dV = 3 \cdot \frac{4\pi R^3}{3} = 3 \cdot \frac{4}{3}\pi = 4\pi \quad (f)$$

02/02/22

es 6

$$\underline{F} = \begin{pmatrix} x-y-2z \\ 2x+3y-z \\ x+2y+2z \end{pmatrix} \qquad \int_\gamma \underline{F}\cdot d\underline{l} = ?$$

$$\gamma: \begin{cases} x^2+y^2+z^2-8x-4y-2z+19=0 & \Sigma \\ x-y-z=1 & \alpha \end{cases}$$



$$\int_\gamma \underline{F}\cdot d\underline{l} \underset{\text{Stokes}}{=} \int_S \underline{\nabla\wedge F}\cdot \underline{N}\, dS = I$$

$\gamma = \partial S$

$$\underline{\nabla \wedge F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x-y-z & 2x+3y-z & x+2y+2z \end{vmatrix} = \begin{pmatrix} 2+1 \\ -1-1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$$

$$\underline{N} = \underline{N}_\alpha = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$I = \int_S (3+2-3)\, dS = 2\, A(S) = 2\pi R^2 = 2\pi R^2(\gamma)$$

$K(\Sigma) = (4,2,1)$ centro di Σ

$K \in \alpha: \quad 4-2-1=1 \Rightarrow \gamma$ circonf equatoriale

$$R(\gamma) = R(\Sigma) = \frac{1}{2}\sqrt{64+16+4-76} = \frac{1}{2}\cdot 2\sqrt{2} = \sqrt{2}$$

$$I = 2\pi \cdot 2 = 4\pi = L$$

30/06/20
n. 3

$$\underline{F} = \begin{pmatrix} 6x+6y \\ 6x+3y^2 \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

In $\mathbb{R}^2$: F irrotazionale se $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$

Se $\mathrm{Dom}\,F$ è semplicemente connesso
F irrotazionale $\Leftrightarrow$ F conservativo

a) $\frac{\partial F_y}{\partial x} = 6 = \frac{\partial F_x}{\partial y}$ $\Rightarrow$ F irrotazionale
$\mathrm{Dom}\,f = \mathbb{R}^2$ semplicemente connesso
$\Rightarrow$ F conservativo

$\underline{F} = \underline{\nabla} U$: $F_x = \frac{\partial U}{\partial x} = 6x+6y \Rightarrow U = 3x^2 + 6xy + \psi(y)$

$F_y = \frac{\partial U}{\partial y} \Rightarrow \cancel{6x}+3y^2 = \cancel{6x} + \psi'(y) \Rightarrow \psi(y) = y^3 + c$

$U(x,y) = 3x^2 + 6xy + y^3 + c$
$U(0,0) = 0 \Rightarrow c = 0$

$U(x,y) = 3x^2 + 6xy + y^3$

b) $\underline{F} = \underline{\nabla} U = \underline{0}$

$$\begin{cases} 6x+6y=0 \\ 6x+3y^2=0 \end{cases} \quad \begin{cases} y=-x \\ y^2-2y \end{cases} \quad \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=-2 \\ y=+2 \end{cases}$$

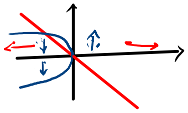
$O(0,0)$ $A(-2,2)$

$\frac{\partial^2 U}{\partial x^2} = \frac{\partial F_x}{\partial x} = 6$

$\frac{\partial^2 U}{\partial y^2} = \frac{\partial F_y}{\partial y} = 6$

$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial F_y}{\partial x} = 6$

$$H_U = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$$



$\frac{\partial U}{\partial x} > 0 \Rightarrow y \geq -x$

$\frac{\partial U}{\partial y} > 0 \Rightarrow x \geq -y^2/2$

$O(0,0)$ sella di U

Alternativamente $U(0,0) = 0$
ma U assume sia valori positivi che negativi in $U(0,0)$
$\Rightarrow$ $(0,0)$ sella di U

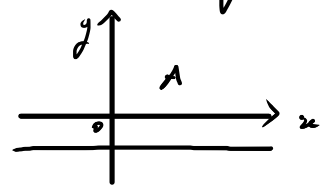
Analogamente per A

13/02/14

es 4

$$\underline{F}(\underline{x}) = \begin{pmatrix} \dfrac{2\cos(2x)}{3y+1} \\ -\dfrac{3\sin(2x)}{(3y+1)^2} \end{pmatrix}$$

$$A = \{ \underline{x} \in \mathbb{R}^2 : 3y+1>0 \}$$



A sempl. conn.

a) $$\frac{\partial F_y}{\partial x} = - \frac{6\cos(2x)}{(3y+1)^2}$$

$$\frac{\partial F_x}{\partial y} = - \frac{2\cos(2x)}{(3y+1)^2} \cdot 3$$

F è irrotazionale
$\Downarrow$ A s. conn.
F è conservativo

$$\frac{\partial U}{\partial x} = F_x = \frac{2\cos(2x)}{3y+1} \Rightarrow \frac{+\sin(2x)}{3y+1} + \psi(y)$$

$$\frac{\partial U}{\partial y} = F_y \Rightarrow \frac{-3\sin(2x)}{(3y+1)^2} = \frac{-3\sin(2x)}{(3y+1)^2} + \psi'(y) \Rightarrow \psi'(y) = 0 \Rightarrow$$

$$\Rightarrow \psi(y) = c$$

$$U = \frac{\sin(2x)}{3y+1} + c$$

b) $$\gamma(t) = \begin{pmatrix} 2^t \\ \ln(1+t^4(e-1)) \end{pmatrix}$$

$t \in [0,1]$

$$\gamma(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\gamma(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\int_\gamma \underline{F} \cdot d\underline{l} = U(\gamma(1)) - U(\gamma(0)) =$$

$$= \frac{\sin(4)}{4} - \sin(2)$$

30/06/20
es. 4

$$\underline{F} = \begin{pmatrix} x^2 \\ 2xz \\ z \end{pmatrix} \qquad \Sigma: \begin{cases} x^2+y^2+z^2 = 9 \\ z > 0 \end{cases}$$

$$\phi = \int_\Sigma \underline{F} \cdot d\underline{S}$$

$$\bar\Sigma: \begin{cases} x^2+y^2+z^2 \le 9 \\ z \ge 0 \end{cases}$$

$$\partial\bar\Sigma = \Sigma \cup S$$

I modo: direttamente

$$\Sigma: \quad S(\varphi,\vartheta) = \begin{pmatrix} 3\cos\varphi\cos\vartheta \\ 3\cos\varphi\sin\vartheta \\ 3\sin\varphi \end{pmatrix}$$

$\vartheta \in [0, 2\pi)$
$\varphi \in (0, \pi/2)$

$$S_\varphi = \begin{pmatrix} -3\sin\varphi\cos\vartheta \\ -3\sin\varphi\sin\vartheta \\ 3\cos\varphi \end{pmatrix}$$

$$S_\vartheta = \begin{pmatrix} -3\cos\varphi\sin\vartheta \\ 3\cos\varphi\cos\vartheta \\ 0 \end{pmatrix}$$

$$G = \begin{bmatrix} 9 & 0 \\ 0 & 9\cos^2\varphi \end{bmatrix} \qquad dS = 9\cos\varphi \, d\vartheta \, d\varphi$$

$$\underline{N}|\underline{N}| = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ -3\sin\varphi\cos\vartheta & -3\sin\varphi\sin\vartheta & 3\cos\varphi \\ -3\cos\varphi\sin\vartheta & 3\cos\varphi\cos\vartheta & 0 \end{vmatrix} = \begin{pmatrix} 9\cos^2\varphi\cos\vartheta \\ -9\cos^2\varphi\sin\vartheta \\ -9\cos^2\vartheta\cos\varphi\sin\varphi - 9\sin^2\vartheta\cos\varphi\sin\varphi \end{pmatrix}$$

$$|\underline{N}|\underline{N} = 9\begin{pmatrix} \cos^2\varphi\cos\vartheta \\ -\cos^2\varphi\sin\vartheta \\ -\cos\varphi\sin\varphi \end{pmatrix}$$

$$|\underline{N}| = 9\sqrt{\cos^4\varphi + \cos^2\varphi\sin^2\varphi} = 9\cos\varphi$$

$$|N| = \begin{pmatrix} \cos\vartheta\cos\varphi \\ -\sin\vartheta\cos\varphi \\ -\sin\varphi \end{pmatrix}$$

Il calcolo è molto complesso

II metodo: teorema di Gauss

$$\underline{F} = \begin{pmatrix} x^2 \\ 2xz \\ z \end{pmatrix} \qquad \Sigma : x^2+y^2+z^2 = 9 \ ; \ z>0$$

$$\int_\Sigma \underline{F}\cdot \underline{N}\, dS \underset{\text{Gauss}}{=} \int_\Sigma \nabla \underline{F}\, dV - \int_S \underline{F}\cdot \underline{N}\, dS$$

N costante

$$\underline{\nabla}\cdot\underline{F} = 2x+1$$

$$\int_\Sigma (2x+1)\, dV = \int_\Sigma (2\cdot \rho\cos\varphi\cos\vartheta + 1)\, \rho^2\cos\varphi\, d\rho\, d\vartheta\, d\varphi =$$

$$= \int_\Sigma \rho^2\cos\varphi\, d\rho\, d\vartheta\, d\varphi + 2\int_\Sigma \rho^3\cos^2\varphi\cos\vartheta\, d\rho\, d\vartheta\, d\varphi =$$

$$= \frac{2}{3}\pi\cdot 27 + 2\int_0^{2\pi} d\vartheta \int_0^{\pi/2} d\varphi \left[\frac{1}{4}\rho^4\right]_0^3 \cos^2\varphi\cos\vartheta =$$

$$= 18\pi + \frac{81}{2}\int_0^{\pi/2}\cos^2\varphi\, d\varphi \left[\sin\vartheta\right]_0^{2\pi} = 18\pi$$

$$\int_S \underline{F}\cdot\underline{N}\, dS = \qquad \underline{N} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad F(x,y,0) = \begin{pmatrix} x^2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \int_S 0\, dS = 0$$

Pertanto $\int_\Sigma \underline{F}\cdot\underline{N}\, dS = \int_\Sigma \underline{\nabla}\cdot\underline{F}\, dV = 18\pi$