CURVE E SUPERFICI DIFFERENZIABILI (PARAMETRICHE)

Def $\gamma : [0,1] \longrightarrow \mathbb{R}^3 \qquad \gamma \in C^1([0,1])$

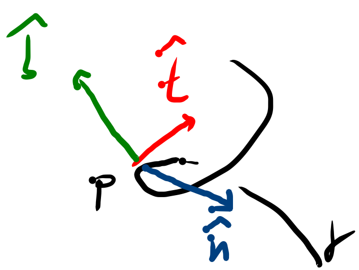
γ chiusa : $\gamma(0) = \gamma(1)$

γ semplice : γ iniettiva

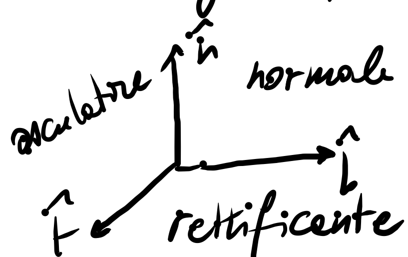
γ regolare : $\dot{\gamma}(t) = \frac{d}{dt}\gamma \neq \underline{0} \quad \forall t \in [0,1]$

γ biregolare : $\forall t \in [0,1] \quad \dot{\gamma}(t) \wedge \ddot{\gamma}(t) \neq \underline{0}$

Def (Terna di Frenet)



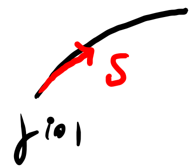
$\hat{t}$ tangente, $\hat{n}$ normale, $\hat{b}$ binormale



Def $\quad l(\gamma) = \int_0^1 |\dot\gamma(t)|\,dt$ $\quad$ lunghezza di γ

Parametro arco

$$s(t) = \int_0^t |\dot\gamma(u)|\,du$$



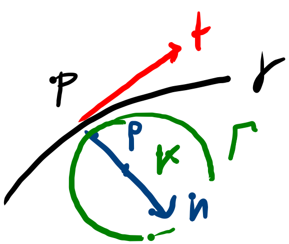
Considero $\gamma = \gamma(s)$ $\qquad \frac{d}{ds}\gamma = \gamma'(s)$

$\hat t = \gamma'(s)$ $\quad$ vettore tangente

$\hat n = \dfrac{\gamma''(s)}{|\gamma''(s)|}$ $\quad$ vettore normale

$|\gamma''(s)| = k$ $\quad$ curvatura

$\hat b = \hat t \wedge \hat n$ $\quad$ vettore binormale



$\rho = 1/k$ $\quad$ raggio di curvatura

Γ circonf. osculatrice : centro su $\hat n$

raggio : $\rho = 1/k$

centro $K = P + \rho\,\hat n$

Matrice di Frenet

$$\begin{pmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{pmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix}$$

k curvatura

τ torsione

$k=0 \Rightarrow \gamma$ retta

$\tau=0 \Rightarrow \gamma$ piana

τ definita $\quad \hat{b}' = -\tau \hat{n}$

Oss In parametrizzazione generica

$$\hat{t}(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} \qquad \hat{b}(t) = \frac{\dot{\gamma}(t) \wedge \ddot{\gamma}(t)}{|\dot{\gamma}(t) \wedge \ddot{\gamma}(t)|}$$

$$\hat{n}(t) = \hat{b}(t) \wedge \hat{t}(t)$$

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$$I = \int_\gamma \frac{1}{(1+3x+y+z)^{3/2}}\, ds$$

$$\gamma(t) = \begin{pmatrix} t+t^2/2 \\ 1-t \\ t^2/2 \end{pmatrix} \qquad t \in [0,1]$$

<u>Def</u> $\int_\gamma f(x,y,z)\, ds = \int_0^1 f(x(t), y(t), z(t))\, |\dot\gamma(t)|\, dt$

a) $\pi: x+y+z=1$

$\gamma \subset \pi$? $\quad \cancel{t}+t^2/2 + \cancel{1}-\cancel{t} + \frac{t^2}{2} \equiv \cancel{1}$ $\quad$ <u>no</u>

$t^2=0$

b) $\pi: x-y+z=1$

$\gamma \subset \pi$? $\quad t+t^2/2 - 1+t + \frac{t^2}{2} \equiv 1$

$2t + t^2 - 2 \equiv 0$ $\quad$ <u>no</u>

c) $\pi: x+y-z=1$

$\gamma \subset \pi$? $\quad t+\frac{t^2}{2} + 1-t - \frac{t^2}{2} \equiv 1$ $\quad \gamma \subset \pi$

$0 \equiv 0$ $\quad$ sì

d) γ è piana

e) γ è regolare?

$$\gamma(t) = \begin{pmatrix} t + t^2/2 \\ 1 - t \\ t^2/2 \end{pmatrix}$$

$$\dot{\gamma}(t) = \begin{pmatrix} 1+t \\ -1 \\ t \end{pmatrix} \neq \underline{0} \qquad \forall t \in [0,1]$$

γ regolare

esiste $\hat{t}$

f) γ è biregolare?

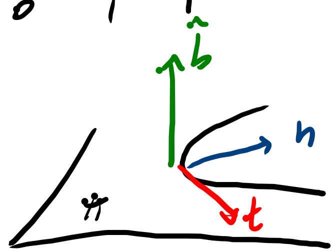
$$\ddot{\gamma}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

esiste $\hat{n}$

$$\dot{\gamma} \wedge \ddot{\gamma} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1+t & -1 & t \\ 1 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -t - 1 + t \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

esiste $\hat{b}$

γ biregolare



Osserviamo:

$$|\hat{b}|\,\hat{b} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Vettore normale di π: $x + y - z = 1$

$$I = \int_\gamma \frac{1}{(1+3x+y+z)^{3/2}}\, ds \qquad \gamma(t) = \begin{pmatrix} t+t^2/2 \\ 1-t \\ t^2/2 \end{pmatrix} \quad t\in[0,1]$$

$$I = \int_0^1 \frac{\sqrt{t^2+2t+1+1+t^2}}{\left(1+3t+\frac{3}{2}t^2+1-t+\frac{t^2}{2}\right)^{3/2}}\,|\dot\gamma(t)|\,dt \qquad \dot\gamma = \begin{pmatrix} 1+t \\ -1 \\ t \end{pmatrix}$$

$$I = \int_0^1 \frac{\sqrt{2t^2+2t+2}}{(2t^2+2t+2)^{3/2}}\,dt = \frac{1}{2}\int_0^1 \frac{dt}{t^2+t+1} =$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}\cdot\int_0^1 \frac{dt}{\frac{3}{4}\left[\left(\frac{2}{\sqrt3}t+\frac{1}{\sqrt3}\right)^2+1\right]} =$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{4}}{3}\cdot\frac{\sqrt3}{\cancel{2}}\int_0^1 \frac{\frac{2}{\sqrt3}\,dt}{1+\left(\frac{2}{\sqrt3}t+\frac{1}{\sqrt3}\right)^2} =$$

$$= \frac{1}{\sqrt3}\left[\arctan\left(\frac{2}{\sqrt3}t+\frac{1}{\sqrt3}\right)\right]_0^1 = \frac{1}{\sqrt3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt3}$$

CORRETTO

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$$\gamma(\theta) = \begin{pmatrix} \theta + \cos\theta \\ \theta + \sin\theta \\ \theta \end{pmatrix} \qquad \theta \in \mathbb{R}$$

a) $\gamma(\theta_1) = \gamma(\theta_2) \underset{\alpha = \theta}{\Rightarrow} \theta_1 = \theta_2$ $\quad\gamma$ semplice

$$\dot\gamma(\theta) = \begin{pmatrix} 1 - \sin\theta \\ 1 + \cos\theta \\ 1 \end{pmatrix} \neq \underline{0} \qquad \gamma \text{ regolare}$$

b) $P \in \gamma$, $P(1,0,0) \Rightarrow \theta = 0$

$$\hat{t} = \frac{\dot\gamma(0)}{|\dot\gamma(0)|} = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{6}} = \begin{pmatrix} 1/\sqrt6 \\ 2/\sqrt6 \\ 1/\sqrt6 \end{pmatrix}$$

$$\ddot\gamma(\theta) = \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \qquad \ddot\gamma(0) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{b} = \frac{\dot\gamma(0) \wedge \ddot\gamma(0)}{|\dot\gamma(0) \wedge \ddot\gamma(0)|} = \frac{1}{\sqrt5} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{vmatrix} = -(\hat{j} - 2\hat{k}) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\hat{b} \wedge \hat{t} = \frac{1}{\sqrt{30}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \frac{1}{\sqrt{30}} (-5\hat{i} + 2\hat{j} + \hat{k}) =$$

$$= \frac{1}{\sqrt{30}} \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \hat{n}$$

c) $k = \dfrac{|\dot{\gamma}(0) \wedge \ddot{\gamma}(0)|}{|\dot{\gamma}(0)|^3} = \dfrac{\left|\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right|}{(\sqrt{6})^3} = \dfrac{\sqrt{5}}{(\sqrt{6})^3}$

$\rho = \dfrac{6\sqrt{6}}{5}$

CORRETTO

$K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \dfrac{\rho^{\cancel{2}}}{\sqrt{30}} \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ $\qquad K = P + \rho^2 \hat{n}$

d) $I = \int_{\gamma'} \sqrt{2 + x - y}\, ds$ $\qquad \gamma' = \gamma|_{[0, 2\pi]}$

$I = \int_0^{2\pi} \sqrt{2 + \cos\theta - \sin\theta}\; |\dot{\gamma}(\theta)|\, d\theta =$

$\dot{\gamma}(\theta) = \begin{pmatrix} 1 - \sin\theta \\ 1 + \cos\theta \\ 1 \end{pmatrix}$

$= \int_0^{2\pi} \sqrt{2 + \cos\theta - \sin\theta}\; \sqrt{1 - 2\sin\theta + 1 + 2\cos\theta + 1 + 1}\; d\theta =$

$= \int_0^{2\pi} \sqrt{2 + \cos\theta - \sin\theta}\; \sqrt{4 + 2\cos\theta - 2\sin\theta}\; d\theta =$

$= \sqrt{2} \int_0^{2\pi} (1 + \cos\theta - \sin\theta)\, d\theta = \sqrt{2}\left(2\pi + \left[\sin\theta + \cos\theta\right]_0^{2\pi}\right) =$

$= 4\sqrt{2}\,\pi$

e) $\underline{F} = \begin{pmatrix} 2y \\ -2x \\ x^2+y^2+z^2 \end{pmatrix}$ $\qquad \mathrm{Dom}\,F = \mathbb{R}^3$

$\underline{F}$ conservativo se $\underline{\nabla} \wedge \underline{F} = 0$ $\qquad$ (F irrotazionale)

$$\underline{\nabla} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -2x & x^2+y^2+z^2 \end{vmatrix} =$$

$$= \hat{i}\,(2y) - \hat{j}\,(2x) + \hat{k}\,(-2-2) \neq \underline{0}$$

F non conservativo

f) $\qquad \gamma = \begin{pmatrix} \theta + \cos\theta \\ \theta + \sin\theta \\ \theta \end{pmatrix}$

$$L = \int_{\gamma'} \underline{F} \cdot d\underline{s} =$$

$\qquad \dot{\gamma} = \begin{pmatrix} 1 - \sin\theta \\ 1 + \cos\theta \\ 0 \end{pmatrix}$

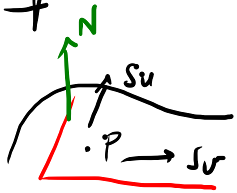
$$= \int_0^{2\pi} \underline{F}(\theta) \cdot \dot{\gamma}(\theta)\, d\theta =$$

$$= \int_0^{2\pi} 2(\theta + \sin\theta)(1 - \sin\theta) - 2(\theta + \cos\theta)(1 + \cos\theta)\, d\theta =$$

e si conclude

SUPERFICI DIFFERENZIABILI
(PARAMETRICHE)

Def



$S: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad S \in C^1(D)$

S regolare

$$\left[\begin{array}{ll} i) \ \cancel{\underline{S}_u = \dfrac{\partial S}{\partial u} \neq \underline{0}} & \\ ii) \ \cancel{\underline{S}_v = \dfrac{\partial S}{\partial v} \neq \underline{0}} & \forall (u,v) \in D \end{array}\right]$$

In linea con la notazione dei corsi assumiamo solo questa condizione ai fini della regolarità →

iii) $\underline{N} = \dfrac{\underline{S}_u \wedge \underline{S}_v}{|\underline{S}_u \wedge \underline{S}_v|} \neq \underline{0}$

vettore normale

Piano tangente in P $= \langle \underline{S}_u(P), \underline{S}_v(P) \rangle$

I forma fondamentale: $\quad G = \begin{bmatrix} \underline{S}_u \cdot \underline{S}_u & \underline{S}_u \cdot \underline{S}_v \\ \underline{S}_u \cdot \underline{S}_v & \underline{S}_v \cdot \underline{S}_v \end{bmatrix}$

$dS = \sqrt{\det G}\, du\, dv$

elemento di superficie

Def flusso di $\underline{E}$ attraverso S

$$\int_S \underline{E} \cdot \underline{N}\, dS$$

02/02/22

$\mathrm{n.}\ 7$

$$\underline{F} = \begin{pmatrix} x+y+z \\ x-y+z \\ xy \end{pmatrix} \qquad \underline{S} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$$

$u, v \in [0, 1]$

$$\underline{S}_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{S}_v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{N} = \underline{S}_u \wedge \underline{S}_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad dS = \sqrt{\det G}\ du\,dv = du\,dv$$

$$\int_S \underline{F} \cdot \underline{N}\, dS = \int_0^1\!\!\int_0^1 uv\ du\,dv = \int_0^1 u\,du \int_0^1 v\,dv =$$

$$= \left[\frac{u^2}{2}\right]_0^1 \left[\frac{v^2}{2}\right]_0^1 = 1/4$$

13/01/22

ns.

$$\Sigma : \quad \underline{S}(u,v) = \begin{pmatrix} u \\ u\sin v \\ u\cos v \end{pmatrix} \qquad u \in [0,1] \quad v \in [0,\pi]$$

$$\underline{S}_u = \begin{pmatrix} 1 \\ \sin v \\ \cos v \end{pmatrix} \qquad \underline{S}_v = \begin{pmatrix} 0 \\ u\cos v \\ -u\sin v \end{pmatrix}$$

$$G = \begin{bmatrix} 2 & 0 \\ 0 & u^2 \end{bmatrix} \qquad dS = \sqrt{\det G}\, du\, dv = \sqrt{2u^2}\, du\, dv = $$

$$= u\sqrt{2}\, du\, dv$$

$$A(\Sigma) = \int_0^1 \int_0^{\pi} u\sqrt{2}\, du\, dv = \int_0^1 u\sqrt{2}\, du \int_0^{\pi} dv =$$

$$= \pi \left[\frac{u^2}{2}\sqrt{2} \right]_0^1 = \frac{\pi\sqrt{2}}{2}$$

09/11/21
es 7

$$I=\int_{\gamma}\sqrt{x^2+y^2+4z}\;ds \qquad \gamma(t)=\begin{pmatrix} t\cos t-\sin t\\ t\sin t+\cos t\\ t+t^2\end{pmatrix} \qquad t\in[0,1]$$

$$\gamma'(t)=\begin{pmatrix} \cancel{\cos t}-t\sin t-\cancel{\cos t}\\ \cancel{\sin t}+t\cos t-\sin t\\ 1+2t\end{pmatrix}=\begin{pmatrix} -t\sin t\\ t\cos t\\ 1+2t\end{pmatrix}$$

$$I=\int_0^1\sqrt{t^2\cos^2 t+\sin^2 t\cancel{-2t\sin t\cos t}+t^2\sin^2 t+\cos^2 t\cancel{+2t\sin t\cos t}+4(t+t^2)}\;|\gamma'(t)|\,dt.$$

corretto

$$I=\int_0^1\sqrt{5t^2+4t+1}\;\sqrt{t^2\sin^2 t+t^2\cos^2 t+1+4t+4t^2}\;dt=$$

$$=\int_0^1\sqrt{5t^2+4t+1}\;\sqrt{5t^2+4t+1}\;dt=\int_0^1(5t^2+4t+1)\,dt=$$

$$=\frac{5}{3}+2+1=\frac{14}{3}$$