MASSIMI E MINIMI

Massimi e minimi liberi — punti di non differenziabilità / punti stazionari

$f : \mathrm{Dom} f \subset \mathbb{R}^2 \to \mathbb{R}$ $\qquad$ $f \in C^2(U(\underline{x_0}))$

<u>CN</u> $\quad \nabla f(\underline{x_0}) = \underline{0}$

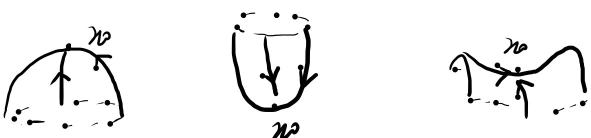
$$Hf(\underline{x_0}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(\underline{x_0}) & \frac{\partial^2 f}{\partial x \partial y}(\underline{x_0}) \\ \frac{\partial^2 f}{\partial x \partial y}(\underline{x_0}) & \frac{\partial^2 f}{\partial y^2}(\underline{x_0}) \end{bmatrix}.$$

<u>Lemma</u> (di Schwarz) $\quad f \in C^2(U(\underline{x_0})) \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(\underline{x_0}) = \frac{\partial^2 f}{\partial y \partial x}(\underline{x_0})$

$\underline{x_0}$ p. di max f $\qquad Hf(\underline{x_0}) < 0 \qquad$ $\det Hf(\underline{x_0}) > 0$, $\mathrm{tr}\, Hf(\underline{x_0}) < 0$

$\underline{x_0}$ p. di min f $\qquad Hf(\underline{x_0}) > 0 \qquad$ $\det Hf(\underline{x_0}) > 0$, $\mathrm{tr}(Hf(\underline{x_0})) > 0$

$\underline{x_0}$ p. di sella $\qquad Hf(\underline{x_0})$ <u>non</u> def $\qquad$ $\det Hf(\underline{x_0}) < 0$



02/02/22

ασ.2.

$f(x,y) = f(\underline{x}) = x^2y + xy^2 + xy =$
$= xy(x+y+1)$

$f(x,y) = f(y,x)$

$\underline{\nabla f} = \underline{0}$: $\begin{cases} \frac{\partial f}{\partial x} = 2xy + y^2 + y = 0 \\ \frac{\partial f}{\partial y} = x^2 + 2xy + x = 0 \end{cases}$

$\begin{cases} y(2x+y+1) = 0 \\ x(x+2y+1) = 0 \end{cases} \Rightarrow y = 0 \lor y = -2x-1$

$\begin{cases} y = 0 \\ x(x+1) = 0 \end{cases}$ $\lor$ $\begin{cases} y = -2x-1 \\ x(x-4x-2+1) = 0 \end{cases}$

$x = 0 \lor x = -1$

$x = 0 \lor x = -1/3$

$O(0,0)$

$A(-1,0)$

$\begin{cases} y = -1 \\ x = 0 \end{cases} \lor \begin{cases} y = -1/3 \\ x = -1/3 \end{cases}$

$B(0,-1)$ $C(-1/3, -1/3)$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy + y^2 + y = 0 \\[2ex] \dfrac{\partial f}{\partial y} = x^2 + 2xy + x = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x + 2y + 1 \qquad \frac{\partial^2 f}{\partial y^2} = 2x$$

$O(0,0)$: $H_f(O) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \det H_f(O) = -1 < 0$

$O(0,0)$ sellon

$C\left(-\frac{1}{3}, -\frac{1}{3}\right)$: $H_f(C) = \begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -\frac{2}{3} \end{bmatrix}$ $\quad \det H_f(C) = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0$

$\operatorname{tr} H_f(C) = -\frac{4}{3} < 0$

$C(-1/3, -1/3)$ max f

$A(-1, 0)$: $H_f(A) = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$ $\quad \det H_f(A) = -1 < 0$

$A(-1, 0)$ sellon

$B(0, -1)$: $H_f(B) = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$ $\quad \det H_f(B) = -1 < 0$

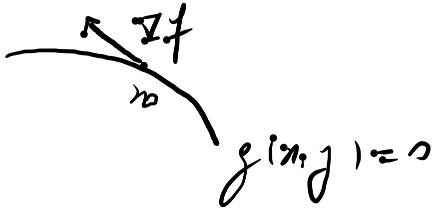
$B(0, -1)$ sellon

MAX E MIN VINCOLATI

$f : \text{Dom} f \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$g(x,y) = 0$ vincolo

$g \in C^1$



$\mathcal{L}(x,y) = f(x,y) + \lambda g(x,y)$

lagrangiana

CN $\quad x_0$ è max f o min f se

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x}(\underline{x_0}) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y}(\underline{x_0}) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda}(\underline{x_0}) = 0 \end{cases}$$

$\Rightarrow \quad \underline{\nabla} f(\underline{x_0}) = -\lambda \nabla g(\underline{x_0})$ (dalle prime due equazioni)

$\Rightarrow \quad g(\underline{x_0}) = 0$

La tipologia di $\underline{x_0}$ si definisce con altre considerazioni

16/02/18 itinere es. 6

$f(x,y) = f(\underline{x}) = (3x+2y)^2$

$g(x,y) = 4x^2+y^2-4=0$ vincolo

$$\mathcal{L} = (3x+2y)^2 + \lambda(4x^2+y^2-4)=0$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 3(3x+2y) + 8\lambda x = 0 \Rightarrow (3x+2y) = -\frac{8\lambda x}{3} \\ \frac{\partial \mathcal{L}}{\partial y} = \cancel{2}(3x+2y) + \cancel{2}\lambda y = 0 \Rightarrow (3x+2y) = -\lambda y \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 4x^2+y^2-4=0 \end{cases}$$

$$\begin{cases} 8\lambda x + 3\lambda y = 0 \Rightarrow \lambda(8x+3y)=0 \\ 3x+2y+\lambda y = 0 \\ 4x^2+y^2-4=0 \end{cases}$$

$$\begin{cases} \lambda=0 \\ 3x+2y=0 \Rightarrow y=-\frac{3}{2}x \\ 4x^2+\frac{9}{4}x^2-4=0 \ (*) \end{cases} \quad \vee \quad \begin{cases} x=-3y/8 \\ -9y/8+2y+\lambda y=0 \ (**) \\ 4x^2+y^2-4=0 \end{cases}$$

$(*)$ $25x^2-4=0$

$x=\pm\frac{2}{5}$

$$\begin{cases} \lambda=0 \\ y=\mp\frac{3}{5} \\ x=\pm\frac{2}{5} \end{cases}$$

$A\left(\frac{2}{5}, -\frac{3}{5}\right)$

$B\left(-\frac{2}{5}, \frac{3}{5}\right)$

$(**)$ $\left(\frac{7}{8}+\lambda\right)y=0$

$$\begin{cases} x=0 \\ y=0 \\ -4=0 \end{cases} \text{ imposs.} \quad \vee \quad \begin{cases} y=-8/3\,x \\ \lambda=-7/8 \\ 4x^2+\frac{64}{3}x^2-4=0 \end{cases}$$

$\Downarrow$

$\overset{25}{\cancel{100}}x^2-\overset{9}{\cancel{36}}=0$

$x=\pm\frac{3}{5}$

$$\begin{cases} \lambda=-7/8 \\ x=\pm 3/5 \\ y=\mp 8/5 \end{cases}$$

$C\left(\frac{3}{5}, -8/5\right)$

$D\left(-\frac{3}{5}, 8/5\right)$

$f(x,y) = (3x+2y)^2 \qquad \forall (x,y) \in \mathbb{R}^2 \quad f(x,y) \geq 0$

$f(A) = \left(3 \cdot \frac{2}{5} - 2 \cdot \frac{3}{5}\right) = 0$ min f

$f(B) = \left(-3 \cdot \frac{2}{5} + 2 \cdot \frac{3}{5}\right) = 0$ min f

$f(C) = \left(\frac{9}{5} - \frac{16}{5}\right)^2 = \left(-\frac{7}{5}\right)^2 = \frac{49}{25}$ max f

$f(D) = \left(-\frac{9}{5} + \frac{16}{5}\right)^2 = \left(\frac{7}{5}\right)^2 = \frac{49}{25}$ max f per teor. di Weierstrass

$\Omega = \{ (x,y) \in \mathbb{R}^2 : 4x^2 + y^2 - 4 = 0 \}$ compatto

Teor (di Weierstrass):

f cont. su Ω compatto $\Rightarrow \exists\ \underline{x_0}, \underline{x_1} \in \Omega$:

$\underline{x_0}$ di max f

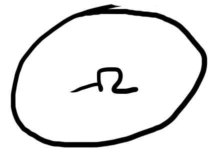
$\underline{x_1}$ di min f

Oss (Vincoli con disequazione)

$f(x,y)$ di cui cerchiamo max e min

$g(x,y) \leq 0$ vincolo

$$\Omega = \{ (x,y) \in \mathbb{R}^2 : g(x,y) \leq 0 \}$$



nell'interno di Ω : $\underline{x} \in \Omega^\circ$

$\downarrow$

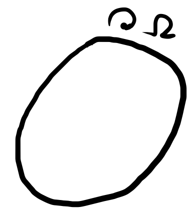
cerco i max e min liberi

$(\underline{\nabla} f = \underline{0} \ + \ \mathrm{sgn}\, H_f)$

sul bordo di Ω : $\underline{x} \in \partial\Omega$

$\downarrow$

cerco i max e min vincolati



$$\left(\mathcal{L} = f + \lambda g \quad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial y} = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \right)$$

9/11/21

es. 1

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(\underline{x}) = \begin{cases} \dfrac{\sqrt{x^6+y^6}}{x^2+y^2} & \underline{x} \neq \underline{0} \\ 0 & \underline{x} = \underline{0} \end{cases}$$

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) \underset{\text{coord. polari}}{\sim} \frac{\rho^{\cancel{3}}\sqrt{\cos^6 x + \sin^6 x}}{\cancel{\rho^2}} \xrightarrow[\rho\to 0]{} 0 \quad f \text{ cont in } \underline{x} = \underline{0}$$

coord. polari

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

$$\frac{\partial f}{\partial x}(\underline{0}) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \sim \frac{|h|^3}{h^2} \to 0$$

$\frac{\partial f}{\partial y}(\underline{0}) = 0$ per simmetria $\quad f$ derivabile in $\underline{x} = \underline{0}$

piano tangente in $\underline{x}=\underline{0}$

$$\frac{f(x,y) - f(0,0) - \underline{\nabla} f(\underline{0}) \cdot \binom{x}{y}}{\left\| \binom{x}{y} \right\|} \xrightarrow[\underline{x}\to\underline{0}]{} 0$$

$$\lim_{\underline{x}\to\underline{0}} \underbrace{\frac{\sqrt{x^6+y^6}}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}}}_{g(x,y)} \sim \frac{\cancel{\rho^3}\sqrt{\cos^6\vartheta + \sin^6\vartheta}}{\cancel{\rho^2}\cdot\cancel{\rho}} \quad \text{non esiste}$$

$$\lim_{x\to 0} g(x,x) = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^6 + \left(\frac{\sqrt{2}}{2}\right)^6} \qquad \vartheta = \pi/4$$

$$\lim_{x\to 0} g(x,0) = 1 \qquad \vartheta = 0$$

f non differenziabile in $\underline{x} = \underline{0}$ $\Rightarrow$ f non è $C^1(U(\underline{0}))$

$$f(x,y) = \frac{\sqrt{x^6+y^6}}{x^2+y^2} \geq 0 \qquad \forall (x,y) \in \mathbb{R}^2$$

$\Rightarrow$ $(0,0)$ __non__ è un max f

__non__ è una sella

$\Rightarrow$ $(0,0)$ è un punto di min f

22/07/09
es. 2

$f(\underline{x}) = x^3 + y^2 - x$

$D = \{ \underline{x} \in \mathbb{R}^2 : |\underline{x}| \leq 1 \}$

f dispari risp. x

f pari risp. y

$\underline{\nabla f} = \underline{0}$: $\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 - 1 = 0 \\ \dfrac{\partial f}{\partial y} = 2y = 0 \end{cases}$ $A \begin{cases} x = \sqrt{3}/3 \\ y = 0 \end{cases} \vee B \begin{cases} x = -\sqrt{3}/3 \\ y = 0 \end{cases}$

$\dfrac{\partial^2 f}{\partial x^2} = 6x \qquad \dfrac{\partial^2 f}{\partial x \partial y} = 0 \qquad \dfrac{\partial^2 f}{\partial y^2} = 2$

$H_f(A) = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2 \end{bmatrix}$. $\det H_f(A) = 4\sqrt{3} > 0$

$\operatorname{tr} H_f(A) = +2+2\sqrt{3} > 0$

$(\sqrt{3}/3, 0)$ è p. di min$f$

$H_f(B) = \begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & 2 \end{bmatrix}$ $\det H_f(B) = -4\sqrt{3} < 0$

$(-\frac{\sqrt{3}}{3}, 0)$ è sella

$A, B \in D$ $\quad f(\frac{\sqrt{3}}{3}, 0) = \dfrac{\sqrt{3}}{9} - \dfrac{\sqrt{3}}{3} = -\dfrac{2}{3}\sqrt{3}$

∂D : $g(x,y) = x^2 - y^2 - 1 = 0$

$\mathcal{L} = f(x,y) - \lambda g(x,y) = x^3 + y^2 - x + \lambda(x^2 - y^2 - 1)$

$$\mathcal{L} = f(x,y) - \lambda g(x,y) = x^3 + y^2 - x + \lambda(x^2 - y^2 - 1)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 3x^2 - 1 + 2\lambda x = 0 \\ \dfrac{\partial f}{\partial y} = 2y - 2\lambda y = 0 \quad \Rightarrow \quad y = 0 \ \vee \ \lambda = 1 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$\begin{cases} 3 - 1 + 2\lambda = 0 \\ y = 0 \\ x = 1 \end{cases} \vee \begin{cases} 3 - 1 - 2\lambda = 0 \\ y = 0 \\ x = -1 \end{cases} \vee \begin{cases} 3x^2 + 2x - 1 = 0 \quad (*) \\ \lambda = 1 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$\begin{cases} \lambda = -1 \\ y = 0 \\ x = 1 \end{cases} \vee \begin{cases} \lambda = 1 \\ y = 0 \\ x = -1 \end{cases}$$

$$(*) \quad x = \frac{-1 \pm \sqrt{1+3}}{3} \quad \begin{matrix} x = -1 \\ x = 1/3 \end{matrix}$$

stessa

$$\begin{cases} x = -1 \\ \lambda = 1 \\ y = 0 \end{cases} \vee \begin{cases} x = 1/3 \\ \lambda = 1 \\ y = \pm\sqrt{\frac{2}{3}} \end{cases}$$

$A'(1,0)$ $\qquad B'(-1,0)$ $\qquad C'\left(\frac{1}{3}, \sqrt{\frac{2}{3}}\right)$ $\qquad D'\left(\frac{1}{3}, -\sqrt{\frac{2}{3}}\right)$

$$f(1,0) = 0 = f(-1,0)$$

$$f\left(\frac{1}{3}, \pm\sqrt{\frac{2}{3}}\right) = \frac{1}{27} + \frac{2}{3} - \frac{1}{3} = \frac{10}{27}$$

Per teor di Weierstrass $\exists$ punti di max e min in D

Punto di minimo assoluto : $A\left(\frac{\sqrt{3}}{3}, 0\right)$ $\qquad \min f$

$f(A) < 0$

Punti di massimo assoluto : $C, D \left(\frac{1}{3}, \pm\sqrt{\frac{2}{3}}\right)$ $\qquad \max f$

$$f(C) = f(D) = \frac{10}{27}$$