13/02/19

$$f(x) = xy\, e^{-2x-y}$$

$$\frac{\partial f}{\partial x} = y\left(\cancel{e^{-2x-y}} - 2x\cancel{e^{-2x-y}}\right) = 0$$

$$\frac{\partial f}{\partial y} = x\left(\cancel{e^{-2x-y}} - y\cancel{e^{-2x-y}}\right) = 0$$

$$\begin{cases} y(1-2x) = 0 \\ x(1-y) = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} x = 1/2 \\ y = 1 \end{cases}$$

$$O(0,0) \qquad A(1/2, 1)$$

$$\frac{\partial^2 f}{\partial x^2} = y\left(-2e^{-2x-y} - 2e^{-2x-y} + 4x e^{-2x-y}\right) = 4y e^{-2x-y}(-1+x)$$

$$\frac{\partial^2 f}{\partial x \partial y} = (1-2x)\left[e^{-2x-y} - y e^{-2x-y}\right] = (1-2x) e^{-2x-y}(1-y)$$

$$\frac{\partial^2 f}{\partial y^2} = x\left(-e^{-2x-y} - e^{-2x-y} + y e^{-2x-y}\right) = x e^{-2x-y}(-2+y)$$

$$H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det H_f(0,0) = -1$$

$(0,0)$ punto di sella

$$H_f\left(\tfrac{1}{2}, 1\right) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & -\frac{1}{2}e^{-2} \end{bmatrix}$$

$$\det H_f\left(\tfrac{1}{2}, 1\right) = e^{-4} > 0$$

$$\operatorname{tr} H_f(1/2, 1) = e^{-2} \cdot \left(-\tfrac{5}{2}\right) < 0$$

$A\left(\frac{1}{2}, 1\right)$ punto di max f

2) a) $\begin{cases} y' = e^x y^2 = f(x,y) \\ y(0) = 1 \end{cases}$

$f \in C^0(0,1) \Rightarrow \exists$ soluzione locale

$\left|\frac{\partial f}{\partial y}\right| = 2e^x |y|$ limitata in $U(0,1)$ $\Rightarrow \exists!$ soluzione locale

$$\frac{dy}{y^2} = e^x\, dx \Rightarrow -\frac{1}{y} = e^x + k$$

$y(0)=1$: $\quad -1 = 1 + k \Rightarrow k = -2$

$$-\frac{1}{y} = e^x - 2 \Rightarrow y = \frac{-1}{e^x-2} = g(x)$$

b) C.E: $e^x - 2 \neq 0 \Rightarrow e^x \neq 2 \Rightarrow x \neq \ln 2$

$0 \quad \ln 2$

Dom g = $(-\infty, \ln 2)$

c) $y = \dfrac{-1}{e^x-2}$ $\qquad y(0) = 1$

$\dfrac{dy}{dx} = \dfrac{e^x}{(e^x-2)^2}$ $\qquad y'(0) = 1$

$$\frac{d^2y}{dx^2} = \frac{e^x(e^x-2)^{\cancel{2}} - 2\cancel{(e^x-2)}e^{2x}}{(e^x-2)^{\cancel{4}\,3}} = \quad y''(0) = 1 - 2(-1) = 3$$

$$= \frac{-e^{2x} - 2e^x}{(e^x-2)^3}$$

$$\frac{d^3y}{dy^3} = \frac{(-2e^{2x}-2)(e^x-2)^3 + (e^{2x}+2e^x)\cdot 3(e^x-2)^2 e^x}{(e^x-2)^6}$$

$$y'''(0) = -4\cdot(-1) + 3\cdot 3 = $$
$$= 13$$

$$y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + o(x^3) =$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + o(x^3)$$

3)
$$\underline{F} = \begin{pmatrix} 4x^3 + 3y^2 + 2yz \\ 6xy + 2k\,xz \\ 2xy - 1 \end{pmatrix}$$

$\mathrm{Dom}\, F = \mathbb{R}^3$ semplicemente connesso:

F conservativo sse $\underline{\nabla} \wedge \underline{F} = \underline{0}$

$$\underline{\nabla} \wedge \underline{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^3+3y^2+2yz & 6xy+2kxz & 2xy-1 \end{vmatrix} =$$

$$= \begin{pmatrix} 2x - 2kx \\ -2y + 2y \\ 6y + 2kz - (6y + 2z) \end{pmatrix} = \underline{0} \quad \text{sse } k = 1$$

$$\underline{F} = \begin{pmatrix} 4x^3 + 3y^2 + 2yz \\ 6xy + 2k\,xz \\ 2xy - 1 \end{pmatrix}$$

$$\frac{\partial U}{\partial x} = 4x^3 + 3y^2 + 2yz \Rightarrow U = x^4 + 3xy^2 + 2xyz + \psi(y, z)$$

$$\frac{\partial U}{\partial y} = \cancel{6xy} + \cancel{2xz} = \cancel{6}xy + \cancel{2}xz + \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \psi(z)$$

$$\frac{\partial U}{\partial z} = 2xy - 1 = 2xy + \frac{\partial \psi}{\partial z} \Rightarrow \frac{\partial \psi}{\partial z} = -1 \Rightarrow \psi = -z + C$$

$$U = x^4 + 3xy^2 + 2xyz - z + C$$

$$\gamma(t) = \begin{pmatrix} 1 + e^t(1-3t+2t^2) \\ 2 - e^t(1-t^2) \\ 1 + te^t(1-t) \end{pmatrix} \qquad \gamma : [0,1] \to \mathbb{R}$$

$$\mathcal{L} = \int_\gamma \underline{E} \cdot \dot{\gamma}\, dt = U(\gamma(1)) - U(\gamma(0))$$

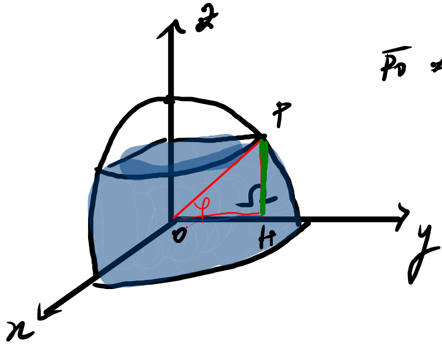
$$\gamma(0) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \gamma(1) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathcal{L} = U(1,2,1) - U(2,1,1) = \cancel{1} + \cancel{12} + \cancel{4} - \cancel{1} - (\cancel{16} + 6 + 4 - 1) = -9$$

$$U = x^4 + 3xy^2 + 2xyz - z + C$$

4) $\quad F = \begin{pmatrix} 2xyz + y^2 - x \\ x + 5y - 2y^2 z \\ xy^2 + yz^2 - z \end{pmatrix}$

$\Sigma = \partial\Omega \qquad \Omega = \{ \underline{x} \in \mathbb{R}^3 : |\underline{x}| \leq 1,\ x \geq 0,$
$y \geq 0,\ 0 \leq z \leq \frac{\sqrt{2}}{2} \}$



$\overline{PO}\ \sin\varphi = \overline{PH}$

$$\Phi_\Sigma(F) = \int_\Sigma \underline{F} \cdot d\underline{S} = \underset{\text{Gauss-Green}}{=} \int_\Omega \underline{\nabla} \cdot \underline{F}\, dv =$$

$$= \int_\Omega (\cancel{2yz} - 1 + 5 - \cancel{4yz} + \cancel{2yz} - 1)\, dv =$$

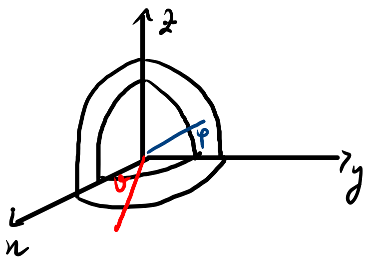
$$= 3\int_\Omega dv \underset{\text{coord sferiche}}{=} 3\int_\Omega \rho^2 \cos\varphi\, d\rho\, d\theta\, d\varphi =$$

(vedi soluzione)

29/08/19

4) $I = \int_{\Omega} (x^2+y^2+z^2)\sqrt{x^2+y^2}\, dx\,dy\,dz$

$\Omega = \{ \underline{x} \in \mathbb{R}^3 : \quad 1 \le |\underline{x}| \le 2 \,, \quad z \ge 0 \}$



$$\begin{cases} x = \rho\cos\varphi\cos\theta \\ y = \rho\cos\varphi\sin\theta \\ z = \rho\sin\varphi \end{cases}$$

$dx\,dy\,dz = \rho^2\cos\varphi\, d\rho\, d\theta\, d\varphi$

$I = \int_{\Omega} \rho^2\sqrt{\rho^2\cos^2\varphi}\;\rho^2\cos\varphi\, d\rho\, d\theta\, d\varphi =$

$= \int_{\Omega} \rho^5\cos^2\varphi\, d\rho\, d\theta\, d\varphi =$

$= \int_1^2 \rho^5\, d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2\varphi\, d\varphi =$

$= \left[\frac{\rho^6}{6}\right]_1^2 \, 2\pi \cdot \left[\frac{\cos\varphi\sin\varphi + \varphi}{2}\right]_0^{\pi/2} = \left(\frac{32}{3} - \frac{1}{6}\right)\cancel{2}\pi\left(\frac{\pi}{\cancel{4}2}\right) = \frac{63}{12}\pi^2 =$

$= \frac{21}{4}\pi^2$

$\int \cos^2\varphi\, d\varphi = \cos\varphi\sin\varphi + \int \sin^2\varphi\, d\varphi =$

$= \cos\varphi\sin\varphi + \varphi - \int\cos^2\varphi\, d\varphi$

$\int \cos^2\varphi\, d\varphi = \frac{\cos\varphi\sin\varphi + \varphi}{2} + k$

2) $y'' - 2(k+1)y' + 4ky = 2e^{-2x}$

$y = y_0 + \bar{y}$

y_0: $\lambda^2 - 2(k+1)\lambda + 4k = 0$

$$\lambda = k+1 \pm \sqrt{\underbrace{k^2+2k+1-4k}_{k^2-2k+1}} = k+1 \pm |k-1|$$

$\lambda_1 = 2k \vee \lambda_2 = 2$

$y_0 = A e^{2kx} + Be^{2x}$ se $k \neq 1$; se $k=1$: $y_0 = Ae^{2x} + Bxe^{2x}$

$\bar{y}$: i) se $k \neq -1$:

$\bar{y} = \alpha e^{-2x}$

$\bar{y}' = -2\alpha e^{-2x}$

$\bar{y}'' = +4\alpha e^{-2x}$

$$4\alpha \cancel{e^{-2x}} - 2(k+1)(-2\alpha \cancel{e^{-2x}}) + 4k\alpha \cancel{e^{-2x}} = 2\cancel{e^{-2x}}$$

$$4\alpha + 4\alpha(k+1) + 4\alpha k = 2$$

$$4\alpha(1+k+1+k) = 2$$

$$\alpha = \frac{1}{4(k+1)}$$

$$\bar{y} = \frac{1}{4(k+1)} e^{-2x}$$

ii) $k = -1$: $y'' - 4y = 2e^{-2x}$

$\bar{y} = \beta x e^{-2x}$:

$\bar{y}' = \beta(e^{-2x} - 2xe^{-2x})$

$\bar{y}'' = \beta(-2e^{-2x} - 2e^{-2x} + 4xe^{-2x}) =$

$= \beta(-4e^{-2x} + 4xe^{-2x})$

$$4\beta \cancel{e^{-2x}}(x-1) - 4\beta x \cancel{e^{-2x}} = 2\cancel{e^{-2x}}$$

$-4\beta = 2 \Rightarrow \beta = -1/2$

Pertanto

se $k \neq \pm 1$: $y = Ae^{2kx} + Be^{2x} + \frac{1}{4(k+1)} e^{-2x}$

se $k = -1$: $y = Ae^{-2x} + Be^{2x} - \frac{1}{2} e^{-2x}$

se $k = 1$: $y = Ae^{2x} + Bxe^{2x} + \frac{1}{8} e^{-2x}$

CORRETTO

1) $f(\underline{x}) = e^x + e^y - e^{x+y}$ $\qquad f(x,y) = f(y,x)$

a) $\begin{cases} \dfrac{\partial f}{\partial x} = e^x - e^{x+y} = 0 \\ \dfrac{\partial f}{\partial y} = e^y - e^{x+y} = 0 \end{cases}$ $\qquad \begin{cases} 1 - e^y = 0 \\ 1 - e^x = 0 \end{cases}$

$\begin{cases} y = 0 \\ x = 0 \end{cases}$ $\qquad O(0,0)$ p. stazionario

$\dfrac{\partial^2 f}{\partial x^2} = e^x - e^{x+y}$ $\qquad \dfrac{\partial^2 f}{\partial y^2} = e^y - e^{x+y}$

$\dfrac{\partial^2 f}{\partial x \partial y} = -e^{x+y}$

$H_f(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\det H_f(0,0) = -1 < 0$

$(0,0)$ sella

b) $\underline{x_0} = (\ln 2, \ln 3)$

$T_{\underline{x_0}} f: \; z - f(\underline{x_0}) = \underline{\nabla} f(\underline{x_0}) \cdot (\underline{x} - \underline{x_0})$

$z - 5 + e^{\ln 3 + \ln 2} = \underline{\nabla} f(\underline{x_0}) \begin{pmatrix} x - \ln 2 \\ y - \ln 3 \end{pmatrix}$

$\underline{\nabla} f(\underline{x_0}) = \begin{pmatrix} 2 - e^{\ln 3 + \ln 2} \\ 3 - e^{\ln 3 + \ln 2} \end{pmatrix} = \begin{pmatrix} 2 - 6 \\ 3 - 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

$z - 1 = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x - \ln 2 \\ y - \ln 3 \end{pmatrix}$

CORRETTO

$T_{\underline{x_0}} f \qquad z - 1 = -4x + 4\ln 2 + 3y + 3\ln 3$

3) $\underline{v} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ $\qquad f$ è diff. in $\underline{x_0}$

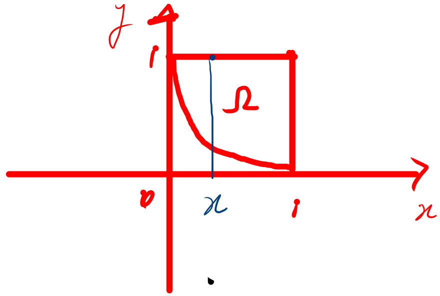
$\dfrac{\partial f}{\partial \underline{v}}(\underline{x_0}) = \dfrac{\sqrt{2}}{2} \dfrac{\partial f}{\partial x}(\underline{x_0}) + \dfrac{\sqrt{2}}{2} \dfrac{\partial f}{\partial y}(\underline{x_0}) =$

$= \dfrac{\sqrt{2}}{2} \left(-4 - 3 \right) = -\dfrac{7}{2} \sqrt{2}$

3) $\underline{F} = \begin{pmatrix} e^x - x^2 y \\ x y^2 + e^y \end{pmatrix}$ $\qquad \gamma = \partial\Omega$

$$\Omega = \left\{ \underline{x} \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \le 1, \quad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \right\}$$

CORRETTO



$$\oint_\gamma \underline{F} \cdot d\underline{l} \underset{\text{Stokes}}{=} \int_\Omega \underline{\nabla \wedge F} \cdot d\underline{s}$$

$$\oint_\gamma \underline{F} \cdot d\underline{l} = \int_\Omega \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) dx\,dy =$$

$$= \int_\Omega (x^2 + y^2)\, dx\,dy =$$

$$= \int_0^1 dx \int_{1-\sqrt{2x-x^2}}^{1} x^2 + y^2 \, dy = \dots$$

etc

$(y-1)^2 = 1 - (x-1)^2$

$y = 1 - \sqrt{2x - x^2}$

$0 \le y \le 1$