30/01/17

1) $\sum_{1}^{\infty} a_n \qquad a_n = \frac{n + \sqrt{n^4+n}}{(n^3+1)(\sqrt{n^2+n+1})}$

$$a_n \sim \frac{n^2 + o(n^2)}{n^3 \cdot n + o(n^4)} \sim \frac{1}{n^2}$$ termini di serie conv.

Richiamo: $\sum \frac{1}{n^\alpha \ln^\beta n}$ converge se $\begin{cases} \alpha > 1 & \forall \beta \\ \alpha = 1 & \beta > 1 \end{cases}$

altrimenti diverge

$\sum a_n$ conv. assolutamente e semplicemente

2)
$$\gamma(\vartheta) = \begin{pmatrix} 2\vartheta\cos\vartheta \\ 2\vartheta\sin\vartheta \\ \vartheta^3/3 \end{pmatrix} \qquad \vartheta \in [0, 2\pi]$$

$$\dot\gamma = \begin{pmatrix} 2\cos\vartheta - 2\vartheta\sin\vartheta \\ 2\sin\vartheta + 2\vartheta\cos\vartheta \\ \vartheta^2 \end{pmatrix}$$

$$l(\gamma) = \int_0^{2\pi} |\dot\gamma(\vartheta)|\, d\vartheta =$$

$$= \int_0^{2\pi} \Big(4\cos^2\vartheta + 4\vartheta^2\sin^2\vartheta \cancel{- 8\vartheta\cos\vartheta\sin\vartheta} + 4\sin^2\vartheta + 4\vartheta^2\cos^2\vartheta \cancel{+ 8\vartheta\sin\vartheta\cos\vartheta} + \vartheta^4 \Big)^{1/2} d\vartheta =$$

$$= \int_0^{2\pi} \left(4 + 4\vartheta^2 + \vartheta^4 \right)^{1/2} d\vartheta =$$

$$= \int_0^{2\pi} (\vartheta^2 + 2)\, d\vartheta = \left[\frac{1}{3}\vartheta^3 + 2\vartheta \right]_0^{2\pi} =$$

$$= \frac{8}{3}\pi^3 + 4\pi$$

3) $f(\underline{x}) = \begin{cases} \dfrac{x^4+y^6}{x^2+y^2} & \underline{x} \neq \underline{0} \\ 0 & \underline{x} = \underline{0} \end{cases}$

Ricordiamo:

f differenziabile in $\underline{x_0}$ $\Rightarrow$ f derivabile in $\underline{x_0}$ $\Rightarrow$

$\Rightarrow$ f continua in $\underline{x_0}$

a) $f(\underline{x}) \geq 0$ $\quad$ $(0,0)$ è min f assoluto

$\forall \underline{x} \in \mathbb{R}^2$

b) $\lim_{\underline{x} \to \underline{0}} f(\underline{x}) \underset{\substack{| \\ \text{coord. polari}}}{\sim} \dfrac{\rho^{\cancel{4}\,2}(\sin^4\vartheta + \cos^4\vartheta)}{\cancel{\rho^2}} \xrightarrow[\rho \to 0]{} 0$

f continua in $(0,0)$

c) $\dfrac{\partial f}{\partial x}(\underline{0}) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} \sim \dfrac{h^4}{h^2} \cdot \dfrac{1}{h} \sim h \to 0$

$\dfrac{\partial f}{\partial x}(\underline{0}) = 0$

Or $f(x,y) = f(y,x)$ $\Rightarrow$ $\dfrac{\partial f}{\partial y}(\underline{0}) = 0$

$\nabla f(\underline{0}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

d) Ricordiamo

f differenziabile in $\underline{x_0}$: $\quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\frac{\partial f}{\partial \underline{v}}(\underline{x_0}) = v_1 \frac{\partial f}{\partial x}(\underline{x_0}) + v_2 \frac{\partial f}{\partial y}(\underline{x_0})$$

(ma non vale il viceversa)

$$\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad a^2+b^2=1$$

$$\frac{\partial f}{\partial \underline{v}}(0) = \lim_{h\to 0} \frac{f(ha, hb) - f(0,0)}{h} \sim$$

$$\sim \frac{h^4(a^4+b^4)}{h^2\underbrace{(a^2+b^2)}_{=1}} \cdot \frac{1}{h} \sim h(a^4+b^4) \to 0$$

$\forall a, b$

e) $$\lim_{\underline{x}\to\underline{0}} \frac{\overbrace{f(x,y) - f(0,0) - \underline{\nabla} f(0) \cdot \begin{pmatrix} x \\ y \end{pmatrix}}^{T_0 f}}{\sqrt{x^2+y^2}} \sim$$

$\sim$ coord. polari $\quad \dfrac{\rho^4(\sin^4\vartheta + \cos^4\vartheta)}{\rho^2 \underbrace{(\sin^2\vartheta+\cos^2\vartheta)}_{=1} \rho} \sim \rho\,(\sin^4\vartheta + \cos^4\vartheta) \xrightarrow{\rho\to 0} 0$

f è differenziabile in $(0,0)$

4) a) $\underline{F} = \begin{pmatrix} y^2 + x^2 y \\ x^2 - xy^2 \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$

Ricordiamo:

Se Dom $\underline{F}$ è semplicemente connesso

$\underline{F}$ conservativo sse $\underline{F}$ irrotazionale

F irrotazionale $\quad \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$

$\frac{\partial F_x}{\partial y} = 2y + x^2$

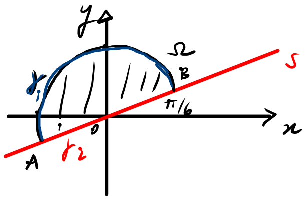
$\frac{\partial F_y}{\partial x} = 2x - y^2$

F non è irrotazionale $\Rightarrow$

$\Rightarrow F$ non conservativo

$\underline{\nabla} \wedge \underline{F} = 2(y-x) + x^2 + y^2$

b) $\Omega = \{ \underline{x} \in \mathbb{R}^2 : |x| \le r, \quad x \le \sqrt{3} y \}$



$s: \ y \ge \frac{x}{\sqrt{3}}$

$A\left(-\frac{r\sqrt{3}}{2}, -\frac{r}{2} \right)$

$B\left(\frac{r\sqrt{3}}{2}, \frac{r}{2} \right)$

c) $\gamma = \partial\Omega$ $\qquad L = \int_\gamma \underline{F} \cdot d\underline{l}$ $\qquad \gamma_1(\vartheta) = \begin{pmatrix} r\cos\vartheta \\ r\sin\vartheta \end{pmatrix}$

$\vartheta \in \left[\frac{\pi}{6}, \frac{7}{6}\pi \right]$

$\gamma = \gamma_1 \cup \gamma_2$

$\gamma_2(t) = \begin{pmatrix} t \\ t/\sqrt{3} \end{pmatrix}$

$t \in \left[-\frac{\sqrt{3}}{2} r, \frac{\sqrt{3}}{2} r \right]$

$L = \int_{\gamma_1} \underline{F} \cdot d\underline{l} + \int_{\gamma_2} \underline{F} \cdot d\underline{l}$

$\dot{\gamma}_1(\vartheta) = \begin{pmatrix} -r\sin\vartheta \\ r\cos\vartheta \end{pmatrix}$ $\qquad \dot{\gamma}_2(t) = \begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix}$

$L = \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \underline{F}(r\cos\vartheta, r\sin\vartheta) \cdot \dot{\gamma}_1(\vartheta) \, d\vartheta + \int_{-\frac{\sqrt{3}}{2} r}^{\sqrt{3}/2 \, r} \underline{F}(t, t/\sqrt{3}) \cdot \dot{\gamma}_2(t) \, dt \Rightarrow$

$\underline{F} = \begin{pmatrix} y^2 + x^2 y \\ x^2 - xy^2 \end{pmatrix}$

$\Rightarrow L = \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \left[(r^2 \sin^2\vartheta + r^3 \cos^2\vartheta \sin\vartheta)(-r\sin\vartheta) + \right.$
$\left. + (r^2\cos^2\vartheta - r^3 \cos\vartheta \sin^2\vartheta)(r\cos\vartheta) \right] d\vartheta +$

$+ \int_{-\frac{\sqrt{3}}{2} r}^{\sqrt{3}/2 \, r} \left[\left(\frac{t^2}{3} + \frac{t^3}{\sqrt{3}} \right) + \left(t^2 - \frac{t^3}{3} \right) \frac{1}{\sqrt{3}} \right] dt =$

$= r^3 \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \left(-\sin^3\vartheta - \cos^2\vartheta \sin^2\vartheta + \cos^3\vartheta - \cos^2 \sin^2\vartheta \right) d\vartheta +$

$+ \int_{-\frac{\sqrt{3}}{2} r}^{\frac{\sqrt{3}}{2} r} \left(\frac{t^2}{3} + \frac{t^3}{\sqrt{3}} + \frac{t^2}{\sqrt{3}} - \frac{t^3}{3\sqrt{3}} \right) dt =$

Calcolo complesso

Teor. di Stokes

$$L = \oint_\gamma \underline{F} \cdot d\underline{l} = \int_\Omega \underline{\nabla} \wedge \underline{F} \cdot d\underline{s}$$

$$L = \int_\Omega \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) \cdot d\underline{s} =$$

$$= \int_\Omega 2(y-x) + (x^2+y^2) \, dx\,dy =$$

$$= \int_\Omega (2\rho(\sin\vartheta - \cos\vartheta) + \rho^2)\, \rho \, d\rho \, d\vartheta =$$

$$dS = \rho \, d\rho \, d\vartheta$$

$$= \int_{\frac{\pi}{6}}^{7/6\pi} d\vartheta \int_0^r 2\rho^2 (\sin\vartheta - \cos\vartheta) + \rho^3 \, d\rho =$$

$$= \int_{\frac{\pi}{6}}^{7/6\pi} (\sin\vartheta - \cos\vartheta) \left. \frac{2}{3}\rho^3 \right|_0^r d\vartheta + \int_{\pi/6}^{7/6\pi} \left. \frac{1}{4}\rho^4 \right|_0^r d\vartheta =$$

$$= \frac{2}{3} r^3 \left[-\cos\vartheta - \sin\vartheta \right]_{\pi/6}^{7/6\pi} + \frac{\pi}{4} r^4 =$$

$$= \frac{2}{3} r^3 \left[\frac{\sqrt{3}}{2} + \frac{1}{2} - \left(-\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] + \frac{\pi}{4} r^4 =$$

$$= \frac{2}{3} r^3 (\sqrt{3} + 1) + \frac{\pi}{4} r^4$$

Richiamo

$$\int_S \underline{F} \cdot d\underline{S} = \int_V \underline{\nabla} \cdot \underline{F} \, dv \qquad \partial V = S$$

Gauss-Green

$$\oint_\gamma \underline{F} \cdot d\underline{\ell} = \int_S \underline{\nabla} \wedge \underline{F} \cdot d\underline{S} \qquad \partial S = \gamma$$

Stokes

16/01/19

1) $y'' - 4y' + 3y = 18t - 2e^t$ ← errore nel testo

$$y = y_0 + \bar{y}$$

y_0: $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 3 \vee \lambda = 1$

$$y_0 = Ae^t + Be^{3t}$$

$\bar{y}$:

$$\bar{y} = ate^t + bt + c$$

$$\bar{y}' = ae^t + ate^t + b$$

$$\bar{y}'' = ae^t + ae^t + ate^t$$

$$2ae^t + \cancel{ate^t} - 4ae^t - \cancel{4ate^t} - 4b + \cancel{3ate^t} + 3bt + 3c = 18t - 2e^t$$

$$\begin{cases} -2a = -2 \\ 3b = 18 \\ -4b + 3c = 0 \end{cases} \qquad \begin{cases} a = 1 \\ b = 6 \\ c = \frac{4}{3}b = 8 \end{cases}$$

$$y = Ae^t + Be^{3t} + te^t + 6t + 8$$

2) $f(x,y) = xy\, e^{x-y}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y e^{x-y} + xy\, e^{x-y} = 0 \\[2mm] \dfrac{\partial f}{\partial y} = x e^{x-y} - xy\, e^{x-y} = 0 \end{cases}$$

$y(1+x) = 0$

$$\begin{cases} y + xy = 0 \\ x - xy = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} x = -1 \\ y = 1 \end{cases}$$

$O(0,0)$

$A(-1,1)$

$$\frac{\partial^2 f}{\partial x^2} = y e^{x-y} + y e^{x-y} + xy \cdot e^{x-y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = (e^{x-y} - y e^{x-y})(x+1)$$

$$\frac{\partial^2 f}{\partial y^2} = -x e^{x-y} - x e^{x-y} + xy\, e^{x-y}$$

$$H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det H_f(0,0) = -1 < 0$

$(0,0)$ sella di f

$$H_f(-1,1) = \begin{bmatrix} e^{-2} & 0 \\ 0 & +e^{-2} \end{bmatrix}$$

$\det H_f(-1,1) = e^{-4} > 0$

$\operatorname{tr} H_f(-1,1) = 2e^{-2} > 0$

$A(-1,1)$ min f

Ricordiamo:

$\det H_f(\underline{x_0}) < 0$ sella

$\det H_f(\underline{x_0}) > 0$

- $\operatorname{tr}(H_f(\underline{x_0})) > 0$ min f
- $\operatorname{tr}(H_f(\underline{x_0})) < 0$ max f