FUNZIONI IN PIU' VARIABILI

$f : \mathrm{Dom} f \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\underline{x}_0 = (x_0, y_0) \in \mathrm{Dom} f$

Def i) f è continua in $\underline{x}_0$ se $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$

es

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \underline{x} \neq (0,0) \\ 0 & \underline{x} = (0,0) \end{cases}$$

$$\lim_{\underline{x} \to \underline{0}} f(\underline{x}) \sim \frac{\cancel{\rho^2} \cos\theta \sin\theta}{\cancel{\rho^2}} \underset{\rho \to 0}{\sim} \cos\theta \sin\theta \rightarrow \nexists$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$\theta = \pi/4$

$x = y$: $\quad f(x,x) = \dfrac{x^2}{2x^2} \xrightarrow[x \to 0]{} 1/2$ $\qquad \nexists \lim_{\underline{x} \to \underline{0}} f(\underline{x})$

$x = 0$: $\quad f(0,y) = 0 \xrightarrow[y \to 0]{} 0$

$\theta = \pi/2$

f non cont. in $\underline{x} = \underline{0}$

ii) <u>Derivabilità</u> :

f è derivabile in $\underline{x_0}$ se $\exists$ $\underline{\nabla} f(\underline{x_0})$

iii) <u>Differenziabilità</u>

f è differenziabile in $\underline{x_0}$ se $\exists$ $T_{x_0} f$ (piano tangente)

$$\lim_{\underline{x} \to \underline{x_0}} \frac{\overbrace{f(\underline{x}) - f(\underline{x_0}) - \underline{\nabla} f(\underline{x_0}) \cdot (\underline{x} - \underline{x_0})}^{T_{x_0} f}}{|\underline{x} - \underline{x_0}|} = 0$$

$$T_{x_0} f : z - f(x_0) = \underline{\nabla} f(\underline{x_0}) \cdot (\underline{x} - \underline{x_0})$$

iv) $C^1(U(\underline{x_0}))$

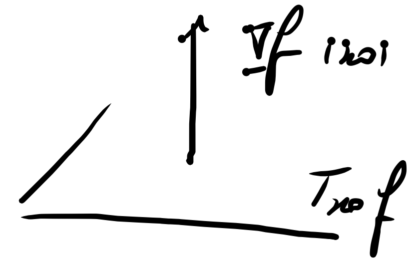
$f \in C^1(U(x_0))$ se $\exists$ e sono continue in $\underline{x} = \underline{x_0}$ le derivate parziali $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

teor. del differenziale totale

Prop $f \in C^1(U(\underline{x_0})) \Rightarrow f$ diff. in $\underline{x_0} \Rightarrow f$ derivabile in $\underline{x_0} \Rightarrow$

$\Rightarrow f \in C^0(U(x_0))$

continua in $\underline{x_0}$



Prop f differenziabile in $\underline{x_0}$

$|\underline{v}| = 1$ $\qquad \underline{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

Allora: $\dfrac{\partial f}{\partial \underline{v}}(\underline{x_0}) = D_{\underline{v}} f(\underline{x_0}) = \underline{v} \cdot \underline{\nabla} f(x_0) =$

$= v_x \dfrac{\partial f}{\partial x}(x_0) + v_y \dfrac{\partial f}{\partial y}(x_0)$

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$$f(\underline{x}) = \begin{cases} xy\, e^{2xy/x^2+y^2} & \underline{x} \neq \underline{0} \\ 0 & \underline{x} = \underline{0} \end{cases}$$

a) $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) \underset{\substack{\text{coord.} \\ \text{polari}}}{\sim} \rho^2 \cos\vartheta \sin\vartheta \; e^{\frac{2\cancel{\rho^2}\cos\vartheta\sin\vartheta}{\cancel{\rho^2}}} \xrightarrow[\rho\to 0]{} 0$

f cont. in $\underline{x} = \underline{0}$

b) $\frac{\partial f}{\partial x}(\underline{0}) = \lim_{h\to 0} \frac{f(h,0) - \overset{\overset{0}{\shortparallel}}{f(\underline{0})}}{h} = 0$ $\qquad$ f derivabile in $\underline{x} = \underline{0}$

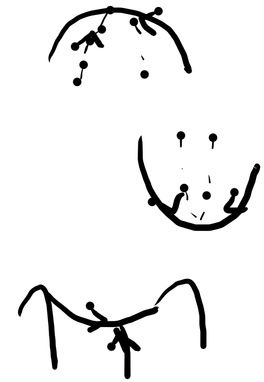
$\frac{\partial f}{\partial y}(\underline{0}) = \lim_{h\to 0} \frac{f(0,h) - \overset{\overset{0}{\shortparallel}}{f(\underline{0})}}{h} = 0$

c) $\lim_{\underline{x}\to\underline{0}} \frac{f(\underline{x}) - \overset{\overset{0}{\shortparallel}}{f(\underline{0})} - \overset{\overset{0}{\shortparallel}}{\nabla f(\underline{0})} \cdot \binom{x}{y}}{\sqrt{x^2+y^2}} \underset{\text{coord. polari}}{\sim}$

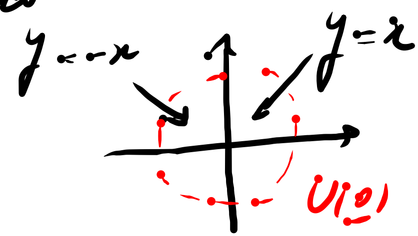
$\sim \frac{\cancel{\rho^2}\, e^{\frac{\cancel{\rho^2}\cos\vartheta\sin\vartheta}{\cancel{\rho^2}}}}{\cancel{\rho}} \xrightarrow[\rho\to 0]{} 0$ $\qquad$ f è differenziabile in $\underline{x} = \underline{0}$

d) $\underline{x}_0$ è stazionario sse $\nabla f(\underline{x}_0) = \underline{0}$

$$\underline{x}_0 \text{ stazionario} \begin{cases} \underline{x}_0 \text{ p. di max } f \\ \underline{x}_0 \text{ p. di min } f \\ \underline{x}_0 \text{ p. di sella} \end{cases}$$



$\nabla f(\underline{0}) = 0$ $\qquad$ $\underline{x} = \underline{0}$ stazionario



$\underline{x} \neq \underline{0}$: $\quad f(x,y) = xy\, e^{2xy/(x^2+y^2)}$

In $U(\underline{0})$:

$$\left.\begin{aligned} f(x,x) &= x^2 e^{2x^2/2x^2} > 0 \\ f(x,-x) &= -x^2 e^{-2x^2/2x^2} \leq 0 \\ f(0,0) &= 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ $(0,0)$ sella per f

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$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+xy+y^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a) $\lim_{x\to 0} f(x) \sim \dfrac{\rho^2 \cos\theta \sin\theta}{\sqrt{\rho^2(1+\cos\theta\sin\theta)}} \sim \dfrac{\rho^{\cancel{2}} \cos\theta \sin\theta}{\cancel{\rho}\sqrt{1+\cos\theta\sin\theta}} \to 0$

f continua in $x=0$

b) $\dfrac{\partial f}{\partial x}(0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0$ $\quad f$ è derivabile in $x=0$

$\dfrac{\partial f}{\partial y}(0) = \lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = 0.$

c) $\nabla f(0) = 0$ $\quad x=0$ p. stazionario

d)
$$\left.\begin{aligned} & f(0,0) = 0 \\ & f(x,x) = \frac{x^2}{\sqrt{3x^2}} \geq 0 \quad \text{in } U(0) \\ & f(x,-x) = \frac{-x^2}{\sqrt{x^2}} \leq 0 \quad \text{in } U(0) \end{aligned}\right\} \Rightarrow x=0 \text{ sella}$$

e) $\lim_{x\to 0} \dfrac{f(x,y) - \overset{0}{\overset{\shortparallel}{f(0,0)}} - \overset{0}{\overset{\shortparallel}{\nabla f(0)}} \binom{x}{y}}{\sqrt{x^2+y^2}} \sim$

$\sim_{\rho\to 0} \dfrac{\cancel{\rho} \cos\theta \sin\theta}{\sqrt{1+\cos\theta\sin\theta}} \cdot \dfrac{1}{\cancel{\rho}}$ $\quad$ non esiste il limite al variare di θ

$g(x,y) = \dfrac{f(x,y)}{\sqrt{x^2+y^2}}$

$g(x,x) = \dfrac{x^2}{x\sqrt{3x^2}} \sim \dfrac{x^{\cancel{2}}}{\cancel{x}|x|\sqrt{3}}$ $\quad \nexists$ limite per $x \to 0$

f NON è differenziabile in $x=0$

f) f non diff in $x=0 \Rightarrow f$ non $C^1(U(0))$

Def $\quad f \in C^\infty(U(\underline{x_0})) \qquad f: \mathrm{Dom} f \subset \mathbb{R}^2 \to \mathbb{R}$

Serie di Taylor $\quad \sum_{|\alpha|=0}^{n} \frac{1}{|\alpha|!} \frac{\partial^\alpha f}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (x-x_0)^{\alpha_1} (y-y_0)^{\alpha_2}$

$n=2$: $\quad P(x,y) = f(\underline{x_0}) + \nabla f(\underline{x_0}) \cdot (\underline{x}-\underline{x_0}) +$

$+ \frac{1}{2!} \Big[\frac{\partial^2 f}{\partial x^2}(\underline{x_0})(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0)(x-x_0)(y-y_0) +$

$+ \frac{\partial^2 f}{\partial y^2}(\underline{x_0})(y-y_0)^2 \Big]$

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$f(x,y) = e^x \ln(1-y) + \sin(3xy) \qquad$ in $\underline{x_0} = (0,0)$

$f(0,0) = 0$

$\frac{\partial f}{\partial x}(\underline{0}) = \Big[e^x \ln(1-y) + \cos(3xy)\, 3y \Big]_{|\underline{x}=\underline{0}} = 0$

$\frac{\partial f}{\partial y}(\underline{0}) = \Big[e^x \left(\frac{-1}{1-y}\right) + \cos(3xy)\, 3x \Big]_{|\underline{x}=\underline{0}} = -1$

$\frac{\partial^2 f}{\partial x^2}(\underline{0}) = \Big[\ln(1-y)\, e^x - 9y^2 \sin(3xy) \Big]_{|\underline{x}=\underline{0}} = 0$

$\frac{\partial^2 f}{\partial x \partial y}(\underline{0}) = \Big[\frac{-e^x}{1-y} - 9xy \sin(3xy) + 3\cos(3xy) \Big]_{|\underline{x}=\underline{0}} =$

$= -1 + 3 = 2$

$\frac{\partial^2 f}{\partial y^2}(\underline{0}) = \Big[-\frac{e^x}{(1-y)^2} - 9x^2 \sin(3xy) \Big]_{|\underline{x}=\underline{0}} = -1$

$p(x,y) = -y + \frac{1}{2}\Big[2xy - y^2 \Big] \qquad$ polinomio di Taylor centrato in $\underline{x_0} = \underline{0}$

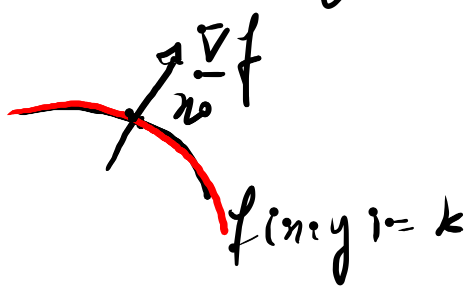
Oss $z = f(x,y)$ $f : \mathrm{Dom} f \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x,y) = k$ curve di livello

$\nabla f \perp$ curve di livello $\Rightarrow$

$\Rightarrow$ direzione di max crescita $\underline{\nabla} f$

direzione di min crescita $-\underline{\nabla} f$



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$f(x,y) = \cos\left(\frac{x}{y}\right)$ $\qquad T(\pi, 4)$

$\mathrm{Dom} f = \{ (x,y) \in \mathbb{R}^2 \mid y \neq 0 \}$

Differenziabilità in $(\pi, 4)$: ($f \in C^1(\mathrm{Dom} f) \Rightarrow f$ diff.)

$$\lim_{(x,y)\to(\pi,4)} \frac{\cos\left(\frac{x}{y}\right) - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{8}(x-\pi) - \frac{\sqrt{2}}{16}\pi(y-4)}{\sqrt{(x-\pi)^2+(y-4)^2}} = 0$$

$$\frac{\partial f}{\partial x} = -\sin\left(\frac{x}{y}\right)\frac{1}{y} \qquad \nabla f(T) = \begin{pmatrix} -\frac{\sqrt{2}}{2}\cdot\frac{1}{4} \\ \frac{\sqrt{2}}{2}\cdot\frac{\pi}{8} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/8 \\ \frac{\sqrt{2}\pi}{16} \end{pmatrix}$$

$$\frac{\partial f}{\partial y} = +\sin\left(\frac{x}{y}\right)\frac{x}{y^2}$$

$T_T f: \quad z - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{8}(x-\pi) - \frac{\sqrt{2}}{16}\pi(y-4) = 0$ piano tangente